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Basins of Attraction for Two Species Competitive Model with quadratic terms and the singular Allee Effect

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Abstract

We consider the following system of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_n^2}{B_1 x_n^2 + C_1 y_n^2} \\ y_{n+1} &= \frac{y_n^2}{A_2 + B_2 x_n^2 + C_2 y_n^2}, \quad n = 0, 1, \dots, \end{aligned}$$

where B_1, C_1, A_2, B_2, C_2 are positive constants and $x_0, y_0 \geq 0$ are initial conditions. This system has interesting dynamics and it can have up to seven equilibrium points as well as a singular point at $(0, 0)$, which always possesses a basin of attraction. We characterize the basins of attractions of all equilibrium points as well as the singular point at $(0, 0)$ and thus describe the global dynamics of this system. Since the singular point at $(0, 0)$ always possesses a basin of attraction this system exhibits Allee's effect.

Keywords: Allee effect, basin, competition, difference equation, global asymptotic stability, invariant manifold, stable manifold.
AMS 2000 Mathematics Subject Classification: 39A10, 39A11

1 Introduction

The following difference equation is known as the Beverton-Holt model

$$x_{n+1} = \frac{a\,x_n}{1+x_n}, \quad n = 0, 1, \dots \tag{1}$$

where $a > 0$ is the rate of change (growth or decay) and x_n is the size of the population at the n-th generation.

This model was introduced by Beverton and Holt in 1957. It depicts density dependent recruitment of a population with limited resources which are not shared equally. The model assumes that the *per capita* number of offspring is inversely proportional to a linearly increasing function of the number of adults.

The Beverton-Holt model is well studied and understood and exhibits the following properties:

(a) Equation (1) has two equilibrium points 0 and $a-1$ when $a > 1$.

(b) All solutions of Eq.(1) are monotonic (increasing or decreasing) sequences.

(b) If $a \leq 1$, then the zero equilibrium is a global attractor, that is, $\lim_{n\to\infty} x_n = 0$, for all $x_0 \geq 0$.

(c) If $a > 1$, then the equilibrium point $a-1$ is a global attractor, that is, $\lim_{n\to\infty} x_n = a-1$, for all $x_0 > 0$.

(d) Both equilibrium points are globally asymptotically stable in the corresponding regions of parameters $a \leq 1$ and $a > 1$, that is, they are global attractors with the property that small changes of initial condition x_0 result in small changes of the corresponding solution $\{x_n\}$.

All these properties can be derived from the explicit form of the solution of Eq.(1):

$$\begin{aligned} x_n &= \tfrac{1}{1/(a-1)+(1/x_0-1/(a-1))1/a^n} \quad \text{if} \quad a \neq 1 \\ x_n &= \tfrac{1}{n+1/x_0}, \quad \text{if} \quad a = 1. \end{aligned} \tag{2}$$

See [28, 29, 46].

The following difference equation

$$x_{n+1} = \frac{a\,x_n^2}{1+x_n^2}, \quad n = 0, 1, \dots, \tag{3}$$

which was introduced by Thompson [47] as a depensatory generalization of the Beverton-Holt stock-recruitment relationship used to develop a set of constraints designed to safeguard against overfishing, see [16] for further references. In view of the sigmoid shape of the function $f(u) = \frac{a\,u^2}{1+u^2}$ Equation (3) is called the Sigmoid Beverton-Holt model. A very important feature of the Sigmoid Beverton-Holt model is that, it exhibits the Allee effect, that is zero equilibrium has a substantial basin of attraction, as we can see from the following results:

(a) Equation (3) has a unique zero equilibrium when $a < 2$;

(b) Equation (3) has a zero equilibrium and the positive equilibrium $\bar{x} = 1/2$, when $a = 2$;

(c) There exist a zero equilibrium and two positive equilibria, $\overline{x}_-$ and $\overline{x}_+$, when $a > 2$;

(d) All solutions of Eq. (3) are monotonic (increasing or decreasing) sequences.

(e) If $a < 2$, then the equilibrium point 0 is a global attractor, that is, $\lim_{n\to\infty} x_n = 0$.

(f) If $a = 2$, then the equilibrium point 0 is a global attractor, with the basin of attraction $B(0) = (0, \bar{x})$ and $\bar{x} = 1/2$ is a non-hyperbolic equilibrium point with the basin of attraction $B(\bar{x}) = [\bar{x}, \infty)$;

(g) If $a > 2$, then zero equilibrium and $\bar{x}_+$ are locally asymptotically stable, while $\bar{x}_-$ is repeller and the basins of attraction of the equilibrium points are given as:

$B(0) = \{x_0 : 0 \le x_0 < \overline{x}_-\}$

$B(\overline{x}_+) = \{x_0 : \overline{x}_- < x_0 < \infty\}$.

In other words, the smaller positive equilibrium serves as the boundary between two basins of attraction. The zero equilibrium has the basin of attraction $B(0)$ and the model exhibits the Allee effect;

(h) The equilibrium points 0 and $\overline{x}_+$ are globally asymptotically stable in the corresponding basins of attractions $B(0)$ and $B(\overline{x}_+)$.

The two dimensional analogue of Eq. (1) is the uncoupled system

$$\begin{aligned} x_{n+1} &= \frac{ax_n}{1+x_n} \\ y_{n+1} &= \frac{by_n}{1+y_n}, \quad n = 0, 1, \ldots, \end{aligned} \tag{4}$$

where a, b are positive parameters. The dynamics of System (4) can be derived from dynamics of each equation. Therefore, this system has an explicit solution given by (2).

Two species can interact in several different ways through competition, cooperation or host-parasitoid interactions. For each of these interactions, we obtain variations of System (4) all of which may require different mathematical analysis.

One such variation that exhibits competitive interaction is the following model, known as the Leslie-Gower model, which was considered in Cushing et al. [10]:

$$\begin{aligned} x_{n+1} &= \frac{ax_n}{1+x_n+c_1y_n} \\ y_{n+1} &= \frac{by_n}{1+c_2x_n+y_n}, \quad n = 0, 1, \ldots, \end{aligned} \tag{5}$$

where all parameters are positive and the initial conditions are non-negative. The global dynamics of System (5) was completed in [31]. Several variations of System (5) where the competition of two species was modeled by linear fractional difference equations were considered in [7, 8, 23, 24, 34, 36, 37]. An interesting fact is that none of these models exhibited the Allee effect.

The two dimensional analogue of System (3) is the following uncoupled system

$$\begin{aligned} x_{n+1} &= \frac{ax_n^2}{1+x_n^2} \\ y_{n+1} &= \frac{by_n^2}{1+y_n^2}, \quad n = 0, 1, \ldots, \end{aligned} \tag{6}$$

where a, b are positive parameters. The dynamics of System (6) can be derived from the dynamics of each equation in the system. Since each equation in System (6) has three possible dynamic scenarios, then System (6) possesses nine dynamic scenarios.

A variation of System (6) that exhibits competitive interactions is the system:

$$\begin{aligned} x_{n+1} &= \frac{x_n^2}{B_1x_n^2+C_1y_n^2} \\ y_{n+1} &= \frac{y_n^2}{A_2+B_2x_n^2+C_2y_n^2}, \quad n = 0, 1, \ldots, \end{aligned} \tag{7}$$

where $B_1, C_1, A_2, B_2, C_2 > 0$. This system will be considered in the remainder of this paper. We will show that System (7) has similar but more complex dynamics than System (6). We will see that like System (6) the coupled system (7) may possess $1, 3, 5$, or 7 equilibrium points in the hyperbolic case and $2, 4$, or 6 equilibrium points in the non-hyperbolic case. In each of these cases we will show that the Allee effect is present, although $(0, 0)$ is outside of the domain of definition of System (7). We will precisely describe the basins of attraction of all equilibrium points and the singular point $(0, 0)$. We will show that the boundaries of the basins of attraction of the equilibrium points are the global stable manifolds of the saddle or the non-hyperbolic equilibrium points. See [2, 3, 4, 23, 24, 32, 36, 37] for related results and [25] for dynamics of competitive system with a singular point at the origin. The biological interpretation of a related system is given in [40, 41] and similar system is treated in [6]. The specific feature of our results is that no equilibrium point in the interior of the first quadrant is computable and so our analysis is based on geometric analysis of the equilibrium curves.

2 Preliminaries

Our proofs use some recent general results for competitive systems of difference equations of the form:

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n), \end{array} \right. \tag{8}$$

where f and g are continuous functions and $f(x, y)$ is non-decreasing in x and non-increasing in y and $g(x, y)$ is non-increasing in x and non-decreasing in y in some domain A.

Competitive systems of the form (8) were studied by many authors in [10, 8, 13, 14, 15, 17, 19, 20, 22, 22, 27, 30, 31, 35, 36, 37, 39, 43, 45, 48, 49] and others.

Here we give some basic notions about monotonic maps in the plane.

We define a *partial order* $\preceq_{se}$ on $\mathbf{R}^2$ (so-called South-East ordering) so that the positive cone is the fourth quadrant, i.e. this partial order is defined by:

$$\left(\begin{array}{c} x^1 \\ y^1 \end{array} \right) \preceq_{se} \left(\begin{array}{c} x^2 \\ y^2 \end{array} \right) \Leftrightarrow \left\{ \begin{array}{c} x^1 \leqslant x^2 \\ y^1 \geqslant y^2. \end{array} \right. \tag{9}$$

Similarly, we define North-East ordering as:

$$\left(\begin{array}{c} x^1 \\ y^1 \end{array} \right) \preceq_{ne} \left(\begin{array}{c} x^2 \\ y^2 \end{array} \right) \Leftrightarrow \left\{ \begin{array}{c} x^1 \leqslant x^2 \\ y^1 \leqslant y^2. \end{array} \right. \tag{10}$$

A map F is called *competitive* if it is non-decreasing with respect to $\preceq_{se}$, that is, if the following holds:

$$\left(\begin{array}{c} x^1 \\ y^1 \end{array} \right) \preceq \left(\begin{array}{c} x^2 \\ y^2 \end{array} \right) \Rightarrow F \left(\begin{array}{c} x^1 \\ y^1 \end{array} \right) \preceq F \left(\begin{array}{c} x^2 \\ y^2 \end{array} \right). \tag{11}$$

For each $\mathbf{v} = (v^1, v^2) \in \mathbf{R}^2_+$, define $\mathcal{Q}_i(\mathbf{v})$ for $i = 1, .., 4$ to be the usual four quadrants based at v and numbered in a counterclockwise direction, e.g., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbf{R}^2_+ : v^1 \leq x, v^2 \leq y\}$.

For $S \subset R^2_+$ let S° denote the *interior* of S.

The following definition is from [45].

Definition 1 *Let R be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : R \to R$ is said to satisfy condition (O+) if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition (O−) if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by DeMottoni-Schiaffino [11] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [43].

Theorem 1 *Let R be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in R$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead (O−) holds, then for all $x \in R$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in R, then its omega limit set is either a period-two orbit or a fixed point.*

It is well known that a stable period-two orbit and a stable fixed point may coexist, see Hess [18].

The following result is from [45], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 2 *Let $R \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : R \to R$ be a C' competitive map. If T is injective and* $\det J_T(x) > 0$ *for all $x \in R$ then T satisfies (O+). If T is injective and* $\det J_T(x) < 0$ *for all $x \in R$ then T satisfies (O−).*

Theorems 1 and 2 are quite applicable as we have shown in [5], in the case of competitive systems in the plane consisting of rational equations.

The following result is from [32], which generalizes the corresponding result for hyperbolic case from [31]. Related results have been obtained by H. L. Smith in [43].

Theorem 3 *Let $\mathcal{R}$ be a rectangular subset of $\mathbb{R}^2$ and let T be a competitive map on $\mathcal{R}$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $(\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}$ has nonempty interior (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$).*

Suppose that the following statements are true.

a. *The map T is strongly competitive on* $\mathrm{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R})$.

b. *T is C^2 on a relative neighborhood of $\overline{x}$.*

c. *The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $|\lambda| < \mu$, where λ is stable and the eigenspace E^λ associated with λ is not a coordinate axis.*

d. *Either $\lambda \geq 0$ and*

$$T(x) \neq \overline{x} \quad \textit{and} \quad T(x) \neq x \quad \textit{for all } x \in \mathrm{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}),$$

or $\lambda < 0$ and

$$T^2(x) \neq x \quad \textit{for all } x \in \mathrm{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}).$$

Then there exists a curve $\mathcal{C}$ in $\mathcal{R}$ such that

(i) *$\mathcal{C}$ is invariant and a subset of $\mathcal{W}^s(\overline{x})$.*

(ii) *the endpoints of $\mathcal{C}$ lie on $\partial\mathcal{R}$.*

(iii) *$\overline{x} \in \mathcal{C}$.*

(iv) *$\mathcal{C}$ the graph of a strictly increasing continuous function of the first variable,*

(v) *$\mathcal{C}$ is differentiable at $\overline{x}$ if $\overline{x} \in \mathrm{int}(\mathcal{R})$ or one sided differentiable if $\overline{x} \in \partial\mathcal{R}$, and in all cases $\mathcal{C}$ is tangential to E^λ at $\overline{x}$,*

(vi) $\mathcal{C}$ *separates* $\mathcal{R}$ *into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \text{ with } x \preceq y\}$$

and

$$\mathcal{W}_+ := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \text{ with } y \preceq x\}.$$

(vii) $\mathcal{W}_-$ *is invariant, and* $\text{dist}(T^n(x), \mathcal{Q}_2(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_-$.

(viii) $\mathcal{W}_+$ *is invariant, and* $\text{dist}(T^n(x), \mathcal{Q}_4(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_+$.

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [31] and [18], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 1 *If the nonnegative cone of* $\preceq$ *is a generalized quadrant in* $\mathbb{R}^n$*, and if* T *has no fixed points in the ordered interval*$I(u_1, u_2)$ *other than* u_1 *and* u_2*, then the interior of* $I(u_1, u_2)$ *is either a subset of the basin of attraction of* u_1 *or a subset of the basin of attraction of* u_2.

The next results gives the existence and uniqueness of invariant curves emanating from a non-hyperbolic point of unstable type, that is a non-hyperbolic point where second eigenvalue is outside interval $[-1, 1]$. Similar result for a non-hyperbolic point of stable type that is a non-hyperbolic point where second eigenvalue is in the interval $(-1, 1)$ follows from Theorem 3.

Theorem 4 *Let* $\mathcal{R} = (a_1, a_2) \times (b_1, b_2)$*, and let* $T : \mathcal{R} \to \mathcal{R}$ *be a strongly competitive map with a unique fixed point* $\bar{\mathbf{x}} \in \mathcal{R}$*, and such that* T *is continuously differentiable in a neighborhood of* $\bar{\mathbf{x}}$*. Assume further that at the point* $\bar{\mathbf{x}}$ *the map* T *has associated characteristic values* μ *and* ν *satisfying* $1 < \mu$ *and* $-\mu < \nu < \mu$.

Then there exist curves $\mathcal{C}_1$*,* $\mathcal{C}_2$ *in* $\mathcal{R}$ *and there exist* $\mathbf{p}_1, \mathbf{p}_2 \in \partial\mathcal{R}$ *with* $\mathbf{p}_1 <<_{se} \bar{\mathbf{x}} <<_{se} \mathbf{p}_2$ *such that*

(i) *For* $\ell = 1, 2$*,* $\mathcal{C}_\ell$ *is invariant, north-east strongly linearly ordered, such that* $\bar{\mathbf{x}} \in \mathcal{C}_\ell$ *and* $\mathcal{C}_\ell \subset \mathcal{Q}_3(\bar{\mathbf{x}}) \cup \mathcal{Q}_1(\bar{\mathbf{x}})$*; the endpoints* $\mathbf{q}_\ell$*,* $\mathbf{r}_\ell$ *of* $\mathcal{C}_\ell$*, where* $\mathbf{q}_\ell \preceq_{ne} \mathbf{r}_\ell$*, belong to the boundary of* $\mathcal{R}$*. For* $\ell, j \in \{1, 2\}$ *with* $\ell \neq j$*,* $\mathcal{C}_\ell$ *is a subset of the closure of one of the components of* $\mathcal{R} \setminus \mathcal{C}_j$*. Both* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are tangential at* $\bar{\mathbf{x}}$ *to the eigenspace associated with* ν.

(ii) *For* $\ell = 1, 2$*, let* B_ℓ *be the component of* $\mathcal{R} \setminus \mathcal{C}_\ell$ *whose closure contains* $\mathbf{p}_\ell$*. Then* B_ℓ *is invariant. Also, for* $\mathbf{x} \in B_1$*,* $T^n(\mathbf{x})$ *accumulates on* $\mathcal{Q}_2(\mathbf{p_1}) \cap \partial\mathcal{R}$*, and for* $x \in B_2$*,* $T^n(\mathbf{x})$ *accumulates on* $\mathcal{Q}_4(\mathbf{p_2}) \cap \partial\mathcal{R}$.

(iii) *Let* $\mathcal{D}_1 := \mathcal{Q}_1(\bar{\mathbf{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ *and* $\mathcal{D}_2 := \mathcal{Q}_3(\bar{\mathbf{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$.
Then $\mathcal{D}_1 \cup \mathcal{D}_2$ *is invariant.*

Corollary 2 *Let a map* T *with fixed point* $\bar{\mathbf{x}}$ *be as in Theorem 4. Let* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *be the sets as in Theorem 4. If* T *satisfies* (O_+)*, then for* $\ell = 1, 2$*,* $\mathcal{D}_\ell$ *is invariant, and for every* $\mathbf{x} \in \mathcal{D}_\ell$*, the iterates* $T^n(\mathbf{x})$ *converge to* $\bar{\mathbf{x}}$ *or to a point of* $\partial\mathcal{R}$*. If* T *satisfies* (O_-)*, then* $T(\mathcal{D}_1) \subset \mathcal{D}_2$ *and* $T(\mathcal{D}_2) \subset \mathcal{D}_1$*. For every* $\mathbf{x} \in \mathcal{D}_1 \cup \mathcal{D}_2$*, the iterates* $T^n(\mathbf{x})$ *either converge to* $\bar{\mathbf{x}}$*, or converge to a period-two point, or to a point of* $\partial\mathcal{R}$.

3 Local Stability of Equilibrium Points

First we present the local stability analysis of the equilibrium points. It is interesting that the local stability analysis is the more difficult part of our analysis.

The equilibrium points of system (7) satisfy the following system of equations:

$$\begin{aligned} \overline{x} &= \frac{\overline{x}^2}{B_1\overline{x}^2+C_1\overline{y}^2} \\ \overline{y} &= \frac{\overline{y}^2}{A_2+B_2\overline{x}^2+C_2\overline{y}^2}, \quad n=0,1,\ldots. \end{aligned} \tag{12}$$

All solutions of system (12) with at least one zero component are given as: $E_{\overline{x}}(\overline{x},0)$ where $\overline{x}=\frac{1}{B_1}$, $E_{\overline{y}}(0,\overline{y})$ where $\overline{y}=\frac{1}{2C_2}$, and $E_{\overline{y}_\pm}\left(0,\overline{y}_\pm\right)$ where $\overline{y}_\pm=\frac{1\pm\sqrt{1-4C_2A_2}}{2C_2}$. The equilibrium point $E_{\overline{y}}(0,\overline{y})$ exists when $1=4C_2A_2$, and $E_{\overline{y}_\pm}\left(0,\overline{y}_\pm\right)$ exists when $1>4C_2A_2$.

The equilibrium points with strictly positive coordinates satisfy the following system of equations

$$\begin{aligned} B_1x^2+C_1y^2-x=0 \\ A_2+B_2x^2+C_2y^2-y=0. \end{aligned} \tag{13}$$

From (13) we have that all real solutions of the system (13) belong to the positive quadrant, since $B_1x^2+C_1y^2=x>0$ and $A_2+B_2x^2+C_2y^2=y>0$. By eliminating y from (13) we obtain

$$\begin{aligned} x^4(B_2C_1-B_1C_2)^2+2C_2x^3(B_2C_1-B_1C_2)+x^2\left(2A_2B_2C_1^2+B_1(C_1-2A_2C_1C_2)+C_2^2\right) \\ +C_1x(2A_2C_2-1)+A_2^2C_1^2=0. \end{aligned} \tag{14}$$

The next result gives the necessary and sufficient conditions for Eq.(14), and so System (12) to have between zero and 4 solutions. As we show in Section 4.2 the global dynamics depends on the number of the equilibrium points with positive coordinates.

Lemma 1 *Let*

$$\begin{aligned} \Delta_3 = {} & 16A_2^2B_1^4C_1^2(1-4A_2C_2)^2-4B_1^3C_1(4A_2C_2-1)\left(32A_2^3B_2C_1^2-8A_2^2C_2^2+6A_2C_2-1\right) \\ & +B_1^2\left(256A_2^4B_2^2C_1^4+128A_2^3B_2C_2^2C_1^2-8A_2\left(3B_2C_1^2+C_2^3\right)+16A_2^2\left(4B_2C_1^2C_2+C_2^4\right)+C_2^2\right) \\ & +2B_2B_1C_1\left(4A_2\left(-64A_2^2B_2C_2C_1^2+4A_2\left(3B_2C_1^2+4C_2^3\right)-13C_2^2\right)+9C_2\right) \\ & +B_2\left(256A_2^3B_2^2C_1^4+B_2C_1^2(16A_2C_2(9-8A_2C_2)-27)+4C_2^3(4A_2C_2-1)\right) \end{aligned} \tag{15}$$

$$\begin{aligned} \Delta_2 = {} & -2B_1^3C_1(2A_2C_2-1)(4A_2C_2-1)+B_1^2\left(32A_2^2B_2C_2C_1^2-4A_2\left(3B_2C_1^2+C_2^3\right)+C_2^2\right) \\ & -4B_2B_1C_1\left(A_2\left(4A_2B_2C_1^2+C_2^2\right)-C_2\right)-B_2\left(B_2C_1^2(9-8A_2C_2)+2C_2^3\right) \end{aligned} \tag{16}$$

and

$$\Delta_1=4A_2B_1C_1C_2-2C_1(2A_2B_2C_1+B_1)+C_2^2.$$

Assume that $B_2C_1\neq B_1C_2$. Then the following holds:

a) If $\Delta_3>0$, $\Delta_2>0$, and $\Delta_1>0$, then the Eq. (14) has four simple real roots.

b) If $\Delta_3>0$ and $\Delta_2\leq 0\vee(\Delta_2>0\wedge\Delta_1\leq 0)$ then the Eq. (14) has no real roots.

c) If $\Delta_3<0$ then Eq. (14) has two simple real roots.

d) If $\Delta_3=0$ and $\Delta_2<0$ then the Eq. (14) has one real double roots.

e) If $\Delta_3=0$ and $\Delta_2>0$ then the Eq. (14) has two real simple roots and one real double root.

f) If $\Delta_3 = 0$, $\Delta_2 = 0$ and $\Delta_1 > 0$ then the Eq. (14) has two real double roots.

g) If $\Delta_3 = 0$, $\Delta_2 = 0$ and $\Delta_1 < 0$ then the Eq. (14) has no real roots.

h) If $\Delta_3 = 0$, $\Delta_2 = 0$ and $\Delta_1 = 0$ then the Eq. (14) has one real root of multiplicity four.

Proof. The discrimination matrix [50] of $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ and $f'(x)$ is given by

$$Discr(f, f') = \begin{pmatrix} A & B & C & D & E & 0 & 0 & 0 \\ 0 & 4A & 3B & 2C & D & 0 & 0 & 0 \\ 0 & A & B & C & D & E & 0 & 0 \\ 0 & 0 & 4A & 3B & 2C & D & 0 & 0 \\ 0 & 0 & A & B & C & D & E & 0 \\ 0 & 0 & 0 & 4A & 3B & 2C & D & 0 \\ 0 & 0 & 0 & A & B & C & D & E \\ 0 & 0 & 0 & 0 & 4A & 3B & 2C & D \end{pmatrix}.$$

Let D_k denote the determinant of the submatrix of $Discr(\tilde{f}, \tilde{f}')$, formed by the first $2k$ rows and the first $2k$ columns, for $k = 1, 2, 3, 4$ where

$$\tilde{f}(x) = x^4 \left(B_2C_1 - B_1C_2\right)^2 + 2C_2x^3 \left(B_2C_1 - B_1C_2\right) + x^2 \left(2A_2B_2C_1^2 + B_1 \left(C_1 - 2A_2C_1C_2\right) + C_2^2\right) + C_1x \left(2A_2C_2 - 1\right) + A_2^2C_1^2. \quad (17)$$

So, by straightforward calculation one can see that

$$\begin{aligned} D_1 &= 4\left(B_2C_1 - B_1C_2\right)^4, \\ D_2 &= 4\Delta_1 \left(B_2C_1 - B_1C_2\right)^6, \\ D_3 &= 4\Delta_2 C_1^2 \left(B_2C_1 - B_1C_2\right)^6, \\ D_4 &= \Delta_3 C_1^4 \left(B_2C_1 - B_1C_2\right)^6. \end{aligned}$$

The rest of the proof follows in view of Theorem 1 [50]. □

Geometrically solutions of System (13) are intersections of two ellipses that satisfy the equations

$$\frac{\left(x - \frac{1}{2B_1}\right)^2}{\frac{1}{4B_1^2}} + \frac{y^2}{\frac{1}{4B_1C_1}} = 1, \quad \frac{x^2}{\frac{1}{4B_2C_2} - \frac{A_2}{B_2}} + \frac{\left(y - \frac{1}{2C_2}\right)^2}{\frac{1}{4C_2^2} - \frac{A_2}{C_2}} = 1 \quad (18)$$

with respective vertices $\left(\frac{1}{2B_1}, 0\right)$ and $\left(0, \frac{1}{2C_2}\right)$. See Figure 1.

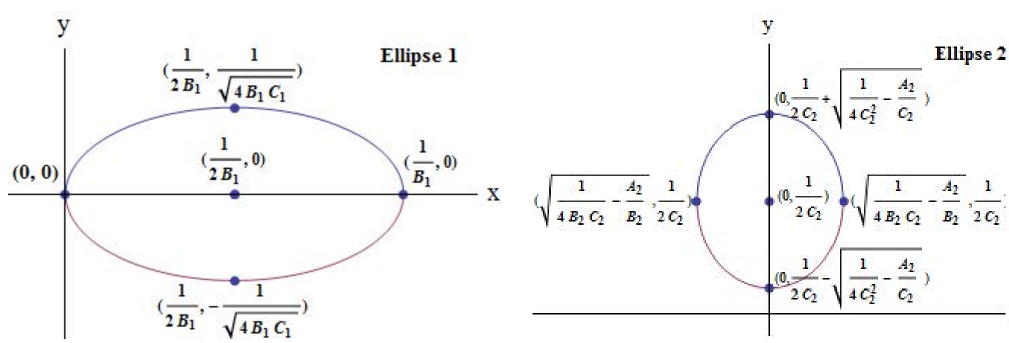


Figure 1: The equilibrium curves of System (7).

Consequently when $1 > 4C_2A_2$, in addition to the three equilibium points on the axes, System (7) may have $1, 2, 3$ or 4 positive equilibrium points. We will refer to these equilibrium points as $E_{SW}(\overline{x}, \overline{y})$ (southwest), $E_{SE}(\overline{x}, \overline{y})$(southeast), $E_{NW}(\overline{x}, \overline{y})$ (northwest), and $E_{NE}(\overline{x}, \overline{y})$ (northeast) where

$$E_{NW} \preceq_{se} E_{NE} \preceq_{se} E_{SE}, \quad E_{SW} \preceq_{ne} E_{NW}.$$

When a positive equilibrium point is non-hyperbolic we will refer to it as $E_N(\overline{x}, \overline{y})$.

The map associated with System (7) has the form:

$$T\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} \frac{x^2}{B_1x^2+C_1y^2} \\ \frac{y^2}{A_2+B_2x^2+C_2y^2} \end{array}\right). \tag{19}$$

The Jacobian matrix of T is

$$J_T\left(x, y\right) = \left(\begin{array}{cc} \frac{2C_1xy^2}{(B_1x^2+C_1y^2)^2} & -\frac{2C_1x^2y}{(B_1x^2+C_1y^2)^2} \\ -\frac{2B_2xy^2}{(A_2+B_2x^2+C_2y^2)^2} & \frac{2A_2y+2B_2x^2y}{(A_2+B_2x^2+C_2y^2)^2} \end{array}\right), \tag{20}$$

and the Jacobian matrix of T evaluated at an equilibrium $E(\overline{x}, \overline{y})$ with positive coordinates has the form:

$$J_T(\overline{x}, \overline{y}) = \left(\begin{array}{cc} \frac{2C_1\overline{y}^2}{\overline{x}} & -2C_1\overline{y} \\ -2B_2\overline{x} & \frac{2A_2+2B_2^2\overline{x}}{\overline{y}} \end{array}\right). \tag{21}$$

The determinant and trace of (21) are:

$$\det J_T(\overline{x}, \overline{y}) = \frac{4A_2C_1\overline{y}}{\overline{x}}, \quad \operatorname{tr} J_T(\overline{x}, \overline{y}) = \frac{2C_1\overline{y}^2}{\overline{x}} + \frac{2A_2 + 2B_2\overline{x}^2}{\overline{y}}. \tag{22}$$

It is worth noting that $\det J_T(\overline{x}, \overline{y})$ and $\operatorname{tr} J_T(\overline{x}, \overline{y})$ of (21) are both positive.

Using the equilibrium condition (13), we may rewrite the determinant and trace in the more useful form:

$$\begin{aligned} \det J_T(\overline{x}, \overline{y}) &= 4\overline{x}\overline{y}B_1C_2 - 4\overline{y}C_2 - 4\overline{x}B_1 - 4\overline{x}\overline{y}B_2C_1 + 4, \\ \operatorname{tr} J_T(\overline{x}, \overline{y}) &= 4 - 2\overline{y}C_2 - 2\overline{x}B_1. \end{aligned} \tag{23}$$

The characteristic equation of the matrix (21) is

$$\lambda^2 - \operatorname{tr} J_T(\overline{x}, \overline{y})\lambda + \det J_T(\overline{x}, \overline{y}) = 0, \tag{24}$$

which solutions are the eigenvalues

$$\begin{aligned} \lambda &= \frac{\operatorname{tr} J_T(\overline{x}, \overline{y}) - \sqrt{(\operatorname{tr} J_T(\overline{x}, \overline{y}))^2 - 4\det J_T(\overline{x}, \overline{y})}}{2} \\ \mu &= \frac{\operatorname{tr} J_T(\overline{x}, \overline{y}) + \sqrt{(\operatorname{tr} J_T(\overline{x}, \overline{y}))^2 - 4\det J_T(\overline{x}, \overline{y})}}{2}. \end{aligned} \tag{25}$$

The corresponding eigenvectors of (25) are

$$\begin{aligned} E_\lambda &= \left(\frac{1}{2xB_2}\left(xB_1 - yC_2 + \sqrt{\left(xB_1 - yC_2\right)^2 + 4B_2C_1xy}\right), 1\right), \\ E_\mu &= \left(-\frac{1}{2xB_2}\left(yC_2 - xB_1 + \sqrt{\left(xB_1 - yC_2\right)^2 + 4B_2C_1xy}\right), 1\right). \end{aligned} \tag{26}$$

We will now consider two lemmas that will be used to prove the local stability character of the positive equilibrium points of System (7). The nonzero coordinates, $(\overline{x}, \overline{y})$, of all equilibrium points will subsequently be designated with the subscripts: r (repeller), a (attractor), s, s_1, s_2 (saddlepoint), ns (non-hyperbolic of the stable type) and nu (non-hyperbolic of the unstable type).

Lemma 2 *The following conditions hold for the coordinates of the positive equilibium points, $E(\overline{x}, \overline{y})$, of System (7).*

(i) For $E_{SW}(\overline{x}_r, \overline{y}_r)$ and $E_N(\overline{x}_{nu}, \overline{y}_{nu})$,

$$\overline{x} < \frac{1}{2B_1} \text{ and } \overline{y} < \frac{1}{2C_2}. \tag{27}$$

(ii) For $E_{NW}\left(\overline{x}_{s_1}, \overline{y}_{s_1}\right)$,

$$\overline{x} < \frac{1}{2B_1} \text{ and } \overline{y} > \frac{1}{2C_2}. \tag{28}$$

(iii) For $E_{NE}(\overline{x}_a, \overline{y}_a)$, $E_{NE}(\overline{x}_s, \overline{y}_s)$, and $E_N(\overline{x}_{ns}, \overline{y}_{ns})$,

$$\overline{x} > \frac{1}{2B_1} \text{ and } \overline{y} > \frac{1}{2C_2}. \tag{29}$$

(iv) For $E_{SE}\left(\overline{x}_{s_2}, \overline{y}_{s_2}\right)$,

$$\overline{x} > \frac{1}{2B_1} \text{ and } \overline{y} < \frac{1}{2C_2}. \tag{30}$$

Proof. This is clear from geometry. See Figure 2. □

Lemma 3 *The following conditions hold for the coordinates of the positive equilibium points, $E(\overline{x}, \overline{y})$, of System (7).*

(i) For $E_{SW}(\overline{x}_r, \overline{y}_r)$ and $E_{NW}\left(\overline{x}_{s_1}, \overline{y}_{s_1}\right)$,

$$4\overline{xy}B_1C_2 - 4B_2C_1\overline{xy} + 1 > 2\overline{y}C_2 + 2\overline{x}B_1. \tag{31}$$

(ii) For $E_{NE}(\overline{x}_a, \overline{y}_a)$, $E_{NE}(\overline{x}_s, \overline{y}_s)$, and $E_{SE}\left(\overline{x}_{s_2}, \overline{y}_{s_2}\right)$,

$$4\overline{xy}B_1C_2 - 4B_2C_1\overline{xy} + 1 < 2\overline{y}C_2 + 2\overline{x}B_1. \tag{32}$$

(iii) For $E_N(\overline{x}_{ns}, \overline{y}_{ns})$ and $E_N(\overline{x}_{nu}, \overline{y}_{nu})$,

$$4\overline{xy}B_1C_2 - 4B_2C_1\overline{xy} + 1 = 2\overline{y}C_2 + 2\overline{x}B_1. \tag{33}$$

Proof.

(i) Let m_{E1} be the slope of the tangent line to ellipse E_1 at $E(\overline{x}, \overline{y}) = E_{SW}(\overline{x}_r, \overline{y}_r)$ and let m_{E2} be the slope of the tangent line to ellipse E_2 at $E(\overline{x}, \overline{y}) = E_{SW}(\overline{x}_r, \overline{y}_r)$. It is clear from geometry that

$$m_{E1} > m_{E2} > 0.$$

See Figure 2. It follows that

$$\frac{dy}{dx}|_{E_1}(\overline{x},\overline{y}) > \frac{dx}{dy}|_{E_2}(\overline{x},\overline{y}) > 0,$$

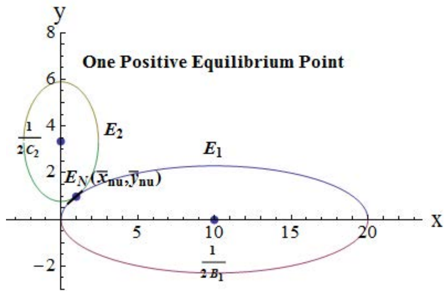
and in turn

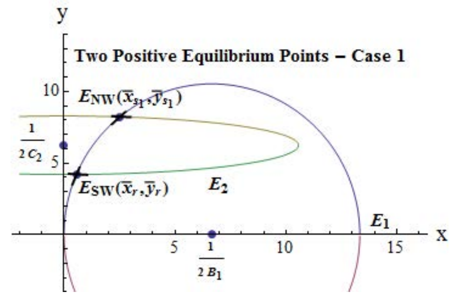
$$\frac{1-2B_1\overline{x}}{2C_1\overline{y}} > \frac{2B_2\overline{x}}{1-2C_2\overline{y}} > 0.$$

Therefore

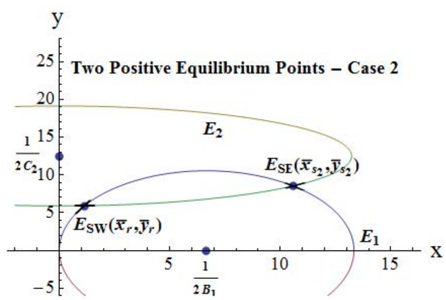
$$4\overline{xy}B_1C_2 - 4B_2C_1\overline{xy} + 1 > 2\overline{y}C_2 + 2\overline{x}B_1.$$

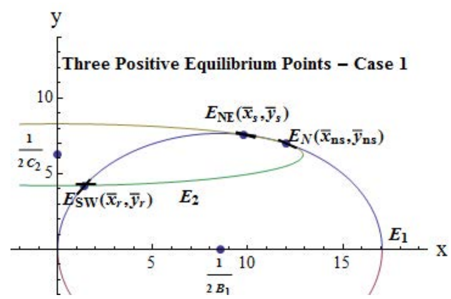
The proofs for the remaining case in (i) and all cases in (ii) and (iii) are similar, and will be omitted. $\square$



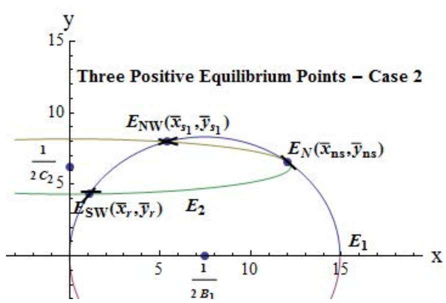


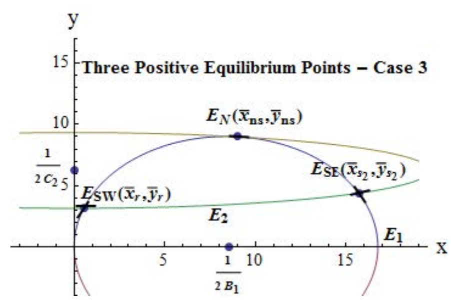
$B_1 = \frac{5}{100}, C_1 = \frac{95}{100}, A_2 = \frac{13}{19}, B_2 = \frac{63}{380}, C_2 = \frac{15}{100}$ $\quad$ $B_1 = .075, C_1 = .03, A_2 = 2.79, B_2 = .003, C_2 = .08$



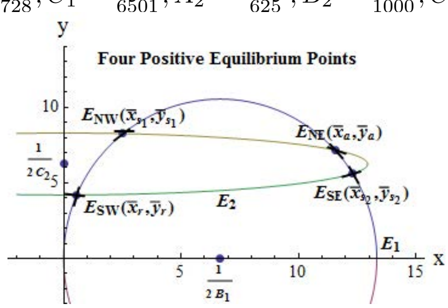


$B_1 = .075, C_1 = .03, A_2 = 4.5, B_2 = .01, C_2 = .04$ $\quad$ $B_1 = \frac{83}{1416}, C_1 = \frac{30}{413}, A_2 = \frac{349}{125}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$





$B_1 = \frac{317}{4728}, C_1 = \frac{350}{6501}, A_2 = \frac{1767}{625}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$ $\quad$ $B_1 = \frac{64}{1071}, C_1 = \frac{55}{1071}, A_2 = \frac{1179}{500}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$



$B_1 = .075, C_1 = .03, A_2 = 2.79, B_2 = .002, C_2 = .08$

Figure 2: Local Stability

Theorem 5 *The following conditions hold for the equilibium points $E(\overline{x}, \overline{y})$ of System (7).*

(i) $E_{\overline{x}}(\overline{x}_a, 0)$ *is a locally asymptotically stable;*

(ii) $E_{\overline{y}}(0, \overline{y}_{ns})$ *is non-hyperbolic of the stable type;*

(iii) $E_{\overline{y}_+}\left(0, \overline{y}_{+a}\right)$ *is locally asymptotically stable and* $E_{\overline{y}_-}\left(0, \overline{y}_{-s}\right)$ *is a saddle point;*

(iv) $E_{SW}(\overline{x}_r, \overline{y}_r)$ *is a repeller;*

(v) $E_{NW}(\overline{x}_{s_1}, \overline{y}_{s_1})$, $E_{SE}(\overline{x_{s_2}}, \overline{y}_{s_2})$, *and* $E_{NE}(\overline{x}_s, \overline{y}_s)$ *are saddle points;*

(vi) $E_{NE}(\overline{x}_a, \overline{y}_a)$ *is a locally asymptotically stable;*

(vii) $E_N(\overline{x}_{ns}, \overline{y}_{ns})$ *is non-hyperbolic of the stable type;*

(viii) $E_N(\overline{x}_{nu}, \overline{y}_{nu})$ *is non-hyperbolic of the unstable type.*

Proof.

(i) The eigenvalues of (20), evaluated at $E_{\overline{x}}(\overline{x}_a, 0)$, are $\lambda = 0$ and $\mu = 0$.

(ii) The eigenvalues of (20), evaluated at $E_{\overline{y}}(0, \overline{y}_{ns})$, are $\lambda = 0$ and $\mu = 1$ when $1 = 4C_2A_2$.

(iii) The eigenvalues of (20), evaluated at $E_{\overline{y}_+}\left(0, \overline{y}_{+a}\right)$ and $E_{\overline{y}_-}\left(0, \overline{y}_{-s}\right)$ respectively, are $\lambda = 0$ and $\mu_\pm = \frac{2A_2}{\overline{y}_\pm}$ when $1 > 4C_2A_2$.

(a) Note that when $1 > 4C_2A_2$,

$$\overline{y}_+ = \frac{1 + \sqrt{1 - 4C_2A_2}}{2C_2} > \frac{1}{2C_2} > 2A_2.$$

Therefore $\mu_+ = \frac{2A_2}{\overline{y}_+} < 1$.

(b) Note that when $1 > 4C_2A_2$, $\sqrt{1 - 4A_2C_2} > 1 - 4A_2C_2$. Therefore

$$\mu_- = \frac{2A_2}{\overline{y}_-} = \frac{4A_2C_2}{1 - \sqrt{1 - 4A_2C_2}} > \frac{1 - \sqrt{1 - 4A_2C_2}}{1 - \sqrt{1 - 4A_2C_2}} = 1.$$

In both cases, the conclusion follows.

(iv) We need to show that $|\operatorname{tr} J_T(\overline{x}, \overline{y})| < |1 + \det J_T(\overline{x}, \overline{y})|$ and $|\det J_T(\overline{x}, \overline{y})| > 1$ when $E(\overline{x}, \overline{y}) = E_{SW}(\overline{x}_r, \overline{y}_r)$. Since $\operatorname{tr} J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y})$ are both positive, our conditions become $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y}) > 1$. We will first show that $\det J_T(\overline{x}, \overline{y}) > 1$. By (31) we have

$$\begin{aligned} &\det J_T(\overline{x}\overline{y}) - 1 \\ &= 4\overline{x}\overline{y}B_1C_2 - 4\overline{x}\overline{y}B_2C_1 - 4\overline{y}C_2 - 4\overline{x}B_1 + 4 - 1 \\ &> 2\overline{y}C_2 + 2\overline{x}B_1 - 1 - 4\overline{y}C_2 - 4\overline{x}B_1 + 4 - 1 \\ &= 1 - 2\overline{y}C_2 + 1 - 2\overline{x}B_1. \end{aligned}$$

By (27) we have $1 - 2\overline{y}C_2 + 1 - 2\overline{x}B_1 > 0$.

Therefore $\det J_T(\overline{x}, \overline{y}) > 1$. We will next show that $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$.

By (31) we have,

$$
\begin{aligned}
&1 + \det J_T(\overline{x}, \overline{y}) - \operatorname{tr} J_T(\overline{x}, \overline{y}) \\
&= 1 + (4\overline{x}\overline{y}B_1C_2 - 4\overline{y}C_2 - 4\overline{x}B_1 - 4\overline{x}\overline{y}B_2C_1 + 4) - (4 - 2\overline{y}C_2 - 2\overline{x}B_1) \\
&= 4\overline{x}\overline{y}B_1C_2 - 4\overline{x}\overline{y}B_2C_1 + 1 - 2\overline{y}C_2 - 2\overline{x}B_1 \\
&> 2\overline{y}C_2 + 2\overline{x}B_1 - 2\overline{y}C_2 - 2\overline{x}B_1 = 0.
\end{aligned}
$$

Therefore $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$.

(v) We need to show that $|\operatorname{tr} J(\overline{x}, \overline{y})| > |1 + \det J_T(\overline{x}, \overline{y})|$ when $E(\overline{x}, \overline{y}) = E_{NW}(\overline{x}_{s_1}, \overline{y}_{s_1})$. Since $\operatorname{tr} J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y})$ are both positive, our condition becomes $\operatorname{tr} J_T(\overline{x}, \overline{y}) > 1 + \det J_T(\overline{x}, \overline{y})$. By (31) we have,

$$
\begin{aligned}
&\operatorname{tr} J_T(\overline{x}, \overline{y}) - (1 + \det J_T(\overline{x}, \overline{y})) \\
&= 4 - 2\overline{y}C_2 - 2\overline{x}B_1 - (1 + 4\overline{x}\overline{y}B_1C_2 - 4\overline{y}C_2 - 4\overline{x}B_1 - 4\overline{x}\overline{y}B_2C_1 + 4) \\
&= 2\overline{x}B_1 + 2\overline{y}C_2 - 4\overline{x}\overline{y}B_1C_2 + 4\overline{x}\overline{y}B_2C_1 - 1 \\
&> 4\overline{x}\overline{y}B_1C_2 - 4B_2C_1\overline{x}\overline{y} + 1 - 4\overline{x}\overline{y}B_1C_2 + 4\overline{x}\overline{y}B_2C_1 - 1.
\end{aligned}
$$

Therefore $\operatorname{tr} J_T(\overline{x}, \overline{y}) > 1 + \det J_T(\overline{x}, \overline{y})$. The proofs that $E_{SE}(\overline{x}_{s_2}, \overline{y}_{s_2})$ and $E_{NE}(\overline{x}_s, \overline{y}_s)$ are saddle points are similar and will be omitted.

(vi) We need to show that $|\operatorname{tr} J_T(\overline{x}, \overline{y})| < 1 + \det J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y}) < 1$ when $E(\overline{x}, \overline{y}) = E_{NE}(\overline{x}_a, \overline{y}_a)$. Since $\operatorname{tr} J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y})$ are both positive, our conditions become $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$ and $\det J_T(\overline{x}, \overline{y}) < 1$. We will first show that $\det J_T(\overline{x}, \overline{y}) < 1$. By (32) we have.

$$
\begin{aligned}
&\det J_T(\overline{x}, \overline{y}) - 1 \\
&= (4\overline{x}\overline{y}B_1C_2 - 4\overline{y}C_2 - 4\overline{x}B_1 - 4\overline{x}\overline{y}B_2C_1 + 4) - 1 \\
&= 4\overline{x}\overline{y}B_1C_2 - 4\overline{x}\overline{y}B_2C_1 - 4\overline{y}C_2 - 4\overline{x}B_1 + 3 \\
&> 2\overline{y}C_2 + 2\overline{x}B_1 - 1 - 4\overline{y}C_2 - 4\overline{x}B_1 + 3 \\
&= 1 - 2\overline{y}C_2 + 1 - 2\overline{x}B_1.
\end{aligned}
$$

By (29) we have $1 - 2\overline{y}C_2 + 1 - 2\overline{x}B_1 < 0$.

Therefore $\det J_T(\overline{x}, \overline{y}) < 1$. We will next show that $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$. By (32) we have,

$$
\begin{aligned}
&1 + \det J_T(\overline{x}, \overline{y}) - \operatorname{tr} J_T(\overline{x}, \overline{y}) \\
&= (1 + 4\overline{x}\overline{y}B_1C_2 - 4\overline{y}C_2 - 4\overline{x}B_1 - 4\overline{x}\overline{y}B_2C_1 + 4) - (4 - 2\overline{y}C_2 - 2\overline{x}B_1) \\
&= 4\overline{x}\overline{y}B_1C_2 - 4\overline{x}\overline{y}B_2C_1 + 1 - 2\overline{y}C_2 - 2\overline{x}B_1 \\
&> 2\overline{y}C_2 + 2\overline{x}B_1 - 2\overline{y}C_2 - 2\overline{x}B_1
\end{aligned}
$$

Therefore $\operatorname{tr} J_T(\overline{x}, \overline{y}) < 1 + \det J_T(\overline{x}, \overline{y})$.

(vii) By (23) and (25) we have

$$\lambda = \frac{(4-2yC_2-2xB_1)-\sqrt{(4-2yC_2-2xB_1)^2-4(4xyB_1C_2-4yC_2-4xB_1-4xyB_2C_1+4)}}{2}$$
$$\mu = \frac{(4-2yC_2-2xB_1)+\sqrt{(4-2yC_2-2xB_1)^2-4(4xyB_1C_2-4yC_2-4xB_1-4xyB_2C_1+4)}}{2}.$$

By (33), we have $\lambda = 3-2\overline{y}C_2-2\overline{x}B_1$ and $\mu = 1$. By (29), we have $\lambda < 1$. The conclusion follows.

(viii) The proof of (viii) is similar to the proof of (vii) and will be omitted.

□

4 Global Results

In this section we combine the results from Sections 2 and 3 to prove the global results for System (7). First, we present the behavior of the solutions of system (7) on coordinate axes and then we prove that the map T which corresponds to System (7) is injective and that it satisfies $(O+)$.

4.1 Convergence of Solutions on the Coordinate Axes; Injectivity and $(O+)$.

When $y_n = 0$, System (7) becomes

$$x_{n+1} = \frac{1}{B_1}, \quad y_{n+1} = 0, \quad n = 0, 1, \ldots. \tag{34}$$

When $x_n = 0$, System (7) becomes

$$x_{n+1} = 0, \quad y_{n+1} = \frac{y_n^2}{A_2 + C_2 y_n^2}, \quad n = 0, 1, \ldots. \tag{35}$$

It follows from (34) and (35) that solutions of System (7) with initial conditions on the x-axis remain on the x-axis and solutions of system (7) with initial conditions on the y-axis remain on the y-axis.

Theorem 6 *The following conditions hold for solutions $\{(x_n, y_n)\}$ of System (7) with initial conditions on the x or y-axis.*

(i) $E_{\overline{x}}(\overline{x}_a, 0)$ is a superattractor of all solutions $\{(x_n, y_n)\}$ of system (7) with initial conditions on the x-axis.

(ii) When no equilibrium points exist on the y axis, if $x_0 = 0$, then $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$.

(iii) When $E_{\overline{y}}(0, \overline{y}_{ns})$ exists,

(a) if $x_0 = 0$ and $y_0 > \overline{y}_{ns}$, then $\lim_{n\to\infty}(x_n, y_n) = (0, \overline{y}_{ns})$.

(b) if $x_0 = 0$ and $0 < y_0 < \overline{y}_{ns}$, then $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$.

(iv) When $E_{\overline{y}_+}(0, \overline{y}_{+a})$ and $E_{\overline{y}_-}(0, \overline{y}_{-s})$ exist,

(a) if $x_0 = 0$ and $y_0 > \overline{y}_{+a}$, then $\lim_{n\to\infty}(x_n, y_n) = (0, \overline{y}_{+a})$.

(b) *if* $x_0 = 0$ *and* $\overline{y}_{-s} < y_0 < \overline{y}_{+a}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (0, \overline{y}_{+a})$.

(c) *if* $x_0 = 0$ *and* $0 < y_0 < \overline{y}_{-s}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$.

Proof.

(i) When $y_0 = 0$, it follows directly from (34) that $(x_n, y_n) = (\overline{x}_a, 0)$ for $n > 1$.

(ii) In this case $1 < 4A_2C_2$. By (35) it can be shown that

$$y_{n+1} - y_n = \frac{-y_n\left(C_2\left(y_n - \frac{1}{2C_2}\right)^2 + A_2 - \frac{1}{4C_2}\right)}{A_2 + C_2 y_n^2}. \tag{36}$$

By (36), when $1 < 4A_2C_2$, it is clear that $\{y_n\}$ is a stricly decreasing sequence, and so is convergent. It follows that $\{y_n\}$ converges to 0.

(iii) In this case, $1 = 4A_2C_2$, and we may rewrite (36) as

$$y_{n+1} - y_n = \frac{-y_n\left(C_2\left(y_n - \overline{y}_{ns}\right)^2\right)}{A_2 + C_2 y_n^2}. \tag{37}$$

By (37) it is clear that $\{y_n\}$ is a stricly decreasing sequence, and so is convergent. It follows that $\{y_n\}$ converges to $\overline{y}_{ns}$ when $y_0 > \overline{y}_{ns}$, and $\{y_n\}$ converges to 0 when $0 < y_0 < \overline{y}_{ns}$.

(iv) In this case, $1 > 4A_2C_2$. By (35), it can be shown that

$$y_{n+1} - y_n = \frac{-C_2 y_n\left(y_n - \overline{y}_{+a}\right)\left(y_n - \overline{y}_{-s}\right)}{A_2 + C_2 y_n^2}. \tag{38}$$

By (38), it is clear that $\{y_n\}$ is a stricly decreasing sequence (and so is convergent) when $y_0 > \overline{y}_{+a}$ and when $0 < y_0 < \overline{y}_{-s}$, and a strictly increasing sequence (and so is convergent) when $\overline{y}_{-s} < y_0 < \overline{y}_{+a}$. It follows that $\{y_n\}$ converges to $\overline{y}_{+a}$ when $y_0 > \overline{y}_{+a}$ and when $\overline{y}_{-s} < y_0 < \overline{y}_{+a}$, and converges to 0 when $0 < y_0 < \overline{y}_{-s}$.

□

Theorem 7 *The map T which corresponds to System (7) is injective.*

Proof. Indeed,

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \frac{x_1^2}{B_1x_1^2 + C_1y_1^2} \\ \frac{y_1^2}{A_2 + B_2x_1^2 + C_2y_1^2} \end{pmatrix} = \begin{pmatrix} \frac{x_2^2}{B_1x_2^2 + C_1y_2^2} \\ \frac{y_2^2}{A_2 + B_2x_2^2 + C_2y_2^2} \end{pmatrix}$$

which is equivalent to

$$y_2^2 x_1^2 = y_1^2 x_2^2, \quad y_1 = y_2. \tag{39}$$

This immediatly implies $x_1 = x_2$. □

Theorem 8 *The map T which corresponds to System (7) satisfies (O^+). All solutions of System (7) converge to either an equilibrium point or to $(0, 0)$.*

Proof.

Assume that

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \preceq_{ne} T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{x_1^2}{B_1x_1^2+C_1y_1^2} \\ \frac{y_1^2}{A_2+B_2x_1^2+C_2y_1^2} \end{pmatrix} \preceq_{ne} \begin{pmatrix} \frac{x_2^2}{B_1x_2^2+C_1y_2^2} \\ \frac{y_2^2}{A_2+B_2x_2^2+C_2y_2^2} \end{pmatrix}.$$

The last inequality is equivalent to

$$y_2^2x_1^2 \leq y_1^2x_2^2, \quad y_1 \leq y_2 \tag{40}$$

Suppose $x_2 < x_1$. Then $y_1^2x_2^2 < y_2^2x_1^2$, which contradicts (40). Consequently $x_1 \leq x_2$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

Thus we conclude that all solutions of System (7) are eventually monotonic for all values of parameters. Furthermore it is clear that all solutions are bounded. Indeed every solution of (7) satisfies

$$x_n \leq \frac{1}{B_1}, \quad y_n \leq \frac{1}{C_2} \tag{41}$$

Consequently, all solutions of System (7) converge to an equilibrium point or to $(0,0)$.

□

4.2 Global Dynamics

Theorem 9 *Assume that* $1 < 4A_2C_2$. *Then System* (7) *has one equilibrium point* $E_{\bar{x}}$ *which is locally asymptotically stable. The singular point* $E_0(0,0)$ *is global attractor of all points on* y*-axis and every point on* x*-axis is attracted to* $E_{\bar{x}}$. *Furthermore, every point in the interior of the first quadrant is attracted to* E_0 *or* $E_{\bar{x}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. In view of Theorem 6, every solution that starts on the y-axis converges to 0 in a decreasing manner and every solution that starts on x-axis is equal to $E_{\bar{x}}$ in a single step. Let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$ and $T(0, y_0) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. In view of Theorems 6 and 8 $T^n(x_0, y_0) \to E_{\bar{x}}$ or $T^n(x_0, y_0) \to E_0$ as $n \to \infty$. □

Theorem 10 *Assume that* $1 = 4A_2C_2$. *Then System* (7) *has two equilibrium points,* $E_{\bar{x}}$ *which is locally asymptotically stable and* $E_{\bar{y}}$ *which is non-hyperbolic of the stable type. The singular point* E_0 *is global attractor of all points on the* y*-axis, which start below* $E_{\bar{y}}$. *Furthermore, every point in the interior of the first quadrant below* $\mathcal{W}^s(E_{\bar{y}})$ *is attracted to* $E_0(0,0)$ *or* $E_{\bar{x}}$ *and every point in the first quadrant which starts above* $\mathcal{W}^s(E_{\bar{y}})$ *is attracted to* $E_{\bar{y}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. In view of Theorem 6, every solution that starts on the y-axis below $E_{\bar{y}}$ converges to 0 in a decreasing manner and every solution that starts on the x-axis is equal to $E_{\bar{x}}$ in a single step. In addition, every solution that starts on the y-axis above $E_{\bar{y}}$ converges to $E_{\bar{y}}$ in a decreasing way. Let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant below $\mathcal{W}^s(E_{\bar{y}})$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$ which implies $T(0, y_0) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. If $y_0 > \bar{y}$ then $T^n(x_0, y_0)$ will eventually enter the ordered interval $I(E_{\bar{y}}, E_{\bar{x}}) = \{(x, y) : 0 < x \leq \bar{x}, 0 < y \leq \bar{y}\}$. In view of Theorems 6 and 8, $T^n(x_0, y_0) \to E_{\bar{x}}$ or $T^n(x_0, y_0) \to E_0$ as $n \to \infty$.

Now, let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant above $\mathcal{W}^s(E_{\bar{y}})$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}})$. This implies $T(0, y_0) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, y_W)$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$. Since $T^n(0, y_0) \to E_{\bar{y}}, T(x_0, y_W) \to E_{\bar{y}}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0) \to E_{\bar{y}}$ as $n \to \infty$.

□

Theorem 11 *Assume that $1 > 4A_2C_2$ and System (7) has three equilibrium points, $E_{\bar{x}}$ and $E_{\bar{y}_+}$ which are locally asymptotically stable and $E_{\bar{y}_-}$ which is a saddle point. The singular point $E_0(0,0)$ is global attractor of all points on y-axis, which start below $E_{\bar{y}_-}$. The basins of attraction of two equilibrium points are given as:*

$B(E_{\bar{y}_+}) = \{(x_0,y_0) : \text{points above } \mathcal{W}^s(E_{\bar{y}_-})\}$,

$B(E_{\bar{y}_-}) = \mathcal{W}^s(E_{\bar{y}_-})$,

where $\mathcal{W}^s(E_{\bar{y}_-})$ denotes the global stable manifold guaranteed by Theorem 3. Furthermore, every initial point below $\mathcal{W}^s(E_{\bar{y}_-})$ is attracted to $E_0(0,0)$ or $E_{\bar{x}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. The existence of the global stable manifold is guaranteed by Theorem 3 in view of Theorem 7.

By Theorem 6, every solution that starts on the y-axis below $E_{\bar{y}_-}$ converges to E_0 in a decreasing manner and every solution that starts on the x-axis is equal to $E_{\bar{x}}$ in a single step. In addition, every solution that starts on the y-axis above $E_{\bar{y}_-}$ converges to $E_{\bar{y}_+}$ in a monotonic way.

Let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant below $\mathcal{W}^s(E_{\bar{y}_-})$. Then $(x_0, y_W) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$ which implies $T(x_0, y_W) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$ and so $T^n(x_0, y_W) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. Since $T^n(x_0, y_W) \to E_{\bar{y}_-}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_{\bar{y}_-}, E_{\bar{x}}) = \{(x, y) : 0 < x \le \bar{x}, 0 < y \le \bar{y}_-\}$, in which case it converges to $E_{\bar{x}}$ or $E_0(0,0)$.

Finally, let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant above $\mathcal{W}^s(E_{\bar{y}_-})$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}_-})$. Thus $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$, which by $T^n(x_0, y_W) \to E_{\bar{y}_-}$ as $n \to \infty$, implies that $T^n(x_0, y_0)$ eventually lands on the part of y-axis above $E_{\bar{y}_-}$ and so it converges to $E_{\bar{y}_+}$.

□

Theorem 12 *Assume that $1 > 4A_2C_2$ and System (7) has four equilibrium points, $E_{\bar{x}}$ and $E_{\bar{y}_+}$ which are locally asymptotically stable, $E_{\bar{y}_-}$ which is a saddle point and E_N which is non-hyperbolic of the unstable type. The singular point $E_0(0,0)$ is global attractor of all points on the y-axis, which start below $E_{\bar{y}_-}$. The basins of attraction of three of the equilibrium points are given as:*

$\{(x_0,y_0) : \text{points below } C_l \text{ such that } x_0 \ge x_N\} \subset B(E_{\bar{x}})$,

$B(E_{\bar{y}_+}) = \{(x_0,y_0) : \text{points above } \mathcal{W}^s(E_{\bar{y}_-}) \cup C_u\}$,

$B(E_N) = \{(x_0,y_0) : \text{points between } C_l \text{ and } C_u$,

$B(E_{\bar{y}_-}) = \mathcal{W}^s(E_{\bar{y}_-})$,

where $\mathcal{W}^s(E_{\bar{y}_-})$ denotes the global stable manifold guaranteed by Theorem 3 and C_l, C_u are continuous non-decreasing curves emanating from E_N, which existence and properties are guaranteed by Corollary 2. Furthermore, every initial point below $\mathcal{W}^s(E_{\bar{y}_-})$ is attracted to $E_0(0,0)$ or $E_{\bar{x}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. The existence of the global stable manifold is guaranteed by Theorems 3 and 7.

By Theorem 6, every solution that starts on the y-axis below $E_{\bar{y}_-}$ converges to E_0 in a decreasing manner and every solution that starts on the x-axis is equal to $E_{\bar{x}}$ in a single step.

In addition, every solution that starts on y-axis above $E_{\bar{y}_-}$ converges to $E_{\bar{y}_+}$ in a monotonic way.

Let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant below $\mathcal{W}^s(E_{\bar{y}_-})\} \cup C_l$. Assume that $x_0 \geq \bar{x}_N$. Then $(x_0, y_W) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$ and so $T(x_0, y_W) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$, where $(x_0, y_W) \in C_l$ and so $T^n(x_0, y_W) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. Since $T^n(x_0, y_W) \to E_N$ and $T^n(x_0, 0) \to E_{\bar{x}}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_N, E_{\bar{x}})$, in which case, in view of Corollary 1, it converges to $E_{\bar{x}}$.

Next, assume that $0 < x_0 < \bar{x}_N$. Then $(x_0, y_W) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}_-})$ and so $T(x_0, y_W) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$ and so $T^n(x_0, y_W) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. Since $T^n(x_0, y_W) \to E_{\bar{y}_-}$ and $T^n(x_0, 0) \to E_{\bar{x}}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_{\bar{y}_-}, E_{\bar{x}})$, in which case, by Theorems 6 and 8, $T^n(x_0, y_0) \to E_{\bar{x}}$ or $T^n(x_0, y_0) \to E_0$ as $n \to \infty$.

Now, let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant above $\mathcal{W}^s(E_{\bar{y}_-}) \cup C_u$. Assume that $x_0 > \bar{x}_N$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$. Assume that $(x_0, y_W) \in C_u$. Thus $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$, which by and $T^n(0, y_0) \to E_{\bar{y}_+}$ and $T^n(x_0, y_W) \to E_N$ as $n \to \infty$, implies that $T^n(x_0, y_0)$ eventually the ordered interval $I(E_{\bar{y}_+}, E_N)$, in which case, in view of Corollary 1, it converges to $E_{\bar{y}_+}$.

Next, assume that $0 < x_0 \leq \bar{x}_N$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}_-})$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$. Since $T^n(x_0, y_W) \to E_{\bar{y}_-}$ and $T^n(0, y_0) \to E_{\bar{y}_+}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ converges to $E_{\bar{y}_+}$.

Finally, let (x_0, y_0) be an arbitrary initial point between C_l and C_u. Then $T^n(x_0, y_0)$ stays between C_l and C_u for all n and in view of Corollary 2 it must converge to E_N.

□

Conjecture 1 *Based on our numerical simulations we believe that $C_l = C_u$ in Theorem 12.*

Theorem 13 *Assume that $1 > 4A_2C_2$ and System (7) has five equilibrium points, $E_{\bar{x}}, E_{\bar{y}_+}$ which are locally asymptotically stable, $E_{\bar{y}_-}$ and E_{NW} (resp. E_{SE}) which are saddle points and E_{SW} which is a repeller. The singular point $E_0(0,0)$ is global attractor of all points on the y-axis, which start below $E_{\bar{y}_-}$. The basins of attraction of four of the equilibrium points are given as:*

$\{(x_0, y_0) : \text{points below } \mathcal{W}^s(E_{NW})\} \subset B(E_{\bar{x}})$,

$B(E_{\bar{y}_+}) = \{(x_0, y_0) : \text{points above } \mathcal{W}^s(E_{\bar{y}_-}) \cup \mathcal{W}^s(E_{NW})$,

$B(E_{NW}) = \mathcal{W}^s(E_{NW})$,

$B(E_{\bar{y}_-}) = \mathcal{W}^s(E_{\bar{y}_-})$,

where $\mathcal{W}^s(E_{\bar{y}_-})$ and $\mathcal{W}^s(E_{NW})$ denote the global stable manifolds which existence is guaranteed by Theorem 3. Furthermore, every initial point below $\mathcal{W}^s(E_{\bar{y}_-})$ is attracted to E_0 or $E_{\bar{y}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. We present the proof in the case of the equilibrium point E_{NW}. The proof in the case of the equilibrium point E_{SE} is similar.

The existence of the global stable manifold is guaranteed by Theorems 3 and 7.

By Theorem 6, every solution that starts on the y-axis below $E_{\bar{y}_-}$ converges to E_0 in a decreasing manner and every solution that starts on the x-axis is equal to $E_{\bar{x}}$ in a single step. In addition, every solution that starts on the y-axis above $E_{\bar{y}_-}$ converges to $E_{\bar{y}_+}$ in a monotonic way.

Let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant below $\mathcal{W}^s(E_{\bar{y}_-}) \cup \mathcal{W}^s(E_{NW})$. Assume that $x_0 > \bar{x}_{SW}$. Then $(x_0, y_W) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$ which implies $T(x_0, y_W) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$, where $(x_0, y_W) \in \mathcal{W}^s(E_{NW})$ and so $T^n(x_0, y_W) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. Since $T^n(x_0, y_W) \to E_{NW}$ and $T^n(x_0, 0) \to E_{\bar{x}}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_{NW}, E_{\bar{x}})$, in which case, in view of Corollary 1, it converges to $E_{\bar{x}}$.

Next, assume that $0 < x_0 \le \bar{x}_{SW}$. Then $(x_0, y_W) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}_-})$. Thus $T(x_0, y_W) \preceq_{se} T(x_0, y_0) \preceq_{se} T(x_0, 0) = E_{\bar{x}}$ and so $T^n(x_0, y_W) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0) = E_{\bar{x}}$. Since $T^n(x_0, y_W) \to E_{\bar{y}_-}$ and $T^n(x_0, 0) \to E_{\bar{x}}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the interior of the ordered interval $I(E_{\bar{y}_-}, E_{\bar{x}})$, in which case, it converges to E_0 or $E_{\bar{x}}$.

Now, let (x_0, y_0) be an arbitrary initial point in the interior of the first quadrant above $\mathcal{W}^s(E_{\bar{y}_-}) \cup \mathcal{W}^s(E_{NW})$. Assume $x_0 > \bar{x}_{SW}$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{NW})$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$. Since $T^n(0, y_0) \to E_{\bar{y}_+}$ and $T^n(x_0, y_W) \to E_{NW}$ as $n \to \infty$, then $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_{\bar{y}_+}, E_{NW})$, in which case, in view of Corollary 1, it converges to $E_{\bar{y}_+}$.

Next, assume that $0 < x_0 \le \bar{x}_{SW}$. Then $(0, y_0) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_W)$, where $(x_0, y_W) \in \mathcal{W}^s(E_{\bar{y}_-})$ and so $T^n(0, y_0) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_W)$. Since $T^n(x_0, y_W) \to E_{\bar{y}_-}$ and $T^n(0, y_0) \to E_{\bar{y}_+}$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ converges to $E_{\bar{y}_+}$.

□

Theorem 14 *Assume that $1 > 4A_2C_2$ and System (7) has six equilibrium points, $E_{\bar{x}}, E_{\bar{y}_+}$ which are locally asymptotically stable, $E_{\bar{y}_-}$ and E_{NE} (resp. E_{SE} or E_{NW}) which are saddle points, E_{SW} which is a repeller and E_N which is non-hyperbolic of the stable type. The singular point $E_0(0,0)$ is global attractor of all points on the y-axis, which start below $E_{\bar{y}_-}$. The basins of attraction of five of the equilibrium points are given as:*

$\{(x_0, y_0) :$ points below $\mathcal{W}^s(E_N) \subset B(E_{\bar{x}})$,

$B(E_{\bar{y}_+}) = \{(x_0, y_0) :$ points above $\mathcal{W}^s(E_{\bar{y}_-}) \cup \mathcal{W}^s(E_{NE})$,

$B(E_N) = \{(x_0, y_0) :$ region bounded by $\mathcal{W}^s(E_N)$ and $\mathcal{W}^s(E_{NE})\}$,

$B(E_{\bar{y}_-}) = \mathcal{W}^s(E_{\bar{y}_-})$,

$B(E_{NE}) = \mathcal{W}^s(E_{NE})$,

where $\mathcal{W}^s(E_{\bar{y}_-}), \mathcal{W}^s(E_N)$, and $\mathcal{W}^s(E_{NE})$ denote the global stable manifolds which existence is guaranteed by Theorem 3. Furthermore, every initial point below $\mathcal{W}^s(E_{\bar{y}_-})$ is attracted to E_0 or $E_{\bar{x}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. We present the proof in the case of the equilibrium point E_{NE}. The proof in the case of the equilibrium points E_{SE} and E_{NW} is similar.

The existence of the global stable manifolds are guaranteed by Theorems 3 and 7.

The proofs of the basins of attractions $B(E_{\bar{x}}), B(E_{\bar{y}_+})$ are the same as the proofs for the corresponding basins of attraction in Theorem 13, so we will only give the proof for $B(E_N)$. Indeed, $B(E_N)$ is an invariant set and $T^n(B(E_N))$ is a subset of the interior of the ordered interval $I(E_{NE}, E_N)$ for n large. In view of Corollary 1 the interior of the ordered interval $I(E_{NE}, E_N)$ is attracted to E_N.

□

Theorem 15 *Assume that $1 > 4A_2C_2$ and System (7) has seven equilibrium points, $E_{\bar{x}}, E_{\bar{y}_+}$, E_{NE} which are locally asymptotically stable, $E_{\bar{y}_-}, E_{SE}, E_{NW}$ which are saddle points and E_{SW} which is a repeller. The singular point $E_0(0,0)$ is global attractor of all points on y-axis, which start below $E_{\bar{y}_-}$. The basins of attraction of six of the equilibrium points are given as:*

$\{(x_0, y_0) :$ points below $\mathcal{W}^s(E_{SE}) \subset B(E_{\bar{x}})$,

$B(E_{\bar{y}_+}) = \{(x_0, y_0) :$ points above $\mathcal{W}^s(E_{\bar{y}_-}) \cup \mathcal{W}^s(E_{NW})$,

$B(E_{NE}) = \{(x_0, y_0) :$ region bounded by $\mathcal{W}^s(E_{SE})$ and $\mathcal{W}^s(E_{NW})\}$,

$B(E_{\bar{y}_-}) = \mathcal{W}^s(E_{\bar{y}_-})$,

$B(E_{SE}) = \mathcal{W}^s(E_{SE})$,

$B(E_{NW}) = \mathcal{W}^s(E_{NW})$,

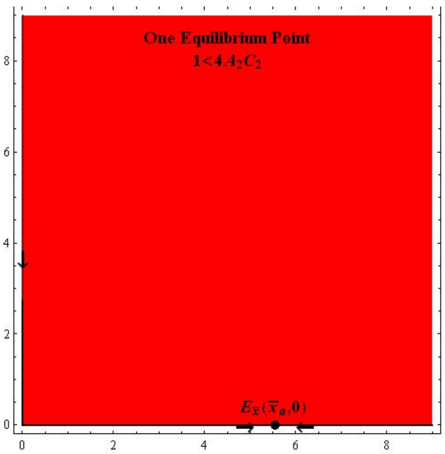
where $\mathcal{W}^s(E_{\bar{y}_-}), \mathcal{W}^s(E_{NW})$, and $\mathcal{W}^s(E_{SE})$ denote the global stable manifolds which existence is guaranteed by Theorem 3. Furthermore, every initial point below $\mathcal{W}^s(E_{\bar{y}_-})$ is attracted to E_0 or $E_{\bar{x}}$.

Proof. Local stability of all equilibrium points follows from Theorem 5. Proofs of the basins of attractions $B(E_{\bar{x}}), B(E_{\bar{y}_+})$ are same as the proofs for corresponding basins of attraction in Theorem 13. So we only give the proof for $B(E_{NE})$. Indeed, $B(E_{NE})$ is an invariant set and $T^n(B(E_{NE}))$ is a subset of the interior of the ordered interval $I(E_{NW}, E_{SE})$ for n large. In view of Corollary1 the interior of the ordered interval (E_{NW}, E_{SE}) is attracted to E_{NE}.

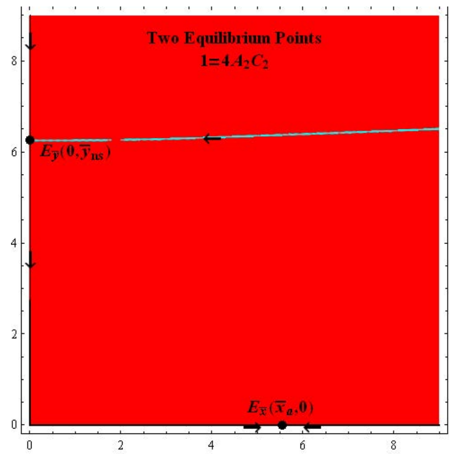
□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.



$B_1 = .18, C_1 = .062, A_2 = 3.25, B_2 = .002, C_2 = .08$



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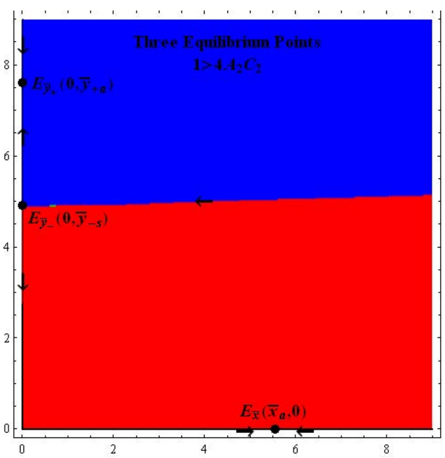

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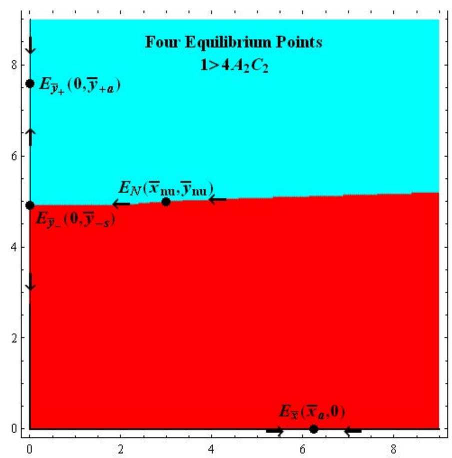
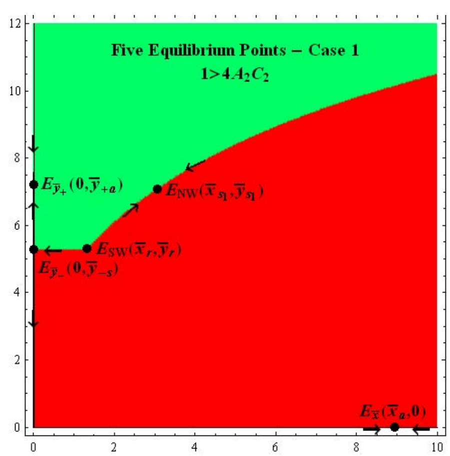
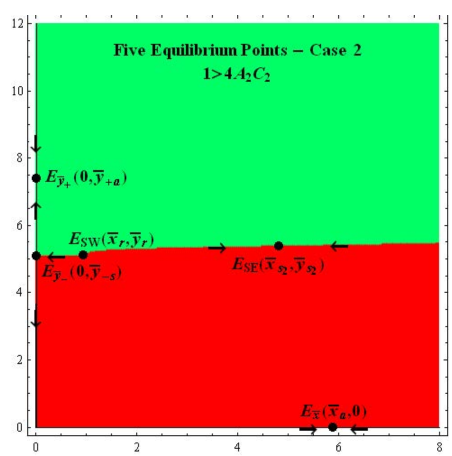


$B_1 = \frac{116}{723}, C_1 = \frac{15}{241}, A_2 = \frac{1491}{500}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$

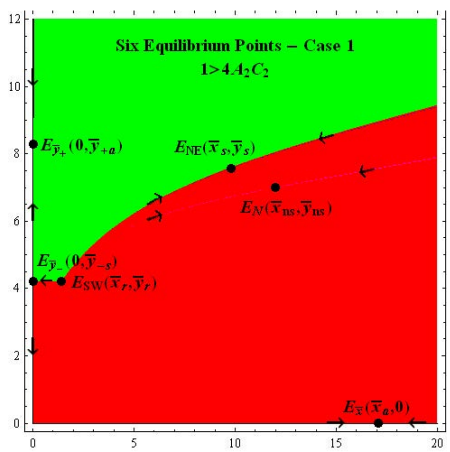


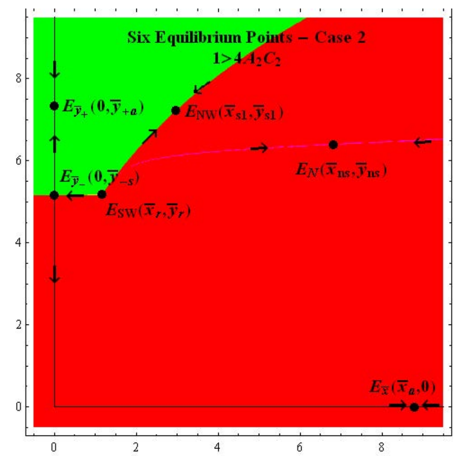
$B_1 = .112, C_1 = .04, A_2 = 3.05, B_2 = .002, C_2 = .08$



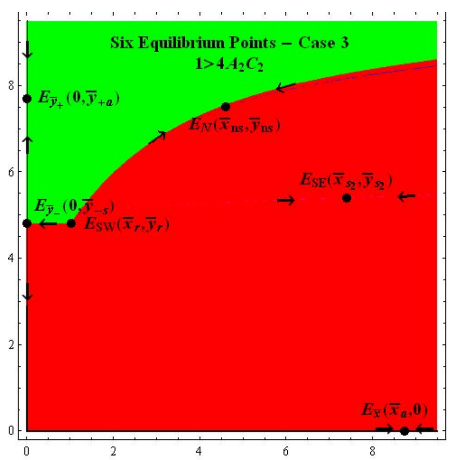
$B_1 = .17, C_1 = .03, A_2 = 3.02, B_2 = .002, C_2 = .08$

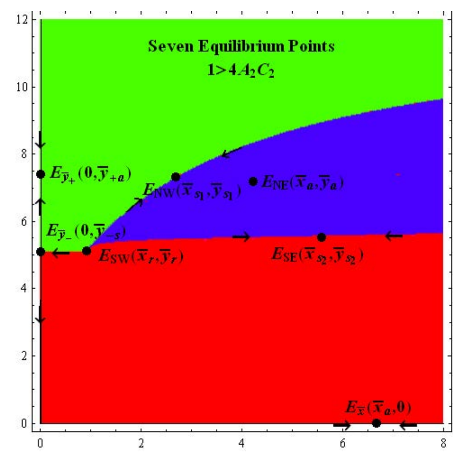
Figure 3: Global Stability





$B_1 = \frac{83}{1416}, C_1 = \frac{30}{413}, A_2 = \frac{349}{125}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$ $\qquad$ $B_1 = \frac{2045}{17986}, C_1 = \frac{1275}{33856}, A_2 = \frac{9471}{3125}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$





$B_1 = \frac{52165}{455584}, C_1 = \frac{575}{14856}, A_2 = \frac{36971}{12500}, B_2 = \frac{2}{1000}, C_2 = \frac{8}{100}$ $\qquad$ $B_1 = .15, C_1 = .03, A_2 = 3.02, B_2 = .002, C_2 = .08$

Figure 4: Global Stability

References

[1] L. J. S. Allen, An Introduction to Mathematical Biology, Prentice Hall, (2006).

[2] S. Basu and O. Merino, On the global behavior of solutions to a planar system of difference equations. *Comm. Appl. Nonlinear Anal.* 16 (2009), 89–101.

[3] A. Brett and M. R. S. Kulenović, Basins of attraction of equilibrium points of monotone difference equations, *Sarajevo J. Math.*, 5(18) (2009), no. 2, 211-233.

[4] Dž. Burgić, S. Kalabušić, and M. R. S. Kulenović, Nonhyperbolic Dynamics for Competitive Systems in the Plane and Global Period-doubling Bifurcations, *Adv. Dyn. Syst. Appl.* 3(2008), 229-249.

[5] E. Camouzis, M. R. S. Kulenović, O. Merino, and G. Ladas, Rational Systems in the Plane, *J. Difference Equ. Appl.* 15 (2009), 303-323.

[6] Y. Chow and S. Jang, Multiple attractors in a LeslieGower competition system with Allee effects, *J. Difference Equ. Appl.* 20 (2014), 169–187.

[7] D. Clark and M. R. S. Kulenović, On a Coupled System of Rational Difference Equations, *Comput. Math. Appl.* 43 (2002), 849-867.

[8] D. Clark, M. R. S. Kulenović, and J. F. Selgrade, Global Asymptotic Behavior of a Two Dimensional Difference Equation Modelling Competition, *Nonlinear Anal., TMA* 52(2003), 1765-1776.

[9] J. M. Cushing, The Allee effect in age–structured population dynamics, *Mathematical Ecology*, T. G. Hallam, L. J. Gross, and S. A. Levin, eds., World Scientific Publishing Co., Singapore, 1988, 479–505.

[10] J. M. Cushing, S. Levarge, N. Chitnis and S. M. Henson, Some discrete competition models and the competitive exclusion principle , *J. Difference Equ. Appl.* 10(2004), 1139-1152.

[11] P. de Mottoni and A. Schiaffino, Competition systems with periodic coefficients: A geometric approach, *J. Math. Biol.* **11** (1981), 319–335.

[12] E. G. Enciso and E. D. Sontag, Global attractivity, I/O monotone small-gain theorems, and biological delay systems. *Discrete Contin. Dyn. Syst.* 14 (2006), 549–578.

[13] J. E. Franke and A-A. Yakubu, Mutual exclusion versus coexistence for discrete competitive systems, *J. Math. Biol.* 30(1991), 161–168.

[14] J. E. Franke and A-A. Yakubu, Global attractors in competitive systems, *Nonlin. Anal. TMA* 16(1991), 111–129.

[15] J. E. Franke and A-A. Yakubu, Geometry of exclusion principles in discrete systems, *J. Math. Anal. Appl.* 168(1992), 385–400.

[16] A. J. Harry, C. M. Kent and V. L. Kocic, Global behavior of solutions of a periodically forced Sigmoid Beverton–Holt model, *J. Biol. Dynamics*, 2012, 212–234.

[17] M. P. Hassell and H. N. Comins, Discrete time models for two-species competition, *Theor. Pop. Biol.* 9(1976), 202–221.

[18] P. Hess, *Periodic-parabolic boundary value problems and positivity.* Pitman Research Notes in Mathematics Series, 247. Longman Scientific & Technical, Harlow, 1991.

[19] M. Hirsch and H. Smith, Monotone Dynamical Systems, Handbook of Differential Equations, Ordinary Differential Equations (second volume), 239-357, Elsevier B. V., Amsterdam, 2005.

[20] M. Hirsch and H. L. Smith, *Monotone Maps: A Review*, *J. Difference Equ. Appl.* 11(2005), 379-398.

[21] F. Jones and J. Perry, Modelling populations of cyst-nematodes (nematoda: heteroderidae), *J. Appl. Ecology*, 15(1978), 349–371.

[22] H. Jiang and T. D. Rogers, The discrete dynamics of symmetric competition in the plane, *J. Math. Biol.* 25(1987), 573–596.

[23] S. Kalabušić, M. R. S. Kulenović and E. Pilav, Dynamics of a Two-dimensional System of Rational Difference Equations of Leslie-Gower type, *Advances in Difference Equations*, 2011 (2011), 29 pages.

[24] S. Kalabušić , M. R. S. Kulenović and E. Pilav, Multiple Attractors for a Competitive System of Rational Difference Equations in the Plane, *Abstract and Applied Analysis*, Volume 2011,(2011), Article ID 295308, 35 pages.

[25] Y. Kan and H. L. Smith, l Global dynamics of a discrete two-species lottery-Ricker competition model. *J. Biol. Dyn.* 6 (2012), no. 2, 358-376.

[26] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordreht/Boston/London, 1993.

[27] W. Krawcewicz and T. D. Rogers, Perfect harmony: the discrete dynamics of cooperation, *J. Math. Biol.* 28(1990), 383–410.

[28] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations,* Chapman & Hall/CRC, Boca Raton, London, 2001.

[29] M. R. S. Kulenović and O. Merino, *Disrete Dynamical Systems and Difference Equations with Mathematica*, Chapman& Hall/CRC Press, Boca Raton, 2002.

[30] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane, *Discrete Cont. Dyn. Syst.Ser. B* 6(2006), 1141–1156.

[31] M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B* 12(2009), 133-149.

[32] M. R. S. Kulenović and O. Merino, Invariant Manifolds for Competitive Discrete Systems in the Plane, *Int. J. of Bifurcations and Chaos*, 20(2010), 2471-2486.

[33] M. R. S. Kulenović and O. Merino, Invariant Curves of Planar Competitive and Cooperative Maps, (to appear).

[34] M. R. S. Kulenović, O. Merino and M. Nurkanović, Dynamics of Certain Competitive System in the Plane, *J. Difference Equ. Appl.*, 18(2012), 1951–196.

[35] M. R. S. Kulenović and M. Nurkanović, Global asymptotic behavior of a two dimensional system of difference equations modelling cooperation, *J. Difference Equ. Appl.* 9(2003), 149-159.

[36] M. R. S. Kulenović and M. Nurkanović, Asymptotic Behavior of a Linear Fractional System of Difference Equations, *J. Ineq. Appl.*, (2005), 127-144.

[37] M. R. S. Kulenović and M. Nurkanović, Asymptotic behavior of a competitive system of linear fractional difference equations. *Adv. Difference Equ.* 2006, Art. ID 19756, 13 pp.

[38] M. R. S. Kulenović and Yakubu, A.-A., Compensatory versus overcompensatory dynamics in density-dependent Leslie models. *J. Difference Equ. Appl.* 10 (2004), 1251-1265.

[39] Robert M. May, Stability and Complexity in Model Ecosystems, Princeton University Press Princeton, NJ, 2001. xx+292 pp.

[40] G. Livadiotis and S. Elaydi, General Allee effect in two-species population biology, *J. of Biological Dynamics*, 6(2012), 959–973.

[41] G. Livadiotis, L. Assas, S. Elaydi, E. Kwessi and D. Ribble, Competition models with Allee effects, *J. Difference Equ. Appl.*, 20(2014), 1127–1151.

[42] H. Sedaghat, *Nonlinear difference equations. Theory with applications to social science models.* Mathematical Modelling: Theory and Applications. 15., Kluwer Academic Publishers, Dordrecht, 2003.

[43] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations* 64 (1986), 165-194.

[44] H. L. Smith, Periodic solutions of periodic competitive and cooperative systems, *SIAM J. Math. Anal.* **17**(1986), 1289–1318.

[45] H. L. Smith, Planar Competitive and Cooperative Difference Equations, *J. Difference Equ. Appl.* , 3(1998), 335–357.

[46] H. R. Thieme, Mathematics in population biology. Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton, NJ, 2003. xx+543 pp.

[47] G.G. Thomson, A proposal for a threshold stock size and maximum fishing mortality rate, in Risk Evaluation and Biological Reference Points for Fisheries Management, *Canad. Spec. Publ. Fish. Aquat. Sci.*, Vol. 120, S. J. Smith, J. J. Hunt, and D. Rivard, eds., 1993, pp. 303320.

[48] A-A. Yakubu, The effect of planting and harvesting on endangered species in discrete competitive systems, *Math. Biosci.* 126(1995), 1–20.

[49] A-A. Yakubu, A discrete competitive system with planting, *J. Difference Equ. Appl.,* 4(1998), 213–214.

[50] L. Yang, X. Hou, and Z. Zeng. Complete discrimination system for polynomials. *Science in China (Series E)*, 39(6):628-646, 1996.