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THE TREE-DEPTH AND CRITICALITY OF UNICYCLIC GRAPHS

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THE TREE-DEPTH AND CRITICALITY OF UNICYCLIC GRAPHS

BY

LILITH WAGSTROM

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

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MASTER OF SCIENCE THESIS

OF

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ABSTRACT

The tree-depth of a graph G is defined as the smallest k for which there exists a proper labeling L of G such that if $L(x) = L(y)$ then every x, y-path must contain a vertex z with $L(z) > L(x)$. The graph G is k-critical if it has tree-depth k and every proper minor of G has tree-depth at most $k - 1$.

We investigate when unicyclic graphs are critical. Despite unicyclic graphs' relatively simple structure it is surprisingly difficult to classify when they are critical. Part of this difficulty arises from the large variance of structures in unicyclic graphs. We present several families of critical unicyclic graphs that vary greatly in structure. In addition to these results, we present patterns present in the optimal feasible tree-depth labelings of cycles and general graphs.

ACKNOWLEDGMENTS

I would like to give my deepest appreciation to my major adviser Michael Barrus for his help and support on this thesis. I would also like to thank my committee members for their support and advice.

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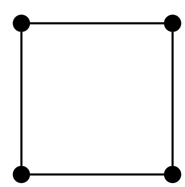
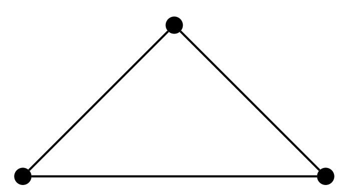
CHAPTER 1

Introduction

In this paper, all graphs are nonempty, finite simple graphs unless otherwise noted. In general, we follow West [1] for terminology and notation. We use P_n and C_n to respectively denote a path and cycle on n vertices. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$, and for $m \in \mathbb{N}$ we write $[m]$ for the set $\{1, \ldots, m\}$. The graph H is a *minor* of a graph G if H can be obtained from edge contraction on a subgraph of G. A *proper minor* of G is a minor different from G. For instance if we let $G = C_4$,

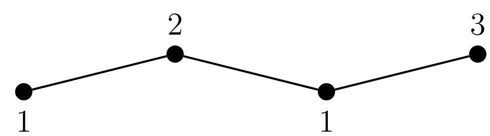


a proper minor of G would be C_3.

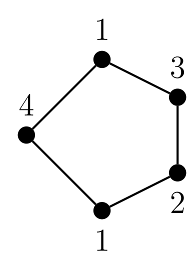


For many vertex labeling problems, we want to label the vertices of a graph G with labels from $[n]$ for some n such that no two adjacent vertices have the same label. Such a labeling is called a *proper labeling* or *proper n-labeling.* In the late 1990s, Bodlaender et al. [2] and Katchaliski et al. [3] independently introduced the tree-depth problem (at the time introduced as the vertex ranking problem and ordered coloring problem, respectively) as a more restrictive version of the proper labeling problem. A labeling L of a graph G is called a *feasible labeling* if for all $x, y \in V(G)$ such that $L(x) = L(y)$, every x, y-path in G contains a vertex z with

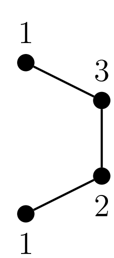
$L(z) > L(x)$. A *feasible k-labeling* is a feasible labeling that uses at most k labels. Note that a feasible labeling is a proper labeling. The *tree-depth* of a graph G is the smallest $k \in \mathbb{N}$ such that a feasible k-labeling exists. We denote the tree-depth of a graph G as $\text{td}(G)$. A feasible labeling of a graph G with $\text{td}(G) = k$ is called *optimal* if the labeling uses k labels. For instance the following is an example of an optimal feasible labeling of P_4.



We are also able to view tree-depth as an elimination ordering. Here we iteratively remove the vertices with the highest label. Feasible labelings are equivalent to elimination orderings where at each step we remove at most one vertex from each component. For instance we consider C_5 with the following labeling.



If we delete the vertex labeled with 4 we get a copy of P_4.



Then if we delete the vertex labeled with 3 we get the disjoint union of P_2 and P_1. We can keeping doing this for the vertices labeled with 2 and then 1. After we delete the vertices labeled with 1 we should have the empty graph.

The notion of tree-depth has received much attention since being introduced. Part of this interest in tree-depth has been directed to what we call *tree-depth critical* graphs. In the literature (such as in [4]), there are three different ways we can define a graph to be tree-depth critical. The first and most common is minor tree-depth critical. A graph G with $\mathrm{td}(G) = k$ is *minor tree-depth k-critical* if all proper minors H of G are such that $\mathrm{td}(H) < k$. We define *subgraph tree-depth critical* graphs and *induced subgraph tree-depth critical* graphs similarly. In this paper when we refer to a graph as being *k-critical* we mean that the graph is minor tree-depth k-critical. The k-critical graphs for $k \leq 4$ are well understood from the results of Dvórak, Giannopoulou and Thilikos [4, 5]. However, these authors also showed that trying to classify all k-critical graphs for $k \geq 5$ is much more difficult. For instance, they were able to find 1,044 different graphs that are 5-critical.

The paper [4] proposed the following conjecture:

Conjecture 1. *Every k-critical graph has as most 2^{k-1} vertices.*

Barrus and Sinkovic [6] proposed the following:

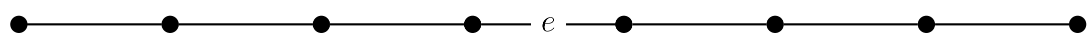
Conjecture 2. *Every k-critical graph has maximum degree at most $k-1$.*

These two conjectures combined with the difficulty of trying to classify all k-critical graphs for $k \geq 5$ generate interest in the structure of critical graphs. Dvórak et al. [4] showed the following.

Theorem 3. *Let G and H be k-critical graphs. Also let $x \in V(G)$ and $y \in V(H)$. Let G' be the graph obtained by adding the edge xy to the graph disjoint union $G + H$. Then G' is $(k+1)$-critical.*

However, not all critical graphs can be generated in this manner. In 2015 Barrus and Sinkovic [6] showed that another form of construction is possible. For their construction we let H be a s-critical graph with vertices $v_1, \ldots, v_j$ and $L_1, \ldots, L_j$ be $(r+1)$-critical graphs. We then form a graph G by taking vertices w_i for each L_i and identifying w_i and v_i for all $i \in [j]$. Barrus and Sinkovic were able to show that this construction is $(r+s)$-critical.

Since there are multiple ways to construct critical graphs it raises the question of if there are still ways to easily and quickly classify critical graphs. In regards to this, Dvorak et al. [4] showed a simple way to identify critical trees. They showed that if T is a k-critical tree then there exists an edge e in T such that $T - e$ is the disjoint union of two $(k-1)$-critical trees. So for instance, suppose we have the tree P_8. Note that P_8 is 4-critical. Suppose we remove the edge e pictured below.



After we remove e we are left with two disjoint copies of P_4 which is 3-critical.

Corollary 4. *Let T be a k-critical tree. Then the order of T is exactly* 2^{k-1}.

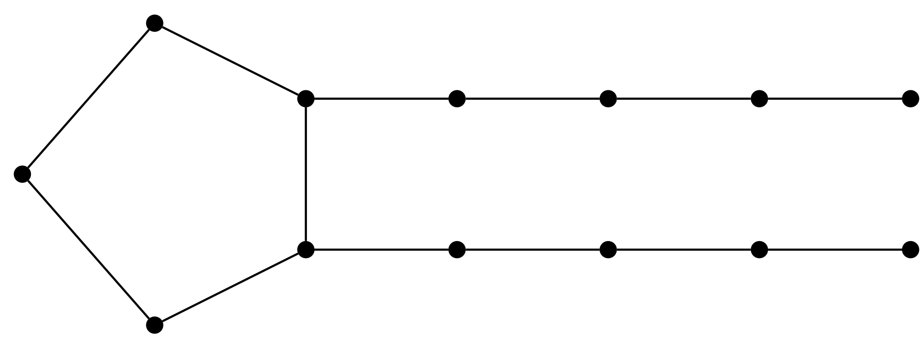
It is of interest to see if there are simple ways to see classify other families of critical graphs. So we can look at families of graphs that are not very complex structurally and try to find ways to easily determine when they are critical. Since we fully understand when trees are critical we want to look at unicyclic graphs since they have a fairly simple structure. Dvorak et al. [4] were able to classify all critical cycles. They showed that for $k \geq 3$ that $C_{2^{k-2}+1}$ is the only k-critical cycle. However, there are more unicyclic graphs than just the cycles and from both ways to construct critical graphs we have seen we care about the tree-depth of the graphs we are building upon. Since we can view every unicyclic graph as building upon a cycle it is valuable to understand the tree-depth of every cycle.

From Dvorak et al. [4] results on critical cycles we can derive the tree-depth of all other cycles.

Theorem 5. *Let $n \geq 3$ and $k \in \mathbb{N}$. If $2^{k-2}+1 \leq n < 2^{k-1}+1$ then* $\mathrm{td}(C_n) = k$.

Proof. For $n < 2^{k-2}+1$ we know that C_n is a minor of $C_{2^{k-1}+1}$ so we get that the $\mathrm{td}(C_n) \leq k$. Since for $n \geq 2^{k-2}+1$ contain $C_{2^{k-2}+1}$ as a minor we have that $\mathrm{td}(C_n) \geq k$. Thus we see that for $2^{k-2}+1 \leq n < 2^{k-1}+1$ that $\mathrm{td}(C_n) = k$. □

In 2021 Barrus and Sinkovic [7] found more critical unicylic graphs. They showed that if we connect two paths of the same length to adjacent vertices in a cycle then, depending on the length of the paths, the graph you obtain can be critical. For example, the graph below is 5-critical.



Due to these results, we seek to further explore critical unicyclic graphs in this thesis.

CHAPTER 2

Tree-depth of Unicylic Graphs

Cycles can be seen as the base that all unicylic graphs are built upon. As Theorem 5 shows, the tree-depth of cycles is something that is already well understood. However, there are patterns present in the optimal tree-depth labelings of cycles that may still be explored. The first pattern that we observe is that of the optimal tree-depth labelings of $C_{2^{k-1}}$. This graph is notable due to it being the largest cycle with tree-depth k.

Theorem 6. *Up to symmetries there is a unique feasible optimal tree-depth labeling of $C_{2^{k-1}}$ with $[k]$. Let v_i (with $i \in [2^{k-1}]$) be the vertices of the $C_{2^{k-1}}$ labeled clockwise. The unique labeling, L, is given as follows: $L(v_i) = c$ if $i \equiv 2^{c-1} \pmod{2^c}$.*

Proof. Note that since $C_{2^{k-1}+1}$ is $(k+1)$-critical we know that $C_{2^{k-1}}$ has tree-depth k. We then label the vertex v_i with c corresponding to the position of the last 1 in the binary expansion of i counted backwards from the end. Note that this is equivalent to the labeling L in the statement of the theorem. Also note that there is only one vertex labeled with k and one with $k-1$. Consider any vertex v_n labeled with $c < k-1$. Note that $n \equiv 2^{c-1} \pmod{2^c}$. Let v_m be a closest vertex to v_n that is labeled with c. Note that $m \equiv (n \pm 2^c) \pmod{2^{k-1}}$. There exist two v_n, v_m-paths in the cycle. For the path that contains $v_{2^{k-1}}$ there exists a vertex with a label greater than c.

So we consider the other v_n, v_m-path. Without loss of generality we suppose that $m = n + 2^c$. Since $m > n$, by the binary expansion definition of the labeling, in the expansion of n the cth binary digit from the end is 1 and every following binary digit is 0. This is also true of the binary expansion of m, but as we consider the expansions of $n+1, n+2, \ldots, m-1$, there must be a point where the cth binary digit from the end and every following binary digit is 0. Thus there must exist a vertex with label greater than c on this v_n, v_m-path. Therefore the labeling is feasible.

We now want to show this optimal feasible labeling is unique up to symmetry. Suppose that we have an arbitrary optimal feasible labeling of $C_{2^{k-1}}$. At this point we adopt the elimination ordering viewpoint of tree-depth. We delete the vertices labeled with k and $k-1$. The graph should be isomorphic to the disjoint union of two copies of $P_{2^{k-2}-1}$. If this is not the case then the graph must contain a copy of $P_{2^{k-2}}$ which has tree-depth $k-1$. Thus we are unable to eliminate the rest of the graph in the remaining $k-2$ steps. Now suppose that for some j when we remove the vertices labeled with $k, \ldots, k-j$ that we end up with $j+1$ components all isomorphic to $P_{2^{k-j}-1}$. We remove the vertices labeled with $k-j-1$ from the $j+1$ copies of $P_{2^{k-j}-1}$. For the sake of contradiction suppose that we don't end up with a $j+2$ components isomorphic to $P_{2^{k-j-1}-1}$. It must be the that there exists a component that contains a graph isomorphic to $P_{2^{k-j-1}}$. However, since $P_{2^{k-j-1}}$ has tree-depth $k-j-1$ we would then be unable to delete the graph with the $k-j-2$ steps that remain. Thus the graph we obtain from removing the vertices labeled with $k, \ldots, k-j-1$ has $j+1$ components all isomorphic to $P_{2^{k-j}-1}$. Continuing inductively we see that there is a unique feasible optimal tree-depth labeling for $C_{2^{k-1}}$ with $[k]$. Note that the labeling L is the unique labeling. □

From now on we will refer to the unique feasible optimal labeling of $C_{2^{k-1}}$ as the *canonical labeling* of the graph. The sequence of numbers in the labeling starting at v_1 and moving left is also called the ruler sequence [8].

Through the canonical labeling of $C_{2^{k-1}}$ we are able to derive every possible optimal tree-depth labeling of smaller cycles.

Theorem 7. *For positive integers n, k every optimal feasible tree-depth labeling of C_n for $2^{k-2}+1 \leq n \leq 2^{k-1}$ with $[k]$ can be obtained from the unique labeling of $C_{2^{k-1}}$ by edge contractions, where at each step the new vertex created through edge-contraction gets the larger label of the endpoint of the contracted edge.*

Proof. From results of Dvorak et al. [4] we know that this method of adjusting labelings during edge contraction generates feasible labelings. So we just need to show that this generates every possible labeling.

The desired result is obvious when $n = 2^{k-1}$. So we may assume that the desired result holds for $n \geq 2^{k-1} - m$ where $m \geq 0$. So suppose that we have $n = 2^{k-1} - m - 1$. Let $G = C_n$ and (c'_i) be the sequence of labels between the vertices labeled with k and $k-1$ on one of the possible paths in G. Note that at least one of the paths in G between the vertices labeled with k and $k-1$ by L has fewer than $2^{k-2}-1$ vertices besides the endpoints. Let v be the first vertex along the path whose label c'_j differs from the corresponding term c_j of (c_i) when starting from the left most vertex and moving right. Since the labels $(c'_i : 1 \leq i \leq j)$ are a feasible labeling of the corresponding path, this sequence must be squarefree (a sequence is *squrefree* if it contains no ww where w is a subsequence). Since the ruler sequence has the property that the first j terms comprise the least squarefree sequence of length j using positive integers (see [8]), we conclude that $c_j < c'_j$. Let G' be the graph obtained by subdividing the edge of the path in G preceding vertex v, and let x be the created vertex. Label x with the label c_j. This yields a feasible labeling of G', because the labels on vertices preceding x, back to the vertex labeled with k or $k-1$, match the feasible labeling (c_i), and because the label on v is greater than the label on x, no conflict is introduced. Therefore by the inductive hypothesis we know that the labeling on G' can be obtained by contracting along edges of $C_{2^{k-1}}$. Contracting along the edge xv gives the labeling on G, as desired.

$\square$

CHAPTER 3

General results of Tree-depth

In this chapter we deduce patterns involving subsets of vertices that lead to the tree-depth increasing when appendages are added at those vertices. Specifically, the set of vertices we look at is such that a vertex in the set has a certain label in every optimal labeling of the graph. The first result we obtain concerns the label 1.

Theorem 8. *Let G be a graph with $\operatorname{td}(G) = k$. Suppose that $S \subseteq V(G)$. Let G' be obtained by appending a finite number of pendants (a pendant is a vertex of degree 1) to every vertex in S. Then $\operatorname{td}(G') > k$ if and only if every optimal feasible tree-depth labeling of G with $[k]$ assigns a label of 1 to a vertex in S.*

Proof. To start we suppose that $\operatorname{td}(G') > k$. For the sake of contradiction suppose that there exists an optimal feasible tree-depth labeling L of G with $[k]$ such that no vertex in S is labeled with 1. Then we can construct a feasible tree-depth labeling of G' by using the labeling L for G and then labeling the vertices on the pendants with 1. Such a labeling only uses k labels, a contradiction.

Now we prove the contrapositive of the other direction so we suppose that $\operatorname{td}(G') = k$. For the sake of contradiction suppose that for every optimal tree-depth labeling of G there exists a vertex in S labeled with 1. Let L' be an optimal feasible tree-depth labeling of G' with $[k]$. If for all $v \in S$ we have that $L'(v) \neq 1$ then there exists a labeling of G such that no vertex in S is labeled with 1. So we may assume that there exists some $x \in S$ such that $L'(x) = 1$. Let y be a pendant vertex adjacent to x. Note that any path between y and any other vertex labeled with $L'(y)$ must pass through a higher label. Since 1 is the smallest label that we can use we can construct an optimal feasible tree-depth labeling of G' by labeling

x with $L'(y)$, y with 1, and all remaining vertices with what they are labeled with according to L'. Since x is arbitrary we are able to do this for all vertices in S labeled with 1. Thus we can construct a labeling of G such that no vertex in S is labeled with 1 a contradiction. □

If we relax the restriction of the desired label being 1 we are able to generalize one of the directions as seen in the following theorem.

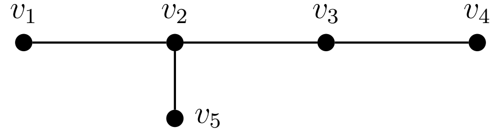
Theorem 9. *Let G be a graph. Suppose that $S = \{v_1, \ldots . v_n\} \subseteq V(G)$ and that $\operatorname{td}(G) = k$. Also, let $\mathcal{H}_i$ be a finite collection of graphs with tree-depth at most c. Let G' be obtained by identifying v_i with an arbitrary vertex in each graph in $\mathcal{H}_i$ for all $i \in [n]$. If $\operatorname{td}(G') > k$ then every optimal feasible tree-depth labeling of G with $[k]$ has a vertex in S that is labeled with at most c.*

Proof. To start we suppose that $\operatorname{td}(G') > k$. For the sake of contradiction suppose that there exists an optimal feasible tree-depth labeling of G with $[k]$, L, such that no vertex in S is labeled with at most c. Then we can construct a feasible tree-depth labeling of G' by using the labeling L for G and then labeling the vertices on each copy of $H_i - v_i$ with an optimal feasible tree-depth labeling with $[c]$. Such a labeling only uses k labels, a contradiction. □

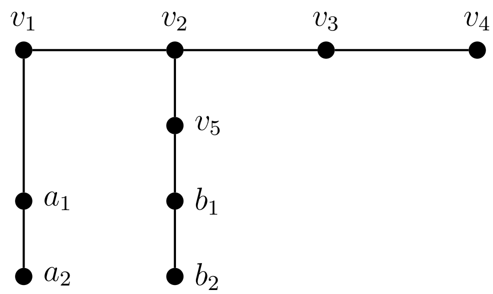
However, when we try to generalize the other direction we start to run into issues as we see in the following observation.

Observation 10. *In general, the converse of Theorem 9 does not hold.*

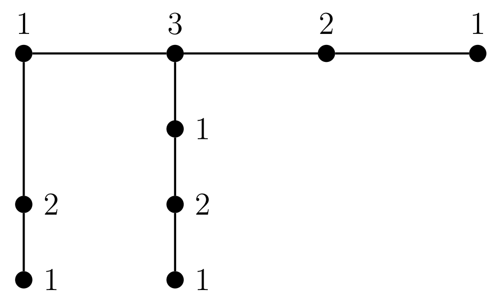
Proof. As an example we consider the following graph G given by



and $S = \{v_1, v_5\}$. It is easy to verify that $\text{td}(G) = 3$ and in any optimal feasible tree-depth labeling of G with $[k]$ vertices v_1 and v_5 must each have a label of either 1 or 2. So if we append a copy of P_2 to each of v_1 and v_5 we get the following graph:



We can then label that graph as such



Which shows that the tree-depth does not increase. □

Thus we see if we want to generalize the other direction we have to add more restrictions to do so. This leads to the following question.

Question 11. *Let G be a graph with $\text{td}(G) = k$ and $S \subseteq V(G)$, and suppose that every feasible k-labeling of G has a vertex in S labeled with something at most c. What graphs satisfy the following?*

> The graph obtained by joining graphs of tree-depth c to the vertices in S has tree-depth greater than k.

CHAPTER 4

Critical Unicyclic Graphs

Theorem 12. *Let $k \geq 3$ and G be a copy of $C_{2^{k-1}}$. Suppose we label the vertices of G with $v_1, \ldots, v_{2^{k-1}}$ in order. If we append pendants to the two vertices v_i, v_j where $i \equiv 1 \pmod 2$ and $j \equiv 0 \pmod 2$ then the graph obtained is $k+1$-critical.*

Proof. Let G' be the graph obtained by appending pendants to G at v_i and v_j. By Theorem 6 we know that in any labeling of G' either v_i or v_j must be labeled with 1. So by Theorem 8 we know that $\text{td}(G') \geq k+1$.

Let x be the pendant vertex adjacent to v_i and let y be the pendant vertex adjacent to v_j. Consider the labeling

$$L(v) = \begin{cases} c \text{ if } v = v_n, n \equiv 2^{c-1} \pmod{2^c} \text{ for some } c \in [k] \\ 1 \text{ if } v = y \\ k+1 \text{ if } v = x \end{cases}.$$

Note that the coloring restricted to the cycle is an optimal feasible tree-depth labeling with $[k]$. Also note that $L(v_j) \neq 1$ so we are able to label y with 1. Therefore we see that L is an optimal feasible tree-depth labeling of G' with $[k+1]$. Therefore $\text{td}(G') = k+1$.

Now we prove criticality. If we remove one of the pendants from G' through edge contraction, vertex deletion, or edge deletion, we can construct a labeling similar to the previous labeling with only k labels. So we may restrict our attention to making changes to the cycle. Suppose H is the graph obtained by deleting an edge or vertex on the cycle in G'. Note that H is a forest and that

$$|V(H)| \leq 2^{k-1} + 2 < 2^k.$$

Thus by Corollary 4 we see that H does not have enough vertices to have tree-depth $k+1$. Therefore the tree-depth must go down. Now suppose that H is the graph obtained by contracting along the edge $v_m v_{m+1}$ on the cycle in G'. By Theorem 7 we can construct a k-labeling of the cycle in H so that both of the vertices that the vertices adjacent to the pendant vertices have labels greater than 1. Therefore we can label the pendant vertices with 1 to create an optimal feasible tree-depth labeling of H with $[k]$. Therefore we see that G' is $(k+1)$-critical.

□

Now we will move on to another nice family of critical unicyclic graphs.

Theorem 13. *Let $G = C_{2^{k-1}-a}$. Suppose that the vertices of G are labeled $v_1, \ldots, v_{2^{k-1}-a}$ in order. If we append one pendant to v_i for each $i \in [a+2]$ then the graph obtained is $(k+1)$-critical.*

Proof. Note that by Theorem 12 the desired result holds for $a = 0$. Let G' be the graph obtained by appending pendants to G at v_i for $i \in [a+2]$. Let $S = \{v_i : i \in [a+2]\}$. We will now show that for every optimal feasible labeling of G at least one vertex in S is labeled with 1. By Theorem 7 we can generate every possible feasible labeling of G from edge contraction on $C_{2^{k-1}}$ and keeping the highest label. If there exists $v_m, v_{m+1} \in S$ such that the edge $v_m v_{m+1}$ is in $C_{2^{k-1}}$ then either v_m or v_{m+1} is labeled with 1. Otherwise, none of the vertices in S are adjacent in $C_{2^{k-1}}$. Thus there exists a v_1, v_{a+2} path of length $2a+2$ in $C_{2^{k-1}}$ that contains every vertex in S. Such a path must contain $a+1$ vertices labeled with 1. Therefore at least one of the vertices of S must be labeled with 1. Therefore by Theorem 8 we know that $\text{td}(G') \geq k+1$. Through a similar labeling to that used in Theorem 12 we can show that $\text{td}(G') = k+1$.

So all that remains is to show criticality. Note that if we delete an edge or vertex from the cycle in G' we obtain a forest with at most $2^{k-1}+2$ vertices. So by Corollary 4 we know that the forest does not contain enough vertices to have tree-depth of $k+1$. Therefore tree-depth goes down. Now every other case of edge contraction or vertex and edge deletion either causes the size of the cycle to go down by one or the number of pendants to go down by one. In the case the size of the cycle goes down by 1 we can construct a labeling of the cycle so that none of the vertices in S are labeled with 1 since if we contract along $a+1$ edges then we can construct a path on $a+2$ vertices that has no vertex labeled with 1. So by Theorem 8 we know that the tree-depth of such a graph is at most k. In the case that the number of pendants goes down by 1 we can find a labeling of the cycle so that none of the vertices with a pendant appended to them is labeled with 1. Thus by Theorem 8 we know that the tree-depth of the resulting graph is at most k. Therefore G' is critical. □

Now, what if we want a more complex critical unicyclic graph? If we glue two trees of tree-depth $k-1$ and a pendant to $C_{2^{k-2}+1}$ we are able to obtain another family of critical graphs.

Theorem 14. *Let $G = C_{2^{k-2}+1}$ for some $k > 3$ and v_1, v_2, v_3 be consecutive vertices in G. Let H_1 and H_2 be $(k-1)$-critical trees. Also let $x \in V(H_1)$ and $y \in V(H_2)$. Suppose that G' is the graph obtained by appending a pendant to v_3, and identifying v_1 as x and v_2 as y. Then G' is $(k+1)$-critical.*

Proof. We being by showing that $\text{td}(G') = k+1$. To start we suppose that $\text{td}(G') = k$. Note that a k-labeling of G' must produce k-labelings of G, H_1 and H_2. Note that there are distinct vertices in G labeled with k and $k-1$. Also note that H_1 and H_2 must each contain a vertex labeled with k or $k-1$. For $i \in [2]$, let x_i be the vertex with the highest label in H_i. Since $\text{td}(H_i) = k-1$ the label on x_i is either $k-1$ or k. If $x_i \neq v_i$ then there exist a path between x_i and the vertex in G with the same label that does not pass through a vertex with a larger label. Thus we see that v_i is labeled with k or $k-1$. Since v_1 and v_2 are adjacent one has to be labeled with k and the other with $k-1$.

Note that by Theorem 7 the labeling of G can be obtained through edge contraction on $C_{2^{k-1}}$. Since we have a labeling with k and $k-1$ on adjacent vertices and we contracted along $2^{k-2}-1$ edges, it must be that we contracted along every edge in one of the paths in $C_{2^{k-1}}$ between the vertices labeled with k and $k-1$. The labels on the other path remain unchanged and match the labeling in $C_{2^{k-1}}$. Thus v_3 must be labeled with 1.

Let u be the pendant vertex attached to v_3. Let d be the label of u. Since v_1 and v_2 are labeled with k and $k-1$ it must be that $d < k-1$. However since the path between v_1 and v_3 not containing v_2 follows the ruler sequence we must come across another vertex labeled with d before we reach a vertex with a higher label. Thus u cannot be labeled with anything from $[k]$, a contradiction. Therefore we see that $\mathrm{td}(G') > k$ and moreover that $\mathrm{td}(G') = k+1$. We will now refer to this labeling of G' as L.

We now show that G' is critical. Note that

$$|V(G')| = 2^{k-2} + 1 + 1 + 2(2^{k-2} - 1) = 2^{k-1} + 2^{k-2}.$$

So if we delete an edge or vertex on the cycle we would have a forest with $2^{k-1}+2^{k-2}$ vertices which is not enough vertices to have tree-depth $k+1$ since $k > 3$. So the tree-depth goes down. Now every other single edge contraction or vertex and edge deletion either lowers the size of the cycle by one or modifies H_1, H_2, or the pendant. In the case when the size of the cycle decreases we are able to label v_3 with something different from 1 by Theorem 7. Thus we can label u with 1 and every other vertex according to L.

In the case that we modify the pendant, either u is removed from the graph or the edge uv_3 is removed. In both cases we label every vertex in $G' - u$ according to L and then label u with 1 if it does exist. So now we consider the case when we modify H_i where $i \in [2]$. Since H_i is critical, when we modify it the part of H_i that remains attached to the cycle G must have tree-depth at most $k-2$. Thus if we modify H_i we are able to label v_i with a label at most $k-2$ and then label the vertices adjacent to it in G with k and $k-1$; we are able to obtain such a labeling of G by Theorem 7.

□

Based on Conjecture 1 we seek to put a bound on the order of critical unicyclic graphs. We start by putting a lower bound on the order.

Theorem 15. *Suppose that G is a k-critical unicyclic graph for some $k \geq 3$. Then $|V(G)| \geq 2^{k-2}+1$.*

Proof. For the sake of contradiction suppose that $|V(G)| \leq 2^{k-2}$. Let e be an edge on the cycle in G. Since G is critical, $\mathrm{td}(G-e) = k-1$. Note that $G-e$ is a forest with $|V(G-e)| \leq 2^{k-2}$. Since $\mathrm{td}(G-e) = k-1$ we know that $|V(G-e)| = |V(G)| = 2^{k-2}$. Since $\mathrm{td}(G-e) = k-1$ the forest must have a $(k-1)$-critical minor, which has 2^{k-2} vertices by Corollary 4. Thus $G-e$ is critical which implies that for all $v \in V(G-e)$ there exists a labeling, L, of $G-e$ such that $L(v) = k-1$. Thus we can extend such a labeling to G by labeling one of the endpoints of e with $k-1$. Such a labeling only uses $k-1$ labels a contradiction. □

Moreover, there is a unique k-critical unicyclic graph with order $2^{k-2}+1$.

Theorem 16. *Let $k \geq 3$. $G = C_{2^{k-2}+1}$ is the only k-critical unicyclic graph with $|V(G)| = 2^{k-2}+1$.*

Proof. Suppose that G is a k-critical unicylic graph such that $|V(G)| = 2^{k-2}+1$. Let v be a vertex on the cycle in G. Note that $G-v$ is a forest with $\mathrm{td}(G-v) = k-1$

and $|V(G-v)| = 2^{k-2}$. As in the proof of Theorem 15 $G-v$ must be a $(k-1)$-critical tree. Note that $\deg_G(v) = 2$ since otherwise G would have at least two cycles. Since v was arbitrary it must be that every vertex in G has degree 2. Thus $G = C_{2^{k-2}+1}$. □

Conjecture 1 puts an upper bound on the order of critical graphs which is currently something we lack for critical unicyclic graphs. However, using currently known critical unicyclic graphs we can have an idea of what the upper bound could be.

Proposition 17. *Let $k \geq 3$. There exists a k-critical unicyclic graph G with $2^{k-2}+1 \leq |V(G)| \leq 2^{k-1}-1$.*

Proof. For $0 \leq t \leq k-2$ let $R_{k,t}$ be the graph obtained by taking a path with $2^{k-2}+1+t$ vertices and adding an edge between the two vertices at distance t from the endpoints. By the results of [7] we know that the graph $R_{k,t}$ is a k-critical unicyclic graph when $0 \leq t \leq 2^{k-2}-2$. Note that $R_{k,0} = C_{2^{k-2}+1}$ which has an order of $2^{k-2}+1$ and $R_{k,1}, \ldots R_{k,2^{k-2}-2})$ are k critical unicyclic graphs with orders equal to every number in the desired inequality. □

Based on this proposition and Theorem 15 we conjecture bounds on the order of k-critical unicyclic graph.

Conjecture 18. *Let $k \geq 3$. If G is a k-critical unicyclic graph, then $2^{k-2}+1 \leq |V(G)| \leq 2^{k-1}-1$.*

It is important to note here that if this conjecture does hold that $2^{k-1}-1 < 2^{k-1}$ as in Conjecture 1. Also the simple method from Chapter 1 for determining if a tree is critical leads to the following question:

Question 19. *Is there a simple way to determine if a unicyclic graph is critical or not?*

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