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08. Central Force Motion I

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Abstract

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Central Force Motion: Two-Body Problem

Mechanical system with six degrees of freedom:
Consider two masses $m_1, m_2$ interacting via a central force.
Central-force potential: $V(r_1, r_2) \equiv V(|r_1 - r_2|)$.
Lagrangian of two-body problem: $L = \frac{1}{2}m_1 \dot{r}_1^2 + \frac{1}{2}m_2 \dot{r}_2^2 - V(|r_1 - r_2|)$.
Conservation laws inferred from translational and rotational symmetries:
- Energy: $E = \frac{1}{2}m_1 \dot{r}_1^2 + \frac{1}{2}m_2 \dot{r}_2^2 + V(|r_1 - r_2|)$.
- Linear momentum: $P = p_1 + p_2 = m_1 \dot{r}_1 + m_2 \dot{r}_2$.
- Angular momentum: $L = r_1 \times p_1 + r_2 \times p_2$.

Reduction to three degrees of freedom:
Center-of-mass position vector: $R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$.
Distance vector: $r = r_2 - r_1$.
Total mass: $M = m_1 + m_2$.
Reduced mass: $m = \frac{m_1 m_2}{m_1 + m_2}$.
Lagrangian (after point transformation):
\[ L = L_M(\dot{R}) + L_m(r, \dot{r}) = \frac{1}{2}M \dot{R}^2 + \frac{1}{2}m \dot{r}^2 - V(|r|). \]
Center-of-mass motion: $L_M(\dot{R}) = \frac{1}{2}M \dot{R}^2$.
- $R_x, R_y, R_z$ are cyclic coordinates.
- Conserved center-of-mass momentum: $P = M \dot{R} = \text{const}$.
- Uniform rectilinear center-of-mass motion: $R(t) = R_0 + \frac{P}{M} t$.
Effective one-body problem: $L_m(r, \dot{r}) = \frac{1}{2}m \dot{r}^2 - V(|r|)$.
- Three degrees of freedom.
- Particle of mass $m$ moving in a stationary central potential $V(|r|)$. 
Central Force Motion: One-Body Problem

Reduction to one degree of freedom:
Consider a particle of mass \( m \) moving in a central potential:

Lagrangian: \( L(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 - V(|r|) \).

Conservation of angular momentum: \( \mathbf{L} = r \times m \dot{\mathbf{r}} = \text{const.} \)

- Case \( \mathbf{L} = 0 \): One degree of freedom.
  - Purely radial motion: \( r \parallel \dot{r} \Rightarrow L(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 - V(r) \).
  - Energy conservation: \( E(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 + V(r) \).
  - Reduction to quadrature (see [mln4]).

- Case \( \mathbf{L} \neq 0 \): Two separable degrees of freedom.
  - Motion in plane perpendicular to \( \mathbf{L} \).
  - Transformation to polar coordinates: \( x = r \cos \vartheta, \ y = r \sin \vartheta \).
  - Lagrangian: \( L(r, \dot{r}, \dot{\vartheta}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\vartheta}^2) - V(r) \).
  - Cyclic coordinate: \( \dot{\vartheta} \).
  - Conserved angular momentum: \( \ell = \frac{\partial L}{\partial \dot{\vartheta}} = m r^2 \dot{\vartheta} = \text{const.} \)
  - Routhian: \( R(r, \dot{r}; \ell) = L - \ell \dot{\vartheta} = \frac{1}{2} m \dot{r}^2 - \frac{\ell^2}{2 m r^2} - V(r) \).
  - Effective potential for radial motion: \( \tilde{V}(r; \ell) = V(r) + \frac{\ell^2}{2 m r^2} \).
  - Conserved energy: \( E(r, \dot{r}; \ell) = \frac{1}{2} m \dot{r}^2 + \tilde{V}(r; \ell) \).
  - Reduction to quadrature (see [mln4]).
  - Integral for angular motion: \( \vartheta(t) = \vartheta_0 + \frac{\ell}{m} \int_0^t \frac{dt}{m r^2(t)} \).

Central Force Problem: Formal Solution

Lagrangian: \[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\vartheta}^2 \right) - V(r). \]

Lagrange equations (coupled 2nd order ODEs):
\[ m \ddot{r} = mr \dot{\vartheta}^2 - \frac{\partial V}{\partial r}, \quad \frac{d}{dt} \left( mr^2 \dot{\vartheta} \right) = 0. \]

Integrals of the motion (angular momentum and energy):
\[ [A] \quad \ell = mr^2 \dot{\vartheta} = \text{const}, \quad [B] \quad E = \frac{1}{2} mr^2 + \frac{\ell^2}{2mr^2} + V(r) = \text{const}. \]

Motion in time (solution by quadrature):
\[ [B] \quad \frac{dr}{dt} = \pm \sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]} \quad \Rightarrow \quad t = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]}} \quad \Rightarrow \quad r(t) = \ldots \]
\[ [A] \quad \frac{d\vartheta}{dt} = \frac{\ell}{mr^2} \quad \Rightarrow \quad \vartheta(t) = \frac{\ell}{m} \int_{t_0}^{t} \frac{dt}{r^2(t)} + \vartheta_0. \]

Integration constants: \( E, \ell, r_0, \vartheta_0. \)

Orbital integral: eliminate \( t \) from \( r(t), \vartheta(t) \) to obtain \( r(\vartheta) \) or \( \vartheta(r) \).
\[ \sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]} = \frac{dr}{dt} = \frac{dr}{d\vartheta} \frac{d\vartheta}{dt} = \frac{dr}{d\vartheta} \frac{\ell}{mr^2}. \]
\[ \Rightarrow \quad \int_{r_0}^{r} dr \sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]} = \int_{\vartheta_0}^{\vartheta} d\vartheta = \vartheta - \vartheta_0 \quad \Rightarrow \quad \vartheta(r) = \vartheta_0 + \ldots \]

Orbital integral for power-law potentials \( V(r) = -\frac{\kappa}{r^\alpha} \): set \( u = 1/r \).
\[ \vartheta - \vartheta_0 = - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2m\kappa}{\ell^2} u^\alpha - u^2}}. \]

For the cases \( \alpha = 6, 4, 3, 2, 1, -1, -2, -4, -6 \), the orbit can be expressed in terms of elementary functions.
Orbits of Power-Law Potentials

\[ E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}mr^2 + \tilde{V}(r), \quad \tilde{V}(r) = V(r) + \frac{\ell^2}{2mr^2}, \quad E > \tilde{V}(r) > V(r). \]

\[ E - V(r) = \frac{1}{2}mv^2, \quad E - \tilde{V}(r) = \frac{1}{2}mr^2, \quad \tilde{V}(r) - V(r) = \frac{1}{2}mr^2\dot{\vartheta}^2. \]

Particle speed: \( v \propto \sqrt{E - V} \).
Radial speed: \( |\dot{r}| \propto \sqrt{E - \tilde{V}} \).
Angular speed: \( r|\dot{\vartheta}| \propto \sqrt{\tilde{V} - V}. \)

(i) \( V(r) = -\frac{\kappa}{r^\alpha}, \quad 0 < \alpha < 2 : \)
\( \tilde{V}(r) \) has minimum at \( r_0 = (\ell^2/\alpha\kappa m)^{1/(\alpha - 2)}. \)
\( E = E_1: \) unbounded orbit, turning point (\( \dot{r} = 0 \)) at \( \tilde{V}(r_{\text{min}}) = E_1. \)
\( E = E_3: \) bounded orbit, turning points at \( \tilde{V}(r_{\text{min}}) = \tilde{V}(r_{\text{max}}) = E_3. \)
\( E = E_4: \) circular orbit at \( r_0: \dot{r} = 0, \dot{\vartheta} = \text{const}. \)

(ii) \( V(r) = -\frac{\kappa}{r^\alpha}, \quad \alpha > 2 : \)
\( \tilde{V}(r) \) has maximum at \( r_0 = (\alpha\kappa m/\ell^2)^{1/(\alpha - 2)}. \)
\( E < \tilde{V}(r_0) \) and large \( r: \) unbounded orbit at \( r > r_2, \) where \( \tilde{V}(r_2) = E. \)
\( E < \tilde{V}(r_0) \) and small \( r: \) bounded orbit at \( r < r_1, \) where \( \tilde{V}(r_1) = E. \)
\( E > \tilde{V}(r_0): \) Unbounded orbit with particle spiraling through center.
\( E = \tilde{V}(r_0): \) Unstable circular orbit exists.

(iii) \( V(r) = \kappa' r^{\alpha'}, \quad \kappa' = -\kappa > 0, \quad \alpha' = -\alpha > 0 : \)
\( \tilde{V}(r) \) has minimum at \( r_0 = (\ell^2/\alpha'\kappa' m)^{1/(\alpha' + 2)}. \)
All orbits are bounded: \( r_1 < r < r_2, \) where \( \tilde{V}(r_1) = \tilde{V}(r_2) = E \)
\( E = \tilde{V}(r_0): \) circular orbit exists.
(i) $\alpha = 1$ (gravitation):

(ii) $\alpha = 3$:

(iii) $\alpha' = 2$ (harmonic oscillator):

[Goldstein 1981]
Unstable circular orbit

The central force potential $V(r) = -\kappa/r^4$ has an unstable circular orbit of radius $R$ centered at the center of force. (a) Find the angular momentum $\ell$, the energy $E$, and the period $\tau$ of this circular orbit. (b) Find a second orbit $r(\vartheta)$ for the same values of $E$ and $\ell$ which starts at the center of force and approaches the circular orbit of radius $R$ asymptotically.

Solution:
[mex46] Orbit of the inverse-square potential at large angular momentum

Consider the central force potential $V(r) = -\kappa/r^2$. If $\kappa < \ell^2/2m$, all orbits are unbounded and have energies $E > 0$. (a) Show that the orbits can be expressed in the form

$$\frac{1}{r} = \sqrt{\frac{2mE}{\ell^2 - 2m\kappa}} \cos \left( \sqrt{1 - \frac{2m\kappa}{\ell^2}} \right).$$

(b) Determine the total angle an orbit describes between the incoming and outgoing asymptotes.

Solution:
Consider the central force potential $V(r) = -\kappa/r^2$. If $\kappa > \ell^2/2m$, all orbits at $E > 0$ are unbounded and all orbits at $E < 0$ are bounded. (a) Show that these orbits can be expressed in the form

$$ E > 0 : \frac{1}{r} = \sqrt{\frac{2mE}{2m\kappa - \ell^2}} \sinh \left( \sqrt{\frac{2m\kappa}{\ell^2}} - 1 \right), \quad E < 0 : \frac{1}{r} = \sqrt{\frac{2m|E|}{2m\kappa - \ell^2}} \cosh \left( \sqrt{\frac{2m\kappa}{\ell^2}} - 1 \right). $$

(b) Determine the time it takes the particle to move along the bounded orbit from $r_{max}$ to the center of force ($r = 0$).

Solution:
In search of some hyperbolic orbit

A particle of unit mass \((m = 1)\) moves from infinity along a straight line which, if continued, would allow it to pass a distance \(d = b\sqrt{2}\) from a point \(P\). Instead, the particle is attracted toward \(P\) by the central force \(F(r) = -k/r^5\). If the angular momentum of the particle relative to \(P\) is \(\ell = \sqrt{k}/b\), show that the orbit is \(r(\theta) = b\coth(\theta/\sqrt{2})\).

Solution:
Virial Theorem

Consider a system of interacting particles in bounded motion.
Newton’s equations of motion: \( \dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, \ldots, N. \)
\( \mathbf{F}_i \): sum of external and interaction forces acting on particle \( i \).

Definition: \( G(t) = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i. \)
For bounded motion \( G(t) \) is finite.

Time derivative: \( \frac{dG}{dt} = \sum_i (\mathbf{p}_i \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{p}}_i \cdot \mathbf{r}_i) = \sum_i m_i |\dot{\mathbf{r}}_i|^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i. \)

Kinetic energy: \( T = \sum_i \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2. \)

Time average: \( \frac{dG}{dt} = \frac{1}{\tau} \int_0^\tau dt \frac{dG}{dt} = \frac{1}{\tau} \left[ G(\tau) - G(0) \right] \xrightarrow{\tau \to \infty} 0. \)
\( \Rightarrow 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = 0. \)

Virial: \( \overline{T} = -\frac{1}{2} \sum_i \overline{\mathbf{F}_i \cdot \mathbf{r}_i}. \)

Application to particle in bounded orbit of central-force motion.
Power-law central force potential: \( V(r) = -\frac{k}{r^\alpha}. \)

\[ \overline{T} = -\frac{1}{2} \left( -r \frac{dV}{dr} \right) = -\frac{1}{2} \alpha \overline{V}. \]

- Gravity (\( \alpha = 1 \)): \( \overline{T} = -\frac{1}{2} \overline{V}. \)
- Harmonic oscillator (\( \alpha = -2 \)): \( \overline{T} = \overline{V}. \)
A satellite orbits the Earth in a circular orbit of radius $r_0$, traveling with velocity $v_0$. Then a rocket on the satellite fires such that it acquires an additional velocity $v_1$ of the same magnitude as $v_0$ in a very short time. Give a detailed description of the nature of the subsequent orbit of the satellite for the four cases with different directions of $v_1$ as shown.

Solution:
A particle of mass $m$ moves in a circular orbit of radius $r_0$ in a central force potential $V(r) = -\kappa/r$. Suddenly the value of $\kappa$ decreases to half its original value and the particle changes its orbit as a result of the reduced attractive force. Give a detailed description of the new orbit.

Solution:
Bounded Orbits Open or Closed

Consider an effective potential \( \tilde{V}(r) = V(r) + \ell^2/(2mr^2) \) for the radial part of a central force motion as shown.

The radial coordinate \( r \) oscillates between \( r_P \) (periapsis) and \( r_A \) (apsis).

Between successive instances of \( r = r_P \) and \( r = r_A \) the angular coordinate \( \vartheta \) always advances the same amount \( \Delta \vartheta \).

Apsidal vectors: position vectors \( r \) with \( |r| = r_P \) or \( |r| = r_A \).

Orbits are reflection symmetric at apsidal vectors. Hence the complete orbit can be constructed from one segment between successive apsidal vectors.

Apsidal angle: \( \Delta \vartheta = \int_{r_P}^{r_A} dr \frac{\ell/mr^2}{\sqrt{2m \left[ E - V(r) - \ell^2/(2mr^2) \right]}} \).

Condition for closed orbit: \( \Delta \vartheta/2\pi \) must be a rational number.

Examples of closed bounded orbits:

- \( V(r) = -\frac{\kappa}{r} \quad \Rightarrow \quad \vartheta - \vartheta_0 = \arccos \left( \frac{\ell^2/m\kappa}{\sqrt{1 + 2E\ell^2/m\kappa^2}} \right) \quad \Rightarrow \quad \Delta \vartheta = \pi. \)

- \( V(r) = \frac{1}{2} kr^2 \quad \Rightarrow \quad \vartheta - \vartheta_0 = \frac{1}{2} \arccos \left( \frac{\ell^2/mr^2 - E}{\sqrt{E^2/\ell^2 - k/m}} \right) \quad \Rightarrow \quad \Delta \vartheta = \frac{\pi}{2}. \)

Bertrand’s theorem [mln44] proves that only for these two potentials are all bounded orbits closed.
Bertrand’s Theorem

The only central force potentials $V(r)$ for which all bounded orbits are closed are the following:

- Kepler system: $V(r) = -\frac{\kappa}{r}$ (ellipses with $r = 0$ at one focus)
- Harmonic oscillator: $V(r) = \kappa' r^2$ (ellipses with $r = 0$ at center)

J. Bertrand’s proof of 1873 is based on a 2nd order perturbation calculation about stable circular orbits. The following derivation follows Arnold [1989] and rests on five lemmas:

1. The central force potential $V(r)$ has a circular orbit at $r = R$ if $V'(R) = \frac{\ell^2}{mR^3}$. This circular orbit is stable if $V''(R) + \frac{3}{R} V'(R) > 0$. [mex53] [mex125]
2. For a central force potential $V(r)$ with a circular orbit at $r = R$, the apsidal angle for orbits in the vicinity of this circular orbit is $\Delta \vartheta = \pi \sqrt{V'(R)/[3V'(R) + RV''(R)]}$. [mex126]
3. The only central force potentials for which the apsidal angle of nearly circular orbits is independent of the radius are the power-law potentials $V(r) = -\kappa/r^\alpha, \alpha < 2, \alpha \neq 0$ and the logarithmic potential $V(r) = \kappa \ln r$. The value of the apsidal angle is $\Delta \vartheta = \pi/\sqrt{2 - \alpha}$, where the value $\alpha = 0$ pertains to the logarithmic potential. [mex127]
4. For central force potentials with $\lim_{r \to \infty} V(r) = \infty$, the apsidal angle has the property $\lim_{E \to \infty} \Delta \vartheta = \pi/2$. [mex128] [mex129]
5. For power-law central force potentials $V(r) = -\kappa/r^\alpha, 0 \leq \alpha < 2$, the apsidal angle has the property $\lim_{E \to \infty} \Delta \vartheta = \pi/(2 - \alpha)$. [mex130]

Proof of Bertrand’s theorem:

- Closed orbits require $\Delta \vartheta = 2\pi(m/n)$ for integer $m, n$.
- Lemma 3 restricts the class of potentials with no open bounded orbits to potentials (a) $V(r) = \kappa' r^{-\alpha}, \alpha < 0$, (b) $V(r) = -\kappa/r^\alpha, 0 < \alpha < 2$, (c) $V(r) = \kappa \ln r$ (representing $\alpha = 0$).
- For the cases $\alpha < 0$, lemma 4 requires $\pi/\sqrt{2 - \alpha} = \pi/2$, which rules out all exponents except $\alpha = -2$ (harmonic oscillator). The apsidal angle is $\Delta \vartheta = \pi/2$ for all orbits of this system.
- For the cases $0 \leq \alpha < 2$, lemma 5 requires $\pi/\sqrt{2 - \alpha} = \pi/(2 - \alpha)$, which rules out all exponents except $\alpha = 1$ (Kepler system). The apsidal angle is $\Delta \vartheta = \pi$ for all orbits of this system.
Consider a particle of mass $m$ and angular momentum $\ell$ subject to a central force $F(r) = -V'(r)$.

(a) Show that the condition for the existence of a circular orbit at radius $R$ is $F(R) + \ell^2/mR^3 = 0$.

(b) Show that the stability condition of this circular orbit is $F'(R) + (3/R)F(R) < 0$.

Solution:
[mex125] Small oscillations of radial coordinate about circular orbit

Consider a particle of mass \( m \) and angular momentum \( \ell \) subject to a central force \( F(r) = -V'(r) \). Under the conditions stated in [mex53] that a stable orbit at radius \( r = R \) exists, show that on an orbit starting at radius \( r = R + x \) with \( |x| \ll R \) next to a stable circular orbit of radius \( R \), the radial coordinate oscillates about \( R \) with angular frequency \( \omega_0^2 = -3F(R)/mR - F'(R)/m \).

Solution:
Angle between apsidal vectors for nearly circular orbits

Consider a particle of mass \( m \) and angular momentum \( \ell \) subject to a central force \( F(r) = -V'(r) \) and moving in a stable circular orbit of radius \( r = R \). Show that nearly circular orbits in the immediate vicinity have an apsidal angle

\[
\Delta \theta = \pi \sqrt{\frac{V'(R)}{3V''(R) + RV'''(R)}}.
\]

Solution:
Robustness of apsidal angles

(a) Given the result of [mex126], namely that nearly circular orbits at radius \( r = R \) of a central force potential \( V(r) \) have apsidal angle \( \Delta \vartheta = \pi \sqrt{V''(R) / [3V'(R) + RV''(R)]} \), show that the only cases for which this apsidal angle is independent of the radius are the power-law potentials \( V(r) = -\kappa / r^\alpha, \alpha < 2, \alpha \neq 0 \) and the logarithmic potential \( V(r) = \kappa \ln r \). (b) Show that the value of the apsidal angle is \( \Delta \vartheta = \pi / \sqrt{2 - \alpha} \), where the value \( \alpha = 0 \) pertains to the logarithmic potential.

Solution:
[mex128] Apsidal angle reinterpreted

Consider a particle of mass $m$ in a bounded orbit with energy $E$ and angular momentum $\ell$ of a central force potential $V(r)$. Show that the angle $\Delta \vartheta$ between successive apsidal vectors (between pericenter and apocenter) is related to the period $T$ of the oscillatory motion of a fictitious particle in a 1D potential $W(x)$ as investigated in [mex5]:

$$
\Delta \vartheta \equiv \int_{r_{\text{min}}}^{r_{\text{max}}} dr \frac{\ell}{m r^2} \sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]} = \frac{T}{2\sqrt{m}}, \quad T = 2 \int_{x_{\text{min}}}^{x_{\text{max}}} dx \frac{\sqrt{\frac{2}{m} [E - W(x)]}}{\sqrt{\frac{2}{m} [E - W(x)]}}.
$$

Find the relation between the variables $r$ and $x$ and determine the function $W(x)$.

Solution:
Apsidal angle at very high energies

Use the result of [mex128] to show that for a central force potential with the property \( \lim_{r \to \infty} V(r) = \infty \), the apsidal angle of orbits with given angular momentum approaches a universal value at very high energy:

\[
\lim_{E \to \infty} \Delta \vartheta = \frac{\pi}{2}.
\]

Solution:
Use the result of [mex128] to show that for a power-law central force potential \( V(r) = -\frac{\kappa}{r^\alpha} \), 0 ≤ \( \alpha < 2 \) the apsidal angle of orbits with given angular momentum \( \ell \) approaches an \( \ell \)-independent value at very low energy:

\[
\lim_{E \to -\infty} \Delta \vartheta = \frac{\pi}{2 - \alpha}.
\]

Hint: Consider first the case \( \alpha = 1 \).

Solution: