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01. Introduction: Maps

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract
Part one of course materials for Nonequilibrium Statistical Physics (Physics 626), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

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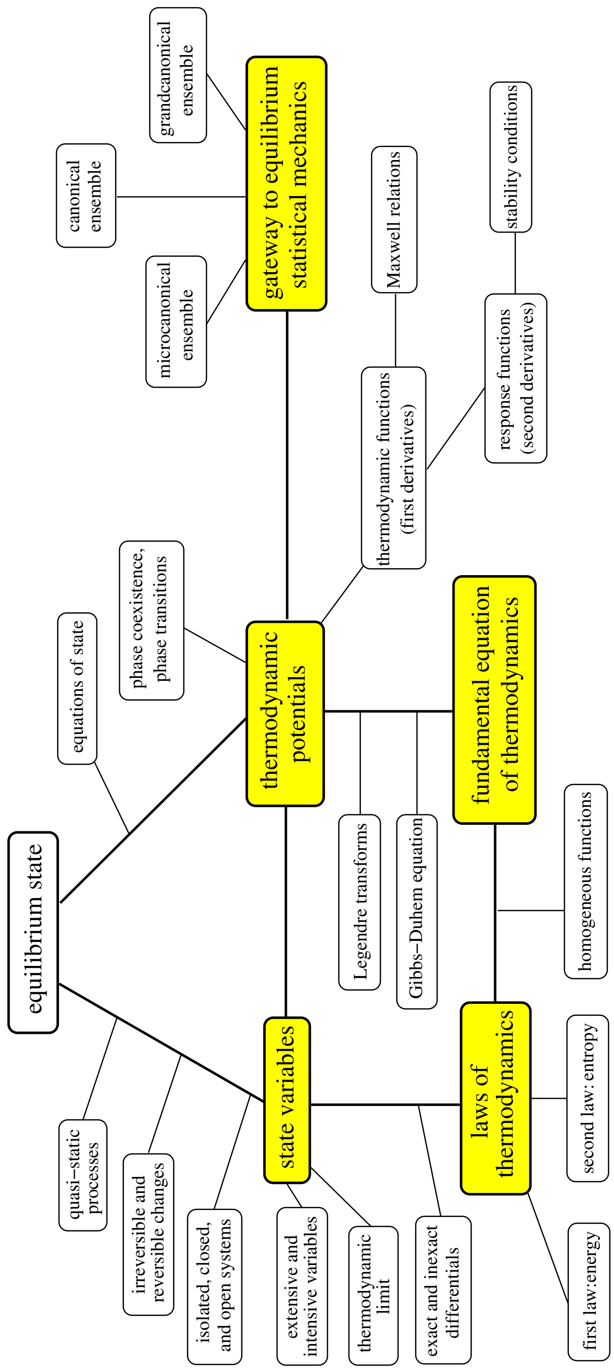
Contents of this Document [ntc1]

Equilibrium Thermodynamics Overview [nln6]



Thermal Equilibrium and Nonequilibrium [nln1]

Equilibrium State	Nonequilibrium State	
at equilibrium	**near equilibrium**	**far from equilibrium**
principle of maximum entropy	principle of minimum entropy production	nonlinear dynamics and fluctuations
quasi–static processes	irreversible processes	stochastic processes
fluctuations at equilibrium static and dynamic correlation functions	linear response to static and dynamic external fields	
(computational probes)	(experimental probes)	

fluctuation–dissipation theorem

Distinguish independently between

- *equilibrium* and *nonequilibrium* situations,
- *time-independent* and *time-dependent* phenomena.

	equilibrium situation	**nonequilibrium situation**
time–independent phenomena	equal–time correlations	equal–time correlations in steady states
time–dependent phenomena	delayed–time correlations	delayed–time correlations in steady states any correlations in non–steady states

Levels of Description in Statistical Physics [nln2]

microscopic level	**kinetic level**	**thermodynamic level**
N−particle phase space	1−particle phase space	configuration space
Liouville equation generalized Langevin equation	Boltzmann equation Fokker−Planck equation Langevin equation	hydrodynamic equations master equation Fokker−Planck equation Langevin equation
no contraction	some contraction	more contraction
deterministic time evolution	probabilistic time evolution	

Contraction – memory – time scales [nln15]

microscopic dynamics	$\Rightarrow$ **contraction** $\Rightarrow$	**stochastic dynamics**
future state determined by present state alone	focus on subset of dynamical variables	future state determined by present and past states
deterministic time evolution of dynamic variables	$\Downarrow$	ignoring memory of past makes dynamics of selected variables probabilistic
	judicious choice: slow variables and long time scales	deterministic time evolution of probability distributions and mean values
	$\Downarrow$ $\Rightarrow$	short memory of fast variables has little impact on dynamics of slow variables at long times

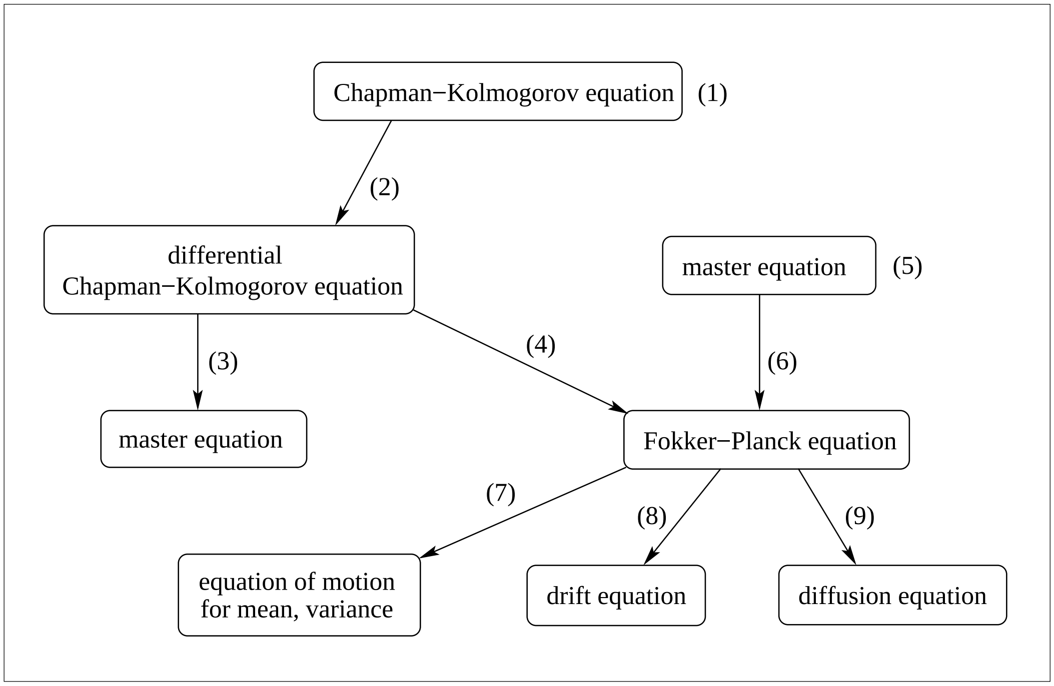
Degrees of memory:

- **Future independent of present and past.**
  Completely factorizing process.
  $P_j(x_1, t_1; x_2, t_2; \ldots) = P_s(x_1, t_1)P_s(x_1, t_1)\cdots$.

- **Future dependent on present only.**
  Markov process.
  $P_c(x_1, t_1; x_2, t_2; \ldots |\bar{x}_1, \bar{t}_1; \bar{x}_2, \bar{t}_2, \ldots) = P_c(x_1, t_1; x_2, t_2; \ldots |\bar{x}_1, \bar{t}_1)$
  for $t_1 \geq t_2 \geq \cdots \geq \bar{t}_1 \geq \bar{t}_2 \geq \cdots$.

- **Future dependent on present and past.**
  Non-Markovian process.
  $P_c(x_1, t_1; x_2, t_2; \ldots |\bar{x}_1, \bar{t}_1; \bar{x}_2, \bar{t}_2, \ldots)$.

Markov processes: map of specifications [nln16]



(1) Chapman-Kolmogorov equation imposes restrictions on permissible functions $P(x,t|x_0)$ but does not suggest a classification of processes.

(2) Particular solutions that are specified by

- $A(x,t)$ describing drift,
- $B(x,t)$ describing diffusion,
- $W(x|x';t)$ describing jumps.

(3) Jump processes exclusively.

(4) Processes with continuous sample paths, satisfying Lindeberg criterion (drift and diffusion, no jumps).

(5) Master equation with any $W(x|x';t)$ specifies a Markov process. Natural starting point for processes with discrete stochastic variables.

(6) Transition rates $W(x|x';t)$ of master equation approximated by two jump moments provided they exist. Approximation captures drift and diffusion parts (on some scale).

(7) Drift and diffusion determine mean $\langle\langle x\rangle\rangle$ and variance $\langle\langle x^2\rangle\rangle$ via equations of motion for jump moments.

(8) Deterministic process have no diffusive part: $B(x,t)=0$.

(9) Purely diffusive processes have no drift: $A(x,t)=0$.

Brownian motion: panoramic view [nln23]

- Levels of contraction (horizontal)
- Modes of description (vertical)

$\longrightarrow$ contraction $\longrightarrow$

relevant space	N-particle phase space	1-particle phase space	configuration space
dynamical variables	$\{\mathbf{x}_i,\, \mathbf{p}_i\}$	$\mathbf{x}$, $\mathbf{p}$	$\mathbf{x}$
theoretical framework	Hamiltonian mechanics	Langevin theory	Einstein theory
... for dynamical variables	generalized Langevin equation	Langevin equation (for $dt \ll \tau_R$)	Langevin equation (for $dt \gg \tau_R$)
... for probability distribution	quant./class. Liouville equation	Fokker-Planck equation (Ornstein-Uhlenbeck process)	Fokker-Planck equation (diffusion process)

- Here dt is the time step used in the theory and τ_R is the relaxation time associated with the drag force the Brownian particle experiences.
- The generalized Langevin equation is equivalent to the Hamiltonian equation of motion for a generic classical many-body system and equivalent to the Heisenberg equation of motion for a generic quantum many-body system.

Linear response and equilibrium dynamics [nln24]