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THE PHILOSOPHICAL PROBLEM OF ENTAILMENT

BY

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IN
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ABSTRACT

This thesis is concerned with the problem of entailment. Entailment is a form of implication, perhaps the strongest found in logical calculi. The development of entailment logics is rather new in the history of logic. The entailment concept has however, been employed for the designation of a strong implication relation between antecedent and consequent. It has been used by some to designate deducibility. There are, however, alternative systems of implication which have been formed and equated by some to the concept of entailment. A central question then, is what conditions shall be given to an acceptable statement of entailment, and how does entailment differ from alternative implication systems?

The properties which are inherent in logical implication are shown by a historical sketch in this thesis. The first form of implication discussed is 'material implication.' Material implication as found in Russell and Whitehead's Principia Mathematica is discussed. Following upon this the evolution of strict implication and the 'S' systems of C.I. Lewis and E.H. Langford are analyzed. In the context of these discussions both the common and diverse properties are discussed. It is found that both material and strict implication have internal weakness in so far as both produce implicational paradoxes.

The question then becomes, can an entailment logic overcome these difficulties, or rather must an entailment logic avoid the paradoxes peculiar to the systems cited?

With the development of system E by Alan Anderson and Noel Belnap, we find a paradox-free system of implication. The system is designated 'E' for entailment by the authors. The system E is formed by combining the semantical character of "relevance" with that of necessity. It is shown that the system E does not contain the paradoxical theorems nor any analogues of the theorems which give the paradoxes. In the context of this discussion a proper focus is given for several logical terms used in describing the relations between antecedent and consequent.

The research of this thesis deals with certain definite problems. Is strict implication the same as entailment? According to the research of this thesis this identification is doubtful. Of further concern is whether or rather what logically follows from contradiction. This research rejects the claims that anything whatsoever "follows from" a contradiction. A final concern of this research is the relation of entailment in modal semantics. This problem which is more general is concerned with such developments as modal models on the one hand and the nature of possibility on the other. It is found that an impossible proposition need not have the essential form of a contradiction.

TABLE OF CONTENTS

CHAPTER	PAGE
I. INTRODUCTION	2
II. STRICT IMPLICATION	6
III. PARADOXES OF STRICT IMPLICATION	20
IV. THE SYSTEM 'E' OF ENTAILMENT	37
A. The Development of System E.	37
B. Conclusions of 'E'	48
V. CONCLUSION	55
VI. BIBLIOGRAPHY	65

LIST OF SYMBOLS

$p, q, r, \text{ etc.}$

propositional variables

$A, B, C, \text{ etc.}$

metalinguistic variables

$\vee$

wedge, sign of disjunction

$\cdot$

dot, sign of conjunction

$\supset$

material implication

$\rightarrow$

strict implication, Lewis and Langford

$\rightarrow$

entailment sign, Anderson and Belnap

$\equiv$

equivalence

$\equiv$

strict equivalence

$\exists x$

existential quantifier

$\Box$

necessity, Lewis

L

necessity, (conventional sign)

N

necessity, Carnap

$\Diamond$

possibility, Lewis

M

possibility, (conventional sign)

$\sim$

sign of negation

INTRODUCTION

The purpose of this thesis is to analyze the logical relation of 'entailment'. More precisely what is wanted is an explication of entailment as a form of implication. The problem centers on deciding what conditions are needed to establish an intuitively acceptable system of logical implication. The problem, at least on face value, is not at all clear-cut. For as is commonly known, there are several forms of implication. The most fundamental and most common is 'material implication'. On the other hand, there is the more sophisticated system of 'strict implication', which is developed employing modality. There are furthermore other systems such as the system 'E', where 'E' stands for entailment.

The problem of implication depends on the conditions for establishing the relation in question. The main components employed designate accordingly the antecedent, the logical symbol for implication and the consequent. When presented with some such 'form' as $p \supset q$, certain descriptions become apparent. The antecedent, here p , acts as the premises, while q acts as the conclusion of an argument. The implication symbol designates that p implies q , and that the antecedent is the grounds for the assertion of q . But how shall we read such a formula? The natural response is of course, " p implies q ", but what significance is there to certain other statements as " q follows

from "p", or "q" is derivable from "p", and still further, "p" entails "q"?

The problem is, that in certain formalisms the renderings just cited are not always intuitively acceptable. In the system of material implication it might be helpful for some to read " $p \supset q$ " as "p materially implies q", but such a reading does not address the underlying issues implicit in the philosophical conception of implication. In the system of "material implication" or any system constructed out of " $\supset$ ", where the horseshoe symbol " $\supset$ " is defined as:

$$p \supset q = dF. \sim p \vee q$$

there exists certain fundamental problems. The main form of this problem occurs upon the derivation of "paradoxes". The most notorious being, $p \supset (q \supset p)$. We can read this as follows: if p (is true) then q implies p. Does this mean p "follows from" q in any acceptable sense? The most obvious criticism is that there is a fallacy of relevance, that is to say, there is no obvious connection between the antecedent and the consequent.

The more complicated system or systems of "strict implication" employ a more sophisticated version of implication. The system of "strict implication" employs the symbol $\rightarrow$, called by some the fish-hook. In this system, $p \rightarrow q$, read "p strictly implies q" employs the modal concept of necessity. The definition

is:

$$p \rightarrow q = \text{Def. } \sim M(p, \sim q)$$

The above definition states that for p to strictly imply q , it is not possible for p to be true and q false. The definition so given allows the desirable feature, that there is a necessary connection between antecedent and consequent. The system so constructed does however have its problematical theorems, such as:

$$q \rightarrow (p \vee \sim p).$$

The above says that a necessary proposition is entailed by any proposition. But does a necessary proposition "follow from" any proposition?

In the text of what follows, the system of "strict implication" and pertinent materials will be discussed. The distinctiveness of material and strict implication will be shown by showing that certain theorems of material implication are unprovable as theorems of "strict implication". Following upon this, the "paradoxes" of "strict implication" will be discussed. In the attempt to eliminate the "paradoxes", the development of the system "E" will be given. It will be found that system "E" gives an intuitively acceptable expression to implication, or rather, entailment. We shall find that just as "strict implication" is distinct from "material implication", so analogously is entailment distinct from "strict implication".

In what follows, small letters shall refer to propositional variables while capital letters refer to metalinguistic expressions.

Both "dots" and "brackets" shall be used by convention, where dots replace brackets and are entered from the left.

CHAPTER II

STRICT IMPLICATION

The pioneer work in the development of systems of strict implication was done by C.I. Lewis. Further advancements and progress were made by the collaboration of C.I. Lewis with C.H. Langford in the co-authored book entitled Symbolic Logic. Strict implication is a species of "modal logic", but more precisely, of modern modal logic, inasmuch as the notion of modality, in principle, is nothing new. In the history of philosophy, such thinkers as early as Aristotle had recourse to modal notions,¹ even Kant went so far as to make modality a category.²

Modern logistics, however, had with the development of Principia Mathematica,³ access to more rigorous and in effect more precise machinery for analysis, the principal benefits being the advancement of axiomatic techniques in logic.

In the context of Principia Mathematica⁴ or more precisely in its formal symbolism, there occurred the

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1 Aristotle, Prior Analytics, passim.

2 Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (New York: St. Martin's Press, 1929), p. 239.

3 Alfred North Whitehead, Bertrand Russell, Principia Mathematica to #56 (Cambridge: Cambridge University Press, 1970), preface V.

4 Ibid., p. xvi.

("horseshoe") which was the symbol for implication. It was this concept, or perhaps the inadequacies of this concept of implication, which stimulated such authors as C.I. Lewis and C.H. Langford to explore the nature of implication. In P.M., the definition of implication is given in terms of negation and disjunction, that is at *1.01 in P.M., we find:

$$p \supset q = . \sim p \vee q \text{ Df.}$$

Likewise, we find the statements; (as explanation). "In virtue of the above definition, when " $p \supset q$ " holds, then either p is false or q is true: hence, if p is true, q must be true. Thus, the above definition preserves the essential characteristic of implication: it gives, in fact, the most general meaning compatible with the preservation of this characteristic."⁵ The above-mentioned implication is of course material implication. We should note in passing that P.M. also incorporated the idea of "formal implication" in which we find a class of implications being asserted. This need not cause any immediate difficulties for the discussion which follows, for the simple reason that many of the properties of this relation are analogous.⁶

In what follows, several properties which characterize material implication shall be developed in contra-distinction to what Lewis and Langford have called "strict implication". The main goal, of

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6 Ibid., p. 94.

Ibid., Chaps. 9 & 10, & Chap. III of Principia Mathematica, by Bertrand Russell, for further discussions.

what follows, is an explication of those essential features possessed by strict implication.

The following general statements are offered as aids in distinguishing the two systems of implication.

- 1) "Strict implication" is designed to represent symbolically the relation "q is deducible from p".
- 2) "Material implication" is essentially a relation between two propositions determined by extensional features.
- 3) "Strict implication" is formally distinguished from "material implication" by the following postulate:

$$(\exists p, q): \sim (p \rightarrow q). \sim (p \rightarrow \sim q)$$

The first observation is readily seen to be a rendering of what Lewis and Langford perceive to be the essential feature of strict implication. The idea of deducibility, as the fundamental feature of strict implication is in accord with the basic motive that a proper format for an implicative calculus should capture a strong logical relation between antecedent and consequent. In what follows, we shall observe the development of strict implication, from a formal point of view, with the aid of modal notions. In contrast, however, the second observation lays emphasis on extensional features, that is, truth value. It is of importance to note that material implication rests upon this feature alone and in so far as prime importance is placed on such truth value conditions, only the barest sense of logical consequence can be incorporated. This can be seen from the fact that the symbol " $\supset$ " is a truth functional connective which can be completely defined by a truth table. In this definition, which corresponds to that

of P.M. above, the basic feature of logical consequence corresponds to the most elementary property of implication, that being, that a false proposition not be inferred from a true proposition.

The first two observations above are not sufficient to distinguish the two kinds of implication. Yet as shall become apparent, the system of Lewis and Langford did not have as its object the elimination of extensional features in their system, but rather an effective blending of the general system of truth functions with that of stronger logical notions. The third observation above we shall find is sufficient to distinguish the two forms of implication. To arrive at this postulate, however, we shall have to embark on a more indirect course, so as to amplify, as it were, deducibility and truth functional inference. It shall also be necessary to introduce modal concepts to explicate this postulate.

We may begin by citing two forms which occur in truth functional propositional logic:

- a) B is validly inferred from A.
- b) A materially implies B.

The occurrence of (a) above has an affinity to deducibility. We can extract the condition that by some mode of inference, that A being asserted, we are entitled to assert B. For example, if A were to stand for "p" then "pvq" would validly be inferred as B by the logical law of addition. In two-valued propositional logic this is intuitively obvious. What makes this obvious is the accepted or rather stipulated employment of certain logical

constants. The truth conditions for the " $\vee$ " (wedge) are simply that one of the disjuncts be true for the whole proposition " $p \vee q$ " to be true. The point to all this is simply to show that there is indeed a sense in which deducibility is a property of such truth functional systems. But what is to be further noted is that the point was made presupposing "assertion", that is to say we did not merely say "if p and if p then $p \vee q$, then $p \vee q$." This could lead to the familiar paradox of Lewis Carroll, in which the tortoise insisted to Achilles, "But that's another conditional!"⁸ The problem is: do we have in our system a symbol which represents the relation " B is deducible from A "?

Here we lead to (b) above in which we find an implication form. We might however, in order to broaden the range of this form, modify (b) by eliminating the adjective "material." The result would simply be " A implies B " and stand for a truth functional implication. The conditions upon which such implications rest are not adequate to satisfy the relation " q is deducible from p ," because the truth implication is dependent on a functional relation between the possible values of the propositional variables. The result is that such "truth implications" as C.I. Lewis calls them, must be too inclusive in their meaning to be equivalent to " q is deducible from p ," in as much as the relation holds whenever both p and q are true.⁹

⁸ Lewis Carroll, The Complete Works of Lewis Carroll, ed. A. Woolcott, What The Tortoise Said to Achilles (New York: Random House, Inc.), p. 1225.

⁹ Lewis and Langford, p. 239.

Likewise we can observe similar problems occurring with the definition of implication in the system P.M. mentioned earlier, which is essentially the same. Consider for instance the propositions "Caesar crossed the Rubicon" and "Neil Armstrong walked on the Moon." Let the former be p and the latter q , then in accordance with valid operations we can establish either $\sim p \supset q$, or $\sim q \supset p$. Now it is quite obvious that in the proposition, "If it is not the case that Caesar crossed the Rubicon, then Neil Armstrong walked on the Moon.", the supposed implication indicated by the "if . . . then . . ." fails even to approximate a relation between antecedent and consequent such that "B is deducible from A" yet it does designate in accordance with stipulated conventions, "A materially implies B." Whether material implication is a kind of implication at all as in the statement by Anderson and Belnap, "Material Implication is not a 'kind' of implication, or so we hold: it is no more a kind of implication than a blunderbuss is a kind of buss",¹⁰ shall be left to the reader.

We can now investigate the strengthened form of implication which Lewis and Langford called "strict implication". This form of implication was strengthened by the addition of "modal" features which are non-extensional. That is to say, the sign for strict implication, ' $\rightarrow$ ', the "fishhook", is not defined by a truth table, but rather is defined in terms of "if" 'possibility', where possibility is a monadic operator over truth functional statements.

The "fishhook" sign is a dyadic operator. We should also note that possibility is interdefinable with "necessity" designated "L" by convention, and that a system whose primitive modal notion is possibility is said to be M-based and similarly a system may take necessity as primitive and be L-based."¹¹ Hence we have:

$$Mp \equiv \sim L \sim p$$

$$Lp \equiv \sim M \sim p$$

Lewis and Langford introduce a further modal notion into their system of strict implication, that is, the concept of consistency. The proposition "p is consistent with q", symbolized by "poq" has the intended interpretation that with either p or q serving as a pre-mise it is not possible to deduce the falsity of the other.¹² The following statements are several principle results of the above-mentioned modal notions.* In Lewis and Langford's Symbolic Logic, we have:

$$11.02 \quad p \supset q. = . \sim M(p . \sim q)$$

This in effect is the definition of strict implication, given in terms of negation, possibility and product. To say q is deducible from p says that it is not possible for p to be true and q false.

$$17.01 \quad poq. = . \sim (p \supset \sim q)$$

To say p is consistent with q is to assert that it is false that the negation of q be deduced from p.

¹¹ G.E. Hughes and M.J. Cresswell, Modal Logic. (London: Methuen & Co., Ltd., 1968), p. 26.

¹² Lewis and Langford, p. 153.

* The occurrence of '=' stands for logical equivalence as is $p=q \therefore p \supset q \cdot q \supset p$. The numbers refer to indices for theorems found in L. and L. Symbolic Logic. Likewise, proofs or explanations are to be found under their numbers respectively.

$$17.12 \quad p \rightarrow q. = . \sim (p \sim q)$$

For the condition of q being deducible from p to hold, then p and the negation of q are not consistent.

$$18.1 \quad 'p. = .(p \supset p). = \sim(p \rightarrow \sim p)$$

Levis and Langford point out, that in their system, ' p ' could have been defined in terms of consistency. Thus, ' p ' means " p is self-consistent" likewise, p does not imply its own negation.

$$18.12 \quad \sim 'p. = . \sim (p \supset p) = p \rightarrow \sim p$$

If p is not possible then it is not self-consistent.

$$18.14 \quad \sim M \sim p. = \sim (\sim p \supset p) = \sim p \rightarrow p$$

If p is necessary, then its negation is not self-consistent.

Having introduced the primitive ideas and concepts of modality, attention can now be directed toward three objectives all of which stand as developments of what has preceded. First, we can accentuate the logical consequences of the existence postulate given earlier under (3), that is:

$$20.01 \quad (\exists p, q): \sim (p \rightarrow q). \sim (p \rightarrow \sim q),$$

which categorically distinguishes strict implication from material implication. Following upon this we can secondly analyze the employment of strict implication. Finally, we can proceed to the paradoxes of strict implication and the logical import of such paradoxes in modal calculi.

We may begin by citing a distinctive theorem in the system of strict implication. That is:

$$12.81 \quad p \rightarrow q . \rightarrow . \sim (p \sim q)$$

This theorem states that if p strictly implies q , then that it is not the case that p is true and q false, is likewise strictly implied. What is important about this theorem is that its converse which would be

$$\sim (p \sim q) . \rightarrow . p \rightarrow q$$

cannot hold. If the converse were to hold, then a free interchange of " $\supset$ " for " $\rightarrow$ " would occur, rendering strict implication a redundant form of material implication. Likewise, if the converse of 12.81 were to hold then all the modal notions would be reduced to the simpler extensional constants i.e., $p \supset q$ would be pq , $\sim Mp$ would be $\sim p$, etc...¹³ This however, is contrary to the system, for although we have theorems of the form

$$18.4 \quad p \rightarrow Mp$$

that is, if p is true then p is possible the converse, p is possible implies p is true is certainly not valid. Postulate 20.01 excludes the free interchange of " $\supset$ " for " $\rightarrow$ " and prohibits the reduction of modal operators to the extensional constants. Postulate 20.01 allows that there be two propositions such that, if p be one of them, then it says nothing of the truth or falsity of the other, i.e., q . That is to say, two propositions can at once be consistent

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¹³

Lewis and Langford, p. 173.

and independent, for "p is consistent with q", $p \circ q$ is, by 17.01 given earlier, equivalent to $\sim (p \rightarrow \sim q)$; that p and q are independent is expressed by $\sim (p \rightarrow q)$ which states q is not deducible from p. Contrasting the above distinctions with the theorem

$$15.72 \quad p \supset q . \vee . p \supset \sim q,$$

we find that if strict implication were to be identified with material implication, as in the case in which the converse of 12.81 were to hold, then 15.72 would be an exact contradictory of 20.01.¹⁴ In terms of material implication, two propositions cannot be consistent and independent as indicated by 15.72, ergo the two kinds of implications are distinct if the system is consistent.*

Proceeding to the second objective, that is, the nature and function of strict implication, we may begin by reflecting on two points mentioned earlier. To begin, we observed under (a) and (b) above, two logistic forms which characterize two important points of a logistics calculus. One emphasizes the act of making an assertion as in the case in which a conclusion is asserted on the grounds of the premises, and the other cites a relation of "implication" which holds between the antecedent and consequent. The first pertains to inference and generally speaking does not offer as much difficulty as the second. In fact, most penetrating

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¹⁴

Ibid., p. 179.

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Consistency of 20.01 is proved in Appendix II of Symbolic Logic.

problems of inference revolve around finding criteria for deciding what is to be a valid inference. The problem in this form shall be postponed until the discussion of the paradoxes of strict implication. However, the logical notion of inference is germane to the concept of deducibility, in as much as we shall find it very advantageous to subjugate a) under a modified form of b).

To effect this change, we shall want the concept of a valid inference of B from A to correspond to a valid deduction of B from A. As for b), which was originally "A materially implies B", we shall stipulate the modification, "A strictly implies B," with the intended meaning that B is deducible from A. In accordance with these points, we may now ask, has Lewis and Langford's calculus of strict implication successfully achieved this result? We can give an affirmative answer, in so far as the above-mentioned distinctions give insight into the usage of a certain logical symbol, that is, what was designated by the "fish hook." To explicate the usage of this symbol we shall need to appeal to the concept of tautology. Similarly, we shall find justification for making strict implication an extension of truth functional logic, that is, from Lewis and Langford's points of view.

Informally, we shall define a tautology in the propositional logic as a formula which is logically valid. That is to say, a tautology has the value T, or 1 from a syntactic point of view, for every possible assignment of values to constituent

propositional variables. A tautology is true by virtue of the form of the statement. A tautology is said; therefore, to be a necessary truth which is in contradistinction to a proposition which is contingently true as in the case when some truth function generates a true proposition by a possible assignment of truth values. Consider some truth functional implication, say, $p \supset q$, this need not be tautological but nevertheless if p is asserted, then on the intended interpretation q must be asserted. Let us generalize this scheme as $A \supset B$ where both $A \supset B$ is asserted, and A is asserted as laws of the system, i.e., as tautological, then B likewise must be asserted. What we have is a distinction between $p \supset q$ as true and $A \supset B$ as tautological in which case if we are given two premises p and $p \supset q$, we can infer q , but this is now a special case of $A \supset B$, because $(p (p \supset q)) \supset q$ is now a tautology.¹⁵ This is precisely the point Lewis and Langford have in mind when they state: "Whenever any truth implication, $p \supset q$, expresses a tautology (is necessarily true) the relation $p \rightarrow q$ holds: when $p \supset q$ is true, but does not express a tautology, $p \rightarrow q$ does not hold."¹⁶

Several examples will exhibit more clearly the employment of strict implication. Consider the following two columns for comparison:¹⁷

15

Lewis and Langford, p. 244.

16

Ibid. p. 244.

17

Ibid., p. 246.

I

15.21 $p \supset q \supset p$.
 15.23 $\sim p \supset p \supset q$
 15.31 $\sim(p \supset q) \supset p \supset \sim q$
 15.41 $\sim(p \supset q) \supset q \supset p$

II

15.2 $p \supset q \supset p$.
 15.22 $\sim p \supset p \supset q$.
 15.3 $\sim(p \supset q) \supset p \supset \sim q$
 15.4 $\sim(p \supset q) \supset q \supset p$.

Column I gives propositions derived in the system of Material Implication while propositions under II are derived in the system of Strict Implication. The employments of " $\supset$ " under Column I which are tautological are capable of being deduced in the system of strict implication whereas employments of " $\supset$ " which are not tautological, that is, cases which lack assertion cannot be replaced by " $\supset$ ". In fact, if in Column II, the occurrences of " $\supset$ " were replaced by " $\supset$ " the theorems would be false.¹⁸ What then can be concluded about the nature of strict implication?

With respect to the systems developed by Lewis and Langford we can see an explication of valid deduction. This occurs, however, with respect to extensional systems in so far as the relation of strict implication holds when valid inference in some system is possible. This is the most fundamental use of strict implication, although it would be misleading to limit the notion of strict implication to the expression of properties of truth-functions. That is to say, we would not want to obscure the importance of modal operators in their own right, so to speak. Strict implication does however, give an important distinction between the case in which $p \supset q$ holds and the idea of "q's deducibility from p". Consider for instance 15.21, "If p is true, then q (any

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Ibid., p. 245.

proposition) implies p.", which is one of the familiar paradoxes of material implication. Strict implication does not allow the consequent to be equivalent to "p is deducible from q" although the consequent as a whole is strictly implied by the antecedent as in 15.2 given under Column II above.

CHAPTER III

PARADOXES OF STRICT IMPLICATION

To this point the symbol " $\rightarrow$ " has been interpreted as "strictly implies" but as will soon become obvious there are additional senses associated with this symbol. Firstly, there is the concept of "deducibility" briefly touched upon earlier, and endorsed by many logicians as a meaning for "strictly implies." There are likewise other verbal statements for " $p \rightarrow q$ " such as "q is deducible from p" or "q follows from p" and perhaps the most fundamental of all, "p entails q". A real problem and controversy enters at this point as many prominent logicians take head to head opposing stances. The key to the issue centers on the answers to the question "Is strict implication the same as entailment?" The appearance of the paradoxes of strict implication occupies a central role in this controversy in as much as their occurrence is claimed as evidence for both sides!

We should note that the theorems to be given directly are dependent on a formal context. That is to say, the several expressions which are considered paradoxical occur in some formal system and as such are derivable in some system. Here are four important formulae for consideration:¹⁹

- 1) $(p \cdot \sim p) \rightarrow q$
- 2) $q \rightarrow (p \vee \sim p)$
- 3) $\sim Mp \supset (p \rightarrow q)$
- 4) $L q \supset (p \rightarrow q)$

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¹⁹Hughes and Cresswell, p. 335.

Furthermore, we shall put forth two basic forms of the paradoxes which occur in the systems of Lewis and Langford.

They are

$$19.74 \sim Mp. \supset p \supset q$$

which states that an impossible proposition strictly implies any proposition and,

$$19.75 \sim M\sim p. \supset q \supset p$$

which states that a necessarily true proposition is strictly implied by any proposition.

The formulae (1)-(4) are more general, although basically the same as 19.74 and 19.75 which are somewhat restricted. The difference is dependent upon the modal notion involved. Formula (1) above states that a contradiction entails any proposition while (2) states that a proposition of the form $(p \vee \sim p)$ follows from or is entailed by any proposition. Formula (3) states that an impossible proposition implies any proposition, and (4) states that a necessary proposition is entailed by any proposition. The modal variance between 1-4 and 19.74 and 19.75 rests upon the restricted sense of impossible and necessary. For Lewis and Langford, "p is impossible" means "p is not consistent" while "p is necessary" means "the denial of p is not self-consistent." "Necessary truths so defined, coincide with the class of tautologies, or truths which can be certified by logic alone; and impossible propositions coincide with the class of those which deny some tautology."²⁰ The occurrence of the impossible in (3) admits to a

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Lewis and Langford, p. 249.

broader scope that is, it is not limited to propositions of the form $p \sim p$, likewise in (4) necessary is not limited to $(pv \sim p)$.²¹ Admittedly, this picture does not have the quality of crystal clarity, consequently a minor digression is in order.

The questionable features of this scheme of modality rests upon the interpretation of both impossibility and necessity. We are faced with the problem of deciding whether all impossible propositions are in essence contradictions, and if so, how so. Particularly, we must take notice of the claim cited above that impossible propositions deny some tautology. We shall want then to deal with the context of modal-logical truths on the one hand, and applied modal logic on the other.

Briefly, we may first take note of the fact that there are many modal logics. Logics which deal with necessity and possibility are conveniently called alethic modality, of which the "S" systems are most common. The S systems are constructed by building upon a set of postulates. When further axioms are added, the systems are said to be stronger.²² For instance we have the system S4 by adding to certain postulates the axiom $Lp \supset LLp$ while S5 is obtained by adding the axiom $Mp \supset LMp$ to the same set of postulates. The system S5 however contains the system S4 because the axiom $Lp \supset LLp$ can be derived in S5 therefore S5 is said to be stronger. The development of such systems often results

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²¹

Hughes and Cresswell, p. 336.

²²

Ibid., pp. 341-342.

from a curiosity over the logical structure which is the product of such an inquiry. It is not, as it were, a brilliant intuitive insight into logical truths which is the impetus, rather that the nature of such 'modal laws' generated by some system generally rests in a mist of controversy which is the impetus.

The common view that logical analysis laws bare the structure of reason is not forsaken however, as logicians and the like attempt to find application for many of these newly formed logics. A.N. Prior, for instance, attempted to accommodate his tense logic to the structure of the S4 modal logic.²³ "I shall now show that tense logic as I have described it, is a modal system, and that if we define M (or 'Possibly') as 'It either is or will be the case that', and I (or 'Necessarily') as 'It is and always will be the case that', these operators will meet Lukasiewicz's conditions for being modal operators and, furthermore, will have all the formal properties of the M and L of the Lewis system S4.²⁴ The point we care to make can be represented diagrammatically as follows:

I 1				
	M, L	F, 1 1	$\Rightarrow$	S4
		F, 1 1	$\Rightarrow$	S5
I 2				
	M, L	F, 1 2	$\Rightarrow$	S4
		F, 1 2	$\Rightarrow$	S5

Her I 1 and I 2 are interpretations for the modal operators M and L, while F is some formula constructed in them in accordance with the interpretation. The arrow indicates that some formula in

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²³

Hughes and Cresswell, p. 263.

²⁴

A.N. Prior, Time and Modality (Oxford: Clarendon Press, 1957), p. 17.

question would suit our interpretation, perhaps our logical intuitions, while a bar across the arrow would reject the intended interpretation with respect to the logical structure in question i.e., the S4 or S5. In the tense-logic of A.N. Prior for instance, the first case above would exemplify the rejection of the S5 axiom $Mp \supset LMp$ for on that interpretation we would have, 'whatever is or will be true' is and always will be (true or going to be true) and this is false in the case in which p describes a process which has not yet but will someday stop for good.²⁵ The second case above could be an alternate interpretation which employs the S5 structure and consequently would contain formulae which hold in the S4 and in the S5.*

Now to return to the original line of thought we can attempt to clarify the concept of impossibility. The position given by C.I. Lewis insists that impossible propositions deny some tautology and since a tautology is a statement which is true for all value assignments it is quite obvious that a denial of such a statement shall result in falsehood. Yet it must be kept in mind that this view of modality and entailment (strict implication) was created on an extensional base of propositional logic, consequently impossible propositions

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Ibid., p. 24.

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Note: Prior's use of S4 was eventually rejected by him. Of interest, however, was the conjecture that a proper logic of time must account for time as discrete and time as continuous. The conception of time as discrete presupposes that it is meaningful to speak of time in terms of a next moment while time as continuous assumes that between any two moments there is a third moment. To accommodate this distinction two systems are required and are built upon S4. Time as continuous is formed by adding $L(Lp \supset Lq) \vee L(Lq \supset Lp)$ while time as discrete is formed by adding $((p \supset Lp) \supset p) \supset (MLp \supset Lp)$ to the S4 axiom.

take the form of a contradiction. In tense logics, as in the case of A.N. Prior, it seems tenable that senses of 'impossibility' could have significance while not having the general form of contradiction. This however anticipates the force of the upcoming 'Independent proof' that an impossible proposition entails any proposition.

Lewis and Langford have argued that the paradoxes not only do not tell against the system of strict implication giving expression to entailment or deducibility but in fact state an important principle concerning these concepts. The proof proceeds as follows. We take any proposition which is of the form $p \vee \sim p$ and assume the negation. This sets us on our way, for the denial of $p \vee \sim p$ easily gives us p and $\sim p$. Of course, the lesson to be learned is that if someone should contradict or show inconsistency then we should expect any proposition to follow.

The lesson begins thus:

1	$p \cdot \sim p$	assumption
2	p	from 1, simplification
3	$\sim p$	from 1, simplification
4	$p \vee q$	from 2, addition
5	q	from 3 and 4 by disjunctive syllogism.

The proponents of this argument appeal to the intuitive validity of the component inference moves. The following set of statements gives these rules for inspection.

- A. Any conjunction entails each of its conjuncts.
- B. Given a proposition p , $p \vee q$ can be inferred where q is any proposition, by logical addition.
- C. From the premises $\sim p$ and $p \vee q$ one is entitled to infer q by the rule of disjunctive syllogism.

- D. Implicit to this argument is also the transitivity of entailment, that is p implies q and q implies r , therefore, p implies r .

In what follows, argument will be given in brief for the rejection of some of these rules and their success or failure likewise will be discussed.

Under A we have the logical operation of conjunction which allows true premises to be conjoined and conversely the operation of simplification exemplified by the inference made at steps 1 and 3 above. It has been argued by some that the premises of an argument should function together in entailing a conclusion.²⁶ The import of such a system would be that if two propositions A and B, given conjoined as $A \cdot B$ are said to entail some proposition C then it does so in virtue of the joint force of both or a mutual functioning. $A \cdot B$ is not to be considered simply as A is true and B is true. In this form $A \cdot B$ would not entail A because A entails A, and B would in this case be superfluous. T.J. Smiley has attempted to modify this version, by certain modifications, which ultimately remove the paradoxes; however, this results in an awkward system. Smiley modifies the formal demand that the premises function together.²⁷ Smiley offers a definition of entailment in which the propositions entailed are contingent, likewise in this context the premises are contingent. He offers the following definition:²⁸

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E.J. Nelson, "Intentional Relations", Mind (1930), pp. 440-453.

²⁷

T.J. Smiley, "Entailment and Deducibility", Proceedings of the Aristotelian Society (May, 1959), p. 249.

²⁸

Ibid., p. 244.

$A_1 \dots A_n \vdash B$ if and only if the implication
 $A_1 + \dots + A_n \supset B$ is a tautology and neither B nor
 $\sim (A_1 + \dots + A_n)$ is a tautology.

The system developed employing this definition for entailment
 29 allows $A \bullet B \vdash A$ only when A and B are compatible with one another.
 This line of argument which rejects internally inconsistent
 premises does stop the 'paradoxes' for on this view the impossible
 proposition of line one would be invalid for any proof. There is
 however a severe difficulty which enters, for, on this view
 proof employing reductio ad absurdum is impossible. In the same
 light however, we can entertain Von Wright's claim; "If one could
 not discriminate between propositions which are entailed and propo-
 sitions which are not entailed, by impossible propositions, inverse
 proof could not be validly conducted. For it is essential to such
 proof that an impossible proposition should entail exactly such and
 30 such consequences, and not anything whatever."

The variance of interpretation for contradiction and impossi-
 bility can take on different forms. For the case of Von Wright the
 modal judgement of impossibility is given to a proposition which is
 inconsistent with a body of knowledge. Thus, we cannot infer any-
 thing from a contradiction, on his view, except the denial of the
 contradictory premise. In tense logic, one might desire to use

29

Ibid., p. 244.

30

Georg Henrik Von Wright, Logical Studies (New York: The Humanities
 Press, 1957) p. 174.

'impossibility' to designate, that 'no tires' is a certain proposition true without the self same proposition having the form of a contradiction. With Smiley, above, the contradictory premise is simply excluded at the outset. We may note at this point then, that depending on the interpretation one may argue, that a contradictory premise implies everything, as is the case with Lewis. On the other hand, one can argue, as does Von Wright, that only certain results can be inferred from a contradictory, and still further one can, as does Smiley, that nothing may be inferred from a contradictory.

Anderson and Belnap have put forth the most penetrating criticism of Lewis' independent proof, that a contradiction entails any proposition. According to the above, Lewis must commit either a fallacy of ambiguity or a circular argument by the acceptance of principles B and C above.³¹ That is to say, the inferring ' $p \vee q$ ' from p , a case of logical addition, and the inference of q from $p \vee q$ and $\sim p$, which is a case of disjunctive syllogism, are the focal point of criticism.

The inference scheme of $A \vee B$ from A is dependent on truth functional considerations of ' $\vee$ ', in which the statement $A \vee B$ is understood as, either A is asserted or B is, and since A is in fact given, the condition for $A \vee B$ is fulfilled. Steps 2 and 4 above appeal to this extensional usage of the wedge, ' $\vee$ '. Do we not in fact, often use the wedge, translated as 'or', in an intensional sense and is this not presupposed for the move to 5? This is the contention of Anderson and Belnap. That is to say, in our

³¹ Anderson and Belnap, "Entailment", p. 106.

ordinary reasonings we normally form the connection in question
 32
 when there is some relevance between the disjuncts.

The traditionally accepted rule of disjunctive syllogism, which in the two-valued propositional calculus, appears unquestionable, is rejected by the authors by the imposition of conditions none too obvious. It is apparent that a preliminary account of the rule is usually enough to convince someone, that the rule is acceptable. When given a choice between two disjuncts, such that either the first or the second is true, and following upon this, given that one is not the case, it seems undeniable that the remaining choice must be the case. This, however, is not the point of contention. What is of concern, is whether the proposition 'follows from' the premises, in any logically interesting sense, other than the extensional condition. What actually happens is not the rejection of all instances of disjunctive syllogism but rather such modifications that it fails to be an inference form. The rejection of disjunctive syllogism is in accord with the demand of imposing rigid conditions for the logical notion of 'valid inference'. The demand in question being, that any conclusion inferred from premises must not be totally irrelevant; any valid inference must not commit a fallacy of relevance. As for disjunctive syllogism, the rendering of $A \vee B$ must actually mean $\sim A$ entails B , thereby securing a stronger relation between A and B ; this is not generally satisfied for disjunctive syllogism.

The independent proof of Lewis is accordingly, rejected by Anderson and Belnap. In the proof there is an ambiguity inasmuch as, $A \vee B$ of line 4 is produced by extensional features, while its subsequent use assumes or rather needs the 'intensional features', i.e. relevance, of entailment to be valid.³³

This very controversial position which rejects disjunctive syllogism as a valid mode of inference is often considered too high a price to pay by many. But if the disjunctive syllogism is to be kept, the above argument must be answered. Those who doubt the above argument have done so by several methods. The most obvious rests on the concept of relevance. One might insist, and rightly so, that if the concept of relevance is to form the basis of rejection for such a long accepted principle of inference then there ought to be a rigorous formulation of such an elusive concept. That there is in fact such an adequate and formal expression is to be acknowledged. It will not be discussed here, but rather in upcoming sections, inasmuch as the concept of relevance will be found to be a fundamental feature of 'entailment'. This will be discussed in the development of system 'F'. (E for entailment) of Anderson and Belnap.

Another side to the attack on the call for relevance cited above appeals to methodological considerations. On this view we can cite the criticism thus: "The rules of inference of a formal system whose theorems (on their intended interpretation) are

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Ibid., p. 108

truths of logic ought, on their intended interpretation, to be
³⁴
 valid rules of inference;" This statement, which is a refinement of one aspect of Anderson and Belnap's criticism of disjunctive syllogism, is attacked by Geoffrey Hunter as follows:
 "I regard the rule of inference of PS, not as something that on its intended interpretation has to be a valid rule of inference, but merely as a rule for generating formulas that, on their intended interpretation, express truths of logic. It does not matter for our present purpose how these formulas are generated, provided
³⁵
 that they are all generated (and that nothing else is)." This view has merit if we agree with the basic tenet that logic is a formal matter (perhaps as some would say game). For on this view the call for relevance is irrelevant, in so far as a rule of inference does not have to be a 'valid' rule of inference. This, however, has the appearance of a red herring. To establish methods of generating formulas in accordance with the concept of logical consequence, to establish 'truths of logic' is very noble and interesting indeed, but the issue here is whether disjunctive syllogism is an acceptable rule, because of certain conditions.

In all fairness, we should likewise observe that Anderson and Belnap have developed a system which excludes disjunctive syllogism and likewise modus ponens. The system called AB is semantically complete, consistent and decidable. Not meaning to go too far afield

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 34

 Geoffrey Hunter, Metalogic (Berkeley: University of Calif., 1973),

35 p. 125

Ibid., p. 126

we shall note in passing that the system is constructed employing negation and disjunction in which proofs are constructed by 'trees'. A formula or string of formulas of a branch ends in either the production of an axiom, in which case the formula is a theorem, or in a primitive disjunction in which case it is not a theorem. 36

If a fallacy of ambiguity between the intensional and extensional uses of the wedge is sought to be avoided by an alternate sense of 'valid form of inference', then according to Anderson and Belnap a circular argument may result. The demand given at the outset of the independent proof, was, that the conclusion be reached by some 'valid mode of inference'. If in the independent proof 'strict validity' is the criterion for valid form of inference then the move from $\sim A$ and $A \vee B$ to B will be acceptable. The problem, however, with 'strict validity', which would say the criterion for valid inference is that it is necessary, that either the conclusion be true or the premises false, that the argument employing disjunctive syllogism would be unnecessary. On this view $A \bullet \sim A$ would validate an inference to B , insofar as $A \bullet \sim A$ must be false. This however, renders the independent proof superfluous. If the independent proof is supposed to justify the paradoxical theorem, then it cannot appeal to strict implication which gives the theorem, in the first place. 37

We may now turn our discussion to the final principle of inference listed under D above. Smiley considers transitivity an

36

Ibid., pp. 127-134.

37

Anderson and Belnap, "Entailment", p. 107.

essential factor of a paradox-free system of entailment. He also suggests that transitivity may be fundamental factor for satisfying the call for a 'connection of meanings' (or relevance) in entailment. That is to say, the limiting or restricting of transitivity will count in favor of or correspond to the idea of a connexion of meanings.³⁸

Smiley offers the following definition of entailment:

$A_1, \dots, A_n \vdash B$ if and only if the implication

$A_1 + \dots + A_n \supset B$ is a substitution instance of a tautology

$A'_1 + \dots + A'_n \supset B'$, such that neither $\vdash B'$ nor

$\vdash \sim(A'_1 + \dots + A'_n)$.³⁹

Transitivity in the above view fails in the attempt to establish the middle term between $A \& \sim A \vdash (A \vee B)$ & $\sim A$, along with $(A \vee B)$ & $\sim A \vdash B$. This happens because of the principle of inference which would validate the above, excludes the formation of the middle term. The actual principle would give only, $A \& C \vdash (A \vee B) \& C$ and $(A \vee B) \& \sim A \vdash B$, which have no middle term in common.

In the beginning of this section it was mentioned that there were several renderings for entailment. Depending on one's position one might believe that strict implication and entailment are the same; it has been the aim of previous discussion to cast doubt on this identification. A telling feature of this doubt enters with the 'paradoxes'.

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³⁸

Smiley, p. 237.

³⁹

Ibid., p. 240.

⁴⁰

Ibid., p. 241.

Despite the fact that independent arguments are offered, plausible alternatives have been suggested which eliminate the 'paradoxical' principles. As can be seen, these alternatives, as in the cases developed by Smiley, do so by eliminating certain inferences by conditioning the entailment relation on a new definition. This method employs the strategy of admitting or establishing the essential features desired for the entailment relation. The various schemes establish a definition of entailment in terms of such concepts as logical consequence and derivability.

The definitions of entailment cited above, which are attributable to Smiley, center upon the concept of logical consequence. In such an approach the conditions are selected in accordance with their ability to refine the truth preserving character of entailment.⁴¹ This character can be found in the Lewis systems as the impossibility for $(A \bullet \sim B)$ when A is said to entail B. This condition is undoubtedly necessary but that it is not sufficient, is more primarily the problem. At this stage one might turn to the concept of derivation for a possible solution. Can we add in some way, to the truth preserving features of logical consequence derivational features to arrive at a satisfying account of entailment? It would seem quite to the point to explicate the derivational features of entailment. This certainly appeals to the concept of 'follows from', in an entailment, such that a satisfactory entailment relation will give; that if A entails B then

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Ibid., p. 234.

B follows from A or perhaps more explicitly, B is derivable from A. So where as one might be reluctant to admit that A entails B is the same as A strictly implies B, the suggestion that if A entails B then B follows from or is derivable from A, is easier to accept. But will such a format, though very general, accommodate an acceptable explication of relevance, if the concept of relevance is insisted upon?

The criticism of disjunctive syllogism and logical addition as given in the independent proof discussion above, by Anderson and Belnap, makes clear that the call for relevance is not only apparent in light of the 'paradoxes', but likewise occurs as grounds for refuting the independent proof offered by Lewis and Langford. The answer then is perhaps, that to properly explicate the concept of entailment a suitable yet also rigorous statement of 'relevance' must be integrated into the derivational syntax. But how would such a method compare to those used by Smiley above? The first point to be observed is that although Smiley recognizes the need to satisfy the demand for a 'connection of meanings' or 'relevance' he subjugates this property to the conditions of transitivity. Secondly, Smiley sees fit to establish his system of entailment upon a truth-functional base. The problem with such a technique is evident in his attempt given above, in which the method used to remove the 'paradoxical' laws does so by filtering out certain truth-functional forms.

That the subsequent systems are awkward is undeniable. Yet the basic idea of building the non-extensional modal logic on an extensional system was fundamental to Lewis' approach. It might just be

the case that such a method will not leave room for an adequate expression of entailment. Furthermore, if a system of entailment can be established which satisfies the intuitive demands for relevance and the necessity of logical consequence, and yet is distinct from the system of strict implication then we should expect this to lend support to such a criticism. Likewise the position that the 'paradoxes' are but mere freaks is similarly weakened. If it can be shown that the 'paradoxical' principles are the results of an inadequate expression of entailment, then they will be certainly more than mere freaks.

CHAPTER IV

THE SYSTEM "E" OF ENTAILMENT

1. The Development of System E:

The system to be presented in what follows is a system of "entailment," which solves the problem of the paradoxes of implication. It does so in virtue of a rigorous and specialized sense of implication. The first point to be considered is that, whereas the horseshoe symbol was used to represent material implication and the fishhook for strict implication, the arrow " $\rightarrow$ " will be used to represent the logical relation of entailment. Further, it will be argued that the relation of entailment gives a proper statement of the fundamental philosophical problem of implication. In the course of such a statement, the paradoxical features associated with material and strict implication as given by certain theorems will be found to be unprovable and in fact, rational grounds for rejecting both material and strict implication as kinds of implication at all. It is the opinion of this writer, however, that this does not render the systems previously discussed exercises in futility; on the contrary, what is meant is that it is perhaps simply a mistake to call such relations implication from a philosophical point of view.

The system "E" for entailment was formulated by Alan Ross Anderson and Nuel D. Belnap. In this system, it is argued that there are two essential ingredients for a proper system of entailment. One condition is that there be in any entailment $A \rightarrow B$, a necessary

connection between antecedent and consequent. It will be found that the development of necessity as used in the system of Lewis and Langford discussed earlier is sufficient. On the other hand, there is the condition that if A entails B , then B is relevant to A .

At stake here is the philosophical importance of logical consequence. We should ask ourselves, in this connection, what features we require to employ this concept properly, so as to give proper expression to the above mentioned conditions, while at the same time eliminating the undesirable features in certain alternative systems. The most obvious result yields an interpretation for such key concepts as "follows from", "implies", "is deducible from", "therefore", etc., in our logical locutions about our deductive apparatus. The formula formed $A \supset (B \supset A)$, which can be interpreted if A is true, then A follows from any arbitrary B , is acceptable in the system of material implication; it is not acceptable in the system E . The very language in question is unacceptable, in so far as a true A does not follow from any arbitrary contingent B in any proper sense of "follow from".

In what follows, the system E will be presented with attention especially on the development of the deductive apparatus. In the course of this development, rigorous expression will be given to both necessity and relevance.

1. Deductive Method

The basic method for proof procedure will be a variant of natural deduction. When appropriate, axiomatic versions will be given also. The use of natural deduction techniques is a specialized

method of proof procedure established by stipulating certain rules. Since the particular problem here concerns the implicational features, only these rules will be given. That is to say, the other connections will be omitted, i.e., the conjunction and disjunction as well as the negation sign. Both brackets and "dots" will be used in the formation of formulas. Dots are to be entered by convention from the left.

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The following rules are to be used:

- A. Hypothesis (Hyp.) is the assumption put forth in either the beginning or in a subordinate proof.
- B. Repetition (Rep.), allows the repeating of premises in the same proof.
- C. Reiteration (Reit.), allows the repeating of premises into subproofs.
- D. Entailment Introduction ($\rightarrow$ I), allows the assertion of $A \rightarrow B$ when B is deducible from A in the proof scheme.
- E. Entailment Elimination ($\rightarrow$ E), allows the inference of B from A when we have $A \rightarrow B$.

A brief discussion and a few examples will clarify matters. We may first observe the very simple proof for the law of identity.

1	A	hyp
2	A	1 rep
3 $A \rightarrow A$ 1-2 $\rightarrow$ I		

This simple proof exemplifies the basic use of hypothesis, repetition, and entailment introduction. Anderson and Belnap point out

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Anderson and Belnap, "Entailment", p. 80-84.

the rules given above give a set of theorems which are equivalent to the intuitionist propositional calculus of Heyting 1930, which consists of the following two axioms with entailment elimination.

$$H1 \quad A \rightarrow \cdot B \rightarrow A$$

$$H2 \quad (A \rightarrow \cdot B \rightarrow C) \quad \cdot A \rightarrow B \rightarrow \cdot A \rightarrow C$$

H1 is the familiar paradox and is proved by the above five rules as follows:

1.	A	hyp.
2.	B	hyp.
3.	A	1 reit.
4.	B $\rightarrow$ A	2-3 $\rightarrow$ I
5.	A $\rightarrow$ $\cdot$ B $\rightarrow$ A	1-4 $\rightarrow$ I

Anderson and Belnap point out that this is not an instance of entailment, "For if A is contingent, then $A \rightarrow \cdot B \rightarrow A$ says that an entailment $B \rightarrow A$ follows from or is deducible from a contingent proposition - in defiance of the condition that formal considerations alone validate valid inferences."⁴³ Essentially, the above formula in no way gives any acceptable form of implication. The fact that one uttering such a formula will not produce a falsehood does not justify the use of "implies" in this formula nor any acceptable sense of deducibility of A from B.

Anderson and Belnap at this point attempt to modify the rules so that formulas like the above mentioned will be omitted. The missing element is necessity. That is, the modification must guarantee that logical truths shall be necessary and not contingent, "These considerations suggest that we should be allowed to import into a

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Ibid., p. 86.

deduction (i.e., into a subproof by reiteration) only propositions which, if true at all, are necessarily true: i.e., we should reiterate only entailments."⁴⁴

The system so modified, called by the authors $S4_I^*$, is equivalent to the axiomatic formulation, called by the authors $S4_I$. These axioms form the implicational fragment of Lewis's $S4$.⁴⁵

$$S4_I1 \quad A \rightarrow A$$

$$S4_I2 \quad (A \rightarrow \cdot B \rightarrow C) \rightarrow \cdot A \rightarrow B \rightarrow \cdot A \rightarrow C$$

$$S4_I3 \quad A \rightarrow B \rightarrow \cdot C \rightarrow \cdot A \rightarrow B$$

The system so modified, however, has its own problems as was discussed in previous chapters. The problem being that any necessarily true proposition is entailed by any irrelevant proposition. $B \rightarrow \cdot A \rightarrow A$ is derived from the above by axiom $S4_I1$ and $S4_I3$. An alternate derivation from the modified rules gives:

$$\begin{array}{lll} 1 & \vdash & B \quad \text{hyp} \\ 2 & \vdash & A \rightarrow A \quad \text{axiom} \\ 3 & & B \rightarrow \cdot A \rightarrow A \quad 1-2 \rightarrow 1 \end{array}$$

In the example, the derivation is valid by the rules given. It is unconvincing however, that $A \rightarrow A$ follows, in any interesting sense, from B . Anderson and Belnap suggest that the above proof fails to

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⁴⁴ *Ibid.*, p. 87

⁴⁵

⁴⁵ *Ibid.*, p. 88

take seriously the word "from" in "proof from hypotheses." The method used to overcome this difficulty centers on a technique which would allow the keeping track of premises and their usage, "what is wanted is a system for which there is a provable deduction theorem, to the effect that there exists a proof of B from the hypothesis A if and only if $A \rightarrow B$ is provable."⁴⁶

A subscripting procedure is used to keep track of the premises and to mark their entry into usage. The relevance indices are thereby entered and discharged much the same as proofs and subproofs use the " $\rightarrow$ I" rule. The rules which follow, insure that if $A \rightarrow B$ then A is relevant to B.⁴⁷

1. Each hypothesis is given a distinct numerical subscript k, where k is different than any other previous A (k)
2. From A_a and $(A \rightarrow B)_b$ to infer $B_a \wedge b$
3. From a proof of B_a from hypothesis $A_{(k)}$, where k is in a, to infer $A \rightarrow B_a \wedge (k)$
4. Reiteration and repetition retain subscripts.

The following proof of transitivity is offered as example:

1.	$A \rightarrow B_1$	hyp
2.	$C \rightarrow A_2$	hyp
3.	C_3	hyp
4.	$C \rightarrow A_2$	2 reit.
5.	$A_{3,2}$	4, 3 $\rightarrow$ E

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⁴⁶

Ibid., p. 90.

⁴⁷

Ibid., p. 92.

6.	$A \rightarrow B_1$	1 reit.
7.	$B_{1,2,3}$	$6,5 \rightarrow E$
8.	$\cdot C \rightarrow B_{1,2,}$	$3,7 \rightarrow I$
9.	$C \rightarrow A \rightarrow \cdot C \rightarrow B_1$	$2,3-8 \rightarrow I$
10.	$A \rightarrow B \rightarrow \cdot C \rightarrow A \rightarrow \cdot C B$	$1,2-9 \rightarrow I$

An axiomatic counterpart was developed by Alonzo Church and called "weak positive implicational propositional calculus." The axioms are:⁴⁸

R_I^1	$A \rightarrow A$
R_I^2	$A \rightarrow B \rightarrow \cdot C \rightarrow A \rightarrow \cdot C \rightarrow B$
R_I^3	$(A \rightarrow \cdot B \rightarrow C) \rightarrow \cdot B \rightarrow \cdot A \rightarrow C$
R_I^4	$(A \rightarrow \cdot A \rightarrow B) \rightarrow \cdot A \rightarrow B$

Anderson and Belnap call this system R_I , while the natural deduction version is referred to as R_I^* . The R , of course, is meant to signify relevance.

In either case, the condition of relevance is given formal expression in an acceptable fashion. That this notion of relevance is philosophically acceptable will be considered in what follows. The system R_I^* does have its problems such that it is not properly an entailment system. For in this form a very obvious modal fallacy occurs. The S systems of modal logic discussed in earlier chapters define necessity in terms of logical truth: that is, necessary

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ibid., p. 90.

truths are tautological truths. The system R_1^* ignores such consideration of necessity by the following theorem.

1.		A_1	hyp	
2.			$A \rightarrow A_2$	hyp
3.			A_1	1 reit
4.			$A_{1,2}$	2,3 E
5.		$\cdot A \rightarrow A \rightarrow A_1$	2 - 4	$\rightarrow I$
6.		$A \rightarrow \cdot A \rightarrow A \rightarrow A$	1 - 5	$\rightarrow I$

Its counterpart in the S systems would be $p \rightarrow p$ which is invalid.

$\cdot A \rightarrow A \rightarrow A$ is given by Anderson and Belnap as the definition of necessity. The meaning is that if A is necessary, then A follows from a logical truth. The authors are quick to point out that from a formal point of view, the definition of necessity as $NA = \cdot A \rightarrow A \rightarrow A$ is equivalent to those in the systems of Lewis and Langford.⁴⁹

The most obvious results suggests that the condition of necessity be combined with the relevance condition. This is precisely the method of arriving at the system E.

The system is obtained by combining the restriction for reiteration of S_4T^* and the subscripting technique of R_1^* . The rules so modified are:⁵⁰

1. Hyp. A step may be introduced as the hypothesis of a new subproof, and each new hypothesis receives a unit class $\{K\}$ of numerical subscripts, where k is new.
2. Rep. A_n may be repeated, retaining the relevance indices n .

⁴⁹Anderson and Belnap, "Entailment," p. 95.

⁵⁰

Ibid., p. 93.

3. Reit. $(A \rightarrow B)_a$ may be reiterated, retaining a .
4. $\rightarrow$ E. From A_a and $A \rightarrow B_b$ to infer B_{ab} .
5. $\rightarrow$ I. From a proof of B_a on hypothesis $A_{\{k\}}$ to infer $A \rightarrow B_{a-\{k\}}$, provided k is in a .

The following proofs are given for inspection.

Specialized Assertion

1.	$A \rightarrow B$	hyp.
2.	$A \rightarrow B \rightarrow A \rightarrow B$	hyp.
3.	$A \rightarrow B$	1 reit.
4.	$A \rightarrow B$	2, 3 $\rightarrow$ E
5.	$A \rightarrow B \rightarrow A \rightarrow B \rightarrow A \rightarrow B$	2-4 $\rightarrow$ I
6.	$A \rightarrow B \rightarrow N(A \rightarrow B)$	1-5 $\rightarrow$ I

Transitivity (sufficing)

1.	$D \rightarrow B$	hyp.
2.	$B \rightarrow C$	hyp.
3.	D	hyp.
4.	$D \rightarrow B$	1 reit.
5.	B	3, 4 $\rightarrow$ E
6.	$B \rightarrow C$	2 reit.
7.	C	5, 6 $\rightarrow$ E
8.	$D \rightarrow C$	3-7 $\rightarrow$ I
9.	$B \rightarrow C \rightarrow D \rightarrow C$	2-8 $\rightarrow$ I
10.	$D \rightarrow B \rightarrow B \rightarrow C \rightarrow D \rightarrow C$	1-9 $\rightarrow$ I

Replacement of the Middle

1.	$D \rightarrow B$	hyp.
2.	$B \rightarrow C$	hyp.
3.	D	hyp.
4.	$D \rightarrow B$	1 reit.
5.	B	4, 3 $\rightarrow$ E
6.	$B \rightarrow C$	2 reit.
7.	C	5, 6 $\rightarrow$ E
8.	$D \rightarrow C$	3-7 $\rightarrow$ I
9.	$B \rightarrow C \rightarrow D \rightarrow C$	2-8 $\rightarrow$ I
10.	$A \rightarrow B \rightarrow C$	hyp.
11.	A	hyp.
12.	$A \rightarrow B \rightarrow C$	10 reit.
13.	$B \rightarrow C$	11, 12 $\rightarrow$ E
14.	$B \rightarrow C \rightarrow D \rightarrow C$	9, reit.
15.	$D \rightarrow C$	13, 14 $\rightarrow$ E
16.	$A \rightarrow D \rightarrow C$	11-15 $\rightarrow$ I
17.	$A \rightarrow B \rightarrow C \rightarrow A \rightarrow D \rightarrow C$	10-16 $\rightarrow$ I
18.	$D \rightarrow B \rightarrow (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow D \rightarrow C)$	1-17 $\rightarrow$ I

The third proof, replacement of the middle, is particularly interesting in so far as a form of transitivity must be derived to reach the conclusion in question. It is to be observed in fact, that the proof for transitivity as given in the second proof forms the first nine steps of the third proof. The subscript of the first hypothesis is preserved through to the addition of hypotheses four and five while the second and third hypotheses have been discharged. The first proof embodies the principle that entailments if true are necessarily true.

An axiomatic counterpart exists with the following axiom schema:⁵¹

Entailment.

- E.1 $A \rightarrow A \rightarrow B \rightarrow B$
 E.2 $A \rightarrow B \rightarrow \cdot B \rightarrow C \rightarrow \cdot A \rightarrow C$
 E.3 $(A \rightarrow \cdot A \rightarrow B) \rightarrow \cdot A \rightarrow B$

Conjunction.

- E.4 $(A \wedge B) \rightarrow A$
 E.5 $(A \wedge B) \rightarrow B$
 E.6 $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow \cdot A \rightarrow (B \wedge C)$
 E.7 $NA \wedge NB \rightarrow N(A \wedge B) \quad [NA = df (A \rightarrow A) \rightarrow A]$

Disjunction.

- E.8 $A \rightarrow A \vee B$
 E.9 $B \rightarrow A \vee B$
 E.10 $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B) \rightarrow C$
 E.11 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$

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Alan Ross Anderson, "Some Open Problems Concerning the System E of Entailment", Acta Philosophica Fennica, FASC. XVI (1963), p. 14.

Negation

- E.12 $A \rightarrow \sim A \rightarrow \sim A$
 E.13 $A \rightarrow \sim B \rightarrow B \rightarrow \sim A$
 E.14 $\sim \sim A \rightarrow A$

The above (preceding) fourteen axioms give a version with the truth functional connectives included. The rules of modus ponens and adjunction are employed in this version.

2. Concluding remarks on the system E.

The system E of Anderson and Belnap is designed to explicate the concept of relevance while securing necessity in inference. The authors thereby reject material and strict implication as entailment calculi. The authors have thus attempted to give the conditions for the assertion of $A \rightarrow B$ where the formula is read 'A "entails" B'. A motivating feature is suggested by the following statement.⁵²

The implication $p \supset q$ can be asserted, if and only if we possess a construction r , which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q . In other words, a proof of p , together with r , would form a proof of q . (Heyting).¹

Now, if "the implication of $p \supset q$ can be asserted", means what Heyting says, then the arrow of entailment answer exactly to the notion of "would automatically effect a construction of", whereby "answering exactly" we mean that

$$A \supset B = \text{df. } (\exists r)[r(A \rightarrow B)].$$

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⁵²

Ibid.

The above statement does give an intuitively acceptable statement of what is at stake in the development of the entailment relation. As is seen by the preceeding material a proper concept of relevance was required. For explication of the concept of "relevance" or "connection of meaning" the following two conditions justify an affirmative response, for E, to the question, "Does the formal system establish relevance between antecedent and consequent in an implication?"

1. The subscripting technique expresses that for A to be relevant to B it must be possible to use A in a deduction of B from A.
2. If $A \rightarrow B$ is provable in E then A and B share a variable.⁵³

The first condition is both necessary and sufficient, while the second condition is necessary. The second condition is proved by the following matrices: every axiom takes a designated (+) value for all assignments to its variables and the rules of E preserved this property. If $A \rightarrow B$ is such that A and B share no variable, then there is an undesigned value assignment to $A \rightarrow B$. For example, assign the value $^+1$ to the variables of A while B takes the value $^+2$, then A will be $^+1$ and B will be $^+2$. However, $^+1 \rightarrow ^+2$ takes the undesigned value -3. Hence, if $A \rightarrow B$ then A and B share a variable.⁵⁴

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Anderson and Belnap. "Entailment", pp. 101-103.

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Belnap, Noel D., "Entailment and Relevance", Journal of Symbolic Logic, 1959, pp. 144-145.

$$A \rightarrow B$$

	-3	-2	-1	-0	+0	+1	+2	+3
-3	+3	+3	+3	+3	+3	+3	+3	+3
-2	-3	+2	-3	-2	-3	-3	+2	+3
-1	-3	-3	+1	+1	-3	+1	-3	+3
-0	-3	-3	-3	+0	-3	-3	-3	+3
+0	-3	-2	-1	-0	+0	+1	+2	+3
+1	-3	-3	-1	-1	-3	+1	-3	+3
+2	-3	-2	-3	-2	-3	-3	+2	+3
+3	-3	-3	-3	-3	-3	-3	-3	+3

	$\bar{A}$	NA	MA
-3	+3	-3	-3
-2	+2	-2	-2
-1	+1	-1	-1
-0	+0	-0	-0
+0	-0	+0	+0
+1	-1	+1	+1
+2	-2	+2	+2
+3	-3	+3	+3

$$A \wedge B$$

	-3	-2	-1	-0	+0	+1	+2	+3
-3	-3	-3	-3	-3	-3	-3	-3	-3
-2	-3	-2	-3	-2	-3	-3	-2	-2
-1	-3	-3	-1	-1	-3	-1	-3	-1
-0	-3	-2	-1	-0	-3	-1	-2	-0
+0	-3	-3	-3	-3	+0	+0	+0	+0
+1	-3	-3	-1	-1	+0	+1	+0	+1
+2	-3	-2	-3	-2	+0	+0	+2	+2
+3	-3	-2	-1	-0	+0	+1	+2	+3

$$A \vee B$$

	-3	-2	-1	-0	+0	+1	+2	+3
-3	-3	-2	-1	-0	+0	+1	+2	+3
-2	-2	-2	-0	-0	+2	+3	+2	+3
-1	-1	-0	-1	-0	+1	+1	+3	+3
-0	-0	-0	-0	-0	+3	+3	+3	+3
+0	+0	+2	+1	+3	+0	+1	+2	+3
+1	+1	+3	+1	+3	+1	+1	+3	+3
+2	+2	+2	+3	+3	+2	+3	+2	+3
+3	+3	+3	+3	+3	+3	+3	+3	+3

Designated values +0, +1, +2, +3.

The system E of Anderson and Belnap has been received favorably among contemporary logicians. Most literature on the system shows earnest attempts to find applications for the results obtained. Some critics do however exist. The nature of the criticisms requires two avenues of approach. First there is concern over the omission of disjunctive syllogism. Secondly, there has been concern over the validity of the relevance condition. The criticisms over the rejection of disjunctive syllogism are sufficiently covered in Chapter III.

The relevance condition is discussed by Donald J. Hockney in an article entitled "A Vindication of System E." He gives the following form of the relevance condition:

"If A and B have no variables in common, 'A entails b' is rejected as a theorem of the system."⁵⁵

Critics have attempted to reject this postulate by insisting that the postulate cannot accommodate all entailments. Consider the following case:

"If something is blue, then something is colored."⁵⁶ This is offered as a case which cannot be handled by the above postulate. Hockney points out however, that the relevance postulate is given for a propositional logic, and consequently is not

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Hockney, Donald J., "A Vindication of System E", Logique et Analyse, 1962, p. 480.

56

Ibid., p. 483.

designed to deal with analyzed propositions. There is no reason why a reconstruction of such examples as above cannot be performed prior to the application of the relevance postulate. In fact, to properly extend the use of the entailment sign for predicates logic would seem to necessitate an entire system of definitions to introduce non-logical descriptions. This general method is dealt with by Carnap in Meaning and Necessity.

Robert K. Meyer in "Entailment" has argued that the system E of Anderson and Belnap is fundamentally correct. More particularly, it is the claim that relevance between antecedent and consequent is needed for entailment which is correct.

Meyer is especially impressed with the development of 'relevant implication,' (see previous chapter). To accent what the author sees to be the more important issues of relevance he develops a logic of irrelevance.' This mock-serious system allows any arbitrary 'p' to be a theorem. In this system the author points out that all our "logical intuitions" break down. Analogously, our social intuitions break down over the inference of an irrelevant p from any contradiction whatsoever. The system of irrelevance produces inconsistency and so is apparently useless but systems which incorporate the paradoxes although they do not produce inconsistency are not harmless.⁵⁷

Underlining the problem is the inadequacy of the conditional

57

Meyer, Robert K., "Entailment", The Journal of Philosophy, 1971, p. 810.

relation. For Meyers a system may either "express" or "indicate" entailment. A connective indicates entailment when it is held metalogically true that A entails B if and only if $A \rightarrow B$ is a theorem. A system "expresses" entailment if and only if $A \rightarrow B$ means that B is a logical consequence of A. Meyers analysis rests upon the semantical tool of "metavaluation." "Briefly, a metavaluation V for a logic L is simply a function from sentences of L to T, F that respects truth-functional connectives in the usual way, but which has the property that $V(A \rightarrow B) = T$ if and only if $A \rightarrow B$ is a theorem of L. A logic is coherent if all its theorems come out true on all metavaluations; coherence, in view of the remarks just made, appears to be the least that one would expect of a logic that purports to express entailment."⁵⁸ The system E is coherent, while classical truth in functional logic is not.

The heart of the problem still remains making $A \rightarrow B$ true on any interpretation which makes A false. Meyers in fact points out the associated problem of counterfactuals mentioned in the next chapter. The results seem to be the exclusion of a counterfactual logic at the outset.

Further credibility has been given to Anderson and Belnap's 'E' by the work of Kenneth W. Collier.⁵⁹ In his paper "Physical Modalities" and the system E, Collier has attempted to integrate Von Wright's binary modalities with the entailment forms of E. Collier

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Ibid., p. 815.

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Collier, Kenneth W., "Physical Modalities and the System E", Notre Dame Journal of Symbolic Logic, 1970. p. 185.

has appealed to E to avoid the problems of material implication found in Von Wright's own attempts.

CHAPTER V

CONCLUSION

The purpose of this thesis was to explicate certain fundamental features of implication with special attention to the entailment relation. Insofar as this was done, certain apparent results emerge. In the first place, insight is gained into certain elements of our logical locutions. Secondly, the conditions for the establishment of an intuitively acceptable entailment relation are given. And finally, a constituent of the second result surfaces as a fundamental philosophical-logical issue. That is to say, the problem of the commonality of meaning between antecedent and consequent is satisfied by the concept of relevance, as cited above.

Perhaps the most obvious signs of logic are the most ambiguous. The concept of implication, which is one of the most fundamental concepts of logic, has been the chief concern in this research. Three symbols have been used to represent this relation - they were the material implication sign " $\supset$ ", the strict implication sign " $\rightarrow$ " and the entailment sign, " $\rightarrow$." In each of these, certain peculiar properties are possessed.

Of special significance is the concept of consequence. The conditions for the employment of the above-mentioned symbols limit the use of the respective symbols. The variance of acceptable usage suggests the following observation: the systems discussed

in this research form three classes of formulas given by the respective implication sign employed. The first system, given as the system of material implication, contains such tautologies as $p \supset (q \supset p)$. The second system, the system of strict implication, is a stronger system of implication and does not contain the analogue of $p \supset (q \supset p)$ which would be $p \rightarrow (q \rightarrow p)$. The third system of implication, the system of entailment, differs from the second in a similar fashion. That is to say, such formulas as $\vdash p \rightarrow (q \rightarrow p)$ do not have a provable analogue.

Another point of greater significance surrounds the paradoxical features of the first two classes of implication. It is unnecessary to reiterate here what has already been discussed in the appropriate sections earlier, but what needs to be mentioned is the variance in usage of certain logical locutions. When we say 'A follows from B' we should keep in mind the proper context, if we have an implicational scheme in mind. This does not mean one must have a set of properties say, F_1, F_2, F_3 for the different degrees or senses of implication. The point is simply, that even the advanced student of logic should not forget what the elementary texts of logic say of "fallacies of relevance."

Any attempt to incorporate meaning into logical formalism, as with the case of the calculus of entailment, must come head to head with the vast range of problems clustered around such a concept, as meaning. The analysis of meaning has been recently a common subject of inquiry in many philosophical circles. We could not here survey

all the possibilities and theories and furthermore, need not. But some remarks are indeed called for, since the entailment relation discussed earlier, employs relevance as an essential ingredient. A rudimentary feature of concern, and often objection, centers around the possibility of 'meaning' or 'meanings' which have a timeless, unchanging determinacy. We become faced with the task of establishing the basic form or at least the mode, by which any formal representative of meaning can be assimilated into our intellectual activity. It is to be acknowledged that the above use of the terms 'meaning,' 'relevance', etc. presents many difficulties because of the terms' vagueness, as such, in general discourse but the relevant points pertinent to entailment will become clearer in what follows.

Often such a topic begins by stating that such a subject is dependent on the nature of a proposition. How for instance, does a proposition display, show, or in some way capture the so-called "sense" of a sentence? Indeed, if the propositions are to be related, as in the case of antecedent and consequent for the entailment relation, then one should rightly expect that this relation would be accounted for in logical theory. The analysis, however, of propositional logic proceeds by the employment of unanalyzed propositional variables and the use of truth-functional propositional connectives. The proposition in this context, is concerned with the expression of something true, or false. There is then, a need to clarify how truth and falsity relate to the proposition and likewise to the above mentioned use of 'sense.'

In the body of this research, the notion of a tense model of modal logic was touched upon. The tense model was formed by employing logical modal systems between S4 and S5 with differing interpretations for both M and L. In such a model the concept of changing truth values often appears. The question then is whether changing truth values is incompatible with the nature of a proposition and further whether changing truth values alters the "sense" of a statement? The possibility of the alteration of sense appears excluded at the outset. If the sense, whatever this could mean, was to change over time, we would be involved in a regress of interpretation. For if the sense changed, this change would presuppose a change in time: time, however, is the original factor. The determinacy of what we are to mean by sense, is presupposed independently of, and perhaps logically prior to the influence of time. Since we are concerned with the nature of a proposition, it becomes obvious that the relations between sense and Tense should be accounted for in a theory about the nature of a proposition.

Whatever we are to mean then by sense, it must be determinate in time. The model which allows the values of propositions to change in time must account for the value assignment of a proposition at one point in time to assume the status of an alternative assignment. The direction suggested then is that the sense on the one hand is determinate, while the relations of propositions with respect to truth value may vary. These relations might, of course, be only suggestive; they might be purely fictitious. Changing truth value does not mean that some logical law may move from the realm of necessity to

possibility; this would surely be preposterous. What is at stake here is rather the possibility of a logic, for instance, of counterfactual conditionals. In such a logic, the value assignment is contrary to the factual occurrence of events, but the hypothetical still suggests possible relations. If meaning is assumed to be determinate, then alternative events should still produce ~~d~~eterminate relations. If it is not irrational to consider alternatives in the future, then should we expect it to be irrational to consider them in the subsequent tense changes? Here we have a theory that allows propositions to change value in time; yet given this operation (or deny this operation), the proposition functions according to rules in tense logic. A logic of counterfactuals would seem to suggest that if the nature of a proposition is receptive to such a logic, then tense conditions of this sort are pertinent to the proposition itself, or at least its employment!

A real problem or objection at this point is perhaps given from the nature of a proposition. Is the proposition which has changed value identical with the original proposition? If the proposition is not identical in both occurrences, then the issue of changing truth values would appear to be excluded at the outset. What would count as a criterion for identity is then the issue. The signs, that is the words are certainly the same with the sole exception of copular tense changes. What actually causes the supposed change in value is the factual state. The factual state is not, however, the proposition but only what is represented.

The conditions for identity are perhaps dependent on the form of the proposition. The form of the proposition, it seems, resolves itself into subject and predicates, etc. Consider the case in which there is a change in state of a process, such that the process terminates: at the time t_1 , p is assigned the value T, at t_2 , p is assigned the value F. A logic to deal with such a change would certainly require p , at both t_1 and t_2 , to be considered the same proposition. What such a logic must account for is in fact the mechanics. The resolution of questions of identity actually finds expression in the establishment of conditions or rules for the system. This, however, concerns rules for the relations which a proposition may enter into.

What then shall give a unified theory for entailment? On the one hand, there are serious problems to resolve from the philosophical foundations of logic. On the other hand, certain mechanical operations seem to be undeniable. The answer seems to center upon methodological considerations. The logical theory must be philosophically acceptable and yet be formally rigorous. In the process, the nature of a proposition reveals itself by showing the various types of relations which it may enter into. One immediate aim is to clarify the conception of 'sense' as a meaningful term about meaning. It might be helpful to speak about sentences as opposed to propositions to clarify matters. In this way, following the method of Carnap, (see below), one can speak of the proposition as the 'sense' or intension of the sentence, while leaving value assignment open as the extension.

When the proposition is considered as something besides an unanalyzed propositional variable, the nature of the proposition begins to show itself. Truth is a key point of concern for propositional theory. The concepts of truth, although closely allied to the problems of the nature of a proposition, maintain an independence, as a subject of philosophical inquiry. The variance of the conception of truth allows the subject to be discussed with qualification. There is for instance, 'truth' as an entity, i.e., as in the case in which someone might identify the Deity with truth. On the other hand, there are more restricted usages such as truth value assignments for propositions. And further, there are methodological considerations which give truth as something dependent on determinate conditions of meaning. Such conditions may give rules of truth, for some systems and may allow for such distinctions, as necessary truth on the hand, and factual truth on the other.

This view is suggestive of several of the views of Rudolph Carnap. Fundamental to Carnap's method as given in Meaning and Necessity, is the conception that properties, concepts, and qualities, etc., are real and independent of mental processes.⁵⁶ Furthermore, such 'entities' are expressed in language. A similar analysis applies to proposition, especially concerning the objective status of propositions. This method of semantical analysis is developed by considering meaning as something which is constructed from elementary components. Carnap's view is nothing especially new, for

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56

Rudolf Carnap, Meaning and Necessity, (Chicago: University of Chicago Press, 1970), p. 71.

it is, in fact, an adaptation of several of Frege's methods. What Carnap has done, however, is to replace the traditional technique of the 'name relation,' with his own method of 'intension and extension.' The meaning of any compound sentence is understood as a function of component parts. The component parts themselves are logical constructions, which are analyzed into simpler elements likewise given in terms of extension and intension. The extension of a sentence is its truth value, while the intension of a sentence is the proposition.⁵⁷ Likewise, the extension of an expression would be the 'individual' designated while the intension of an expression would be the concept.⁵⁸

The rigor of Carnap's method suggests a direction to a problem mentioned earlier. The notion that meaning is determinate is perhaps best approached through semantical methods. Support is given for this view in so far as all basic elements are given an interpretation. Especially strong is the notion that a proposition shall maintain an objective status. What is particularly strong in such a view is that the proposition assumes the status of a basic building block of intellection. The motive is to give an organized framework to refer to a basic feature of propositions; the feature is, of course, that propositions are the most acceptable vehicle of thought. The proposition is in a way residual with respect to sentences. For a sentence must presuppose the context of a language system and yet the proposition is abstracted from the sentence.

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⁶⁰ *Ibid.*, p. 27.

⁶¹ *Ibid.*, p. 19.

The proposition maintains an independent status and yet it enters into a variety of logical relations. The sentence which gives the proposition as its intension may from the extensional side, for instance, yield a multivalued system.⁵⁹

Carnap actually calls his method the method of 'logical analysis' in which ambiguous vague expressions are made more precise and rigorous. Any notion thus modified is done so, by 'explication.' Carnap's method, as a method in logic, embraces semantical analysis. The semantical method can be contrasted with what is usually called syntax. Hintikka, whose own semantical methods are especially noteworthy for their logical economy, has suggested that an over-dependence on syntactical methods will lead to unnecessary limitation on achievement. "The methods best suited to increase conceptual clarity are here, as in many other areas of logic, the semantical ones. (in the sense of the term in which it has been applied to Carnap's and Tarski's studies.)"⁶⁰ Hintikka proceeds in clarifying by suggesting, that it is more fruitful to inquire into the conditions of truth for different kinds of sentences. This latter point is, according to Hintikka, representative of the basic method of semantical analysis.⁶¹

In this research, the systems which employed material and strict implication, relied on deductive and axiomatic methods which are syntactical in character. The deductive operation of these systems

⁶² See Prior's treatment of multivalued logic especially in 'The Sea-Battle Tomorrow' argument, Prior, A.N., Formal Logic, 1962, pp. 240-250.

⁶³ Jaakko Hintikka, Models for Modalities, (New York: The Humanities Press, 1969), p. 23.

⁶⁴ Ibid., p. 24

produced theorems which have been open to question. But they have been questionable, for our concern in certain points of the logical relation of implication. It may be the case that the logical structure given in these systems has some acceptable application. The call for a stronger sense of implication, i.e., the case in which the concept of relevance is desired, can be achieved. In the system E above, the inadequacies of the earlier systems are avoided by certain modifications. The relevance concept is established by the notion of indices in the subscripting technique, and by the incorporation of a notion of commonality of meaning, (see conditions given in Chapter IV, part 2). The latter is particularly interesting, in so far as, it appeals to semantical features. The semantics of the latter concerns entailment, "conceived of as a relation of logical consequence,...since it has to do with possible assignments of values to propositional variables."⁶² In system E, the truth of an 'entailment' $A \rightarrow B$, is dependent on both a specific deductive structure and the inclusion of a concept of relevance. The latter concept is given in terms of the sharing of a propositional variable between A and B. Finally, in so far as the vague notion of "commonality of meaning" is made more precise, it has, in the tradition of Carnap, been explicated.

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⁶⁵ Anderson and Belnap, "Entailment", p. 104.

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