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05. Random Variables: Applications

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Abstract
Part five of course materials for Nonequilibrium Statistical Physics (Physics 626), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [ntc5]

[nex14] Reconstructing probability distributions

Determine three probability distributions $P_X(x)$ from the following information:
(a) $\langle X^n \rangle = a^n n!$ for $n \geq 0$,
(b) $\langle\langle X^n \rangle\rangle = a^n (n-1)!$ for $n \geq 1$,
(c) $\langle X^n \rangle = a^n/(n+1)$ for even n and $\langle X^n \rangle = 0$ for odd n.

Solution:

[nex95] Probability distribution with no mean value

Consider the function $P_X(x) = x^{-1}e^{-x}I_1(x)$ for $0 < x < \infty$, where $I_1(x)$ is a modified Bessel function.
(a) Show that $P_X(x)$ is normalized to unity.
(b) Produce a plot of $P_X(x)$ for $0 < x < 6$.
(c) Show that a mean value $\langle x \rangle$ does not exist.
(d) Calculate the median value x_m from $\int_0^{x_m} dx P_X(x) = 1/2$.

Solution:

[nex20] Variances and covariances.

A stochastic variable X can have values $x_1 = 1$ and $x_2 = 2$ and a second stochastic variable Y the values $y_1 = 2$ and $y_2 = 3$. Determine the variances $\langle\langle X^2 \rangle\rangle$, $\langle\langle Y^2 \rangle\rangle$ and the covariance $\langle\langle XY \rangle\rangle$ for two sets of joint probability distributions as defined in [nln7]:

(i) $P(x_1, y_1) = P(x_1, y_2) = P(x_2, y_1) = P(x_2, y_2) = \frac{1}{4}$.

(ii) $P(x_1, y_1) = P(x_2, y_2) = 0, \quad P(x_1, y_2) = P(x_2, y_1) = \frac{1}{2}$.

Solution:

[nex23] Statistically independent or merely uncorrelated?

Consider a classical spin, described by a 3-component unit vector

$$\mathbf{S} = (S_x, S_y, S_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Let us assume that the spin has a completely random orientation, meaning a uniform distribution on the unit sphere. Show that the stochastic variables $\cos\theta, \phi$ are uncorrelated and statistically independent, whereas the stochastic variables S_x, S_z are uncorrelated but not statistically independent. This difference is testimony to the special role of canonical coordinates (here $\cos\theta, \phi$) in statistical mechanics.

Solution:

[nex96] Sum and product of uniform distributions

Consider two independent random variables X_1, X_2, both uniformly distributed on the interval $0 < x_1, x_2 < 1$: $P_i(x_i) = \Theta(x_i)\Theta(1 - x_i)$, $i = 1, 2$. Find range and probability distribution of
(a) the random variable $Y = X_1 + X_2$,
(b) the random variable $Z = X_1 X_2$.
Check the normalization in both cases.

Solution:

[nex79] Exponential integral distribution

Consider two independent random variables X_1, X_2, one exponentially distributed, $P_1(x_1) = e^{-x_1},\ 0 < x_1 < \infty$, and the other uniformly distributed, $P_2(x_2) = 1,\ 0 < x_2 < 1$.
(a) Determine the probability distribution $P_Z(z)$ of the random variable $Z = X_1X_2$ for $0 < z < \infty$.
(b) Determine the asymptotic properties of $P_Z(z)$ for $z \to 0$ and for $z \to \infty$.
(c) Calculate the moments $\langle z^n \rangle$ of $P_Z(z)$.
(d) Plot $P_Z(z)$ for $0 < z < 6$.

Solution:

[nex80] Generating exponential and Lorentzian random numbers

Given is a sequence of uniformly distributed random numbers $x_1, x_2, \ldots$ with $0 < x_i < 1$ as produced by a common random number generator.

(a) Find the transformation $Z = Z(X)$ which produces a sequence of random numbers $z_1, z_2, \ldots$ with an exponential distribution:

$$P_Z(z) = \frac{1}{\zeta}\, e^{-z/\zeta}, \quad \zeta > 0.$$

(b) Find the transformation $Y = Y(X)$ which produces a sequence of random numbers $y_1, y_2, \ldots$ with a Lorentzian distribution:

$$P_Y(y) = \frac{1}{\pi} \frac{a}{y^2 + a^2}, \quad a > 0.$$

Solution:

[nex5] Random chords (Bertrand's paradox)

Consider a circle of unit radius and draw *at random* a straight line intersecting it in a chord of length L
(a) by taking lines through an arbitrary fixed point on the circle with random orientation;
(b) by taking lines perpendicular to an arbitrary diameter of the circle with the point of intersection chosen randomly on the diameter;
(c) by choosing the midpoint of the chord at random in the area enclosed by the circle.
For each *random choice* determine the probability distribution $P(L)$ for the length of the chord and calculate the average length $\langle L \rangle$.

Solution:

[nex8] From Gaussian to exponential distribution

A random variable X has a continuous Gaussian distribution $P_X(x)$ with mean value $\langle X\rangle = 0$ and variance $\langle\langle X^2\rangle\rangle = 1$. Find the distribution function $P_Y(y)$ for the stochastic variable Y with values $y = x_1^2 + x_2^2$, where x_1, x_2 are independent realizations of the random variable X. Calculate the mean value $\langle Y\rangle$ and the variance $\langle\langle Y^2\rangle\rangle$.

Solution:

[nex78] Transforming a pair of random variables

Consider two independent random variables X_1, X_2 that are uniformly distributed on the intervals $0 \leq x_1, x_2 \leq 1$. Show that the transformed variables

$$Y_1 = \sqrt{-2\ln X_1}\cos 2\pi X_2, \quad Y_2 = \sqrt{-2\ln X_1}\sin 2\pi X_2$$

obey independent normal distributions:

$$P_{\mathbf{Y}}(y_1, y_2) = \frac{1}{\sqrt{2\pi}}\, e^{-y_1^2/2}\, \frac{1}{\sqrt{2\pi}}\, e^{-y_2^2/2}.$$

Solution:

[nex3] Gaussian shootist versus Lorentzian shootist

The shots of two marksmen on a square-shaped target of dimensions 20cm×20cm are found to be distributed with probability densities

$$P_1(x,y) = C_1 e^{-(x^2+y^2)}, \qquad P_2(x,y) = \frac{C_2}{1+x^2+y^2},$$

where $r = \sqrt{x^2+y^2}$ is the distance from the center of the target, and C_1, C_2 are normalization constants. Answer the following questions separately for each marksman.
(a) What is the probability that a given shot that hits the target is at least 1cm high ($y > 1$cm)?
(b) Given that a shot that hits the target is at least 1cm high ($y > 1$cm), what is the probability that it is also at least 1cm to the right ($x > 1$cm)?

Solution:

[nex16] Moments and cumulants of the Poisson distribution.

Calculate the generating function $G(z) \equiv \langle z^n \rangle$ and the characteristic function $\Phi(k) \equiv \langle e^{ikn} \rangle$ for the Poisson distribution

$$P(n) = \frac{a^n}{n!}\, e^{-a}, \quad n = 0, 1, 2, \ldots$$

From $\Phi(k)$ calculate the cumulants $\langle\langle n^m \rangle\rangle$. From $G(z)$ calculate the factorial moments $\langle n^m \rangle_f$ and the factorial cumulants $\langle\langle n^m \rangle\rangle_f$.

Solution:

[nex17] Maxwell velocity distribution

In the original derivation of the velocity distribution $f(v_x, v_y, v_z)$ for a classical ideal gas, Maxwell used the following ingredients: (i) The Cartesian velocity components v_x, v_y, v_z (interpreted as stochastic variables) are statistically independent. (ii) The distribution $f(v_x, v_y, v_z)$ is spherical symmetric. (iii) The mean-square velocity follows from the equipartition theorem. Determine $f(v_x, v_y, v_z)$ along these lines.

Solution:

[nex18] Random bus schedules.

Three bus companies A, B, C offer schedules in the form of a probability density $f(t)$ for the intervals between bus arrivals at the bus stop:

$$A: \quad f(t) = \delta(t - T), \qquad B: \quad f(t) = \frac{1}{T}\, e^{-t/T}, \qquad C: \quad f(t) = \frac{4t}{T^2}\, e^{-2t/T}.$$

(i) Find the probability $P_0(t)$ that the interval between bus arrivals is larger than t.
(ii) Find the mean time interval τ_B between bus arrivals and the variance thereof.
(iii) Find the probability $Q_0(t)$ that no arrivals occur in a randomly chosen time interval t.
(iv) Find the probability density $g(t)$ of the time a passenger waits for the next bus from the moment he/she arrives at the bus stop.
(v) Find the average waiting time τ_P of passengers and the variance thereof.

Solution:

[nex106] Life expectancy of the young and the old

The distribution of life times in some population is $f(t) = (4t/T^2)e^{-2t/T}$.
(a) Show that the parameter T is the average life time of individuals.
(b) Calculate the conditional probability distribution $P_c(t|\tau)$ for individuals of age τ.
(c) If we define the *life expectancy* T_τ as the average remaining life time for an individual of age τ calculate T_τ as a function of T and τ.
(d) What is the life-expectancy ratio of the very young and the very old.

Solution:

[nex38] Life expectancy of the ever young

The probability distribution of life times in some population is $f(t)$ with an average life time T for individuals.
(a) Express the conditional probability distribution $P_c(t|\tau)$ for individuals of age τ in terms of $f(t)$. Enforce normalization of $P_c(t|\tau)$ under the assumption that $f(t)$ is normalized.
(b) If we define the *life expectancy* T_τ as the average remaining life time for an individual of age τ express T_τ in terms of $P_c(t|\tau)$.
(c) Find the function $f(t)$ for the case where the life expectancy is independent of the age of the individual, i.e. for the case where $T_\tau = T$ holds. Then infer an explicit expression for $P_c(t|\tau)$.

Solution:

[nex35] Random frequency oscillator

Consider a physical ensemble of classical harmonic oscillators with randomly distributed angular frequencies: $P_\Omega(\omega) = \frac{1}{2}\Theta(1-|\omega|)$. At time $t=0$ all oscillators are excited in phase with unit amplitude: $Y(t) = \cos(\omega t)$.
(a) Find the average displacement $\langle Y(t)\rangle$ and its variance $\langle\langle Y^2(t)\rangle\rangle$ as functions of t. What are the long-time asympotic values of these two quantities?
(b) Find the autocorrelation function $\langle Y(t+\tau)Y(t)\rangle$ for arbitrary t,τ and its asymptotic τ-dependence for $t\to\infty$.
(c) Show that the probability distribution of Y for $m\pi \leq t < (m+1)\pi$ is

$$P(y,t) = \frac{m}{t\sqrt{1-y^2}}\,\Theta(1-|y|) + \frac{1}{t\sqrt{1-y^2}}\,\Theta(y_{max}-y)\Theta(y-y_{min}),$$

where $y_{max}=1$, $y_{min}=\cos t$ if $m=0,2,4,\ldots$ and $y_{max}=\cos t$, $y_{min}=-1$ if $m=1,3,5,\ldots$. Find the asymptotic distribution $P(y,\infty)$.

Solution: