17. Action-Angle Coordinates

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Abstract

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An elegant way of using Hamiltonian mechanics to solve a dynamical problem is to search for a canonical transformation to action-angle coordinates,

\[(q_1, \ldots, q_n; p_1, \ldots, p_n) \rightarrow (\theta_1, \ldots, \theta_n; J_1, \ldots, J_n),\]

such that the Hamiltonian turns into a function of the actions alone:

\[H(q_1, \ldots, q_n; p_1, \ldots, p_n) \rightarrow K(J_1, \ldots, J_n).\]

If such a transformation exists and can be found then the solution of the canonical equations is simple:

\[\dot{J}_i = -\frac{\partial K}{\partial \theta_i} = 0 \quad \Rightarrow \quad J_i = \text{const}.\]

\[\dot{\theta}_i = \frac{\partial K}{\partial J_i} = \omega_i(J_1, \ldots, J_n) = \text{const.} \quad \Rightarrow \quad \theta_i(t) = \omega_i t + \theta_i^{(0)}.\]

The inverse canonical transformation then yields \(q_i(t)\) and \(p_i(t)\).

**Two or more degrees of freedom:**
The existence of a transformation to action-angle coordinates is exceptional. Such systems are named integrable. Nonintegrable systems exhibit symptoms of Hamiltonian chaos (to be discussed later).

**One degree of freedom:**
Integrability is guaranteed. There exists a general prescription for finding the canonical transformation to action-angle coordinates.

The prescription for two modes of bounded motion is discussed in detail:

- libration (oscillation) [mln93],
- rotation [mln94].

The two modes are realized, for example, in the plane pendulum. The rotational motion can also be interpreted as unbounded motion in a periodic potential.
Actions and Angles for Librations [mln93]

Hamiltonian: \( H(q, p) = \frac{p^2}{2m} + V(q) = E = \text{const.} \)

Canonical momentum: \( p(q, E) = \pm \sqrt{2m [E - V(q)]} \).

Action \( J \) and Hamiltonian \( K(J) \) from area \( A \) inside trajectory:

\[
A = \oint dq \, p(q, E) = 2 \int_{q_1}^{q_2} dq \sqrt{2m [E - V(q)]} = \int_0^{2\pi} d\theta \, J = 2\pi J.
\]

\[\Rightarrow J(E) = \frac{1}{\pi} \int_{q_1}^{q_2} dq \sqrt{2m [E - V(q)]} \Rightarrow E = K(J) = H(p, q).\]

Angle variable \( \theta(q, J) \) from area \( dA \) between nearby trajectories:

\[
dA = \int_0^q dq \, [p(q, J + dJ) - p(q, J)] = dJ \int_0^q dq \, \frac{\partial}{\partial J} p(q, J) = dJ \theta(q, J).
\]

\[\Rightarrow \theta(q, J) = \frac{\partial}{\partial J} \int_0^q dq \, p(q, J) = \frac{\partial}{\partial J} \int_0^q dq \, \sqrt{2m [K(J) - V(q)]}.
\]

Time evolution: \( J = \text{const.}, \theta(t) = \omega(J)t + \theta_0, \omega(J) = \frac{dK}{dJ}. \)

\[\Rightarrow q(\theta, J) = q(t) \Rightarrow p(q, J) = p(t).\]

Generating function of the canonical transformation \((q, p) \rightarrow (\theta, J)\):

\[F_2(q, J) = \int_0^q dq \, p(q, J).\]
Actions and Angles for Rotations

Hamiltonian: \( H(q, p) = \frac{p^2}{2m} + V(q) = E = \text{const.} \) with \( V(q + Q_0) = V(q) \).

Canonical momentum: \( p(q, E) = \sqrt{2m[E - V(q)]} \).

Action \( J \) and Hamiltonian \( K(J) \) from area \( A \) under trajectory:
\[
A = \int_0^{Q_0} dq p(q, E) = \int_0^{Q_0} dq \sqrt{2m[E - V(q)]} = \int_0^{2\pi} d\theta J = 2\pi J.
\]
\[
\Rightarrow J(E) = \frac{1}{2\pi} \int_0^{Q_0} dq \sqrt{2m[E - V(q)]} \Rightarrow E = K(J) = H(p, q).
\]

Angle variable \( \theta(q, J) \) from area \( dA \) between nearby trajectories:
\[
dA = \int_0^q dq [p(q, J + dJ) - p(q, J)] = dJ \int_0^q dq \frac{\partial}{\partial J} p(q, J) = dJ \theta(q, J).
\]
\[
\Rightarrow \theta(q, J) = \frac{\partial}{\partial J} \int_0^q dq p(q, J) = \frac{\partial}{\partial J} \int_0^q dq \sqrt{2m[K(J) - V(q)]}.
\]

Time evolution: \( J = \text{const.} \), \( \theta(t) = \omega(J)t + \theta_0 \), \( \omega(J) = \frac{dK}{dJ} \).
\[
\Rightarrow q(\theta, J) = q(t) \Rightarrow p(q, J) = p(t).
\]

In the case of rotations there is no natural boundary for \( J \). Here \( J \) is only determined up to a constant.
Determine the canonical transformation \((q, p) \to (\theta, J)\) which produces the action-angle coordinates for the harmonic oscillator:

\[
H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 \quad \to \quad K(J).
\]

(a) Find the transformed Hamiltonian \(K(J)\). (b) Find the transformation relations \(q(\theta, J), p(\theta, J)\). (c) Reconstruct the generating function \(F_2(q, J)\). (d) Determine from \(F_2\) the generating function \(F_1(q, \theta)\) and verify that it is equal to function \(F_1(q, Q)\) used in [mex86].

Solution:
[mex92] **Action-angle coordinates of an anharmonic oscillator**

Determine the canonical transformation \((q, p) \rightarrow (\theta, J)\) which produces the action-angle coordinates for the anharmonic oscillator:

\[
H(q, p) = \frac{p^2}{2m} + U \tan^2(\alpha q) \rightarrow K(J).
\]

(a) Find the transformed Hamiltonian \(K(J)\) and determine the angular frequency \(\omega(J)\) which determines the linear time evolution \(\theta(t) = \omega(J)t + \theta_0\) of the angle coordinates. (b) Find the transformation relations \(q(\theta, J), p(\theta, J)\), which amount to a solution of the dynamical problem.

**Solution:**
[mex96] Unbounded motion in piecewise constant periodic potential

Consider a particle of mass $m$ moving in the potential $V(q) = 0$ for $0 < |q| < D/2$ and $V(q) = U$ for $D/2 < |q| < D$ with periodicity $V(q + 2D) = V(q)$. For energies $E > U$ the motion is unbounded and can be reinterpreted as a rotational mode of bounded motion. Solve this dynamical problem via transformation $(q,p) \rightarrow (\theta,J)$ to action-angle coordinates for motion with initial conditions $q(0) = 0$, $p(0) > 0$: (a) Find the function $J(E)$, which expresses the action as a function of the energy. (b) Find the period $T \equiv 2\pi/\omega(E)$ of the rotational motion. (c) Find the function $\theta(q,E)$ for $0 < q < 2D$. (c) Plot in one diagram the functions $J = \text{const}$ and $p(t)$ for $0 < t < T$. (d) Plot in a second diagram the functions $q(t)$ and $\theta(t)$ for $0 < t < T$.

Solution:
Consider a particle of mass \( m \) moving in a periodic potential \( V(q) = (U/\pi)|q| \) for \(-\pi \leq q \leq \pi\) and \( V(q + 2\pi) = V(q) \). For energies \( E > U > 0 \), the motion is unbounded and can be reinterpreted as a rotational mode of bounded motion. Solve this dynamical problem via transformation \((q,p) \to (\theta,J)\) to action-angle coordinates by establishing the following relations:

\[
p(q,E) = \sqrt{2m \left[ E - (U/\pi)|q| \right]}, \quad J(E) = \frac{2\sqrt{2m}}{3U} \left[ E^{3/2} - (E - U)^{3/2} \right],
\]

\[
\omega(E) = \frac{1}{\sqrt{2m}} \left[ \sqrt{E} + \sqrt{E - U} \right], \quad \theta(q,E) = \pm \pi \frac{\sqrt{E} - \sqrt{E - (U/\pi)|q|}}{\sqrt{E} - \sqrt{E - U}}, \quad 0 \leq \pm q \leq \pi.
\]

Solution:
Bounded motion in piecewise constant periodic potential

Consider a particle of mass $m$ moving in the potential $V(q) = 0$ for $0 < |q| < D/2$ and $V(q) = U$ for $D/2 < |q| < D$ with periodicity $V(q + 2D) = V(q)$. For energies $E < U$ the motion is bounded. Solve this dynamical problem via transformation $(q,p) \rightarrow (\theta,J)$ to action-angle coordinates for motion with initial conditions $q(0) = 0$, $p(0) > 0$. (a) Find the function $K(J)$, which expresses the Hamiltonian as a function of the action coordinate. (b) Find the period $T \equiv 2\pi/\omega(J)$ of the librational motion. (c) Find the function $q(\theta,J)$ for $0 < \theta < 2\pi$. (d) Plot in one diagram the functions $J = \text{const}$ and $p(t)$ for $0 < t < T$. (e) Plot in a second diagram the functions $q(t)$ and $\theta(t)$ for $0 < t < T$.

Solution:
Poisson Brackets

Definition: \( \{f, g\} = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} \right) \),

where \( f(q_1, \ldots, q_n, p_1, \ldots, p_n) \) and \( g(q_1, \ldots, q_n, p_1, \ldots, p_n) \) are arbitrary dynamical variable expressed as functions of canonical coordinates.

Algebraic properties:

- \( \{f, g\} = -\{g, f\} \)
- \( \{f, c\} = 0 \) if \( c = \text{const.} \)
- \( \{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\} \)
- \( \{f_1 f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\} \)
- \( \frac{\partial}{\partial t} \{f, g\} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\} \)
- \( \{q_j, f\} = \frac{\partial f}{\partial p_j}, \quad \{p_j, f\} = -\frac{\partial f}{\partial q_j} \)

Fundamental Poisson brackets: \( \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij} \)

Invariance under canonical transformations:

\( Q_j = Q_j(q_1, \ldots, q_n, p_1, \ldots, p_n), \quad P_j = P_j(q_1, \ldots, q_n, p_1, \ldots, p_n) \)

\( \Rightarrow \{Q_i, Q_j\}_{q,p} = 0, \quad \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij} \)

Canonical equations: \( \dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\} \).

Jacobi's identity: \( \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \)

Poisson's theorem: \( \frac{d}{dt} \{f, g\} = \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\} \) [mex191]

Implication: If \( f \) and \( g \) are integrals of the motion, then \( \{f, g\} \) is also an integral of the motion.
Specifications of Hamiltonian System

Canonical variables:

- Canonical coordinates: \( q_1, \ldots, q_n; p_1, \ldots, p_n \).
- Hamiltonian: \( H(q_1, \ldots, q_n; p_1, \ldots, p_n) \).
- Canonical equations: \( \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \).
- Dynamical variable: \( f(q_1, \ldots, q_n; p_1, \ldots, p_n) = f(t) \).

Noncanonical variables:

- Elementary dynamical variables: \( u_1, \ldots, u_m \).
- Energy function: \( \bar{H}(u_1, \ldots, u_m) \).
- Symplectic structure: \( \{u_i, u_j\} = B_{ij}(u_1, \ldots, u_m) \).
- Hamilton’s equations: \( \dot{u}_i = \{u_i, \bar{H}\} \).
- Dynamical variable: \( f(u_1, \ldots, u_m) = f(t) \).

Link to quantum mechanics:

- Elementary operators: \( u_1, \ldots, u_m \).
- Hamiltonian operator: \( \bar{H}(u_1, \ldots, u_m) \).
- Commutation relations: \( [u_i, u_j] = A_{ij}(u_1, \ldots, u_m) \).
- Dynamical variable: \( f(u_1, \ldots, u_m) = f(t) \).
- Heisenberg equation: \( \dot{f} = \frac{1}{i\hbar} \{f, \bar{H}\} \).
Poisson’s theorem

Prove Poisson’s theorem for two dynamical variables $f$ and $g$:

$$\frac{d}{dt} \{ f, g \} = \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\}.$$ 

Solution:
[mex192] Poisson brackets of angular momentum variables

Given the fundamental Poisson brackets $\{x_i, x_j\} = 0$, $\{p_i, p_j\} = 0$, $\{x_i, p_j\} = \delta_{ij}$, for the Cartesian position and momentum coordinates, determine the Poisson brackets $\{L_i, x_j\}$, $\{L_i, p_j\}$, $\{L_i, L_j\}$ for the angular momentum variables $L_i = \sum_{m,n=1}^{3} \epsilon_{imn} x_mp_n$, $i = 1, 2, 3$.

Solution:
[mex200] Action-angle coordinates of plane pendulum: librations

Determine the canonical transformation $(\phi, p) \rightarrow (\theta, J)$ which produces the action-angle coordinates for the librational motion of the plane pendulum:

$$H(\phi, p) = \frac{p^2}{2m} + G(1 - \cos \phi), \quad M \doteq mt^2, \quad G \doteq mg \ell. $$

(a) Find the action $J(E)$, the angular frequency $\omega(E)$, and the angle coordinate $\theta(\phi, J)$. (b) Use this result to determine $\phi(t)$ in closed form.

Solution:
A classical dynamical system is specified by the following Hamilton’s equations of motion for three noncanonical variables $A, B, C$:

$$\frac{d}{dt} A = -2BC, \quad \frac{d}{dt} B = -2AC, \quad \frac{d}{dt} C = 4AB.$$ 

The three variables satisfy the mutual Poisson brackets $\{A, B\} = C, \{B, C\} = A, \{C, A\} = B$.

(a) Determine the energy function $\bar{H}(A, B, C)$ of this system.

(b) Show that the function $I(A, B, C) = \sqrt{A^2 + B^2 + C^2}$ is an integral of the motion.

(c) Show that $q = \arctan(B/A), \; p = C$ are a pair of canonical coordinates.

Solution:
Generating a pure Galilei transformation

Demonstrate the canonicity in phase space of a pure Galilei transformation,

\[ \mathbf{R} = \mathbf{r} + \mathbf{v}t \quad \text{with } \mathbf{v} = \text{const.} \]

Find the generating function \( F_2(\mathbf{r}, \mathbf{P}, t) \).

Solution:
[mex199] Exponential potential

Consider a particle of mass $m = 1/2$ moving in a straight line (x-axis) and subject to a force $F(x) = -e^x$. Find the solution $x(t)$, $p(t)$ as follows:

(a) Find a generating function $F_2(x, P)$ which transforms the Hamiltonian $H(x, p) = p^2 + e^x$ into $K(Q, P) = \frac{1}{4}P^2$ and derive canonical transformation relations $Q(x, p)$ and $P(x, p)$ from $F_2(x, P)$.

(b) Solve the canonical equations for $K(Q, P)$ to get $Q(t)$ and $P(t)$ and substitute these solutions into the inverse transformation relations $x(Q, P) = x(t)$ and $p(Q, P) = p(t)$.

(c) State the solutions $x(t), p(t)$ for initial conditions $x(0) = p(0) = 0$. Verify that $x(t)$ and $p(t)$ thus found are indeed solutions of the canonical equations for $H(x, p)$.

Solution: