The Global Dynamics of Some Rational Difference Equations Exhibiting Neimark-Sacker Bifurcation

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DOCTOR OF PHILOSOPHY DISSERTATION

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ABSTRACT

This thesis is presented in manuscript format. The first chapter will introduce preliminary definitions and theorems that will be used in the succeeding chapters.

The second chapter will consider the dynamics of a second-order sigmoid Beverton-Holt equation

\[ x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + d} \]

where the parameters \(a\) and \(d\) are positive numbers and the initial conditions are non-negative numbers. For this equation, we will begin by giving a description of the local dynamics of the equation, and then will also examine the global dynamics, including an investigation into the basins of attraction of the zero and greater positive equilibriums. Furthermore, we will prove the occurrence of Neimark-Sacker bifurcation and give an asymptotic approximation for the resultant invariant manifold produced. This approximation will be computed according to the process developed by K. Murakami. Lastly, we will give asymptotic approximations for the local stable and unstable manifolds of the lesser positive equilibrium and investigate the rates of convergence to the attracting equilibriums.

The third chapter will consist of an investigation into the dynamics of two special cases of the second-order fractional difference equation

\[ x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n} \]

where the parameters \(\beta, \gamma, \delta, B, C, D\) are non-negative and the initial conditions are arbitrary non-negative numbers. Further we assume that \(B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0\) for all \(n \geq 0\). In particular we consider the cases where \(\beta = D = 0\) and \(\gamma = D = 0\).
We prove that in the first case, the equation exhibits supercritical Neimark-Sacker bifurcation. In the second case, we prove that the equation exhibits both supercritical and subcritical Neimark-Sacker bifurcation. In this case, the system probably exhibits Chenciner bifurcation. Again in both cases, we will give an asymptotic approximation for the invariant curve produced by the occurrence of the Neimark-Sacker bifurcation computed according to the method developed by K. Murakami.

The fourth chapter will investigate the behavior of an evolutionary system as a discrete time model of population dynamics. This case study will be based on a sigmoid Beverto-Holt model and will incorporate the dynamics of a single phenotypic trait subject to Darwinian evolution. The system studied in this manuscript will be constructed according to the canonical way to model Darwinian evolution using the dynamics of the system

\[
x_{n+1} = f(x_n, v, u_n)|_{v=\mu_n} x_n
\]

\[
u_{n+1} = u_n + \sigma^2 \frac{\partial F(x_n, v, u_n)}{\partial v}|_{v=\mu_n}
\]

For our case study, we will look specifically at the system

\[
x_{n+1} = b(u_n) \frac{x_n^2}{1 + x_n^2}
\]

\[
u_{n+1} = u_n + \sigma^2 b'(u_n) = F(u_n) \quad n = 0, 1, \ldots
\]

We will study the local and global dynamics of this system and generalize the global results obtained. Lastly, we will give some simulations of this system for different definitions of the function \( b(u) \).
This thesis is dedicated to Dr. J. Crowley, Dr. J. DeLeo and to all haematologist/oncologists, internists, and neurologists working to improve the lives of millions.
This thesis is presented in manuscript format. Chapter 1 introduces some basic theory of discrete dynamical systems including important theorems that are used throughout Chapters 2, 3 and 4. The second chapter consists of the first manuscript which was published in *Discrete Dynamics in Nature and Society* in 2021, while the third chapter consisting of the second manuscript has been submitted to the *Journal of Applied and Computational Mathematics*. The last chapter, composed of the third manuscript, is currently being prepared for submission to an journal for publication.
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CHAPTER 1
Introduction & Preliminaries

This thesis studies the behavior of discrete-time dynamical systems and models in mathematical biology. Here in this chapter, we consider some elementary theory of difference equations that we will use to analyze the dynamics of the equations and systems considered in the following chapters.

Formally, a difference equation is an equality involving the differences between successive values of a function of a discrete variable, which are values separated by some finite amount [1]. In more colloquial terms, we can think of differences equations as rules that will produce a sequence of values where any particular value is determined by previous terms. When we study difference equations, we are studying what we call the dynamics of the equation or system so essentially we are studying what all possible sequences of values satisfying the equation are doing as we add more and more terms to them. This analysis in general hinges on finding equilibrium points, performing local and globally stability analysis and analyzing what kind of bifurcations are occuring in the system.

Now we will consider the following general system of difference equations,

\[ x_{n+1} = f(x_n, y_n) \quad y_{n+1} = g(x_n, y_n) \] (1)

Definition 1 (Equilibrium Point) A point \((\bar{x}, \bar{y}) \in \mathbb{R}^2\) is an equilibrium point, or a fixed point, for a dynamical system if \(f(\bar{x}, \bar{y}) = \bar{x}\) and \(g(\bar{x}, \bar{y}) = \bar{y}\). [2]
When we consider the stability of an equilibrium point, we refer to the following definition;

**Definition 2 (Stability of a Fixed Point)**  
(a) The equilibrium solution \((\bar{x}, \bar{y})\) of (1) is said to be stable if for any \(\epsilon > 0\) there exists \(\delta > 0\) such that for every initial point \((x_0, y_0)\) for which \(||(x_0, y_0) - (\bar{x}, \bar{y})|| < \delta\), the iterates \((x_n, y_n)\) of \((x_0, y_0)\) satisfy \(||(x_n, y_n) - (\bar{x}, \bar{y})|| < \epsilon\) for all \(n > 0\). An equilibrium point \((\bar{x}, \bar{y})\) of (1) is said to be unstable if it is not stable.

(b) An equilibrium point \((\bar{x}, \bar{y})\) of (1) is said to be asymptotically stable if there exists \(r > 0\) such that \((x_n, y_n) \to (\bar{x}, \bar{y})\) at \(n \to \infty\) for all \((x_0, y_0)\) that satisfy \(||(x_0, y_0) - (\bar{x}, \bar{y})|| < r\).

(c) A periodic point \((x_p, y_p)\) of period \(m\) is stable (resp. unstable, asymptotically stable) if \((x_p, y_p)\) is a stable (resp. unstable, asymptotically stable) fixed point of \(F^m\).

One of the most effective tools we have for investigating the local stability of equilibrium points is the use of linearization;

**Definition 3 (Linearization)** Let \((\bar{x}, \bar{y})\) be a fixed point of a map \(F = (f, g)\), where \(f\) and \(g\) are continuously differentiable functions at \((\bar{x}, \bar{y})\). The Jacobian matrix of \(F\) at \((\bar{x}, \bar{y})\) is the matrix

\[
J_F(\bar{x}, \bar{y}) = \begin{pmatrix}
\frac{\partial f}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \\
\frac{\partial g}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial g}{\partial y}(\bar{x}, \bar{y})
\end{pmatrix}
\]

The linear map \(J_F(\bar{x}, \bar{y}) : \mathbb{R}^2 \to \mathbb{R}^2\) given by

\[
J_F(p, q)(\bar{x}, \bar{y}) = \begin{pmatrix}
\frac{\partial f}{\partial x}(\bar{x}, \bar{y})x & \frac{\partial f}{\partial y}(\bar{x}, \bar{y})y \\
\frac{\partial g}{\partial x}(\bar{x}, \bar{y})x & \frac{\partial g}{\partial y}(\bar{x}, \bar{y})y
\end{pmatrix}
\]

is called the linearization of the map \(F\).
Theorem 1 (Linearized Stability Theorem) Let $F = (f, g)$ be a continuously differentiable function defined on an open set $W$ in $R^2$, and let $(\bar{x}, \bar{y})$ in $W$ be a fixed point of $F$.

(a) If all the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x}, \bar{y})$ is asymptotically stable.

(b) If at least one of the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x}, \bar{y})$ is unstable.

We call a point where one of the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ has modulus less than one and the other has modulus greater than one a saddle-point. A point for which at least one eigenvalue has modulus equal to one is called a non-hyperbolic fixed point [2].

In practice, the theorem used most often in the analysis of the systems in chapters 2, 3 and 4, will be the following;

Theorem 2 Consider the trace and determinant of the Jacobian matrix $J_F(\bar{x}, \bar{y})$. Then the stability of an equilibrium point can be determined by examining the solutions of the characteristic equation;

1. An equilibrium point $(\bar{x}, \bar{y})$ of (1) is locally asymptotically stable if and only if every solution of the characteristic equation

\[ \lambda^2 - trJ_F(\bar{x}, \bar{y}) + detJ_F(\bar{x}, \bar{y}) = 0 \]
lies inside the unit circle, that is, if and only if

$$|\text{tr} J_F(\bar{x}, \bar{y})| < 1 + \text{det} J_F(\bar{x}, \bar{y}) < 2$$

2. An equilibrium point \((\bar{x}, \bar{y})\) of (1) is locally a repeller if and only if every solution of the characteristic equation lies outside the unit circle, that is, if and only if

$$|\text{tr} J_F(\bar{x}, \bar{y})| < |1 + \text{det} J_F(\bar{x}, \bar{y})| \text{ and } |\text{det} J_F(\bar{x}, \bar{y})| > 1$$

3. An equilibrium point \((\bar{x}, \bar{y})\) of (1) is locally a saddle-point if and only if the characteristic equation has one root that lies inside the unit circle and one root that lies outside the unit circle, that is, if and only if

$$|\text{tr} J_F(\bar{x}, \bar{y})| > |1 + \text{det} J_F(\bar{x}, \bar{y})|$$

4. An equilibrium point \((\bar{x}, \bar{y})\) of (1) is nonhyperbolic if and only if the characteristic equation has at least one root that lies on the unit circle, that is, if and only if

$$|\text{tr} J_F(\bar{x}, \bar{y})| = |1 + \text{det} J_F(\bar{x}, \bar{y})| \text{ or } \text{det} J_F(\bar{x}, \bar{y}) = 1 \text{ and } |\text{tr} J_F(\bar{x}, \bar{y})| \leq 2$$

In regard to the second and third chapters of this thesis, we consider second-order difference equations. To perform the stability analysis on these equations, we convert the difference equation

$$x_{n+1} = f(x_n, x_{n-1})$$

into the corresponding system

$$u_{n+1} = v_n$$
$$v_{n+1} = f(v_n, u_n)$$

The corresponding map of this system will then be \(F(x, y) = (y, f(x, y))\), and we can then apply our linearization method to this system and obtain the local stability.
results for the equilibrium points of the original equation.

For proving global stability, we cannot solely rely on the results of the local stability analysis, and the main tool that we use in the following chapters is the following theorem;

**Theorem 3 (M&m theorem)** Let \( g : [a, b]^{k+1} \rightarrow [a, b] \) be a continuous function, where \( k \) is a positive integer, and where \([a, b]\) is an interval of real numbers. Consider the difference equation

\[
x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ...
\]

(2)

Suppose that \( g \) satisfied the following conditions;

1. For each integer \( i \) with \( 1 \leq i \leq k + 1 \), the function \( g(z_1, z_2, ..., z_{k+1}) \) is weakly monotonic in \( z_i \) for fixed \( z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1} \).

2. It \((m, M)\) is a solution of the system

\[
m = g(m_1, m_2, ..., m_{k+1}) \quad \text{and} \quad M = g(M_1, M_2, ..., M_{k+1})
\]

then \( m = M \), were for each \( i = 1, 2, ..., k + 1 \) we set

\[
m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases}
\]

and

\[
M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}
\]

Then there exists exactly one equilibrium \( \bar{x} \) of (2), and every solution of (2) converges to \( \bar{x} \).
Additionally, we refer to the bifurcation exhibited by certain equilibrium points at many points in this thesis. By bifurcation, we refer to the occurrence of a change in stability of a particular equilibrium point. In the first two manuscripts presented in this thesis, we concern ourselves mainly with the occurrence of Neimark-Sacker bifurcation, which is elaborated on in the first manuscript.

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CHAPTER 2
The Neimark-Sacker Bifurcation and Global Stability of Perturbation of Sigmoid Beverton-Holt Difference Equation

Published in Discrete Dynamics in Nature and Society, Vol 2021, 2021
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2.1 Introduction and Preliminaries

In this paper we consider the difference equation

\[ x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + d}, \quad n = 0, 1, \ldots, \]  

(3)

where the parameters \( a \) and \( d \) are positive numbers and the initial conditions \( x_{-1} \) and \( x_0 \) are non-negative numbers.

Equation (3) can be considered as a nonlinear perturbation of the Sigmoid Beverton-Holt difference equation

\[ x_{n+1} = \frac{x_n^2}{x_n^2 + d}, \quad d > 0, x_0 \geq 0 \quad n = 0, 1, \ldots, \]  

(4)

which is a major mathematical model in population dynamics and is the simplest model that exhibits Allee’s effect see [1, 2]. A related difference equation of the form

\[ x_{n+1} = \frac{\beta x_n^2}{x_{n-1}^2 + 1}, \quad \beta > 0 \quad n = 0, 1, \ldots, \]  

(5)

where the initial conditions \( x_{-1} \) and \( x_0 \) are non-negative numbers was considered in [3].

Equation (3) is a square version of the well-known Pielou difference equation

\[ x_{n+1} = \frac{x_n}{x_{n-1} + d}, \quad d > 0, x_0 \geq 0 \quad n = 0, 1, \ldots, \]  

(6)

which is another major model in population dynamics [1, 4, 5]. Equation (6) has the same global dynamics as more general difference equation

\[ x_{n+1} = \frac{x_n}{ax_n + x_{n-1} + d}, \quad a, d > 0, x_0 \geq 0 \quad n = 0, 1, \ldots. \]  

(7)

Both equations (6) and (7) exhibit transcritical bifurcation, where the zero equilibrium is globally asymptotically stable up to some critical value where positive equilibrium appears and assume global asymptotic stability.

Another perturbation of (4) is the difference equation studied in [6]

\[ x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + d}, \quad n = 0, 1, \ldots, \]  

(8)
where the initial conditions $x_{-1}$ and $x_0$ are non-negative numbers. Equation (4) exhibits a global asymptotic stability of either zero or positive equilibrium solutions and exchange of stability or transcritical bifurcation. Equation (8) exhibits Neimark-Sacker bifurcation and possibly chaos. If we search for a model that exhibits Allee’s effect, transcritical bifurcation and Neimark-Sacker bifurcation to a periodic solution then Equation (3) is probably the simplest such model. The dynamical difference between Equations (3) and (8) is that Equation (8) cannot exhibit Allee’s effect and has at most two equilibrium solutions.

As we will see in this paper the introduction of more quadratic terms will substantially change the dynamics and will introduce the existence of a locally stable periodic solution and possibly chaos. We will show that local asymptotic stability of the zero equilibrium will also implies its global asymptotic stability. In the case of the positive equilibrium solution we will show that such a statement is true in some subspace of the parametric region of local asymptotic stability and we pose the conjecture that the same property holds in the complete region of local asymptotic stability. The technique used in proving global asymptotic stability of the positive equilibrium solution is based on global attractivity results for maps with invariant boxes, see [7, 5, 8]. Related rational difference equations which exhibit similar behavior were considered in [9, 10].

Now, for the sake of completeness we give the basic facts about the Neimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimensions, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle is generated.

The Neimark-Sacker bifurcation is the discrete counterpart to the Hopf bifurcation for
a system of ordinary differential equations in two or more dimensions, see [11, 12, 13].

It occurs for such a discrete system depending on a parameter, \( \lambda \) for instance, along with a fixed point, the Jacobian matrix of which has a pair of complex conjugate eigenvalues. These eigenvalues \( \mu(\lambda) \) and \( \bar{\mu}(\lambda) \) will cross the unit circle at \( \lambda = \lambda_0 \) transversally. When this occurs in the discrete setting a periodic solution, which will in general be of an unknown period, will appear and this solution will be locally stable.

To represent this periodic solution, we use the Murakami computational approach, see [14], to identify an asymptotic formula for an invariant curve in the phase plane which is locally attracting.

In the discrete setting, the Neimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation. The Neimark-Sacker bifurcation occurs for a discrete system in the plane depending on a parameter, \( \lambda \) say, with a fixed point whose Jacobian matrix has a pair of complex conjugate eigenvalues \( \mu(\lambda), \bar{\mu}(\lambda) \) which crosses the unit circle transversally at \( \lambda = \lambda_0 \). In this case the periodic solution, which is in general, of unknown period appears and is locally stable. In this paper we use Murakami computational approach, see [14] to find an asymptotic formula for an invariant locally attracting curve in the phase plane, which represents a periodic solution.

The following result is referred to as the Neimark-Sacker bifurcation theorem, see [11, 15, 12, 13].

**Theorem 4 (Neimark-Sacker bifurcation)** Let

\[
F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)
\]

be a \( C^4 \) map depending on real parameter \( \lambda \) satisfying the following conditions:

(i) \( F(\lambda, 0) = 0 \) for \( \lambda \) near some fixed \( \lambda_0 \);

(ii) \( \text{Jac}_F(\lambda, 0) \) has two non-real eigenvalues \( \mu(\lambda) \) and \( \bar{\mu}(\lambda) \) for \( \lambda \) near \( \lambda_0 \) with

\[
|\mu(\lambda_0)| = 1;
\]
(iii) \( \frac{d}{d\lambda} |\mu(\lambda)| = d(\lambda_0) < 0 \) at \( \lambda = \lambda_0 \) (transversality condition);

(iv) \( \mu^k(\lambda_0) \neq 1 \) for \( k = 1, 2, 3, 4 \). (nonresonance condition).

Then there is a smooth \( \lambda \)-dependent change of coordinate bringing \( F \) into the form

\[
F(\lambda, x) = \mathcal{F}(\lambda, x) + O(\|x\|^5)
\]

and there are smooth functions \( \alpha(\lambda), \beta(\lambda), \) and \( \omega(\lambda) \) so that in polar coordinates the function \( \mathcal{F}(\lambda, x) \) is given by

\[
\begin{pmatrix}
  r \\
  \theta
\end{pmatrix} = \begin{pmatrix}
  |\mu(\lambda)|r + \alpha(\lambda)r^3 \\
  \theta + \omega(\lambda) + \beta(\lambda)r^2
\end{pmatrix},
\]

(9)

If \( \alpha(\lambda_0) < 0 \), then the Neimark–Sacker bifurcation at \( \lambda = \lambda_0 \) is supercritical and there exists an unique closed invariant curve \( \Gamma(\lambda) \), which is attracting, and bifurcates from \( \bar{x} \) for \( \lambda < \lambda_0 \).

Consider a general map \( F(\lambda_0, x) \) that has a fixed point at the origin with complex eigenvalues \( \mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0) \) and \( \bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0) \) satisfying

\[
\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1 \text{ and } \beta(\lambda_0) \neq 0.
\]

Assume that

\[
F(\lambda_0, x) = A(\lambda_0)x + G(\lambda_0, x)
\]

(10)

where \( A \) is the Jacobian matrix of \( F \) evaluated at the fixed point \((0,0)\), and

\[
G(\lambda_0, x) := \begin{pmatrix}
  g_1(\lambda_0, x_1, x_2) \\
  g_2(\lambda_0, x_1, x_2)
\end{pmatrix}.
\]

Here we denote \( \mu(\lambda_0) = \mu, A(\lambda_0) = A \) and \( G(\lambda_0, x) = G(x) \). We let \( p \) and \( q \) be the eigenvectors of \( A \) associated with \( \mu \) satisfying

\[
Aq = \mu q, \quad pA = \mu p, \quad pq = 1
\]

and \( \Phi = (q, \bar{q}) \). Assume that

\[
G \left( \Phi \left( \frac{z}{\bar{z}} \right) \right) = \frac{1}{2}(g_{20}z^2 + 2g_{11}z\bar{z} + g_{02}\bar{z}^2) + O(|z|^3)
\]
and
\[
\begin{align*}
K_{20} &= (\mu^2 I - A)^{-1}g_{20} \\
K_{11} &= (I - A)^{-1}g_{11} \\
K_{02} &= (\bar{\mu}^2 I - A)^{-1}g_{02}
\end{align*}
\] (11)

Let
\[
G \left( \Phi \left( \frac{\dot{z}}{\bar{z}} \right) + \frac{1}{2} (K_{20} z^2 + 2K_{11} z \bar{z} + K_{02} \bar{z}^2) \right) = \frac{1}{2} (g_{20} z^2 + 2g_{11} z \bar{z} + g_{02} \bar{z}^2) + \frac{1}{6} (g_{30} z^3 + 3g_{21} z^2 \bar{z} + 3g_{12} z \bar{z}^2 + g_{03} \bar{z}^3) + O(|z|^4), \tag{12}
\]
then
\[
a(\lambda_0) = \frac{1}{2} \text{Re}(p\epsilon_1 \bar{\mu}).
\]

The next result of Murakami [14] gives an approximate formula for the periodic solution.

**Corollary 1** Assume \(a(\lambda_0) \neq 0\) and \(\lambda = \lambda_0 + \eta\) where \(\eta\) is a sufficient small parameter. If \(\bar{x}\) is a fixed point of \(F\) then the invariant curve \(\Gamma(\lambda)\) from Theorem 4 can be approximated by
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{x} + 2\rho_0 \text{Re}(q e^{i\theta}) + \rho_0^2 \left( \text{Re}(K_{20} e^{2i\theta}) + K_{11} \right),
\]
where
\[
d = \left. \frac{d}{d\eta} |\mu(\lambda)| \right|_{\lambda = \lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a} \eta}, \quad \theta \in \mathbb{R}.
\]

Here "\(\text{Re}\)" represents the real parts of these complex numbers.

By putting the linear part of such a map into Jordan Canonical form, we may assume \(F\) to have the following form near the origin
\[
F(\lambda_0, \bar{x}) = \begin{pmatrix} \alpha(\lambda_0) & -\beta(\lambda_0) \\ \beta(\lambda_0) & \alpha(\lambda_0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}. \tag{13}
\]
Then the coefficient $a(\lambda_0)$ of the cubic term in Eq.(9) in polar coordinates is equal to

$$a(\lambda_0) = \text{Re} \left[ \frac{(1 - 2\mu(\lambda_0))\bar{\mu}(\lambda_0)}{1 - \mu(\lambda_0)}\xi_{11}\xi_{20} \right] + \frac{1}{2} |\xi_{11}|^2 + |\xi_{02}|^2 - \text{Re}(\bar{\mu}(\lambda_0)\xi_{21}), \quad (14)$$

where

$$\xi_{20} = \frac{1}{8} \left\{ (g_1)x_{1x_1} - (g_1)x_{2x_2} + 2(g_2)x_{1x_2} + i [(g_2)x_{1x_1} - (g_2)x_{x_1} - 2(g_1)x_{1x_2}] \right\},$$

$$\xi_{11} = \frac{1}{4} \left\{ (g_1)x_{1x_1} + (g_1)x_{2x_2} + i [(g_2)x_{1x_1} + (g_2)x_{x_1}] \right\},$$

$$\xi_{02} = \frac{1}{8} \left\{ (g_1)x_{1x_1} - (g_1)x_{2x_2} - 2(g_2)x_{1x_2} + i [(g_2)x_{1x_1} - (g_2)x_{x_2} + 2(g_1)x_{1x_2}] \right\},$$

$$\xi_{21} = \frac{1}{16} \left\{ (g_1)x_{x_1} + (g_1)x_{x_2x_1} + (g_2)x_{x_1} + (g_2)x_{x_2x_2} + i x_{x_1} + (g_2)x_{x_1} + (g_1)x_{x_2} - (g_1)x_{x_2}x_2 \right\}.$$  

Here ”Re” in formula (14) represents the real parts of those complex numbers, and all partial derivatives are evaluated at the fixed point at the origin. The calculation of $a(\lambda_0)$ is given by [16].

The rest of the paper is organized as follows; section 3 presents the computation of Naimark-Sacker bifurcation; section 4 presents the approximations of stable, unstable and center manifolds of the equilibrium solutions of Equation (3); finally, section 5 establishes that the rate of convergence of the solutions that converge to the zero equilibrium is quadratic while the rate of convergence of the solutions that converge to any positive equilibrium solution is linear.

### 2.2 Local and Global Stability

Equation (3) has always the zero equilibrium $\bar{x}_0 = 0$. The positive equilibrium solutions of Equation (3) are the positive solutions of the equation $(a + 1)\bar{x}^2 - \bar{x} + d = 0$, that is

$$\bar{x}_\pm = \frac{1 \pm \sqrt{1 - 4d(a + 1)}}{2(a + 1)}, \quad (15)$$

when

$$4d(a + 1) < 1 \quad (16)$$
and
\[ \ddot{x} = \frac{1}{2(a + 1)}, \]  \hspace{1cm} (17)

when
\[ 4d(a + 1) = 1. \]  \hspace{1cm} (18)

The linearized equation associated with Equation (3) about the equilibrium point \( \bar{x} \) is
\[ z_{n+1} = pz_n + qz_{n-1} \]

where
\[ p = f_u(\bar{x}, \bar{x}) \quad \text{and} \quad q = f_v(\bar{x}, \bar{x}). \]

Now the following results hold:

**Lemma 1** For the equilibrium point \( \bar{x}_0 \) of Equation (3), the equilibrium is always locally asymptotically stable.

The proof of the lemma follows from the fact that linearized equation at \( \bar{x}_0 = 0 \) is \( z_{n+1} = 0 \).

**Lemma 2** Assume that (74) holds. The positive equilibrium \( \bar{x}_+ = \frac{1+\sqrt{1-4d(a+1)}}{2(a+1)} \) of Equation (3) satisfies the following:

(i) If \( d > \frac{1-a}{4} \), the equilibrium point \( \bar{x}_+ \) is locally asymptotically stable.

(ii) If \( d < \frac{1-a}{4} \), the equilibrium point \( \bar{x}_+ \) is a repeller.

(iii) If \( 4d(a + 1) = 1 \) the equilibrium point \( \bar{x} = \frac{1}{2(a+1)} \) is non-hyperbolic of stable type with eigenvalues 1 and \( 4d < 1 \).

**Proof.**
(i) One can see that

\[ p = f_u(x_+, x_+) = \frac{a + 2 - a\sqrt{1 - 4(a + 1)d}}{a + 1} = 2(1 - a\bar{x}_+), \]

and

\[ q = f_v(x_+, x_+) = -\frac{\sqrt{1 - 4(a + 1)d} + 1}{a + 1} = -2\bar{x}_+ < 0, \]

\[ q - p - 1 = \frac{(a - 1)\sqrt{1 - 4(a + 1)d} - 2(a + 1)}{a + 1}, \]

\[ q + p - 1 = \frac{1}{2} \left( 1 - \sqrt{1 - 4(a + 1)d} \right), \]

\[ q + 1 = \frac{a - \sqrt{1 - 4(a + 1)d}}{a + 1}. \]

The rest of the proof follows from Theorem 2.13 [15]. We notice that the linearized equation at any positive equilibrium is

\[ y_{n+1} - 2(1 - 2a\bar{x})y_n + 2\bar{x}y_{n-1} = 0 \quad (19) \]

and the corresponding characteristic equation is:

\[ \lambda^2 - 2(1 - 2a\bar{x})\lambda + 2\bar{x} = 0. \quad (20) \]

(iii) In the case (iii) we have that the characteristic equation at the equilibrium point \( \bar{x} = \frac{1}{2(a + 1)} \) is

\[ \lambda^2 - 2(1 - 2ad)\lambda + 4d = 0 \]

which solutions are 1 and 4d. In view of the condition 4d(a + 1) = 1 we have 4d < 1.
Lemma 3 Assume that (74) holds. The positive equilibrium $\bar{x}_- = \frac{1-\sqrt{1-4d(a+1)}}{2(a+1)}$ is always a saddle point.

Proof. One can see that

$$p = f_u(\bar{x}_-, \bar{x}_-) = \frac{a + 2 + a\sqrt{1 - 4(a + 1)d}}{a + 1},$$

and

$$q = f_v(\bar{x}_-, \bar{x}_-) = -1 + \frac{\sqrt{1 - 4(a + 1)d}}{a + 1},$$

which imply

$$q - p - 1 = \frac{(1 - 1)\sqrt{1 - 4(a + 1)d} - 2(a + 2)}{a + 1},$$

$$q + p - 1 = \frac{-2 + (a + 1)\sqrt{1 - 4(a + 1)d}}{a + 1},$$

$$q + 1 = \frac{a + \sqrt{1 - 4(a + 1)d}}{a + 1}.$$

The rest of the proof follows from Theorem 2.13 [15].

First we give the global asymptotic result for zero equilibrium.

Theorem 5 Assume that

$$4d(a + 1) > 1. \quad (21)$$

Then the zero equilibrium of Equation (3) is globally asymptotically stable.

Proof. Every solution $\{x_n\}$ of Equation (3) satisfies

$$x_{n+1} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + d} \leq \frac{1}{a}, \quad n = 0, 1, \ldots$$

and the function

$$f(u, v) = \frac{u^2}{au^2 + v^2 + d}$$
is increasing in $u$ and decreasing in $v$, with property that it has an invariant and attracting interval $[0, 1/a]$. Now we will employ Theorem 1.4.5 from [5]. Consider the system of equations

$$\begin{align*}
f(M, m) &= M \\
f(m, M) &= m
\end{align*}$$

and prove that $M = m$. This system becomes

$$\begin{align*}
\frac{M^2}{aM^2 + m^2 + d} &= M \\
\frac{m^2}{am^2 + M^2 + d} &= m,
\end{align*}$$

which after eliminating $M$ becomes

$$m (am^2 + d - m) ((a + 1)m^2 - m + d) ((a^3 - a^2 - a + 1)m^2 + (1 - a^2)m + a^2d - 2ad + d + 1) = 0. \quad (22)$$

Now the discriminant of first quadratic polynomial in (22) is $1 - 4ad < 0$ in view of (21) and the discriminant of second quadratic polynomial in (22) is $1 - 4d(a + 1) < 0$ in view of (21). Finally the discriminant of the third polynomial in (22) is

$$-(a - 1)^2(a + 1) \left(4a^2d - 8ad - a + 4d + 3\right)$$

and for $a \neq 1$ is negative in view of (21). If $a = 1$ the third polynomial simply becomes a constant 1. Thus, the only solution of (22) is $m = 0$. The same holds for $M$ in view of symmetry of the considered system. So $m = M = 0$ and by Theorem 1.4.5 in [5] the zero solution of Equation (3) is globally asymptotically stable. \(\square\)

As Figure 1 shows, the boundary of the basins of attraction of two locally asymptotically stable equilibrium solutions $\bar{x}_0$ and $\bar{x}_+$ seem to be the global stable manifold of the smaller equilibrium solution $\bar{x}_-$, which is a saddle point for all values of parameters. In section 3 we will derive the asymptotic formulas for both stable and unstable manifolds based on the functional equations that the two manifolds satisfy. We will visually compare these manifolds with the image of the basin of attraction.
Now we give some results about the basins of attraction of the positive equilibrium solutions. We will show that local asymptotic stability of a positive equilibrium will also imply its global asymptotic stability in a substantial subregion of the parametric space and within the basins of attraction of locally stable equilibrium solutions.

Lemma 4 Assume that (74) holds. If \( \{x_n\} \) is non-zero solution of Equation (3), then the following hold:

(a) If \((x_{-1}, x_0) \in R_1 = \{(x, y) : 0 \leq y \leq x \leq \bar{x}_-\}\), then \(x_1 \leq x_0\). The solution \(\{x_n\}\) is a decreasing sequence and so

\[
\lim_{n \to \infty} x_n = 0.
\]

(b) If \((x_{-1}, x_0) \in R_2 = \{(x, y) : \bar{x}_- \leq x \leq y \leq \bar{x}_+\}\), then \(x_0 \leq x_1\). The solution \(\{x_n\}\) is an increasing sequence and so

\[
\lim_{n \to \infty} x_n = \bar{x}_+.
\]

(c) If \((x_{-1}, x_0) \in R_3 = \{(x, y) : \bar{x}_+ \leq y \leq x\}\), then \(x_1 \leq x_0\). The solution \(\{x_n\}\) is a decreasing sequence and so it converges to one of the equilibrium solutions.

Proof. Set

\[
G(u) = (a + 1)u^2 - u + d.
\]

Then the positive equilibrium solutions \(\bar{x}_\pm\) are solutions of the equilibrium equation \(G(u) = 0\) and \(G(u) < 0\) if and only if \(u \in (\bar{x}_-, \bar{x}_+)\).

(a) Now we have

\[
x_1 = \frac{x_0^2}{ax_0^2 + x_{-1} + d} \leq \frac{x_0^2}{ax_0^2 + x_0^2 + d} = \frac{x_0^2}{x_0 + G(x_0)} \leq \frac{x_0^2}{x_0} = x_0.
\]

By using induction we can prove that the solution \(\{x_n\}\) is a decreasing sequence and since \(x_n < \bar{x}_-\) it can only converge to the zero equilibrium.
(b) Now we have
\[ x_1 = \frac{x_0^2}{ax_0^2 + x_{-1}^2 + d} \geq \frac{x_0^2}{ax_0^2 + x_0^2 + d} = \frac{x_0^2}{x_0 + G(x_0)} \geq \frac{x_0^2}{x_0} = x_0. \]

By using induction we can prove that the solution \( \{x_n\} \) is an increasing and bounded sequence and since \( x_n > \bar{x} \) it can only converge to the larger positive equilibrium \( \bar{x}_+ \).

(c) The proof is identical to the proof of part (a) and it will be omitted.

\[ \Box \]

**Remark 1** An immediate consequence of Lemma 4 is that the set \( R_1 \) is a part of the basin of attraction of the zero equilibrium \( B(0) \) and \( R_2 \) is a part of the basin of attraction of the larger positive equilibrium \( B(\bar{x}_+) \), that is \( R_1 \subset B(0), R_2 \subset B(\bar{x}_+) \).

**Theorem 6** Assume that \( a > 1 \). Then if
\[
\begin{cases}
1 < a \leq 3 \quad \text{and} \quad 0 < d < \frac{1}{4a+4}, \quad \text{or;} \\
a > 3 \quad \text{and} \quad \frac{a-3}{4a^2-8a+4} < d < \frac{1}{4a+4}
\end{cases}
\]

the positive equilibrium of Equation (3) is globally asymptotically stable.

**Proof.** Clearly we can consider solutions of Equation (3) which are positive, that is for which \( x_0 > 0 \). The substitution \( u_n = \frac{D}{x_n} \) transforms Equation (3) into the equation
\[
u_{n+1} = a + du_n^2 + \frac{u_n^2}{u_{n-1}^2} = G(x_n, x_{n-1}), \quad n = 0, 1, \ldots
\]

One can easily show that Equation (24) has a unique positive equilibrium \( \bar{u} = \frac{1}{\bar{x}} \). We will show that \( \bar{u} \) is globally asymptotically stable under the conditions of the theorem.

Our major tool is global asymptotic stability result in [5], more precisely Theorem 1.4.5 [5]. Now we will check the assumptions of this theorem.
Figure 1: Basins of attraction for Equation (3) for parameters \(a = 0.49\) and \(d = 10.125\). Picture is produced by *Dynamica 5* [15].

1. Clearly \(G(x, y)\) is non-decreasing in \(x\) and non-increasing in \(y\).

2. There exists an interval \(I\) such that \(G : I \times I \to I\). Indeed \(I = [a, U]\) where \(U^- \leq U \leq U^+\) given \(U_\pm = \frac{1 \pm \sqrt{1-4(ad+\frac{1}{a})}}{2(d+\frac{1}{a^2})}\) and \(1 - 4(ad + \frac{1}{a}) > 0\).

If \(x, y \in I\) then

\[
G(x, y) = a + dx^2 + \frac{x^2}{y^2} \geq a.
\]

On the other hand for any \(U \geq \frac{1 - \sqrt{1-4(ad+\frac{1}{a})}}{2(d+\frac{1}{a^2})}\) we have

\[
G(U, a) \leq U.
\]

Therefore \(G(x, y) \in I\), which shows that \(I\) is an invariant interval for \(G\).
Next, consider the system of equations
\[
\begin{align*}
\begin{cases}
  f(M, m) = M \
  f(m, M) = m
\end{cases}
\Leftrightarrow
\begin{cases}
  M = a + dM^2 + \frac{M^2}{M^2} \
  m = a + dm^2 + \frac{m^2}{M^2}
\end{cases}
\end{align*}
\]

Using Mathematica, we eliminate the variable $M$ from this system and factor, which gives us
\[
(a + dm^2 - m + 1) \left( a^3 + a^2 dm^2 - a^2 m - a^2 - 2adm^2 - a + dm^2 + m^2 + m + 1 \right)
\]

Then we solve
\[
(a^3 + a^2 dm^2 - a^2 m - a^2 - 2adm^2 - a + dm^2 + m^2 + m + 1) = 0
\]

And we see that
\[
m = \frac{a^2 \pm \sqrt{(1 - a^2)^2 - 4 (a^3 - a^2 - a + 1) (a^2d - 2ad + d + 1) - 1}}{2 (a^2d - 2ad + d + 1)}
\]
When we solve the inequality \((1 - a^2)^2 - 4 \left( a^3 - a^2 - a + 1 \right) \left( a^2 d - 2 a d + d + 1 \right) < 0\)
there is the unique real solution \( m = \frac{a^2}{2 \left( a^2 d - 2 a d + d + 1 \right)} = \bar{u} \).
To see this, we factor to get
\[-(a - 1)^2(a + 1) \left( 4a^2 d - 8a d - a + 4d + 3 \right)\]
Clearly this will be negative when \((4a^2 d - 8a d - a + 4d + 3) > 0\), this will occur under the following conditions:

1. \( 1 < a \leq 3 \) and \( 0 < d < \frac{1}{4a+4} \), or;
2. \( a > 3 \) and \( \frac{a-3}{4a^2 - 8a + 4} < d < \frac{1}{4a+4} \)

Consequently, all conditions of Theorem 1.4.5 [5] are satisfied and every solution of Equation (24) which enters the interval \( I \) must converge to the unique positive equilibrium \( \bar{u} \).
To show that \( \bar{u} \) is globally asymptotically stable, it is sufficient to show that every solution of Equation (24) must enter \( I \). Observe that by Equation (24)
\[ u_{n+1} = a + du^2_n + \frac{u^2_n}{u^2_{n-1}} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \]
and so by the result on difference inequalities, see [4]
\[ u_n \geq \frac{1}{1 - A} - \varepsilon, \quad A = F + \frac{C}{D^2U^2}, \quad \varepsilon > 0 \]
Since \( U \) can be chosen to be large there exists \( N \geq 0 \) such that \( u_n \geq \frac{1}{1 - A} \) for all \( n \geq N \). Furthermore as we can choose \( U \geq \frac{1 - \sqrt{1 - 4(ad + 1)}}{2(d + 1)} \) to be as large as we wish, we can conclude that every solution of Equation (24) must enter and remain in \( I \).
Thus we conclude that the unique positive equilibrium \( \bar{x} \) of Eq.(3) is globally asymptotically stable for the conditions of the theorem. \( \square \)
2.3 The Neimark-Sacker Bifurcation

In this section we bring the system that corresponds to Equation (3) to the normal form which can be used for the computation of the relevant coefficients of the Neimark-Sacker bifurcation.

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{(\bar{x} + y_n)^2}{a(\bar{x} + y_n)^2 + (\bar{x} + y_{n-1})^2 + d} - \bar{x}, \quad n = 0, 1, \ldots$$

(25)

Set $u_n = y_{n-1}$ and $v_n = y_n$ for $n = 0, 1, \ldots$ and write Equation (3) in the equivalent form:

$$u_{n+1} = v_n$$

(26)

$$v_{n+1} = \frac{(\bar{x} + v_n)^2}{a(\bar{x} + v_n)^2 + (\bar{x} + u_n)^2 + d} - \bar{x}.$$

Let $F$ be the corresponding map defined by:

$$F \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} v \\ \frac{(\bar{x} + v)^2}{a(\bar{x} + v)^2 + (\bar{x} + u)^2 + d} - \bar{x} \end{array} \right).$$

(27)

Then $F$ has the unique fixed point $(0, 0)$ and the Jacobian matrix of $F$ at $(0, 0)$ is given by

$$Jac_F(0, 0) = \left( \begin{array}{cc} 0 & 1 \\ -\frac{\sqrt{1-4(a+1)d}+1}{a+1} & \frac{1}{a+1} \end{array} \right).$$

The eigenvalues of $Jac_F(0, 0)$ are $\mu(a)$ and $\overline{\mu(a)}$ where

$$\mu(a) = \frac{a + 2 - a\sqrt{1-4(a+1)d} + i\sqrt{2a^2(2(a+1)d-1) + 2(a^2 + 4a + 2)\sqrt{1-4(a+1)d}}}{2(a+1)}.$$

One can prove that for $a = a_0 = 1 - 4d$ we obtain $|\mu(a_0)| = 1$ and

$$\mu(a_0) = \frac{1}{2} \left( 1 + 4d + i\sqrt{(1-4d)(4f+3)} \right),$$

$$\mu^2(a_0) = 8d^2 + 4d - \frac{1}{2} + \frac{1}{2}i(4d+1)\sqrt{(1-4d)(4d+3)},$$

$$\mu^3(a_0) = 32d^3 + 24d^2 - 1 + 4i(2d+1)\sqrt{(1-4d)(4d+3)d},$$

$$\mu^4(a_0) = 128d^4 + 128d^3 + 16d^2 - 8d - \frac{1}{2} + \frac{1}{2}i(4d+1)(16d^2 + 8d - 1)\sqrt{(1-4d)(4d+3)}.$$
One can see that $\mu^k(a_0) \neq 1$ for $k = 1, 2, 3, 4$ and

$$|\mu(a)|^2 = \frac{\sqrt{1 - 4(a + 1)d + 1}}{a + 1}.$$ 

Furthermore, we get

$$d(a_0) = \frac{d|\mu(a)|}{da} \bigg|_{a=a_0} = -\frac{1}{4(1 - 4d)} < 0.$$ 

The eigenvectors corresponding to $\mu(a)$ and $\mu(a)$ are $q(a)$ and $\overline{q(a)}$, where

$$q = q(a_0) = \left(\frac{1}{2} \left( 1 + 4d - i\sqrt{(1 - 4d)(4d + 3)} \right), 1 \right)^T.$$ 

For $a = a_0$ we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix},$$

(28)

where

$$A = Jac_F(0,0)\big|_{a=a_0} = \begin{pmatrix} 0 & 1 \\ -1 & 4d + 1 \end{pmatrix}$$

and

$$G\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ (v+\frac{1}{2})^2 \end{pmatrix} - (4d + 1)v + u - \frac{1}{2}.$$ 

Hence, for $a = a_0$ system (26) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = A\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G\begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$  

(29)

Define the basis of $\mathbb{R}^2$ by $\Phi = (q, \overline{q})$.

Let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi\begin{pmatrix} z \\ \zbar \end{pmatrix} = (qz + \overline{q}\zbar) = $$

$$= \left( \frac{1}{2} \left( 4d - i\sqrt{(1 - 4d)(4d + 3)} + 1 \right) z + \frac{1}{2} \left( 4d + i\sqrt{(1 - 4d)(4d + 3)} + 1 \right) \zbar \right) \zbar + z.$$ 

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By using this, one can see that

\[
\begin{align*}
g_{20} &= \frac{\partial^2}{\partial z^2} G \left( \Phi \left( \frac{z}{\bar{z}} \right) \right) \bigg|_{z=0} = \left( 16d^2 + (12id - i) \sqrt{(1 - 4d)(4d + 3)} - 3 \right), \\
g_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}} G \left( \Phi \left( \frac{z}{\bar{z}} \right) \right) \bigg|_{z=0} = \left( 0 - 8d \right), \\
g_{02} &= \frac{\partial^2}{\partial \bar{z}^2} G \left( \Phi \left( \frac{z}{\bar{z}} \right) \right) \bigg|_{z=0} = \left( 16d^2 + (i - 12id) \sqrt{(1 - 4d)(4d + 3)} - 3 \right),
\end{align*}
\]

and

\[
\begin{align*}
K_{20} &= (\mu^2 I - A)^{-1} g_{20} = \left( \begin{array}{c} \frac{16d^2 + (12id - i) \sqrt{(1 - 4d)(4d + 3)} - 3}{(2d + 1)(4d - 1)} \frac{16d^2 + 8d + i(4d + 1)}{\sqrt{(1 - 4d)(4d + 3)} - 1} \\
\frac{16d^2 + i\sqrt{(1 - 4d)(4d + 3)} - 1}{4d(4d + 1) - 2} \end{array} \right), \\
K_{11} &= (I - A)^{-1} g_{11} = \left( \begin{array}{c} \frac{8d}{4d - 1} \\
\frac{8d^2}{4d - 1} \end{array} \right), \\
K_{02} &= (\bar{\mu}^2 I - A)^{-1} g_{02} = \bar{K}_{20}.
\end{align*}
\]

By using \( K_{20}, K_{11} \) and \( K_{02} \) we have that

\[
\begin{align*}
g_{21} &= \frac{\partial^3}{\partial z^2 \partial \bar{z}} G \left( \Phi \left( \frac{z}{\bar{z}} \right) \right) + \frac{1}{2} K_{20} z^2 + K_{11} z \bar{z} + \frac{1}{2} K_{02} \bar{z}^2 \bigg|_{z=0} \\
&= \left( -2i \sqrt{\frac{4d + 3}{1 - 4d}} + \frac{20}{3 - 12d} - \frac{74}{6d + 3} + 18 \right).
\end{align*}
\]

Next we have that \( pA = \mu p \) and \( pq = 1 \) where

\[
p = \left( \frac{i}{\sqrt{(1 - 4d)(4d + 3)}}, \frac{-4id + \sqrt{(1 - 4d)(4d + 3)} - i}{2\sqrt{(1 - 4d)(4d + 3)}} \right).
\]

One can see that

\[
\alpha(a_0) = \frac{1}{2} Re (pg_{21} \bar{\mu}) = -\frac{1}{1 - 4d} < 0.
\]

**Theorem 7** Let \( 0 < d < \frac{1}{4(1 + a)} \) and

\[
\bar{x}_+ = \frac{1 + \sqrt{1 - 4(a + 1)d}}{2(a + 1)}.
\]

Then there is a neighborhood \( U \) of the equilibrium point \( \bar{x}_+ \) and some \( \rho > 0 \) such that for

\[
|a - a_0| < \rho \quad (a_0 = 1 - 4d)
\]
and $x_0, x_{-1} \in U$, the ω-limit set of a solution of Equation (3), with initial conditions $x_0, x_{-1}$ is the equilibrium point $\bar{x}_+$ if

$$a > a_0 = 1 - 4d$$

and belongs to a closed invariant $C^1$ curve $\Gamma(a)$ encircling the equilibrium point $\bar{x}_+$ if

$$a < a_0 = 1 - 4d.$$  

Furthermore, $\Gamma(a_0) = 0$ and the invariant curve $\Gamma(a)$ can be approximated by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \bar{x}_+ \\ \bar{x}_+ \end{pmatrix} + 2\rho_0 \text{Re} (q e^{i\theta}) + \rho_0^2 (\text{Re} (K_{20} e^{2i\theta}) + K_{11})$$

where

$$\rho_0 = \frac{1}{2} \sqrt{(1 - 4d) - a}.$$  

**Proof.** The proof follows from above discussion and Theorem 4 and Corollary 1. See Figure 3 for a graphical illustration.

\[ \square \]
2.4 The Invariant Manifolds

In this section we derive the asymptotic formulas for the local stable and unstable manifolds for the equilibrium point $\bar{x}_-$ and provide some numerical examples where we compare visually the local approximations of stable and unstable manifolds and center manifold, obtained by using Mathematica, with the boundaries of the basins of attraction obtained by using the software package Dynamica.

From Lemma 3 it follows that $(\bar{x}_-, \bar{x}_-)$ is a saddle point if $4d(a + 1) < 1$. In order to apply the theorem for the stable and unstable manifolds, we make a change of variable $y_n = x_n - \bar{x}_-$. Then the transformed equation is given by

$$y_{n+1} = \frac{(\bar{x}_- + y_n)^2}{a (\bar{x}_- + y_n)^2 + (\bar{x}_- + y_{n-1})^2 + d} - \bar{x}_-, \quad n = 0, 1, \ldots \quad (33)$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write Equation (3) in the equivalent form:

$$u_{n+1} = v_n \quad (34)$$

$$v_{n+1} = \frac{(\bar{x}_- + v_n)^2}{a (\bar{x}_- + v_n)^2 + (\bar{x}_- + u_n)^2 + d} - \bar{x}_-.$$

Let $G$ be the corresponding map defined by:

$$G \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} f_1(u, v) \\ f_2(u, v) \end{array} \right) = \left( \begin{array}{c} \frac{v}{a(\bar{x}_-+v)^2+(\bar{x}_-+u)^2+d} - \bar{x}_- \end{array} \right). \quad (35)$$

We expand $f_1(u, v)$ and $f_2(u, v)$ as a Taylor series about $(0, 0)$ to write
Assume that the invariant manifold at \((0, 0)\) is locally represented as the graph of a function \(v = h(u)\) such that

\[
h(u) = Au + Bu^2 + Cu^3 + O(|u|^4).
\]

Then, from \(h(f_1(u, h(u))) - f_2(u, h(u)) = 0\), and by using package Mathematica, we obtain

\[
A_\pm = 1 - a\bar{x}_- \pm \sqrt{(\bar{x}_-(a(a\bar{x}_- - 2) - 2) + 1)}
\]

and

\[
B_\pm = \frac{(a + 1)^2 \left( (1 - A_\pm)\bar{x}_-^2 (a(3A_\pm - 1) - A_\pm + 3) - d\bar{x}_-(A_\pm ((3a - 2)A_\pm + 4) + 1) + A^2d^2 \right)}{\bar{x}_- (-a + 1)A_\pm (a(3a - 2A_\pm + 1) - 2) + 2d},
\]

and

\[
C_\pm = C_\pm (a, f, A_\pm, B_\pm).
\]

Then the dynamics restricted to the invariant manifold are given locally by the equation

\[
u_{n+1} = f_1(u_n, h(u_n)) = h(u_n) = A_\pm u_n + B_\pm u_n^2 + C_\pm u_n^3 + O(|u_n|^4).
\]

Note that the Jacobian matrix of \(G\) at \((0, 0)\) is given by

\[
Jac_G(0, 0) = \begin{pmatrix}
0 & 1 \\
\frac{\sqrt{1-4(a+1)d-1}}{a+1} & \frac{1}{a+1}
\end{pmatrix}.
\]

and the eigenvalues of \(Jac_G(0, 0)\) are \(A_\pm\).
Figure 4: The local approximation of the stable (red) and unstable manifold (green) of Equation (3) for $a = 0.49, d = 0.125$

**Theorem 8** Assume that $4d(a + 1) \leq 1$. Then, the equilibrium point $\bar{x}_-$ of Equation (3) is a saddle point if $4d(a + 1) < 1$. The stable manifold $W^s$ and unstable manifold $W^u$ at $(\bar{x}_-, \bar{x}_-)$, are given by

$W^u = \{(x, y) : y = \bar{x}_- + A_+(x-\bar{x}_-) + B_+(x-\bar{x}_-)^2 + C_+(x-\bar{x}_-)^3 + O(|x-\bar{x}_-|^4), x > 0, y > 0\}$

and

$W^s = \{(x, y) : y = \bar{x}_- + A_-(x-\bar{x}_-) + B_-(x-\bar{x}_-)^2 + C_-(x-\bar{x}_-)^3 + O(|x-\bar{x}_-|^4), x > 0, y > 0\}$.

The equilibrium point $(\bar{x}_-, \bar{x}_-)$, is non-hyperbolic if $4d(a + 1) = 1$, and it is semi-asymptotically stable from the right.

**Proof.** The proof of first part of the Theorem follows from above discussion. If $4d(a + 1) = 1$, then $a = \frac{1-4d}{4d} > 0$ and we obtain $A_- = 4d < 1, A_+ = 1$, and

$B_+ = \frac{1}{4d(4d-1)} < 0$. The rest of the proof follows from the fact that the dynamics of Equation (3) are dynamics restricted to the center manifold which is given locally by Equation (38). See Figures 4-5 for a graphical illustration. \qed
2.5 Rate of Convergence

In this section we will shortly discuss the rate of convergence of solutions of Equation (3) that converge to the equilibrium solutions. We will show that the convergence toward the zero equilibrium is quadratic and toward any positive equilibrium is linear. Assume that \(\lim_{n \to \infty} x_n = 0\) for some solution of Equation (3). Then Equation (3) implies

\[
\frac{x_{n+1}}{x_n^2} = \frac{1}{ax_n^2 + x_{n-1}^2 + d}
\]

and so

\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n^2} = \frac{1}{d},
\]

which shows that the convergence toward the zero equilibrium is quadratic.

Let \(\lim_{n \to \infty} x_n = \bar{x} > 0\) for some solution of Equation (3). Then we have

\[
x_{n+1} - \bar{x} = \frac{x_n^2}{ax_n^2 + x_{n-1}^2 + d} - \frac{\bar{x}^2}{(a+1)\bar{x}^2 + d} = \frac{d(x_n^2 - \bar{x}^2) + \bar{x}^2(x_n^2 - x_{n-1}^2)}{(ax_n^2 + x_{n-1}^2 + d)((a+1)\bar{x}^2 + d)}
\]

\[
= \frac{(d + \bar{x}^2)(x_n + \bar{x})}{(ax_n^2 + x_{n-1}^2 + d)((a+1)\bar{x}^2 + d)}(x_n - \bar{x}) - \frac{\bar{x}^2(x_n + \bar{x})}{(ax_n^2 + x_{n-1}^2 + d)((a+1)\bar{x}^2 + d)}(x_{n-1} - \bar{x})
\]
Figure 6: (a) Bifurcation diagram of Equation (3) for \( a = 0.52 \) indicating the convergence to the period 7 solution. (b) Bifurcation diagram of Equation (3) for \( a = 0.52 \) indicating the global attractivity of period 7 solution and period 8 solution.

\[
= g_0(x_n - \bar{x}) + g_1(x_{n-1} - \bar{x}),
\]

where

\[
\lim_{n \to \infty} g_0 = \frac{(d + \bar{x}^2)2\bar{x}}{(a + 1)\bar{x}^2 + d} = \frac{2(d + \bar{x}^2)}{\bar{x}} = 2(1 - a\bar{x})
\]

and

\[
\lim_{n \to \infty} g_1 = -\frac{\bar{x}^22\bar{x}}{(a + 1)\bar{x}^2 + d} = \frac{-2\bar{x}^3}{\bar{x}^2} = -2\bar{x}.
\]

Setting \( y_n = x_n - \bar{x} \) we see that the limiting equation is exactly the linearized equation (19). Now in view of Poincaré-Perron theorem we conclude that

\[
\lim_{n \to \infty} \left| \frac{y_{n+1}}{y_n} \right| = \lim_{n \to \infty} \left| \frac{x_{n+1} - \bar{x}}{x_n - \bar{x}} \right| = |\lambda_{1,2}|,
\]

where \( \lambda_{1,2} \) are roots of the characteristic equation (20), [17, 18]. Thus, in this case the convergence is linear since \( \lambda_{1,2} \neq 0 \).

List of References


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CHAPTER 3

Neimark-Sacker Bifurcation of a Certain Second Order Quadratic Fractional Difference Equation

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3.1 Introduction and Preliminaries

In this paper we study the Neimark-Sacker Bifurcation of the equilibrium of two special cases of the difference equation

\[ x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \quad n = 0, 1, 2, \ldots \] (39)

where the parameters \( \beta, \gamma, \delta, B, C, D \) are nonnegative numbers which satisfy \( B + C + D > 0 \) and the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary nonnegative numbers such that \( B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0 \) for all \( n \geq 0 \).

Equation (39), which has been studied in [1, 2], is a special case of a general second order quadratic fractional equation of the form

\[ x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2 + D x_n + E x_{n-1} + F}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f}, \quad n = 0, 1, \ldots \] (40)

with non-negative parameters and initial conditions such that \( A + B + C > 0, \) \( a + b + c + d + e + f > 0 \) and \( a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f > 0, n = 0, 1, \ldots \) Several global asymptotic results for some special cases of Eq.(40) were obtained in [3, 4, 5, 6].

The change of variable \( x_n = 1/u_n \) transforms Eq.(39) to the difference equation

\[ u_{n+1} = \frac{D u_{n-1}^2 + C u_n + B u_{n-1}}{\delta u_{n-1}^2 + \gamma u_n + \beta u_{n-1}}, \quad n = 0, 1, \ldots \] (41)

where we assume that \( \delta + \beta + \gamma > 0 \) and that the non-negative initial conditions \( u_{-1}, u_0 \) are such that \( \delta u_{n-1}^2 + \gamma u_n + \beta u_{n-1} > 0 \) for all \( n \geq 0 \). Thus the results of this paper extends to Eq.(41).

The first systematic study of global dynamics of a special quadratic fractional case of Eq.(40) where \( A = C = D = a = c = d = 0 \) was performed in [1, 2]. Dynamics of some related quadratic fractional difference equations was considered in the papers [4, 5, 6]. In this paper we will perform the Neimark-Sacker bifurcation analysis of some special cases of Eq.(39) which are obtained when one or more coefficients are set to be zero.
Detailed local stability analysis of Equation (39) was presented in [7] and Neimark-Sacker bifurcation was established for two special cases of Equation (39) in [8]. Some interesting special cases of Eq.(39) which were thoroughly studied in [9] are equations:

(1) Delay Beverton-Holt difference equation when $\gamma = \delta = C = 0$:

$$x_{n+1} = \frac{\beta x_{n-1}}{B x_{n-1} + D}, \quad n = 0, 1, \ldots,$$

which represents the basic discrete model in population dynamics, see [?].

(2) Delay Riccati difference equation when $\gamma = C = 0$:

$$x_{n+1} = \frac{\beta x_{n-1} + \delta}{B x_{n-1} + D}, \quad n = 0, 1, \ldots,$$

(3) Difference equation studied in [9], when $\delta = D = 0$:

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{B x_n + C x_{n-1}}, \quad n = 0, 1, \ldots$$

which represents the discretization of the differential equation model in biochemical networks, see [10].

Now we consider bifurcation of a fixed point of map associated to Eq.(39) in the case where the eigenvalues are complex conjugates and of unit module.

In this manuscript, we will refer to the Neimark-Sacker theorem as presented in Chapter 2 and calculate the asymptotic formula for the invariant curve around the equilibrium using the same method developed by Murakami in [11]. Some of the recent results about Neimark-Sacker bifurcation are discussed in [12, 13].

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\bar{\mu}(\lambda) = \alpha(\lambda) - i\beta(\lambda)$ satisfying $\alpha(\lambda)^2 + \beta(\lambda)^2 = 1$.
and \( \beta(\lambda) \neq 0 \). By putting the linear part of such a map into Jordan Canonical form, we may assume \( F \) to have the following form near the origin

\[
F(\lambda, x) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}
\]  

(43)

Then the coefficient \( a(\lambda_0) \) of the cubic term in Eq(9) in polar coordinate is equal to

\[
a(\lambda_0) = \text{Re} \left[ \frac{(1 - 2\mu(\lambda_0))\bar{\mu}^2(\lambda_0)}{1 - \mu(\lambda_0)} \xi_{11} \xi_{20} \right] + \frac{1}{2} |\xi_{11}|^2 + |\xi_{02}|^2 - \text{Re}(\bar{\mu}(\lambda_0)\xi_{21}),
\]

(44)

where

\[
\begin{align*}
\xi_{20} &= \frac{1}{8} \left\{ (g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i \left[ (g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2} \right] \right\} \\
\xi_{11} &= \frac{1}{4} \left\{ (g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i \left[ (g_2)_{x_1x_1} + (g_2)_{x_2x_2} \right] \right\} \\
\xi_{02} &= \frac{1}{8} \left\{ (g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i \left[ (g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2} \right] \right\} \\
\xi_{21} &= \frac{1}{16} \left\{ (g_1)_{x_1x_1} + (g_1)_{x_1x_2} + (g_2)_{x_1x_1} + (g_2)_{x_2x_2} + i \left[ (g_2)_{x_1x_1} + (g_2)_{x_1x_2} - (g_1)_{x_1x_1} - (g_1)_{x_2x_2} \right] \right\}
\end{align*}
\]

Table 1 gives a basic information about local stability of four equations of type (2,2) which might exhibit Neimark-Sacker bifurcation, see [7]. The second equation was considered in [8] and the fourth equation was discussed in great detail in [14]. In this paper we will study the first and the third equation.

We will show that the first equation exhibits Neimark-Sacker bifurcation which, depending on the values of parameters, can be subcritical or supercritical and that there exists a unique closed invariant curve which is attracting in the subcritical case and repelling in the supercritical case. The simulations indicate that the Chenciner bifurcation may occur, in which case two invariant curves may appear. We will show that the second equation exhibits supercritical Neimark-Sacker bifurcation when the parameter \( \delta \) varies in a small neighborhood of the critical value \( \delta_0 \). We will derive the asymptotic formula for a unique closed invariant curve.
Table 1: Equations of type (2.2)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equilibrium point</th>
<th>Stability of equilibrium point</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{n+1} = \frac{\beta x_n x_{n-1} + \delta x_n}{B x_n x_{n-1} + a x_{n-1}^2}$</td>
<td>$x = \frac{\sqrt{4(B + C) + \beta^2 + \beta}}{2(B + C)}$</td>
<td>LAS for $\beta B^2 &gt; \delta$</td>
<td>$\mu(\delta) = \left(\sqrt{B^2 + \beta} + \beta\right)$</td>
</tr>
<tr>
<td>$x_{n+1} = \frac{\beta x_{n-1} + \delta x_n}{B x_{n-1} + a x_{n-1}^2}$</td>
<td>$x = \frac{\sqrt{4B(\beta + 1) + \beta}}{2B}$</td>
<td>LAS for $\delta &lt; B(\beta + 1)$</td>
<td>$\mu(\delta) = \frac{B(\beta + 1) + \beta}{2B}$</td>
</tr>
<tr>
<td>$x_{n+1} = \frac{\gamma x_n + \delta x_n}{B x_n + a x_n^2}$</td>
<td>$x = \frac{\sqrt{4B(\gamma + \beta) + \beta}}{2B}$</td>
<td>LAS for $\delta &lt; B(\gamma + \beta)$</td>
<td>$\mu(\delta) = \frac{B(\gamma + \beta) + \beta}{2B}$</td>
</tr>
</tbody>
</table>

* $\Delta_1 = 4(B + \beta)^2 (\beta^2 + (B + 2)\delta - (B + \delta)^2 > 0$, $\Delta_2 = (C + \beta + \delta)^2 - 4(C + \beta + \delta)^2 (B + \beta - 2C + 1)$; $\Delta_3 = 4(B + 1)^2 (B + 2\delta - (B + 2)\delta + (B + 2\delta - \delta)^2$; and $\Delta_4 = (\gamma - \beta)^2 - 4(C + \gamma + \beta + \delta)^2 (2C + 1)^2$.

3.2 Neimark-Sacker Bifurcation

3.2.1 The case $\delta, \beta, C > 0$, $B \geq 0$ : $x_{n+1} = \frac{\beta x_n x_{n-1} + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2}$

Now, we verify analytically that the equation undergoes a Neimark-Sacker bifurcation at the positive equilibrium as parameter $\delta$ is varied. The difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2}$$

has the equilibrium point

$$\bar{x} = \frac{\sqrt{4(B + C) + \beta^2 + \beta}}{2(B + C)}.$$

We can write Eq. (45) in the equivalent form:

$$y_{n+1} = \frac{\beta y_n z_n + \delta z_n}{B y_n z_n + C y_n^2}$$

$$z_{n+1} = \frac{\beta y_n z_n + \delta z_n}{B y_n z_n + C y_n^2}$$
where
\[ y_n = x_{n-1} \text{ and } z_n = x_n \text{ for } n = 0, 1, \ldots \]

In order to apply the Theorem 4 we make a change of variable \( u_n = y_n - \bar{x} \) and \( v_n = z_n - \bar{x} \). Then, we write this System (46) in the equivalent form:

\[
\begin{align*}
  u_{n+1} &= v_n \\
  v_{n+1} &= \frac{(\bar{x} + v_n)(\beta \bar{x} + \beta u_n + \delta)}{(\bar{x} + u_n)(\bar{x}(B + C) + Bu_n + Cu_n)} - \bar{x}
\end{align*}
\]

Let \( F_1 \) be the function defined by:

\[
F_1(u, v) = \left( \frac{v}{(\bar{x} + u)(\bar{x}(B + C) + Bu + Cu)} - \bar{x} \right)
\]

Then \( F_1 \) has the unique fixed point \((0, 0)\) and maps \((-\bar{x}, \infty)^2\) into \((-\bar{x}, \infty)^2\). The Jacobian matrix of \( F_1 \) is given by

\[
Jac_{F_1}(u, v) = \begin{pmatrix}
0 & 1 \\
-\frac{B\delta + 2C\delta + C\beta}{(B + C)^2 \bar{x}^2} & \frac{C(\delta + \beta \bar{x})}{(B + C)^2 \bar{x}^2}
\end{pmatrix}
\]

At \((0, 0)\), \( Jac_{F_1}(u, v) \) has the form

\[
J_{01} = Jac_{F_1}(0, 0) = \begin{pmatrix}
0 & 1 \\
-\frac{B\delta + 2C\delta + C\beta}{(B + C)^2 \bar{x}^2} & \frac{C(\delta + \beta \bar{x})}{(B + C)^2 \bar{x}^2}
\end{pmatrix}
\]

**Lemma 5** Assume \( \delta_0 = \frac{\beta B^2}{C^2} \). Then, \((0, 0)\) is an equilibrium point of the map \( F_1 \). If \( \mu_1 \) and \( \mu_2 \) are corresponding eigenvalues of \( A_1 = J_{01}|_{\delta = \delta_0} \), then

\[
\mu_1(\delta_0) = \frac{C + i\sqrt{(2B + 3C)(2B + 3C)}}{2(B + C)}.
\]

Furthermore:

(a) \( |\mu_1(\delta_0)| = 1 \).

(b) \( \mu_1(\delta_0)^k \neq 1 \) for \( k = 1, 2, 3, 4 \).

(c) \( d_1 = d_1(\delta_0) = \left. \frac{d}{d\delta} |\mu_1(\delta)| \right|_{\delta = \delta_0} = \frac{C^3}{2B^2(B + C)(2B + C)} > 0 \).
(d) The corresponding eigenvectors are:

\[ q_1(\delta_0) = \left( \frac{C - i\sqrt{(2B + C)(2B + 3C)}}{2(B + C)}, 1 \right)^T \]

and

\[ p_1(\delta_0) = \left( \frac{i(B + C)}{\sqrt{(2B + C)(2B + 3C)}}, \frac{4B^2 + 8BC + 3C^2 - i\sqrt{4B^2 + 8BC + 3C^2}}{8B^2 + 16BC + 6C^2} \right), \]

where \( A_1 q_1(\delta_0) = \mu_1 q_1(\delta_0) \), \( p_1(\delta_0) A_1 = \mu_1 p_1(\delta_0) \) and \( p_1(\delta_0) q_1(\delta_0) = 1 \).

**Proof.** The eigenvalues of (48) are \( \mu_1(\delta) \) and \( \bar{\mu}_1(\delta) \) where

\[
\mu_1(\delta) = \frac{C\beta\bar{x} + C\delta + i\sqrt{4\bar{x}^2(B + C)^2 (C\beta\bar{x} + B\delta + 2C\delta) - (C\beta\bar{x} + C\delta)^2}}{2\bar{x}^2(B + C)^2},
\]

since

\[
(C\beta\bar{x} + C\delta)^2 - 4\bar{x}^2(B + C)^2 (C\beta\bar{x} + B\delta + 2C\delta) =
\]

\[
- \frac{\sqrt{4B\delta + 4C\delta + \beta^2} (2\beta\delta(B + C) (2B^2 + 8BC + 5C^2) + C\beta^3(4B + 3C))}{2(B + C)^2}
\]

\[
- \frac{4\beta^2\delta(B + C) (B^2 + 6BC + 4C^2) + 2\delta^2(B + C)^2 (4B^2 + 12BC + 7C^2) + C\beta^4(4B + 3C)}{2(B + C)^2} < 0
\]

(50)

One can see that

\[ |\mu_1(\delta)|^2 = \mu_1(\delta)\bar{\mu}_1(\delta) = \frac{C\beta\bar{x} + B\delta + 2C\delta}{\bar{x}^2(B + C)^2} \frac{2\left(2B^2\delta + C\beta\sqrt{4B\delta + 4C\delta + \beta^2} + 6BC\delta + 4C^2\delta + C\beta^2\right)}{(B + C) \left(2\sqrt{4B\delta + 4C\delta + \beta^2} + \beta\right)^2}, \]

from which, by using (3.2.1) we get

\[
\frac{d}{d\delta} |\mu_1(\delta)| = \frac{d}{d\delta} \left( \frac{2\sqrt{2B^2\delta + C\beta\sqrt{4B\delta + 4C\delta + \beta^2} + 6BC\delta + 4C^2\delta + C\beta^2}}{\sqrt{B + C} \left(2\sqrt{4B\delta + 4C\delta + \beta^2} + \beta\right)} \right) =
\]

\[
\frac{\sqrt{2B} \sqrt{(B + C)^3}}{\sqrt{4\delta(B + C) + \beta^2 + \beta}}
\]

\[
= \frac{(B + C)^3 \sqrt{4\delta(B + C) + \beta^2 + \beta}}{\sqrt{4\delta(B + C) + \beta^2 + \beta}}
\]

Let

\[ \delta_0 = \frac{B\beta^2}{C^2}. \]

(51)
Then, we have that
\[
\frac{d|\mu_1(\delta)|}{d\delta}\bigg|_{\delta=\delta_0} = \frac{C^3}{2\beta^2(B+C)(2B+C)} > 0
\]
For \(\delta = \delta_0\) we obtain
\[
\bar{x} = \frac{\beta}{C} \text{ and } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{C}{B+C} \end{pmatrix}
\]
The eigenvalues of \(A_1\) are \(\mu_1(\delta_0)\) and \(\mu_1(\delta_0)\) where
\[
\mu_1(\delta_0) = \frac{C + i\sqrt{(2B+C)(2B+3C)}}{2(B+C)}, \quad |\mu_1(\delta_0)| = 1.
\]
The eigenvectors corresponding to \(\mu_1(\delta)\) and \(\mu_1(\delta)\) are \(q_1(\delta_0)\) and \(q_1(\delta_0)\) where
\[
q_1(\delta_0) = \begin{pmatrix} C - i\sqrt{(2B+C)(2B+3C)} \\ 2(B+C) \end{pmatrix}, \quad 1
\]
One can prove that \(|\mu_1(\delta_0)| = 1\), and
\[
\mu_1^2(\delta_0) = \frac{-2B^2 - 4BC - C^2}{2(B+C)^2} + \frac{iC\sqrt{(2B+C)(2B+3C)}}{2(B+C)^2}
\]
\[
\mu_1^3(\delta_0) = \frac{-C(3B^2 + 6BC + 2C^2)}{2(B+C)^3} + \frac{iB(B + 2C)\sqrt{(2B+C)(2B+3C)}}{2(B+C)^3}
\]
\[
\mu_1^4(\delta_0) = \frac{2B^4 + 8B^3C + 8B^2C^2 - C^4}{2(B+C)^4} - \frac{iC\sqrt{(2B+C)(2B+3C)(2B^2 + 4BC + C^2)}}{2(B+C)^4},
\]
from which follows that \(\mu_1^k(\delta_0) \neq 1\) for \(k = 1, 2, 3, 4\) and \(C > 0\) and \(B \geq 0\).
The rest of the proof is almost immediate, so we skip it.

For sufficiently small \(\varepsilon\), let \(\delta = \delta_0 + \varepsilon\). By using Lemma 5, System (47) can be transformed into the Arnold’s normal form
\[
z_{n+1} = \mu(\varepsilon)z_n + \gamma(\varepsilon)z_n^2\bar{z}_n + O(|z_n|^4).
\]
In the sequel, by procedure described in [11], we compute \(K_{20}\) and \(K_{11}\). We have that
\[
F_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{B\delta + 2C\delta + C\beta}{(B+C)^2\varepsilon^2} & \frac{C(\delta + \beta\varepsilon)}{(B+C)^2\varepsilon^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_{11}(\delta, u, v) \\ f_{12}(\delta, u, v) \end{pmatrix},
\]
and

\[ f_{11}(\delta, u, v) = 0 \]

\[ f_{12}(\delta, u, v) = \frac{(\bar{x} + v)(\beta(\bar{x} + u) + \delta)}{(\bar{x} + u)(B(\bar{x} + v) + C(\bar{x} + u))} + \frac{u(C\beta\bar{x} + B\delta + 2C\delta)}{\bar{x}^2(B + C)^2} - \frac{Cv(\beta\bar{x} + \delta)}{\bar{x}^2(B + C)^2} - \bar{x}. \]

Substituting \( \delta = \delta_0 \) and \( \bar{x} \) into (52) we get

\[ F_{01}(u, v) = \begin{pmatrix} 0 & 1 \\ -1 & B + C \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & B + C \end{pmatrix} \begin{pmatrix} h_{11}(u, v) \\ h_{12}(u, v) \end{pmatrix}, \tag{53} \]

and

\[ h_{11}(u, v) = f_{11}(\delta_0, u, v) = 0 \]

\[ h_{12}(u, v) = f_{12}(\delta_0, u, v) = C \left( \frac{u^2}{C u + \beta} - \frac{C v (B v + C u)}{(B + C)(C(B v + C u) + \beta(B + C))} \right). \]

Hence, for \( \delta = \delta_0 \) system (47) is equivalent to

\[ \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & B + C \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_{11}(u_n, v_n) \\ h_{12}(u_n, v_n) \end{pmatrix}, \tag{54} \]

Let

\[ \begin{pmatrix} u_n \\ v_n \end{pmatrix} = P_1 \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} \]

where

\[ P_1 = \begin{pmatrix} \frac{C}{2(B+C)} & \frac{\sqrt{(2B+C)(2B+3C)}}{2(B+C)} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P_1^{-1} = \begin{pmatrix} 0 & -\frac{1}{2B+C} \\ \frac{2(B+C)}{\sqrt{(2B+C)(2B+3C)}} & \frac{1}{\sqrt{(2B+C)(2B+3C)}} \end{pmatrix}. \]

Then system (47) is equivalent to

\[ \begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} C & -\sqrt{(2B+C)(2B+3C)} \\ \frac{\sqrt{(2B+C)(2B+3C)}}{2(B+C)} & \frac{\sqrt{(2B+C)(2B+3C)}}{2(B+C)} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + G_{01} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}, \tag{55} \]

where

\[ H_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} h_{11}(u, v) \\ h_{12}(u, v) \end{pmatrix} \]

and

\[ G_{01} \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} g_{11}(u, v) \\ g_{12}(u, v) \end{pmatrix} = P_1^{-1} H_1 \begin{pmatrix} u \\ v \end{pmatrix}. \]
By straightforward calculation we obtain that
\[ g_{11}(u, v) = \frac{C}{2(B + C)} \left( \frac{(v\sqrt{(2B + C)(2B + 3C)} + Cu)^2}{Cv\sqrt{(2B + C)(2B + 3C)} + 2\beta(B + C) + C^2u} \right) - 2C\left( \frac{u(2B^2 + 2BC + C^2) + C\sqrt{(2B + C)(2B + 3C)}}{Cu(2B^2 + 2BC + C^2) + C^2v\sqrt{(2B + C)(2B + 3C)} + 2\beta(B + C)^2} \right) \]
\[ g_{12}(u, v) = -\frac{C}{\sqrt{(2B + C)(2B + 3C)}} g_{11}(u, v) \]

If we take the basis of \( \mathbb{R}^2 \) as \( \Phi_1 = (q_1, \bar{q}_1) \), where \( q_1 = q_1(\bar{\delta}_0) \), then we can represent \((u, v)\) as
\[ \begin{pmatrix} u \\ v \end{pmatrix} = \Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (q_1 z + \bar{q}_1 \bar{z}). \]

Let
\[ G_{01} \left( \Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) = \frac{1}{2} (g_{20} z^2 + 2g_{11} z \bar{z} + g_{02} \bar{z}^2) + O(|z|^3) \]

By using package Mathematica, we obtain
\[ g_{20} = \frac{\partial^2}{\partial z^2} G_{01} \left( \Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{C}{\beta(B + C)^3} \left( 2B^3 + 8B^2C + BC \left( 7C + i\sqrt{(2B + C)(2B + 3C)} \right) + 2C^3 \right) \end{pmatrix} \]
\[ g_{11} = \frac{\partial^2}{\partial z \partial \bar{z}} G_{01} \left( \Phi_1 \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{C}{\beta(B + C)^3} \left( 2B^3 + 4B^2C + 4BC^2 + C^3 \right) \end{pmatrix} \]

and
\[ K_{20} = (\mu_1^2 I - A)^{-1} g_{20} = \begin{pmatrix} 2C(B + C) \left( 2B^3 + 8B^2C + BC \left( 7C + i\sqrt{(2B + C)(2B + 3C)} \right) + 2C^3 \right) \\ -\beta(B + C)(B + 2C) \left( 2B^2 + 4BC + C \left( C - i\sqrt{(2B + C)(2B + 3C)} \right) \right) \\ C \left( 2B^3 + 8B^2C + BC \left( 7C + i\sqrt{(2B + C)(2B + 3C)} \right) + 2C^3 \right) \end{pmatrix} \]
\[ K_{11} = (I - A)^{-1} g_{11} = \begin{pmatrix} C \left( 2B^3 + 4B^2C + 4BC^2 + C^3 \right) \\ \frac{\beta(B + C)^2(2B + C)}{\beta(B + C)^2(2B + C)} \end{pmatrix} \]

By using \( K_{20} \) and \( K_{11} \), we have that
The package Mathematica yields

$$\alpha_1(0) = \alpha_1(\delta_0) = \frac{1}{2} Re(p_1 g_{21} \mu_1) = \frac{B^2 C^3 (B^2 - BC - C^2)}{2 \beta^2 (B + C)^4 (2B + C)}.$$

Figure 7: Figures a) and b): Bifurcation diagram in $(\delta - x)$ plane for $\beta = 1.6$, $B = 1.9$, $C = 1$.

**Theorem 9** Assume that $\beta, B, C > 0$ and $\delta_0 = \frac{B \beta^2}{C^2}$. If $\alpha_1(\delta_0) \neq 0$, then System (46) passes through a Neimark-Sacker bifurcation at an unique positive equilibrium point $E_1(\bar{x}, \bar{x}) = \left(\frac{\beta}{C}, \frac{\delta}{C}\right)$ when the parameter $\delta$ varies in small neighborhood of $\delta_0$. If $\alpha_1(\delta_0) < 0$ (resp., $\alpha_1(\delta_0) > 0$), then the Neimark–Sacker bifurcation of model (46) at $\delta = \delta_0$ is supercritical (resp., subcritical) and there exists an unique closed invariant curve $\Gamma(\delta)$, which is attracting (resp., repelling), and bifurcates from $E_1$ for $\delta > \delta_0$ (resp., $\delta < \delta_0$).

Let $\delta = \delta_0 + \varepsilon$. Then for small $\varepsilon$, the curve $\Gamma(\delta)$ can be approximated by

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx (\bar{x}, \bar{x}) + 2 \rho_0 Re(q_1 e^{i\theta}) + \rho_0^2 \left(Re\left(K_{20} e^{2i\theta}\right) + K_{11}\right), \quad \theta \in \mathbb{R}$$

where

$$\rho_0 = \sqrt{-\frac{d_1(\delta_0)}{\alpha_1(\delta_0)} \varepsilon}.$$
Proof. The proof follows from above discussion, Lemma 5 and Theorem 4.

Hence, if \( \delta_0 = \frac{\beta B}{C^2} \) then \( \bar{x} \) is non-hyperbolic equilibrium point such that 
\[
\operatorname{Det}(J_{F_1})(\bar{x}, \bar{x}) = 1 \quad \text{and} \quad 0 < \operatorname{Tr}(J_{F_1})(\bar{x}, \bar{x}) < 2.
\]
Therefore, a Neimark-Sacker bifurcation can occur as parameter \( \delta \) is varied. We have that \( d_1(\delta_0) > 0 \), for all \( \beta, B, C > 0 \). It appears that \( \alpha_1(\delta_0) \) changes its sign, which implies that we have supercritical Neimark-Sacker bifurcation and subcritical Neimark-Sacker bifurcation. If \( \alpha_1(\delta_0) < 0 \) (resp., \( \alpha_1(\delta_0) > 0 \), then the Neimark–Sacker bifurcation of the System (47) at \( \delta = \delta_0 \) is supercritical (resp., subcritical) and there exists an unique closed invariant curve
\[ \Gamma(\delta), \text{ which is attracting (resp., repelling), and bifurcates from } (\bar{x}, \bar{x}) \text{ for } \delta > \delta_0 \text{ (resp., } \delta < \delta_0). \]

In Figure 8 (resp. Figure 9) the typical behavior of the solutions is shown for different parameters for \( \alpha_1(\delta_0) < 0 \) (resp. \( \alpha_1(\delta_0) > 0 \)). Figure 7 (a)-(b) shows a bifurcation diagram in the supercritical case. If \( B_0 \) is such that \( \alpha(\delta_0, B_0) = 0 \) then according to the theory Chenciner bifurcation may occurs. It means, that if \( \delta > \delta_0 \), for \( \delta \) close to \( \delta_0 \), one (inner) or two (inner and outer) invariant curves may occur. The inner curve is stable and the outer curve is unstable. The both stable equilibrium and stable invariant curve are always surrounded by an unstable invariant curve which bounding their basins of attraction, see Figure 10. The parameter values are chosen to illustrate Chenciner bifurcation and dynamics of the the System (47).

\[ \text{Figure 10: Trajectories for a) } \beta = 42.0508, \delta = 11.5072, B = 2.7206, C = 25.3146, \text{ and b) } \beta = 36.1931, \delta = 65.0631, B = 35.3563, C = 25.733, \text{ respectively for the System (47).} \]

### 3.2.2 The case \( \delta, \gamma > 0, \ B \geq 0 \) : \( x_{n+1} = \frac{\gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + x_{n-1}^2} \)

It is easy to see that equation

\[ x_{n+1} = \frac{\gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + x_{n-1}^2} \quad (56) \]

has the equilibrium point

\[ \bar{x} = \frac{\sqrt{4B\delta + \gamma^2 + 4\delta + \gamma}}{2B + 2} \].
We can write Eq. (56) in the equivalent form:

\begin{align*}
y_{n+1} &= z_n \\
z_{n+1} &= \frac{\gamma y_n^2 + \delta z_n}{B y_n z_n + y_n^2}
\end{align*}

where

\( y_n = x_{n-1} \) and \( z_n = x_n \) for \( n = 0, 1, \ldots \)

In order to apply the Theorem 4 we make a change of variable \( u_n = y_n - \bar{x} \) and \( v_n = z_n - \bar{x} \). Then, we write this System (57) in the equivalent form:

\begin{align*}
u_{n+1} &= v_n \\
v_{n+1} &= \frac{\gamma (\bar{x} + u_n)^2 + \delta (\bar{x} + v_n)}{(\bar{x} + u_n)(B(\bar{x} + v_n) + \bar{x} + u_n) - \bar{x}} - \bar{x}
\end{align*}

Let \( F_2 \) be the function defined by:

\[
F_2(u, v) = \left( \frac{v}{(\bar{x} + u_n)(B(\bar{x} + v_n) + \bar{x} + u_n) - \bar{x}} - \bar{x} \right)
\]

Then \( F_2 \) has the unique fixed point \((0, 0)\) and maps \((-\bar{x}, \infty)^2\) into \((-\bar{x}, \infty)^2\). The Jacobian matrix of \( F_2 \) is given by

\[
Jac_{F_2}(u, v) = \begin{pmatrix}
0 & \frac{1}{(\bar{x} + u_n)(B(\bar{x} + v_n) + \bar{x} + u_n) - \bar{x}} \\
\frac{\gamma (\bar{x} + u_n)^2 - 2\delta(\bar{x} + v_n) - B\delta(\bar{x} + v_n)}{(u + \bar{x})^2(u + Bv + B\bar{x} + \bar{x})^2} & \frac{\delta - B\gamma(\bar{x} + u_n)}{(u + Bv + B\bar{x} + \bar{x})^2}
\end{pmatrix}
\]

At \((0, 0)\), \( Jac_{F_2}(u, v) \) has the form

\[
J_{02} = Jac_{F_2}(0, 0) = \begin{pmatrix}
0 & 1 \\
\frac{\delta - B\gamma}{(B+1)^2(\bar{x} + \bar{x})^2} & \frac{\delta - B\gamma}{(B+1)^2(\bar{x} + \bar{x})^2}
\end{pmatrix}
\]

**Lemma 6** Assume \( \delta_0 = 2(2B\gamma^2 + \gamma^2) \). Then, \((0, 0)\) is an equilibrium point of the map \( F_2 \). If \( \mu_2 \) and \( \bar{\mu}_2 \) are corresponding eigenvalues of \( A_2 = J_{02}\big|_{\delta = \delta_0} \), then

\[
\mu_2(\delta_0) = \frac{1 + i\sqrt{(4B + 3)(4B + 5)}}{4(B + 1)}.
\]

Furthermore:
(a) $|\mu_2(\delta_0)| = 1$.

(b) $\mu_2(\delta_0)^k \neq 1$ for $k = 1, 2, 3, 4$.

(c) $d_2 = d_2(\delta_0) = \left. \frac{d}{d\delta} |\mu_2(\delta)| \right|_{\delta=\delta_0} = \frac{1}{4(4B^2 + 7B + 3)\gamma^2} > 0$.

(d) The corresponding eigenvectors are:

$$q_2(\delta_0) = \left( \frac{1 - i\sqrt{(4B + 3)(4B + 5)}}{4(B + 1)}, 1 \right)^T$$

and

$$p_2(\delta_0) = \left( \frac{2i(B + 1)}{\sqrt{(4B + 3)(4B + 5)}} \cdot \frac{16B^2 - i\sqrt{16B^2 + 32B + 15 + 32B + 15}}{32B^2 + 64B + 30} \right),$$

where $A_2 q_2(\delta_0) = \mu_2 q_2(\delta_0)$, $p_2(\delta_0) A_2 = \mu_2 p_2(\delta_0)$ and $p_2(\delta_0) \cdot q_2(\delta_0) = 1$.

**Proof.** The eigenvalues of (59) are $\mu_2(\delta)$ and $\bar{\mu}_2(\delta)$ where

$$\mu_2(\delta) = \frac{i\sqrt{4(B + 1)^2\bar{x}^2 (B\gamma \bar{x} - (B + 2)\delta) + (\delta - B\gamma \bar{x})^2 - B\gamma \bar{x} + \delta}}{2(B + 1)^2\bar{x}^2}. \quad (60)$$

One can see that

$$|\mu_2(\delta)|^2 = \frac{(B + 2)\delta - B\gamma \bar{x}}{(B + 1)^2\bar{x}^2} = \frac{-\gamma\sqrt{4B\delta + \gamma^2 + 4\delta + B\delta + \gamma^2 + 2\delta}}{B\delta + \delta},$$

from which, by using (3.2.2) we get

$$\frac{d}{d\delta} |\mu_2(\delta)| = \frac{\gamma \left( -\gamma\sqrt{4(B + 1)\delta + \gamma^2 + 2(B + 1)\delta + \gamma^2} \right)}{2\delta^{3/2} \sqrt{(B + 1) (4(B + 1)\delta + \gamma^2)}} \left( -\gamma\sqrt{4(B + 1)\delta + \gamma^2 + (B + 2)\delta + \gamma^2} \right).$$

Let

$$\delta_0 = 2(2B + 1)\gamma^2. \quad (61)$$

This implies

$$\left. \frac{d|\mu_2(\delta)|}{d\delta} \right|_{\delta=\delta_0} = \frac{1}{4(4B^2 + 7B + 3)\gamma^2} > 0$$
For $\delta = \delta_0$ we obtain

$$\bar{x} = 2\gamma$$

and

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{2B+2} \end{pmatrix}$$

The eigenvalues of $A_2$ are $\mu_2(\delta_0)$ and $\bar{\mu}_2(\delta_0)$ where

$$\mu_2(\delta_0) = \frac{1 + i\sqrt{(4B + 3)(4B + 5)}}{4(B + 1)}.$$

The eigenvectors corresponding to $\mu_2(\delta)$ and $\bar{\mu}_2(\delta)$ are $q_2(\delta_0)$ and $\bar{q}_2(\delta_0)$ where

$$q_2(\delta_0) = \left( \frac{1 - i\sqrt{(4B + 3)(4B + 5)}}{4(B + 1)}, 1 \right).$$

One can prove that

$$|\mu_2(\delta_0)| = 1$$

$$\mu_2^2(\delta_0) = \frac{i\sqrt{(4B + 3)(4B + 5)}}{8(B + 1)^2} + \frac{1}{8(B + 1)^2} - 1$$

$$\mu_2^3(\delta_0) = -\frac{12B(B + 2) + 11}{16(B + 1)^3} - \frac{i\sqrt{(4B + 3)(4B + 5)(4B(B + 2) + 3)}}{16(B + 1)^3}$$

$$\mu_2^4(\delta_0) = -\frac{i\sqrt{(4B + 3)(4B + 5)(8B(B + 2) + 7)}}{32(B + 1)^4} - \frac{1}{2(B + 1)^2} + \frac{1}{32(B + 1)^4} + 1,$$

from which follows that $\mu_2^k(\delta_0) \neq 1$ for $k = 1, 2, 3, 4$ and $B \geq 0$. The rest of the proof is almost immediate, so we skip it.

In the sequel, by procedure described in [11], we compute $K_{20}$ and $K_{11}$. We have that

$$F_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{B\gamma\bar{x} - (B + 2)\delta}{(B + 1)^2\bar{x}^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_{21}(\delta, u, v) \\ f_{22}(\delta, u, v) \end{pmatrix},$$

and

$$f_{21}(\delta, u, v) = 0$$

$$f_{22}(\delta, u, v) = \frac{\gamma (\bar{x} + u)^2 + \delta (\bar{x} + v)}{(\bar{x} + u)(B(\bar{x} + v) + \bar{x} + u)} - \frac{u (B\gamma\bar{x} - (B + 2)\delta)}{(B + 1)^2\bar{x}^2} - \frac{u (\delta - B\gamma\bar{x})}{(B\bar{x} + \bar{x})^2} - \bar{x}.$$ 

Substituting $\delta = \delta_0$ and $\bar{x}$ into (62) we get

$$F_{02} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{2B+2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_{21}(u, v) \\ h_{22}(u, v) \end{pmatrix},$$

(63)
and

\[ h_{21}(u, v) = f_{21}(\delta_0, u, v) = 0 \]
\[ h_{22}(u, v) = f_{22}(\delta_0, u, v) = -\gamma \left( \frac{2B uv + 4(B + 1)u\gamma + u^2 - 2v\gamma}{(u + 2\gamma)(Bv + 2(B + 1)\gamma + u)} \right) - \frac{v}{2B + 2} + u. \]

Hence, for \( \delta = \delta_0 \) system (58) is equivalent to

\[
\begin{pmatrix}
  u_{n+1} \\
  v_{n+1}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  -1 & \frac{1}{2B+2}
\end{pmatrix}
\begin{pmatrix}
  u_n \\
  v_n
\end{pmatrix} +
\begin{pmatrix}
  h_{21}(u_n, v_n) \\
  h_{22}(u_n, v_n)
\end{pmatrix},
\]

(64)

Let

\[
\begin{pmatrix}
  u_n \\
  v_n
\end{pmatrix} = P_2 \begin{pmatrix}
  \xi_n \\
  \eta_n
\end{pmatrix},
\]

where

\[
P_2 = \begin{pmatrix}
\frac{1}{4(B+1)} & \frac{\sqrt{(4B+3)(4B+5)}}{4(B+1)} \\
1 & 0
\end{pmatrix} \quad \text{and} \quad P_2^{-1} = \begin{pmatrix}
0 & \frac{1}{4(B+1)} \\
\frac{\sqrt{(4B+3)(4B+5)}}{4(B+1)} & -1
\end{pmatrix}.
\]

Then system (64) is equivalent to

\[
\begin{pmatrix}
  \xi_{n+1} \\
  \eta_{n+1}
\end{pmatrix} =
\begin{pmatrix}
  \frac{1}{4B+4} & \frac{\sqrt{(4B+3)(4B+5)}}{4(B+1)} \\
\frac{\sqrt{(4B+3)(4B+5)}}{4(B+1)} & \frac{1}{4B+4}
\end{pmatrix}
\begin{pmatrix}
  \xi_n \\
  \eta_n
\end{pmatrix} + G_2 \begin{pmatrix}
  \xi_n \\
  \eta_n
\end{pmatrix},
\]

(65)

where

\[
H_2 \begin{pmatrix}
  u \\
  v
\end{pmatrix} := \begin{pmatrix}
  h_{21}(u, v) \\
  h_{22}(u, v)
\end{pmatrix}.
\]

and

\[
G_2 \begin{pmatrix}
  u \\
  v
\end{pmatrix} = \begin{pmatrix}
  g_{21}(u, v) \\
  g_{22}(u, v)
\end{pmatrix} = P_2^{-1} H_2 \begin{pmatrix}
  u \\
  v
\end{pmatrix}.
\]

By straightforward calculation we obtain that

\[
g_1(u, v) = \frac{\Lambda(u, v)}{\Upsilon(u, v)}
\]
\[
g_2(u, v) = -\frac{\Lambda(u, v)}{\Upsilon(u, v) \sqrt{(4B + 3)(4B + 5)}};
\]

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where

\[
\Lambda(u, v) = -(2B + 1)^2 u^3 - u^2 \left( \sqrt{(4B + 3)(4B + 5)} v + 4(B + 1)(2B + 1)(8B + 5) \gamma \right)
+ uv \left( (2B + 1)^2(4B + 3)(4B + 5)v - 8(B + 1)\sqrt{(4B + 3)(4B + 5)} \gamma \right)
+ (4B + 3)(4B + 5)v^2 \left( \sqrt{(4B + 3)(4B + 5)} v + 4(B + 1)(2B + 3) \gamma \right)
\]

\[
\Upsilon(u, v) = 4(B + 1) \left( \sqrt{(4B + 3)(4B + 5)} v + 8(B + 1) \gamma + u \right) \left( (2B + 1)^2u + \sqrt{(4B + 3)(4B + 5)} v + 8(B + 1)^2 \gamma \right)
\]

If we take the basis of \( \mathbb{R}^2 \) as \( \Phi_2 = (q_2, \bar{q}_2) \), where \( q_2 = q_2(\delta_0) \), then we can represent \((u, v)\) as

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \Phi_2 \begin{pmatrix}
z \\
\bar{z}
\end{pmatrix} = (q_2 z + \bar{q}_2 \bar{z}).
\]

Let

\[
G_{02} \left( \Phi_2 \begin{pmatrix}
z \\
\bar{z}
\end{pmatrix} \right) = \frac{1}{2} (g_{20} z^2 + 2g_{11} z \bar{z} + g_{02} \bar{z}^2) + O(|z|^3)
\]

By using package Mathematica, we obtain

\[
g_{20} = \left. \frac{\partial^2}{\partial \bar{z}^2} G_{02} \left( \Phi_2 \begin{pmatrix}
z \\
\bar{z}
\end{pmatrix} \right) \right|_{z=0} = \begin{pmatrix} 0 \\ \frac{-8B(2B(B + 4) + 9) + i \sqrt{(4B + 3)(4B + 5)} - 25}{16(B + 1)^3 \gamma} \end{pmatrix}
\]

\[
g_{11} = \left. \frac{\partial^2}{\partial z \partial \bar{z}} G_{02} \left( \Phi_2 \begin{pmatrix}
z \\
\bar{z}
\end{pmatrix} \right) \right|_{z=0} = \begin{pmatrix} 0 \\ \frac{B(8B(B + 3) + 27) + 10}{8(B + 1)^3 \gamma} \end{pmatrix}
\]

and

\[
K_{20} = (\mu_2^2 I - A)^{-1} g_{20} = \begin{pmatrix}
\frac{2(B + 1)}{(2B + 3)(4B + 3) \left( \sqrt{(4B + 3)(4B + 5)} + 7 \gamma \right)}
\end{pmatrix}
\]

\[
K_{11} = (I - A)^{-1} g_{11} = \begin{pmatrix}
\frac{B(8B(B + 3) + 27) + 10}{4(B + 1)^2(4B + 3) \gamma}
\frac{4(B + 1)^2(4B + 3) \gamma}{B(8B(B + 3) + 27) + 10}
\end{pmatrix}
\]

By using \( K_{20} \) and \( K_{11} \), we have that
\[ g_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} G_{02} \left( \Phi_2 \left( \frac{z}{\bar{z}} \right) + \frac{1}{2} K_{20} z^2 + K_{11} \bar{z} \right) \bigg|_{z=0} \].

The package Mathematica yields

\[ \alpha_2(0) = \alpha_2(\delta_0) = \frac{1}{2} Re(p_2 g_{21} \bar{\mu}_2) = -\frac{48B^4 + 168B^3 + 200B^2 + 97B + 16}{64(B + 1)^4(4B + 3)\gamma^2} < 0 \]

Figure 11: Trajectories for \( \gamma = 1.9 \), \( B = 71.6 \), \( \delta_0 = 30.324 \), where \( \alpha_2(\delta_0) \approx -0.0169873 \), and a) \( \delta = 29 \), b) \( \delta = 30.324 \), c) \( \delta = 33.0 \) respectively for the System (47) (supercritical Neimark-Sacker bifurcation) and d) approximation of the invariant curve \( \Gamma_2(\delta) \) for \( \gamma = 1.9 \), \( B = 1.6 \), and \( \delta = 33.00 \).

**Theorem 10** Assume that \( \beta, B, C > 0 \) and \( \delta_0 = (4B + 2)\gamma^2 \). Since \( d_2(\delta_0) > 0 \) and \( \alpha_2(\delta_0) < 0 \), the System (47) passes through a supercritical Neimark-Sacker bifurcation.
at an unique positive equilibrium point $E_2(\bar{x}, \bar{x}) = (2\gamma, 2\gamma)$ when the parameter $\delta$ varies in small neighborhood of $\delta_0$. Furthermore, there exists an unique closed invariant curve $\Gamma_2(\delta)$, which is attracting, and bifurcates from $E_2$ for $\delta > \delta_0$.

Let $\delta = \delta_0 + \varepsilon$. Then for small $\varepsilon$, the curve $\Gamma_2(\delta)$ can be approximated by

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx (\bar{x}, \bar{x}) + 2\rho_0 \text{Re}(q_2 e^{i\theta}) + \rho_0^2 \left( \text{Re}(K_{20} e^{2i\theta}) + K_{11} \right), \quad \theta \in \mathbb{R} \text{ where } \rho_0 = \sqrt{-\frac{d_2(\delta_0)}{\alpha_2(\delta_0)} \varepsilon}.$$ 

**Proof.** The proof follows from above discussion, Lemma 6 and Theorem 4. \qed

**List of References**


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CHAPTER 4
An Evolutionary Sigmoid Beverton-Holt Model

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4.1 Introduction and Preliminaries

Difference equations of the form

\[ x_{n+1} = f(x_n)x_n, \quad n = 0, 1, \ldots \]  

(66)

are widely used to model the discrete time deterministic dynamics of biological populations. Here \( x_n \) is some measure of population density at discrete census times \( n = 0, 1, \ldots \) and the expression \( f(x) \) describes the per capita (or per unit) contribution to the population at the next census time. We refer to \( f \) as the population growth rate, see [1, 2, 3]. In view of this, we assume that the solution \( x_n \) of (66) is non-negative, and the function \( f \) assumes only non-negative values for non-negative values of its argument and that feasible initial conditions \( x_0 \geq 0 \) generates a unique sequence for \( n = 0, 1, 2, \ldots \). The asymptotic properties of the solution \( x_n \) of (66) are often of major interest and these depend on the properties of \( f \). The best known population growth rates are

\[ f(x) = \frac{b}{1 + cx}, b, c > 0, \]  

(67)

\[ f(x) = be^{-cx}, b, c > 0 \]  

(68)

and

\[ f(x) = \frac{bx}{1 + x^2}, b > 0. \]  

(69)

The first population growth rate (67) used in (66) yields the discrete logistic model or Beverton-Holt model, see [1, 4, 3]. For this model, it is well known that the extinction equilibrium \( x_0 = 0 \) is globally asymptotically stable for \( x_0 \geq 0 \) if \( b < 1 \) while the positive equilibrium \( x_+ = (b - 1)/c \) is globally asymptotically stable for \( x_0 \geq 0 \) when \( b > 1 \). This is an example of the fundamental exchange of stability bifurcation that occurs at \( b = 1 \) where the extinction equilibrium destabilizes and, as a result, a stable positive equilibrium is created.
The second population growth rate (68) used in (66) gives the so-called Ricker model, see [1, 5, 3]. It is well-known, that the extinction equilibrium $x_0 = 0$ of this model also destabilizes at $b = 1$ with the result that there exists positive equilibrium $x_+ = \ln b/c$ for $b > 1$. The positive equilibrium is globally stable for $1 < b < e^2$ but unstable for $b > e^2$. As $b$ increases the Ricker model exhibits a period doubling route to chaos.

Thus, both of these basic examples illustrate a fundamental exchange of stability bifurcation: when the extinction equilibrium destabilizes, a stable positive equilibria is created.

The third population growth rate (69) used in (66) yields the sigmoid Beverton-Holt model or Thomson model, see [1, 4]. For this model, it is well known that the extinction equilibrium $x_0 = 0$ is globally asymptotically stable for $x_0 \geq 0$ if $b < 2$ and for $x_0 < x_1 = (b - \sqrt{b^2 - 4})/2$ if $b > 2$. Two positive equilibrium solutions $x_1 = (b - \sqrt{b^2 - 4})/2$ and $x_2 = (b + \sqrt{b^2 - 4})/2$ are borned when $b > 2$. The smaller $x_1$ is a repeller and the larger one $x_2$ is globally asymptotically stable for $b > 2$. This is an example of the Allee’s effect and the case of new type of bifurcation that occurs at $b = 2$ where the extinction equilibrium remains stable and two new equilibrium solutions are created resulting in the Allee’s effect.

If $f$ depends only on the state variable $x$, then it is only the current population density that determines the population density at the next time census and, as a result, the mathematical model (66) is time autonomous. There are many circumstances under which $f$ also depends explicitly on time $n$. For example, in asymptotically constant environment model the model coefficients $b, c$ could be variable with constant limits, while in a seasonally fluctuating environment model coefficients $b, c$ could be periodic sequences.

Another reason that model parameters can change in time is Darwinian evolution, which is a case that will be considered here. The detailed explanation is given in
Suppose $v$ is a quantified, phenotypic trait of an individual that is subject to evolution. If we assume the per capita contribution to the population made by an individual depends on its trait $v$, then $f = f(x, v)$ depends on both $x$ and $v$. It might be the case that this contribution also depends on the traits of other individuals. We can model this situation by assuming that $f$ also depends on the mean trait $u$ in the population so that $f = f(x, v, u)$. A canonical way to model Darwinian evolution is to model the dynamics of $x_n$ and the mean trait $u_n$ by means of the equations

$$x_{n+1} = f(x_n, v, u_n)|_{v = u_n} x_n$$

$$u_{n+1} = u_n + \sigma^2 \frac{\partial F(x_n, v, u_n)}{\partial v}|_{v = u_n},$$

see [10].

Equation (70) asserts that the population dynamics can be modeled by assuming the trait $v$ is equal to the population mean. Equation (71) (called Lander’s or Fisher’s or the breeder’s equation) prescribes that the change in the mean trait is proportional to the fitness gradient, where fitness in this model is denoted by $F(x, v, u)$. The modeler decides on an appropriate measure of fitness, which is often taken to be $f$ or $\ln f$. The constant of proportionality $\sigma^2 \geq 0$ is called the speed of evolution. It is related to the variance of the trait in the population, which is assumed constant in time. Thus, if $\sigma^2 = 0$ no evolution occurs (there is no variability) and one has a one-dimensional difference equation for just population dynamics of the form (66). If evolution occurs $\sigma^2 > 0$ then the model is a two dimensional system of difference equations with state variable $[x_n, u_n]$.

In this paper we consider the system of difference equations

$$x_{n+1} = b(u_n) \frac{x_n^2}{1 + x_n^2},$$

$$u_{n+1} = u_n + \sigma^2 \frac{b'(u_n)}{b(u_n)} = F(u_n) \quad n = 0, 1, \ldots$$

see [10].
Notice that the system (72),(73) is triangular system where equation (73) is first order difference equation which global dynamics can be always established, in which case the equation (72) becomes first order non-autonomous difference equation with either asymptotically constant coefficients or periodic coefficients or almost periodic coefficients, which dynamics can be established by method of inequalities as in [4] or some methods as in Cushing and Elaydi [7, 8, 11] or by using the methods based on contractive maps as in [12].

4.2 Local Stability

The equilibrium solutions $\bar{x}$ of system (72),(73) satisfy the equations

$$\bar{x} = b(\bar{u}) \frac{\bar{x}^2}{1 + \bar{x}^2}$$

and

$$\bar{u} = \bar{u} + \sigma^2 \frac{b'(\bar{u})}{b(\bar{u})}.$$  (75)

The equilibrium solutions will then be the following:

$$E_0(0, \bar{u}), E_{\pm}(\bar{x}_{\pm}, \bar{u})$$

where if $b(\bar{u}) > 2$

$$\bar{x}_{\pm} = \frac{b(\bar{u}) \pm \sqrt{(b(\bar{u})^2 - 4)}}{2}.$$  (76)

Notice that every equilibrium solution $\bar{u}$ is a critical point of function $b(u)$. The most classical examples of $b(u)$ are probability density functions of normal distribution with one or more critical points, see [2, 6] for examples of simulations.

The Jacobian matrix for the map that corresponds to the system (72),(73) has the form

$$J(x, u) = \begin{bmatrix} b(u) \frac{2x}{(1+x^2)^2} & \frac{b'(u) x^2}{(1+x^2)^2} \\ 0 & 1 + \sigma^2 \frac{b(u)b''(u)-(b'(u))^2}{(b(u))^2} \end{bmatrix}$$

whenever $b(u)$ is twice differentiable function.

Now the following local stability results hold:
Theorem 11 The extinction equilibrium $E_0(0, \bar{u})$ is:

(i) locally asymptotically stable if

$$-\frac{2}{\sigma^2} b(\bar{u}) < b''(\bar{u}) < 0; \quad (77)$$

(ii) saddle point if

$$b''(\bar{u}) > 0 \quad \text{or} \quad b''(\bar{u}) < -\frac{2b(\bar{u})}{\sigma^2}; \quad (78)$$

(iii) non-hyperbolic if

$$b(\bar{u}) = 0 \quad \text{or} \quad \frac{b''(\bar{u})}{b(\bar{u})} = -\frac{2}{\sigma^2}. \quad (79)$$

Proof. The proof of the theorem is an immediate application of Theorem 2.12 [13] as the Jacobian matrix (76) for the map that corresponds to the system (72) at $E_0(0, \bar{u})$ is

$$J(0, \bar{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 + \sigma^2 \frac{b'(\bar{u})}{b(\bar{u})} \end{bmatrix} \quad (80)$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1 + \sigma^2 \frac{b'(\bar{u})}{b(\bar{u})}$. The straightforward calculation shows that $|\lambda_2| < 1$ (respectively $|\lambda_2| > 1$ and $|\lambda_2| = 1$) if and only if condition (77) (respectively (78) and (79)) is satisfied. \(\square\)

Theorem 12 Assume that $b(\bar{u}) > 2$. The survival equilibrium $E_+(\bar{x}_+, \bar{u})$ of Equation (73) is:

(i) locally asymptotically stable if (77) is satisfied.

(ii) saddle point if (78) is satisfied.

(iii) non-hyperbolic if (79) is satisfied.

Proof. It follows from Theorem 2.12 [13] as the Jacobian matrix (76) at $E_+(\bar{x}_+, \bar{u})$ is

$$J(\bar{x}_+, \bar{u}) = \begin{bmatrix} b(\bar{u}) \frac{2\bar{x}_+}{(1+\bar{x}_+^2)^2} & 0 \\ 0 & 1 + \sigma^2 \frac{b'(\bar{u})}{b(\bar{u})} \end{bmatrix} \quad (81)$$

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with eigenvalues
\[ \lambda_1 = \frac{2\bar{x}_+}{\sigma^2 b''(\bar{u})}, \quad \lambda_2 = 1 + \sigma^2 \frac{b''(\bar{u})}{b(\bar{u})}. \]

Now we will show that \( \lambda_1 \in (0, 1) \). By using equation (74) we have that
\[ \lambda_1 = \frac{2\bar{x}_+}{(1 + \bar{x}_+^2)^2} = \frac{2}{b(\bar{u})\bar{x}_+}, \]
so the condition \( \lambda_1 \in (0, 1) \) is equivalent to \( b(\bar{u})\bar{x}_+ > 2 \). The last condition is always satisfied since \( b(\bar{u}) > 2 \) and
\[ \bar{x}_+ = \frac{b(\bar{u}) + \sqrt{(b(\bar{u})^2 - 4)}}{2} > 1 \iff b(\bar{u}) - 2 + \sqrt{(b(\bar{u})^2 - 4)} > 0. \]
Thus the stability of \( E_+ \) depends on \( \lambda_2 \) and is equivalent to local stability analysis in Theorem 11.

**Theorem 13** Assume that \( b(\bar{u}) > 2 \). The survival equilibrium \( E_-(\bar{x}_-, \bar{u}) \) of Equation (73) is:

(i) saddle point if (77) is satisfied.

(ii) repeller if (78) is satisfied.

(iii) non-hyperbolic if (79) is satisfied.

**Proof.** In this case the Jacobian matrix (76) evaluated at \( E_-(\bar{x}_-, \bar{u}) \) is
\[ J(\bar{x}_-, \bar{u}) = \begin{bmatrix} \frac{2\bar{x}_-}{(1 + \bar{x}_+^2)^2} & 0 \\ 0 & 1 + \sigma^2 \frac{b''(\bar{u})}{b(\bar{u})} \end{bmatrix}, \]
with eigenvalues
\[ \lambda_1 = \frac{2\bar{x}_-}{(1 + \bar{x}_+^2)^2}, \quad \lambda_2 = 1 + \sigma^2 \frac{b''(\bar{u})}{b(\bar{u})}. \]

Now we will show that \( \lambda_1 > 1 \). By using equation (74) we have that \( \lambda_1 = \frac{2}{b(\bar{u})\bar{x}_-} \) and so \( \lambda_1 > 1 \) if
\[ (b(\bar{u}))^2 - 4 < b(\bar{u})\sqrt{(b(\bar{u}))^2 - 4} \]
which holds.

Thus \( \lambda_1 > 1 \) so \( E_- \) is always unstable. The rest of the proof follows from Theorem 1.12 [13]. \( \Box \)

### 4.3 Global Dynamics

Now we get the following global dynamic result for the equilibrium \( E_0 \).

**Theorem 14** Assume that the function \( F(u) = u + \sigma \frac{b'(u)}{b(u)} : [\alpha, \beta] \to [\alpha, \beta] \) for some \( \alpha < \beta \), has no minimal period-two points and is continuous. Then for every equilibrium solution \( \bar{u} \) of system (72),(73) with \( b(\bar{u}) < 2 \), such that (77) holds, the equilibrium \( E_0 \) is globally asymptotically stable.

**Proof.** The local stability follows from Theorem 11, Part (i). The global attractivity of \( \bar{u} \) follows from Theorem C.3 [5]. Thus, we have

\[
\lim_{n \to \infty} u_n = \bar{u}
\]

for every \( u_0 \in B(\bar{u}) \), the basin of attraction of \( \bar{u} \). The limiting equation of (73) becomes

\[
x_{n+1} = b(\bar{u}) \frac{x_n^2}{1 + x_n^2}.
\]

Using Theorem 13 or Example 17 in [4], we conclude that every solution \( \{x_n\} \) of (72) converges to zero and so

\[
\lim_{n \to \infty} (x_n, u_n) = E_0.
\]

\( \Box \)

**Remark 2** As a direct consequence of Theorem 11, if condition (78) holds, then for every point \( \bar{u} \), the corresponding equilibrium \( E_0(0, \bar{u}) \) is unstable.
Remark 3 The condition that $F(u)$ has no minimal period-two points reduces to the nonexistence of the solution of functional equation
\[ u = F(u) + \sigma^2 \frac{b'F(u)}{b(F(u))} = u + \sigma^2 \frac{b'(u)}{b(u)} + \sigma^2 \frac{b'(u + \sigma^2 \frac{b'(u)}{b(u)})}{b(u + \sigma^2 \frac{b'(u)}{b(u)})} \iff \frac{b'(u)}{b(u)} + \frac{b'(u + \sigma^2 \frac{b'(u)}{b(u)})}{b(u + \sigma^2 \frac{b'(u)}{b(u)})} = 0 \]
different from the equilibrium solutions.

In the case of the equilibrium $E_+$ we have the following results.

**Theorem 15** Assume that the function $F(u)$, defined in Theorem 14, maps some interval $[\alpha, \beta]$ into $[\alpha, \beta], \alpha < \beta$, has no minimal period-two points and is continuous.

Then for every equilibrium solution $\bar{u}$ of system (72),(73) with $b(\bar{u}) > 2$, such that (77) holds, the equilibrium points $E_0$ and $E_+$ are globally asymptotically stable within their respective basins of attraction.

**Proof.** The local stability of the equilibrium points follows from Theorems 11 and 12. The global attractivity of $\bar{u}$ follows from Theorem C.3 in [5].

Thus we have that
\[ \lim_{n \to \infty} u_n = \bar{u}, \quad u_0 \in [a, b]. \]

The limiting equation of equation (72) becomes the difference equation
\[ x_{n+1} = b(\bar{u}) \frac{x_n^2}{1 + x_n^2}. \]

Using Theorem 13 or Example 17 in [4] we conclude that every solution $\{x_n\}$ of equation (73) converges to zero if $x_0 \in [0, \bar{x}_-)$ or to $\bar{x}_+$ if $x_0 \in (\bar{x}_-, \infty)$.

Consequently, we have the following result;

- If $x_0 \in [0, \bar{x}_-)$, then $\lim_{n \to \infty} (x_n, u_n) = E_0$;

- If $x_0 \in (\bar{x}_-, \infty)$, then $\lim_{n \to \infty} (x_n, u_n) = E_+$.

\[ \square \]
Theorem 16 Assume that condition (78) holds. Then for every equilibrium solution \( \bar{u} \) of system (73), the corresponding equilibrium \( E_+(\bar{x}_+, \bar{u}) \) is unstable.

Proof. The local stability of the equilibrium points follows from Theorems 11 and 12. The rest of the proof is the same as the proof of Theorem 15.

Remark 4 We did not discuss global dynamics for equation (72) in non-hyperbolic case. The reason is that dynamics of non-autonomous difference equation with asymptotically constant coefficients is well described by dynamics of limiting equation only in hyperbolic case and it is poorly described by dynamics of limiting equation in non-hyperbolic case, which should be investigated by case to case investigation. See [14, 15].

The presented results can be extended directly to the system consisting of equation (73) and the difference equation

\[ x_{n+1} = b(u_n)G(x_n) \]  

where \( G \) is nondecreasing, differentiable function with the property \( G(0) = 0 \). Such results will include functions such as

\[ G(u) = \frac{u}{1 + u}, \quad G(u) = \frac{u^2}{1 + u^2}, \quad G(u) = 1 - e^{-u}. \]

Here we give generalizations of two results presented here, Theorems 14 and 15. First, we give the generalization of Theorem 14.

Theorem 17 Assume that the function \( F(u) = u + \sigma^2 \frac{b'(u)}{b(u)} : [\alpha, \beta] \to [\alpha, \beta] \) for some \( \alpha < \beta \), has no minimal period-two points and is continuous. Assume that \( G \) is nondecreasing, differentiable function with the property \( G(0) = 0 \) and that 0 is the unique fixed point of \( G \). Then for every equilibrium solution \( \bar{u} \) of system (72),(73) the equilibrium \( E_0 \) is globally asymptotically stable.
Proof. The proof follows from Theorems 11, Theorem C.3 in [5] and Theorem 13 in [4]

\[ \square \]

Second, we give the generalization of Theorem 15.

**Theorem 18** Assume that the function \( F(u) = u + \sigma^2 \frac{b'(u)}{b(u)} : [\alpha, \beta] \rightarrow [\alpha, \beta] \) for some \( \alpha < \beta \), has no minimal period-two points. Assume that \( G \) is nondecreasing, differentiable function with the property \( G(0) = 0 \), that has one positive equilibrium solution \( \bar{x}, G(\bar{x}) = \bar{x} \) for \( b(\bar{u}) > c \) where \( \bar{u} \) is a critical point of \( b \). Further assume that \( \bar{u} \) is a local attractor for \( F \) and that \( \bar{x}_0 = 0 \) is globally asymptotically stable equilibrium of the limiting equation

\[
x_{n+1} = b(\bar{u})G(x_n)
\]

for \( b(\bar{u}) < c \) within its basin of attraction and \( \bar{x} \) is globally asymptotically stable equilibrium of \( (84) \) for \( b(\bar{u}) > c \) within its basin of attraction.

Then for every equilibrium solution \( \bar{u} \) of system \( (72),(83) \) with \( b(\bar{u}) < c \), such that \( (77) \) holds, the equilibrium \( E_0 = (0, \bar{u}) \) is globally asymptotically stable within its basin of attraction, while when \( b(\bar{u}) > c \) the equilibrium \( E_+(\bar{x}_+, \bar{u}) \) is globally asymptotically stable within its basin of attraction.

Proof. The proof follows from Theorems 11, 12, Theorem C.3 in [5] and Theorem 13 in [4]

\[ \square \]

**Remark 5** The conditions on function \( F \) in Theorems 17 and 18 can be replaced by the conditions for global asymptotic stability of fixed point of

\[ F(u) = u + \sigma^2 \frac{b'(u)}{b(u)} : [\alpha, \beta] \rightarrow [\alpha, \beta] \] for some \( \alpha < \beta \) which follows from Singer theorem, see Theorem 2.1 in [11] and [5]. If we assume that \( F \) is \( C^3 \) with an equilibrium point \( \bar{x} \in [\alpha, \beta] \) such that \( F \) satisfies negative feedback condition, that is \( F(x) > x \) if \( x < \bar{x} \).
and \( F(x) < x \) if \( x > \bar{x} \). Assuming that the Schwarzian derivative
\[
Sf(x) = \frac{f'''(x)}{f'(x)} - 3 \left( \frac{f''(x)}{f'(x)} \right)^2 < 0
\]
for all \( x \in [\alpha, \beta] \), then if \( |f'(\bar{x})| \leq 1 \), \( \bar{x} \) is globally asymptotically stable. Now condition \( |f'('x')| \leq 1 \) is equivalent to conditions (77) and (79).

**Remark 6** An interesting problem is to establish global dynamics in the case when (78) holds. In this case Theorems 11, 12, 13 imply that the equilibrium solutions \( E_0 \) and \( E_+ \) are saddle points while the equilibrium \( E_- \) is a repeller. The local stable manifold Theorem implies the existence of the local stable and unstable manifolds in some neighborhood of solutions \( E_0 \) and \( E_+ \) and so both points will have some basins of attractions.

### 4.4 Simulations

We give the following examples with specified expressions of the coefficient \( b = b(v) \).

**Example 1** By defining the coefficient \( b = b(v) \) in a particular way, we can ensure that the corresponding Fisher’s equation will be of a form for which we already know the dynamics. If we take the coefficient
\[
b(v) = \exp \left( \frac{1}{\sigma^2} (cx - c \log(x + 1) + c - \frac{x^2}{2} + \frac{1}{2}) \right),
\]
then the system (72), (73) will be equal to the system
\[
x_{n+1} = b(u_n) \frac{x_n^2}{1 + x_n^2},
\]
\[
u_{n+1} = \sigma^2 \frac{u_n}{1 + u_n}
\]
where Fisher’s equation is a Beverton-Holt equation, the dynamics of which is well known. Then we have the following result
\[
\lim_{n \to \infty} u_n = \begin{cases} 
0 & a \leq 1 \\
\frac{a - 1}{a} & a > 1 
\end{cases}
\]
Using this result for Fisher’s equations and applying monotone operators techniques we have the result

$$\lim_{n \to \infty} (x_n, u_n) = \begin{cases} 
(0, 0) & b(0) < 2, a \leq 1 \text{ or } b(0) > 2, a \leq 1 \text{ and } x_0 < \bar{x}_- \\
(0, a - 1) & b(a - 1) < 2, a > 1 \text{ or } b(a - 1) > 2, a > 1 \text{ and } x_0 < \bar{x}_- \\
(\bar{x}_+, 0) & b(0) > 2, a \leq 1 \text{ and } x_0 > \bar{x}_- \\
(\bar{x}_+, a - 1) & b(a - 1) > 2, a > 1 \text{ and } x_0 > \bar{x}_-
\end{cases}$$

Example 2 Frequently in evolutionary modeling the assumption is made that the function of the phenotypic trait are normally distributed. Considering this let the coefficient $b = b(v)$ be defined as follows

$$b(v) = b_0 \exp \left( \frac{1}{100} \left( -\frac{v^4}{4} + v^3 + 2v^2 \right) \right)$$

![Figure 13: $b(v)$ graphed for $b_0 = 0.5, 1.0$ and $1.3$](image)

Then the system (72), (73) will be equal to the system

$$x_{n+1} = b(u_n) \frac{x_n^2}{x_n^2 + 1},$$

$$u_{n+1} = u_n - \sigma^2 \frac{1}{100} (u_n - 4) u_n (u_n + 1) = f(u_n)$$

In this case, $f(u)$ will have two critical values in the first quadrant

$$\bar{u} = 0, 4$$
Figure 14: a) Defining the equation $f(x, \sigma) = x - \frac{1}{100} \sigma^2 (x - 4)x(x + 1)$ with $\sigma^2 = 14.44$, two iterates of $f(u, \sigma)$ showing that the only intersections of the second iterate and the bisector are exactly the fixed points of the equation and thus that there is no periodic solution; b) Defining the function as in the first case with $\sigma^2 = 6.25$ depicting the intersection of the second iterate of $f(u, \sigma)$ with the bisector at two points besides the equilibrium points, and thus that there exists a periodic solution; c) Recurrence plot of the sequence $u_n$ with $\sigma^2 = 9$; d) Recurrence plot of the sequence $u_n$ with $\sigma^2 = 11$.

where $b(u)$ has a local minimum at $\bar{u} = 0$ and a global maximum at $\bar{u} = 4$ which will represent the ESS (evolutionary stable strategy). The negative feedback condition will be satisfied at the point $\bar{u} = 4$, and so this point will be an attracting fixed point.

The global attractivity of $\bar{u} = 4$ on an interval $(0, \infty)$ follows from Theorem C.3 [5].

$$\lim_{n \to \infty} u_n = 4, \quad u_0 \in (0, \infty) \text{ and } \sigma^2 < 10$$

Evaluating $b(v)$ and $b''(v)$ at the equilibrium $\bar{u} = 4$ gives the following

$$b(4) = b_0 e^{8/25} \quad b''(4) = - \frac{b_0 e^{8/25}}{5}$$

Then for the condition (77) to be satisfied, it must be that $\sigma^2 < 10$. By Theorems (14) and (15), we have the following result

**Theorem 19** Consider the system (72), (73) such that

$$b(u_n) = b_0 \exp \left( \frac{1}{100} \left( -\frac{u_n^4}{4} + u_n^3 + 2u_n^2 \right) \right)$$. Then the global dynamics of the system is given by:
i) Assume $b_0 < 2e^{-8/25}$ and $\sigma^2 < 10$, then every solution $(x_n, u_n)$ of the evolutionary model converges to the equilibrium $(0, 4)$.

ii) Assume $b_0 > 2e^{-8/25}$ and $\sigma^2 < 10$. If $x_0 \in [0, \bar{x}_-)$, then every solution of the evolutionary model converges to $(0, 4)$. If $x_0 \in (\bar{x}_-, \infty)$ then every solution converges to $(\bar{x}_+, 4)$ or

$$\lim_{n \to \infty} (x_n, u_n) = \begin{cases} (0, 4) & b_0 < 2e^{-8/25}, \sigma^2 < 10 \text{ or } b_0 > 2e^{-8/25}, \sigma^2 < 10 \text{ and } x_0 < \bar{x}_- \\ (\bar{x}_+, 4) & b_0 > 2e^{-8/25}, \sigma^2 < 10 \text{ and } x_0 > \bar{x}_- \end{cases}$$

iii) Assume $\sigma^2 > 10$. Then there will be an attracting periodic solution of $u_n$ around $\bar{u} = 4$.

**Remark 7** The parameter $\sigma^2$ will determine the bifurcation of the system where $\sigma^2 = 10$ is the bifurcation value. For $\sigma^2 < 10$ the stability of the equilibrium points $(0, 4)$ and $(\bar{x}_+, 4)$ will be determined in accordance with Theorem 10, however, when $\sigma^2 > 10$, a period-two solution will emerge around the point $\bar{u}$. Then at the bifurcation value, the system will undergo period doubling bifurcation.

**Example 3** Another class of functions often chosen for the function of the phenotypic trait in evolutionary modeling are surge functions. Let $b(v)$ be equal to the following

$$b(v) = \frac{b_0 v}{v^2 + 1}$$

Then the system (72), (73) will be defined as

$$x_{n+1} = \frac{b_0 u_n x_n^2}{(u_n^2 + 1)(x_n^2 + 1)}$$

$$u_{n+1} = u_n + \sigma^2 \frac{1 - u_n^2}{(u_n^2 + 1)u_n}$$
Considering the critical points of the function $b(v)$, the system will have one critical trait at $\bar{u} = 1$, where this point is a global maximum, which will represent the ESS. The global attractivity of $\bar{u} = 1$ on the interval $(0, \infty)$ follows from Theorem C.3 [5], and so

$$\lim_{n \to \infty} u_n = 1, \quad u_0 > 0, \text{ and } \sigma^2 < 2$$

To find the conditions for (77) to hold, we get the following:

$$b(1) = \frac{b_0}{2} \quad b''(1) = -\frac{b_0}{2}$$

To satisfy condition (77), we must have that $\sigma^2 < 2$ and so applying the results from Theorem 14, we get the following result

**Theorem 20** Consider the system (72), (73) such that $b(u_n) = \frac{b_0 u_n}{u_n^2 + 1}$. Then the global dynamics of the system is given by;

i) Assume $b_0 < 4$ and $\sigma^2 < 2$, then every solution $(x_n, u_n)$ of the evolutionary model converges to the equilibrium $(0, 1)$.

ii) Assume $b_0 > 4$ and $\sigma^2 < 2$. Then if $x_o \in [0, \bar{x}_-)$, every solution of the evolutionary model converges to the equilibrium $(0, 1)$. If $x_o \in (\bar{x}_-, \infty)$, then every solution converges to $(\bar{x}_+, 1)$.
Figure 16: a) Defining the equation \( f(x, \sigma) = \sigma^2 \left( \frac{1-x^2}{x(x^2+1)} \right) + x \) with \( \sigma^2 = 0.81 \), two iterates of \( f(u, \sigma) \) showing that the only intersections of the second iterate and the bisector are exactly the fixed points of the equation and thus that there is no periodic solution; b) Defining the function as in the first case with \( \sigma^2 = 2.89 \) depicting the intersection of the second iterate of \( f(u, \sigma) \) with the bisector at two points besides the equilibrium points, and thus that there exists a periodic solution.

\[
\lim_{n \to \infty} (x_n, u_n) = \begin{cases} (0, 1) & \text{if } b_0 < 4, \text{ or } b_0 > 4 \text{ and } x_0 < \bar{x}_- \\ (\bar{x}_+, 1) & \text{if } b_0 > 4, \text{ and } x_0 > \bar{x}_- \end{cases}
\]

iii) Assume \( \sigma^2 > 2 \). Then there will exist a periodic solution of \( u_n \) around \( \bar{u} = 1 \).

For this simulation a quick evaluation of the Schwarzian derivative will show that the Allwright-Singer Theorem [11] is not applicable.

**Example 4** Lastly, consider the following definition of \( b(v) \) as a normally distributed phenotypic trait with five critical traits

\[
b(v) = b_0 \exp\left( -\frac{v^6}{5} + \frac{2v^5}{4} + \frac{22v^3}{3} - 20v^2 \right)
\]

Using this definition of \( b(v) \), the system (72), (73) can be rewritten as

\[
x_{n+1} = b(u_n) \frac{x_n^2}{x_n^2 + 1} \\
u_{n+1} = u_n - \frac{\sigma^2}{1000} (u_n - 4)(u_n - 1)u_n(u_n + 2)(u_n + 5)
\]
In the first quadrant, the sequence $u_n$ will have three critical traits

$$\bar{u} = 0, 1, 4$$

where $\bar{u} = 0$ is a local maximum, $\bar{u} = 1$ is a local minimum and $\bar{u} = 4$ is a global maximum. Both the global and the local maximum will satisfy the negative feedback condition, and so will be attracting.

$$\lim_{n \to \infty} u_n = \begin{cases} 0 & u_0 \in (0, 1) \\ 4 & u_0 > 1 \end{cases}$$

Finding the values of $b(v)$ and $b''(v)$ at the equilibrium points $\bar{u} = 0$ and $\bar{u} = 4$ results in the following

$$b(0) = b_0 \quad b''(0) = -\frac{b_0}{25}$$

$$b(4) = b_0 e^{752/1875} \quad b''(4) = -\frac{81 b_0}{125} e^{752/1875}$$

This will result in three cases to consider: $0 < b_0 < 2 \exp(-752/1875)$ in which both $b(0)$ and $b(4) < 2$; $2 \exp(-752/1875) < b_0 < 2$ in which $b(0) < 2$ and $b(4) > 2$; $b_0 > 2$ in which both $b(0)$ and $b(4) > 2$. In each case, we will have the following results;

**Theorem 21** Consider the system (72), (73) such that
\[
b(u_n) = \frac{b_0}{1000} \exp(-\frac{u_n^6}{6} - \frac{2u_n^5}{5} + \frac{21u_n^4}{4} + \frac{22u_n^3}{3} - 20u_n^2). \]
Suppose that \(\sigma^2 < \frac{250}{81}\). Then the global dynamics of the system is given by:

i) Assume \(0 < b_0 < 2 \exp(-752/1875)\). Then if \(u_o \in (0, 1)\) every solution \((x_n, u_n)\)
will converge to the equilibrium \((0, 0)\). If \(u_0 > 1\), then every solution of the
evolutionary model will converge to the equilibrium \((0, 4)\).

ii) Assume \(2 \exp(-752/1875) < b_0 < 2\). Then the solutions \((x_n, u_n)\) of the
evolutionary model will converge as follows:

\[
\lim_{n \to \infty} (x_n, u_n) = \begin{cases}
(0, 0) & u_0 \in (0, 1) \text{ and } x_0 < \bar{x}_-

(0, 4) & u_0 > 1 \text{ and } x_0 < \bar{x}_-

(\bar{x}_+, 4) & u_0 > 1 \text{ and } \bar{x}_- < x_0

(\infty, 0) & u_0 \in (0, 1) \text{ and } \bar{x}_- < x_0
\end{cases}
\]

iii) Assume \(b_0 > 2\). Then the solutions of \((x_n, u_n)\) of the evolutionary model will
converge as follows:

\[
\lim_{n \to \infty} (x_n, u_n) = \begin{cases}
(0, 0) & u_0 \in (0, 1) \text{ and } x_0 < \bar{x}_-

(0, 4) & u_0 > 1 \text{ and } x_0 < \bar{x}_-

(\bar{x}_+, 0) & u_0 \in (0, 1) \text{ and } \bar{x}_- < x_0

(\bar{x}_+, 4) & u_0 > 1 \text{ and } \bar{x}_- < x_0
\end{cases}
\]

**Remark 8** The conditions on \(\sigma^2\) will again result in the emergence of a periodic
solution around the equilibrium points \(\bar{u} = 0, 4\). For the equilibrium \(\bar{u} = 0\), this
bifurcation value will be \(\sigma^2 = 50\) while for the equilibrium \(\bar{u} = 4\), the bifurcation value
will be \(\sigma^2 < \frac{250}{81}\).
Figure 18: Defining the equation $f(x, \sigma) = x - \frac{\sigma^2(x-4)(x-1)x(x+2)(x+5)}{1000}$, varying the value of $\sigma$ results in the graphs described as follows; a) $\sigma^2 = 25$, two iterates of $f(u, \sigma)$ showing that the only intersections of the second iterate and the bisector are exactly the fixed point $\bar{x} = 0$ of the equation and thus that there is no periodic solution; b) $\sigma^2 = 2.89$ depicting the intersection of the second iterate of $f(u, \sigma)$ with the bisector only at the fixed point $\bar{x} = 4$, and thus that there is no periodic solution; c) $\sigma^2 = 64$ showing that there are two additional intersection points of the second iterate of the function and the bisector and thus that there exists a periodic solution; d) $\sigma^2 = 4.84$ showing the additional two intersection points between the second iterate of the function and the bisector, indicating the existence of a periodic solution.
**Example 5** Consider Equation (72), where the fitness function is

\[ b(u) = e^{-\frac{x^2}{\sigma^2}} - \arctan(x), \]  

(85)

in which case the Fisher's equation becomes:

\[ u_{n+1} = \sigma^2 \frac{u_n^2}{1 + u_n^2}, \quad n = 0, 1, \ldots \]  

(86)

In view of well known dynamics of Equation (86) we can formulate the following result:

**Theorem 22** Consider system consisting of Equations (72) and (86). Then the global dynamics of this system is given as:

(i) Assume \( \sigma^2 \leq 2 \). Then every solution \((u_n, x_n)\) of evolutionary model (72), (86) converges to the equilibrium \((0, 0)\).

(ii) Assume \( \sigma^2 > 2 \). If \( x_0 \in [0, \bar{x}_-) \) then every solution \((u_n, x_n)\) of evolutionary model (72), (86) converges to the equilibrium \((0, 0)\) if \( u_0 \in (0, \bar{u}_-) \) and converges to the equilibrium \((\bar{u}_+, 0)\) if \( u_0 \in (\bar{u}_-, \infty) \) and \( b(\bar{u}_+) < 2 \). If \( x_0 \in (\bar{x}_-, \infty) \) then every solution \((u_n, x_n)\) of evolutionary model (72), (86) converges to the equilibrium \((0, \bar{x}_+)\) when \( u_0 \in (0, \bar{u}_-\) and converges to the equilibrium \((\bar{u}_+, \bar{x}_+)\) if \( u_0 \in (\bar{u}_-, \infty) \) and \( b(\bar{u}_+) > 2 \)

or

\[
\lim_{n \to \infty} (x_n, u_n) = \begin{cases} 
(0, 0) & \sigma^2 \leq 2 \\
(0, 0) & \sigma^2 > 2 \text{ and } u_0 \in (0, \bar{u}_-) \\
(\bar{u}_+, 0) & \sigma^2 > 2 \text{ and } u_0 \in (\bar{u}_-, \infty), b(\bar{u}_+) < 2 \\
(0, \bar{x}_+) & \sigma^2 > 2 \text{ and } u_0 \in (0, \bar{u}_-), x_0 \in (\bar{x}_-, \infty) \\
(\bar{u}_+, \bar{x}_+) & \sigma^2 > 2 \text{ and } u_0 \in (\bar{u}_-, \infty), x_0 \in (\bar{x}_-, \infty), b(\bar{u}_+) > 2.
\end{cases}
\]
Remark 9 The fitness function has the property that it has one critical point when \( \sigma^2 < 2 \), two critical points when \( \sigma^2 = 2 \) and three critical points when \( \sigma^2 > 2 \). Thus the value \( \sigma^2 = 2 \) is the bifurcation value. It can be also be shown that in the case \( \sigma^2 > 2 \) first \((\sigma^2 - \sqrt{\sigma^4 - 4})/2\) and third critical points are local maxima and the second one 0 is a local minimum. The global maximum is attained at the largest critical point \((\sigma^2 + \sqrt{\sigma^4 - 4})/2\) and so this critical point is ESS (evolutionary stable strategy or equilibrium, see [10]). Theorem 22 is a bifurcation result with bifurcation parameter being variance \( \sigma^2 \). Thus the maximum of fitness function \( b(u) \) is attained at critical point \((\sigma^2 + \sqrt{\sigma^4 - 4})/2\) and is equal to

\[
b_{\text{max}} = b \left( \frac{\sigma^2 + \sqrt{\sigma^4 - 4}}{2} \right) = \exp \left( \frac{\sigma^4 + \sqrt{\sigma^4 - 4}\sigma^2 - 4\sigma^2 \tan^{-1} \left( \frac{1}{2} \left( \sqrt{\sigma^4 - 4} + \sigma^2 \right) \right) + 2}{4\sigma^2} \right).
\]

The fitness function as a function of the mean phenotypic trait \( v \), is unimodal with a maximum at the equilibrium point 0 for \( \sigma^2 < 2 \) and is bimodal with a maximum at the equilibrium point \( \bar{u}_+ \) for \( \sigma^2 > 2 \).

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