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# TWO VARIANTS OF COPS AND ROBBERS WITH ASYMMETRIC MOVEMENT RULES

Eric Peterson
*University of Rhode Island*, epeterson11492@my.uri.edu

TWO VARIANTS OF COPS AND ROBBERS WITH ASYMMETRIC

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ERIC M. PETERSON

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ERIC M. PETERSON

APPROVED:

Dissertation Committee:

Major Professor    William Kinnersley

Michael Barrus

Edmund Lamagna

Nasser H. Zawia

DEAN OF THE GRADUATE SCHOOL

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**ABSTRACT**

This dissertation presents two variants of *Cops and Robbers* that feature asymmetric movement, meaning that the cops and robber have different rules for traversing the edges of the graph. The first variant discussed is the *bridge-burning variant*, wherein the game is played on an undirected graph and the robber deletes any edge he traverses. The second variant discussed is the *weak directed variant*, wherein the game is played on a directed graph; here the cops can only move along each edge in the direction it points, whereas the robber can traverse each edge in either direction. In both variants, we will be looking at the *cop numbers* for well-known classes of graphs.

# ACKNOWLEDGMENTS

some way during my graduate studies. This thesis is dedicated to the memory of Dr. Paul Wenger, my advisor during my Master of Science program at RIT. Dr. Wenger was my first mentor as a mathematics student, and I will never forget the many life lessons he was able to impart to me during our time together.

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# DEDICATION

In memory of Dr. Paul Wenger. Thank you for everything.

## PREFACE

This thesis will be presented in the manuscript format. Chapter 1 is the introduction, which will present fundamental definitions used in the main chapters of the thesis. Chapter 2 is the first manuscript, which was submitted for publication to Journal of Combinatorics in December, 2018. Chapter 3 is the second manuscript, which will be submitted soon to Graphs and Combinatorics.

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# CHAPTER 1

## Introduction

This thesis will focus on *Cops and Robbers*, a well-known pursuit-evasion game in graph theory. We begin by outlining some basic definitions and theorems that will be used throughout the study.

A *simple graph* (or just *graph*) is a structure consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, such that each edge is a pair of distinct vertices. Unless stated otherwise, we assume a graph is *undirected*, meaning that each edge is an unordered pair of vertices. It is convention to refer to an edge pair without using "$\{\,,\,\}$"; for example, we might denote the edge joining vertices $u$ and $v$ as $uv$. A *subgraph* of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we use the notation $H \subseteq G$ to state that $H$ is a subgraph of $G$. In a graph $G$, a *clique* is a subgraph of $G$ with all possible edges. We say that two distinct vertices are *adjacent* or *neighbors* if they belong to the same edge, and a vertex is *incident on* an edge if it belongs to that edge. The two vertices that belong to an edge are called the edge's *endpoints*. The *degree* of a vertex is the number of edges containing that vertex. In his analysis of the celebrated Seven Bridges of Könisberğ problem, Euler laid the groundwork for the Handshaking Lemma, one of the most fundamental theorems of graph theory [5].

**Theorem 1.0.1.** *(Handshaking Lemma) Every finite graph has an even number of vertices with odd degree.*

Graphs are often used to model networks, with vertices representing the nodes in the network and edges representing the connections between nodes; as such, the concept of travel through a network can be modeled as movement from one vertex to another along edges of the graph. A *walk* is a sequence of vertices $v_1, v_2, ..., v_n,$

not all necessarily distinct, such that for each $1 \leq i \leq n-1$, there is an edge joining $v_i$ and $v_{i+1}$; a *closed walk* is a walk whose first and last vertices are the same. More specifically, a *path* is a walk with no repeated vertices and a *cycle* is a closed walk with no repeated vertices (except for the first and last). We say that a graph $G$ is *connected* if, for every pair of distinct vertices $u, v \in V(G)$, there exists a path that starts at $u$ and ends at $v$; a graph is *disconnected* if it is not connected.

Some results of this thesis pertain specifically to various classes of graphs. A *tree* is a connected graph containing no cycles. In a tree, a *leaf* is a vertex whose degree is 1, and each leaf's neighbor is its *parent vertex*. The *path graph on $n$ vertices*, denoted $P_n$, is the graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_1 v_2, ..., v_{n-1} v_n\}$. The *cycle graph on $n$ vertices*, denoted $C_n$, is the graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_1 v_2, ..., v_{n-1} v_n, v_n v_1\}$. The *complete graph on $n$ vertices*, denoted $K_n$, is the graph consisting of $n$ vertices and all possible edges. The *$n$-hypercube graph*, denoted $Q_n$, is the graph consisting of $2^n$ vertices corresponding to unique binary $n$-tuples where two vertices are adjacent if and only if they differ in exactly one coordinate. We will sometimes want to consider a particular way of drawing a particular graph in the plane where each vertex is represented by a point and each edge $uv$ is represented by a curve that starts at vertex $u$ and ends at vertex $v$; this is called an *embedding* of the graph. A *planar* graph is one that can be embedded in the plane such that any two edges intersect in the embedding only at a vertex that is an endpoint for both edges. Such an embedding divides the plane into non-overlapping regions called *faces*. A planar graph is *outerplanar* if it can be embedded in the plane such that each vertex lies along the unbounded face. We will also be examining *Cartesian products* of graphs. Given two graphs $G$ and $H$, the *Cartesian product of $G$ and $H$*, denoted $G \,\square\, H$, is the graph where:

- $V(G \square H) = V(G) \times V(H)$.

- Two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either $u_1 = u_2$ and $v_1$ and $v_2$ are adjacent in $H$ or $v_1 = v_2$ and $u_1$ and $u_2$ are adjacent in $G$.

The second manuscript heavily features *directed* graphs in which each edge is an ordered pair of distinct vertices. In a directed graph, an edge from a vertex $u$ to a vertex $v$ is denoted $\overrightarrow{uv}$ (here, we might say the edge $\overrightarrow{uv}$ is *directed away from* $u$, and is *directed towards* $v$). An *oriented graph* is a directed graph $G$ such that for each pair of vertices $u$ and $v$, if there is an edge from $u$ to $v$, then there is no edge from $v$ to $u$. We refer to a vertex's *in-degree* as the number of edges directed towards that vertex and its *out-degree* as the number of edges directed away from that vertex. A *source* is a vertex with an in-degree of 0; a *sink* is a vertex with an out-degree of 0. We can also think of travel in directed graphs. A *directed walk* is a sequence of vertices $v_1, v_2, ..., v_n$, not all necessarily distinct, such that for each $1 \leq i \leq n - 1$, there's an edge directed from $v_i$ to $v_{i+1}$; a *directed closed walk* is a directed walk whose first and last vertices are the same. More specifically, a *directed path* is a directed walk with no repeated vertices and a *directed cycle* is a directed closed walk with no repeated vertices (except for the first and last). A *directed acyclic graph* is a directed graph that contains no directed cycles (note that directed acyclic graphs are therefore oriented graphs). An *arborescence* is a directed graph with a root vertex $u$ such that, for each vertex $v$, there is exactly one directed path from $u$ to $v$. The *directed path graph on n vertices*, denoted $\overrightarrow{P_n}$, is the graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{\overrightarrow{v_1v_2}, ..., \overrightarrow{v_{n-1}v_n}\}$. The *directed cycle graph on n vertices*, denoted $\overrightarrow{C_n}$, is the graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{\overrightarrow{v_1v_2}, ..., \overrightarrow{v_{n-1}v_n}, \overrightarrow{v_nv_1}\}$. A *tournament graph on n vertices* is an oriented graph $G$ consisting of $n$ vertices such that for each pair of distinct vertices $u, v \in V(G)$, either $\overrightarrow{uv} \in E(G)$ or $\overrightarrow{vu} \in E(G)$.

The main focus of this thesis is *Cops and Robbers*, a game used to model pursuit and evasion on a graph $G$. The standard variant involves two players; one player controls a finite number of *cops* (we will refer to the cops using the pronouns she/her/hers) and the other player controls a *robber* (we will refer to the robber using the pronouns he/him/his). The cops and robber all start on vertices of $G$, with the cops choosing their initial positions first and then the robber choosing his initial position; throughout the game the cops and robber move from vertex to vertex along edges of $G$. The game is played in *rounds*; a round consists of the cops taking their turn and then the robber taking his turn. During a cop turn, each cop either moves to an adjacent vertex or remains on her current vertex; during a robber turn, the robber makes the same decision. We say that the cops win the game if some cop ends up on the same vertex as the robber during a player's turn; in this case, we say that the particular cop *captures* the robber. The robber wins the game if he has a strategy to perpetually avoid all capture from cops. For a graph $G$, the *cop number* of $G$, denoted $c(G)$, is the minimum number of cops required to ensure that the cops always win the game on $G$. A graph $G$ where $c(G) = 1$ is said to be *cop-win*. We can also discuss the length of a game of Cops and Robbers in terms of the number of rounds. The *$k$-capture time of $G$*, denoted $\text{capt}_k(G)$, is the minimum number of rounds needed for $k$ cops to win the game on $G$ (where $c(G) \leq k$); the *capture time of $G$*, denoted $\text{capt}(G)$, is the $k$-capture time where $k = c(G)$.

The game of Cops and Robbers was first introduced in the early 1980s by Quilliot [10] and, independently, by Nowakowski and Winkler [9]. In both of these papers, the authors presented a classic characterization of cop-win graphs. The idea for the cop number of a graph was proposed shortly thereafter by Aigner and Fromme [1]. Since its initial examination, many different variants of the game have

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been analyzed. Some of these variants include: the cops and robbers using different edge sets [8], the robber's location not being fully known by the cops [3, 4], and the robber having the ability to move faster than the cops during each round [6, 7, 11]. A standard topic of study in each variant is to analyze the cop number for various classes of graphs. For additional background on Cops and Robbers, see [2].

In this thesis, we introduce two variants of Cops and Robbers that explore asymmetric movement of the cops versus the robber. In Chapter 2, we introduce *bridge-burning Cops and Robbers* (for undirected graphs) in which the robber deletes each edge he traverses. We start by looking at the cop number in this model (denoted $c_b(G)$ for a graph $G$) for paths, cycles, and complete graphs, as well as providing a general upper bound based on the graph's domination number (we will properly define *domination number* in Chapter 2). We then provide a polynomial-time algorithm for computing $c_b(T)$ when $T$ is a tree. We also look at $c_b$ for the Cartesian product of two path graphs, the Cartesian product of two cycle graphs, and $n$-hypercube graphs. The chapter ends with a brief analysis of the capture time for this variant.

In Chapter 3, we introduce *weak directed Cops and Robbers*. In this variant, the game is played on a directed graph where the cops may only traverse each edge in the direction it points; the robber, however, can traverse each edge in any direction he chooses. We start by looking at the cop number in this model (denoted $c_w(G)$ for a directed graph $G$) for directed paths, directed cycles, and tournaments, as well as providing a general upper bound based on the graph's domination number. We then provide a full characterization of cop-win oriented graphs in this variant and a sufficient condition for general directed graphs to be cop-win. We also look at $c_w$ for the Cartesian product of two arborescences, the Cartesian product of two cycles, the Cartesian product of three path graphs, and

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both outerplanar and planar graphs.

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# CHAPTER 2

## Cops, Robbers, and Burning Bridges

William B. Kinnersley and Eric Peterson

Department of Mathematics

University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

**Abstract**

We consider a variant of Cops and Robbers wherein each edge traversed by the robber is deleted from the graph. The focus is on determining the minimum number of cops needed to capture a robber on a graph $G$, called the *bridge-burning cop number* of $G$ and denoted $c_b(G)$. We determine $c_b(G)$ exactly for several elementary classes of graphs and give a polynomial-time algorithm to compute $c_b(T)$ when $T$ is a tree. We also study two-dimensional square grids and tori, as well as hypercubes, and we give bounds on the capture time of a graph (the minimum number of rounds needed for a single cop to capture a robber on $G$, provided that $c_b(G) = 1$).

## 2.1 Introduction

The game of *Cops and Robbers* is a well-studied model of pursuit and evasion. Cops and Robbers is played by two players: one controls a team of one or more *cops*, while the other controls a single *robber*. The cops and robber all occupy vertices of a graph $G$ and take turns moving from vertex to vertex. At the outset of the game, each cop chooses her initial position on $G$, after which the robber does the same. (Multiple cops may occupy a single vertex simultaneously.) Thereafter, the game proceeds in *rounds*, each consisting of a cop turn and a robber turn. On the cop's turn, every cop may either remain in place or move to a neighboring vertex; on the robber's turn, he may do the same. The cops win if some cop ever occupies the same vertex as the robber, at which time we say that cop *captures* the robber. Conversely, the robber wins if he can perpetually avoid capture. The cops and robber know each others' positions at all time.

Many variants of Cops and Robbers have been studied, each modeling pursuit and evasion in a slightly different context. For example, the robber may move faster than the cops [6, 7], or the cops may have only partial information about the robber's location [4, 5], or the two players may have different sets of edges

available to them [11]. In these variants, one typically seeks to determine the minimum number of cops needed to capture a robber on a graph $G$. In the usual model of Cops and Robbers, this quantity is deemed the *cop number* of $G$ and denoted $c(G)$. For more background on Cops and Robbers, we refer the reader to [3].

In this paper, we introduce and study a variant of Cops and Robbers wherein the robber, after traversing an edge, deletes that edge from the graph. For example, perhaps the edges of our graph represent bridges joining various regions, and the robber burns each bridge as he passes over it, denying its future use both to the cops and to the robber. (We require that the robber always burns every edge he uses; he may not elect to leave an edge intact.) Aside from this change, the rules are the same as in the usual model of Cops and Robbers. We refer to this game as *bridge-burning Cops and Robbers* and define the *bridge-burning cop number*, denoted $c_b(G)$, to be the minimum number of cops needed to capture a robber on $G$ in this model.

In general, $c(G)$ and $c_b(G)$ are not directly comparable, and the relationship between the two can be surprising. As the bridge-burning game wears on, the robber deletes more and more edges from $G$ and thus has fewer escape routes. Hence one might expect that generally $c_b(G) \leq c(G)$, and indeed, sometimes this is the case. However, in the usual model of Cops and Robbers, the robber can only play defensively, while in the bridge-burning game, he can adopt an offensive tack: if the robber can disconnect the graph and leave himself in a different component from all of the cops, then he wins. Thus sometimes $c_b(G) > c(G)$, since there must be enough cops to capture the robber before he can pull off this feat.

In this paper, we investigate the bridge-burning game on a variety of graph classes on which the usual model of Cops and Robbers is well-understood. In

Section 2.2, we determine the bridge-burning cop numbers of paths, cycles, and complete graphs. We also generalize the elementary bound $c(G) \leq \gamma(G)$ by giving an upper bound on $c_b(G)$ in terms of the domination number of $G$. In Section 2.3, we give a polynomial-time algorithm to compute $c_b(T)$ when $T$ is a tree. In Section 2.4, we examine square grids and tori. Theorem 2.4.3 states that when $G$ is a $2 \times n$ grid, $c_b(G) = \lceil \frac{n+2}{9} \rceil$, while Theorems 2.4.9 and 2.4.11 state that when $G$ is an $m \times n$ square grid or torus, $\frac{mn}{121} \leq c_b(G) \leq (1 + o(1))\frac{mn}{112}$. We also show in Theorem 2.4.12 that the bridge-burning cop number of the $n$-dimensional hypercube, $Q_n$, is always 1. Finally, in Section 2.5, we briefly consider the concept of *capture time* – the number of rounds needed for a single cop to win on a graph with bridge-building cop number 1. Theorem 2.5.1 shows that among all $n$-vertex graphs $G$, the capture time of $G$ is $O(n^3)$, while Theorem 2.5.2 shows that there exist $n$-vertex graphs having capture time $\Omega(n^2)$. Finally, in Section 3.5, we suggest some directions for future research.

## 2.2 General Bounds

We begin with elementary observations about the bridge-burning game, starting with the value of $c_b(G)$ on several elementary classes of graphs.

**Proposition 2.2.1.**

(a) $c_b(K_n) = 1$ *for all* $n$.

(b) $c_b(C_n) = 1$ *for* $n \geq 3$.

(c) $c_b(P_n) = \begin{cases} 1, & \text{if } n \leq 5 \\ 2, & \text{otherwise} \end{cases}$

*Proof.*

(a) To capture a robber on $K_n$, the cop starts on an arbitrary vertex; no matter where the robber starts, the cop can capture him on her first turn.

(b) On $C_n$, the cop starts on an arbitrary vertex and simply moves closer to the robber on each turn. Once the robber has taken his first step (and hence burnt the corresponding edge), he finds himself at one endpoint of a path. Subsequent moves by the robber only serve to shorten this path, so eventually the cop will reach him.

(c) For $n = 2$, the claim is trivial.

For $3 \leq n \leq 5$, label the vertices of the path $v_1, v_2, \ldots, v_n$ in order. The cop begins on $v_3$. If the robber begins the game on or adjacent to the cop, then the cop wins on her first turn. Otherwise, it must be that the robber begins at one endpoint of the path. On her first turn, the cop moves toward the robber; she is now adjacent to the robber. The robber cannot leave his current vertex, since that would result in immediate capture; however, if he remains in place, then the cop captures him on her next turn. Either way, the cop wins.

For $n \geq 6$, we claim that two cops are necessary and sufficient to capture the robber. It is clear that two cops can capture the robber: one begins at each endpoint, and on each turn they both move closer to the robber. To see that two cops are necessary, we give a strategy for the robber to avoid capture by a single cop. Label the vertices of the path $v_1, v_2, \ldots, v_n$ in order. Since $n \geq 6$, at least one of $v_2$ and $v_{n-1}$ must not be adjacent to the cop's initial position; by symmetry suppose this is true of $v_2$. The robber begins the game on $v_2$. By assumption the cop cannot capture him on her first turn. On the robber's first turn, he moves to $v_1$, thereby burning edge $v_1 v_2$. Now

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$v_1$ is isolated, so the cop can never reach the robber.

$\square$

Note that even on such elementary graphs, the cop number and the bridge-burning cop number can differ: for $n \geq 3$ we have $c(C_n) = 2$ but $c_b(C_n) = 1$, and for $n \geq 6$ we have $c(P_n) = 1$ but $c_b(P_n) = 2$. We will see later (in Theorem 2.3.1) that the difference between $c(G)$ and $c_b(G)$ can be arbitrarily large.

The argument in the proof of Proposition 2.2.1(c) suggests a natural heuristic strategy for the robber: attempt to move in such a way that he ends up in a different component from every cop. Certainly if the robber accomplishes this, then he wins. However, this is not the only way for the robber to win. For example, in the graph shown in Figure 1, the robber can evade a single cop by causing a stalemate. If the cop begins on $v$ or $y$, then the robber can safely begin on $x$; on his first turn the robber moves to $z$ and wins. Likewise, if the cop begins on $x$ or $z$, then the cop begins on $v$ and subsequently moves to $y$. Thus the cop must begin on $u$ or $w$; suppose without loss of generality that she begins on $u$. The robber now begins on $w$. If the cop moves to $v$, then the robber can move to $x$ and subsequently to $z$, thereby winning the game. Likewise, if the cop moves to $x$, then the robber can move to $v$ and from there to $y$. Thus the cop's only reasonable option is to remain at $u$; the robber responds by remaining at $w$. The robber wins if the cop ever leaves $u$, so the cop must remain at $u$ perpetually and thus cannot capture the robber.

It is well-known that in the usual model of Cops and Robbers, we have $c(G) \leq \gamma(G)$, where $\gamma(G)$ denotes the minimum size of a dominating set – that is, a set $S \subseteq V(G)$ such that every vertex in $G$ either belongs to $S$ or has a neighbor in $S$. It is clear that $c_b(G) \leq \gamma(G)$, since placing one cop on each vertex of a dominating set allows the cops to win on their first turn. However, in the context of the
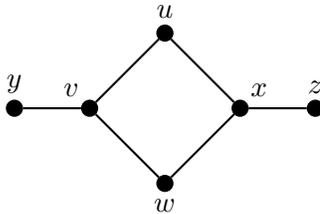
Figure 1.

bridge-burning game, we can strengthen this bound.

**Theorem 2.2.2.** *If there exist cliques $S_1, S_2, \ldots, S_k$ in $G$ such that $S_1 \cup S_2 \cup \cdots \cup S_k$ is a dominating set of $G$, then $c_b(G) \leq k$.*

*Proof.* Let $S_1, S_2, \ldots, S_k$ be cliques whose union dominates $G$, and let $S = S_1 \cup S_2 \cup \ldots S_k$. We show how $k$ cops can capture a robber on $G$. Label the cops $c_1, \ldots, c_k$, and let each cop $c_i$ begin the game on any vertex in $S_i$; throughout the game, she will remain in $S_i$. For $i \in \{1, \ldots, k\}$, the robber cannot start on any vertex in $S_i$ without being captured immediately by $c_i$. Instead, the robber must start on some vertex not in $S$. The cops now play as follows.

Suppose the robber currently occupies vertex $v$. Since $S$ dominates $G$, vertex $v$ must be adjacent to some vertex in $S$, say $u$; suppose $u \in S_i$. Cop $c_i$ moves to $u$, while every other cop remains on her current vertex. The robber cannot remain on $v$ without being captured, and he cannot move to a vertex in $S$, so on his next turn, he must flee to another vertex not in $S$. Since $G$ has only finitely many edges, the robber cannot flee forever; he will eventually be captured. $\square$

**Corollary 2.2.3.** *For all $m$ and $n$, we have $c_b(K_{m,n}) = 1$.*

*Proof.* In $K_{m,n}$, any pair of adjacent vertices forms a dominating set; the result now follows from Theorem 2.2.2. $\square$

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## 2.3 Trees

In this section, we study the bridge-burning game on trees. In the usual model of Cops and Robbers, trees are easy to analyze: it is well-known that for every tree $T$, we have $c(T) = 1$. In the bridge-burning model, things are more complicated; in fact, there exist trees with arbitrarily large bridge-burning cop number. Below, we give a polynomial-time algorithm to determine the bridge-burning cop number of a tree. The key idea underlying the algorithm is the same as that behind Proposition 2.2.1(c): if the robber can safely start on some vertex adjacent to a leaf, then on his next turn he can isolate himself on the leaf and thus win. To prevent this, the cops must ensure that after their initial placement, each leaf is within distance 2 of at least one cop.

We say that a leaf of $v$ of a tree is *guarded* if some cop begins the game within distance 2 of $v$ and *unguarded* otherwise. A cop within distance 2 of $v$ is said to *guard* $v$. Within the next result, we utilize the fact that, for each pair of distinct vertices $u$ and $v$ in a tree graph $T$, there is a unique path from $u$ to $v$.

**Theorem 2.3.1.** *Let $T$ be a tree. Consider the following algorithm:*

1. *Choose an arbitrary root $r$ for $T$.*

2. *Out of all unguarded leaves of $T$, let $v$ be one furthest from the root.*

3. *If $v = r$ or $v \in N(r)$, place a cop at $r$; otherwise, place a cop at the grandparent of $v$.*

4. *Repeat steps 2 and 3 until all leaves of $T$ have been guarded.*

*If $N$ denotes the number of cops placed by the algorithm, then $c_b(T) = N$. Moreover, this algorithm can be executed in polynomial time.*

*Proof.* It is clear that the algorithm can be executed in polynomial time, so we need only show that $c_b(T) = N$. We first show that $c_b(T) \geq N$, i.e. that we need

at least $N$ cops to capture a robber on $T$. If some leaf $v$ of $T$ is unguarded, then the robber can begin the game on some neighbor of $v$ and, on his first turn, move to $v$. This puts the robber on the isolated vertex $v$, so the cops can never capture him. Hence, the cops must ensure that after their initial placement, every leaf of $T$ is guarded. We claim that this requires at least $N$ cops.

Consider the leaf $v$ chosen in the first iteration of step 2 of the algorithm. If $v = r$ or $v \in N(r)$, then all unguarded leaves are within distance 1 of $r$, so placing a cop on $r$ guards all leaves. Suppose instead that $v$ is at least distance 2 from $r$. Let $u$ and $t$ be the parent and grandparent of $v$, respectively. The cops must ensure that some cop guards $v$, which requires placing a cop at $t$, $u$, or some child of $u$. By choice of $v$, no child of $u$ has any unguarded leaves as descendants. Consequently, any unguarded leaf that would be guarded by a cop at $u$ or some child of $u$ would also be guarded by a cop at $t$. Thus, some optimal cop placement (i.e. one that guards all leaves using the fewest possible cops) places a cop at $t$, just as the algorithm does. Repeating this argument, we see that each cop placed by the algorithm is placed optimally with respect to guarding the chosen unguarded leaf, and hence the algorithm produces an optimal cop placement. This completes the proof that $c_b(T) \geq N$.

To show that $c_b(G) \leq N$, we argue that the cops can always capture a robber starting from the initial cop placement produced by the algorithm. Since every leaf is guarded, if the robber starts on a neighbor of a leaf, then the cops can capture him on their first turn. If instead he starts on a leaf, then some cop can move to the neighbor of that leaf, thereby trapping him, and the cops can capture him on their next turn.

Suppose the robber starts on any other vertex in $T$. Each turn, every cop moves one step closer to the robber (if possible). Initially, for every leaf in $T$, the

unique path between the robber and that leaf contains at least one cop. We claim that after every robber turn, the path between the robber and any leaf in the same component of $T$ contains a cop. This property is maintained by the cops' strategy, so we need only consider what happens on the robber's turns. Suppose that it is the robber's turn and that the property holds. If the robber remains on his current vertex, then the property still holds. Otherwise, the robber's move either takes him toward or away from any given leaf; in the former case the path from the robber to that leaf still contains a cop, and in the latter case the robber and leaf are now in different components. Since there must always be a leaf of $T$ in the robber's component, there must always be a cop in the robber's component, and hence the cops eventually win. □

## 2.4   Grids and Hypercubes

In this section, we investigate the bridge-burning game played on two-dimensional square grids and tori. As with trees, these are graphs on which the bridge-burning model is much more difficult to analyze than the standard model. It is known that every two-dimensional grid has cop number at most 2 (see [10]) and every two-dimensional torus has cop number at most 3 (see [8]), but a s we will show, there exist grids and tori having arbitrarily large bridge-burning cop number.

The $m \times n$ *square grid*, which we denote $G_{m,n}$, is the Cartesian product of the paths $P_n$ and $P_m$; the $m \times n$ *square torus*, denoted $T_{m,n}$, is the Cartesian product of $C_n$ and $C_m$. We view the vertex sets of both graphs as the set of ordered pairs $(i,j)$ with $0 \le i \le n-1$ and $0 \le j \le m-1$. For fixed $i$, we say that vertices of the form $(i,k)$ are in *column i* of the grid; similarly, those of the form $(\ell, i)$ are in *row i*. Note that both $G_{m,n}$ and $T_{m,n}$ have $m$ rows and $n$ columns; a vertex's column is indexed by its first coordinate, while its row is indexed by the second coordinate.

We say that vertex $(i, j)$ is *to the left* of $(i', j')$ if $i < i'$ and *to the right* if $i > i'$. Similarly, $(i, j)$ is *above* $(i', j')$ if $j < j'$ and *below* if $j > j'$. When a player moves from vertex $(i, j)$ to $(i + 1, j)$, we say they *move right*; likewise, when they move to $(i - 1, j)$, $(i, j + 1)$, or $(i, j - 1)$ we say that they *move right, move down,* or *move up,* respectively.

We begin with $2 \times n$ grids. In this setting, a robber who starts near the left or right ends of the grid has somewhat more power than one who starts in the middle: the ends of the grid contain vertices of low degree, which makes it easier for the robber to isolate himself. The cops can prevent this by stationing cops "close enough" to the ends of the grid; the following lemma formalizes this idea.

**Lemma 2.4.1.** *Consider the game played on $G_{2,n}$. If a cop starts in column $j$ where $1 \leq j \leq 3$ and the robber starts to the left of the cop, then the cop can capture the robber. Similarly, if a cop starts in column $k$ where $n - 4 \leq k \leq n - 2$ and the robber starts to the right of the cop, then the cop can capture the robber.*

*Proof.* First, we suppose the cop starts on vertex $(3, 0)$ and the robber starts to the left of the cop; a symmetric argument suffices for cops starting on vertices $(3, 1)$, $(n - 4, 0)$, or $(n - 4, 1)$ with the robber to the left, right, or right, respectively. On her first turn, the cop always moves left to $(2, 0)$. Henceforth, the cop plays as explained in the cases below.

- **Case 1:** the robber starts on $(2, 0)$. The cop captures him immediately on her first turn.

- **Case 2:** the robber starts on $(1, 0)$. If the robber remains on $(1, 0)$ on his first turn, then the cop captures him on her next turn. If the robber moves left to $(0, 0)$, the cop moves left to $(1, 0)$. Regardless of the robber's next move, the cop then moves down to $(1, 1)$ and the robber is trapped, so the

cop wins. If instead the robber moves down to $(1,1)$ on his first turn, then the cop moves down to $(2,1)$. The robber must move left to $(0,1)$ to avoid capture. The cop moves back up to $(2,0)$ and traps the robber, ensuring her win.

- **Case 3:** the robber starts on $(0,0)$. Regardless of the robber's first move, the cop moves left to $(1,0)$ on her second turn. If the robber moves right to $(1,0)$ on his first turn, then the cop captures him on her subsequent turn. If the robber instead chooses to stay on $(0,0)$ on his first turn, then on his second turn, he must move down to $(0,1)$ to avoid capture. The cop can now move down to $(1,1)$ and trap the robber. If the robber moves down to $(0,1)$ on his first turn, then he must remain on $(0,1)$ or move right to $(1,1)$ on his second turn; in either case, the cop moves down to $(1,1)$ and either traps or captures the robber.

- **Case 4:** the robber starts on $(2,1)$. Since the cop moves to $(2,0)$ on her first turn, the robber now must move either left or right to avoid capture on the cop's ensuing turn. For the remainder of the game, on each turn, the cop moves horizontally into the same column as the robber. Consequently, on the ensuing robber turn, the robber must continue moving horizontally in the same direction to avoid capture. Since the graph is finite, the robber cannot keep this up forever, so eventually the cop wins.

- **Case 5:** the robber starts on $(1,1)$. If the robber moves right to $(2,1)$, then the cop captures him on her next turn. If the robber remains on $(1,1)$, then the cop moves left to $(1,0)$. The robber must now move either horizontally on his next turn to avoid capture; the cop can now capture him using the

strategy described in Case 4. If the robber moves left to $(0, 1)$, then the cop

moves left to $(1, 0)$ and traps the robber. In any case, the cop wins.

- **Case 6:** the robber starts on $(0, 1)$. Regardless of the robber's first move,
  the cop moves left to $(1, 0)$ on her second turn. If the robber moves right to
  $(1, 1)$ on his first turn, then on his next turn, he must move left or right to
  avoid capture; once again, the cop can now capture him using the strategy
  given in Case 4. If the robber moves up to $(0, 0)$, then he must remain on
  $(0, 0)$ on his next turn, after which the cop moves left to capture him. If
  the robber remains on $(0, 1)$ after his first move, then after the cop's second
  move, the robber must again remain on $(0, 1)$ to avoid capture. The cop now
  moves down to $(1, 1)$, forcing the robber to move up to $(0, 0)$; the cop moves
  up to $(0, 1)$ and traps the robber.

This establishes the claim for the case where the cop starts in columns 3 or

$n - 4$; similar arguments suffice if the cop begins in columns 1 or 2 (or $n - 2$ or

$n - 3$). □

We next consider how to deal with a robber who begins in the middle of the

grid, far from the edges. The cops must be sure not to leave too large of a "gap"

between cops, lest they give the robber enough freedom to isolate himself.

**Lemma 2.4.2.** *Consider the game on $G_{2,n}$, and suppose two cops start in the same*

*row at a distance of $k$ columns apart where $k \leq 9$. If the robber starts between them,*

*then the cops can capture him.*

*Proof.* By giving a cop strategy, we show that regardless of the robber's strategy,

he cannot win unless the separation between cops is at least 10 columns. The

general strategy for the cops will be to move horizontally toward the robber; we

give a full specification below. First, we claim that the robber must make at least two vertical moves in order to win.

Suppose the robber has a winning strategy using fewer than two vertical moves. Note that if the robber ever moves to a vertex directly above or below a cop with the corresponding vertical edge intact, then that cop captures him immediately. If the robber only moves horizontally, then he is eventually captured by one of the cops he starts between. If the robber makes only one vertical move, then he enters a row with all of its edges intact, and as before, the cops can now trap the robber from both sides. Thus, the robber cannot win without making at least two vertical moves.

We now detail the cops' strategy. Let $c_\ell$ (respectively, $c_r$) denote the cop that starts to the left (respectively, the right) of the robber. Without loss of generality, assume both cops start in row 0. In most circumstances, $c_\ell$ and $c_r$ both move horizontally inward towards the robber on every turn, and move vertically only when the robber is directly below them. Exceptions to this are as follows:

- If moving horizontally would cause a cop to enter a vertex with no vertical edge, then the cop instead moves vertically to the other row and continues moving horizontally towards the robber for as long as possible.

- If the robber starts in row 0 and, on his first turn, moves horizontally away from one of the cops, then that cop subsequently moves down to row 1 and henceforth continues moving horizontally toward.

- If the robber starts in row 0 and his first four moves are to cycle back to his starting vertex (meaning that he has moved up, down, left, and right in some order), then only one cop remains in the robber's component. From this point onward, this cop plays as in Lemma 2.4.1, supposing that the she

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and the robber have each taken a single turn, and the robber's first move was vertical. (We will argue below that by the time the robber returns to his starting vertex, the remaining cop is no more than three columns away, so she may indeed employ the strategy in Lemma 2.4.1.)

- If the robber starts in row 1 and moves up on his first turn, then both cops move down in response and, henceforth, move horizontally inward towards the robber on each turn.

We claim that when the cops employ this strategy, the robber can avoid capture only if the cops start at least 10 columns apart. We consider several cases. We may clearly assume that the robber moves to an adjacent vertex on his first turn, since remaining in place only allows the cops to move closer to each other (and to the robber).

**Case 1:** the robber starts in row 0, initially moves horizontally $k_1$ times $(k_1 \geq 0)$, and then moves down. Without loss of generality, assume that the robber initially moves right.

(a) Suppose $k_1 = 0$ (so the robber moves down on his first turn). If the robber remains on this vertex for the remainder of the game, then he clearly loses, so suppose without loss of generality that he eventually moves right. The robber must eventually return to row 0 in order to win, and he can do so no earlier than his third turn. During this time, the cops have moved horizontally inward on each turn. Thus, before returning to row 0, the robber must be at least 5 columns from each cop's starting position to avoid capture by that cop; the claim now follows.

(b) Next suppose $k_1 \geq 1$, and suppose that after moving down, the robber moves right $k_2$ times before moving up (where $k_2 \geq 1$). In total, the robber has

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returned to row 0 after at least $k_1 + k_2 + 2$ turns and in doing so has moved $k_1 + k_2$ columns to the right of his starting position. To avoid capture by $c_r$, who has moved $k_1 + k_2 + 3$ columns to the left during this time, the robber currently must be at least $k_1 + k_2 + 4$ columns away from $c_r$'s starting position; thus he must have started the game at least $2k_1 + 2k_2 + 4$ columns to the left of $c_r$. Additionally, to avoid capture by $c_\ell$, he must have started at least two columns to her right. Consequently, for the robber to avoid capture, the cops must have started at least $2k_1 + 2k_2 + 6$ columns apart, which is at least 10, as claimed.

(c) Finally suppose that $k_1 \geq 1$ and that after moving down, the robber moves left $k_2$ times before moving up (where $k_2 \geq 1$). Since the robber initially moved right, $c_\ell$ has moved right once, down once, and right another $k_1 + k_2$ times during her first $k_1 + k_2 + 2$ turns. Thus, just before moving up, the robber must be at least $k_1 + k_2 + 2$ columns to the right of $c_\ell$'s starting position to avoid capture by $c_\ell$; since the robber is now $k_2 - k_1$ columns to the left of his starting position, he must have started at least $2k_1 + 2$ columns to the right of $c_\ell$. Additionally, to avoid capture by $c_r$ (who moves left on each of his first $k_1 + 1$ turns), the robber must have started at least $2k_1 + 2$ columns to the left of $c_r$. In total, for the robber to avoid capture, the cops must start at least $2k_1 + 2k_2 + 4$ columns apart.

The claim now follows unless $k_1 = k_2 = 1$. In this case, the robber moves right, down, left, and up in his first four moves, thereby returning to his starting vertex. During this time, $c_\ell$ has responded by moving right once, down once, and right three times, moving a total of 4 columns closer to the robber. Thus, if $c_\ell$ starts no more than 6 columns to the left of the robber, then $c_\ell$ will be able to capture the him using the strategy outlined in Lemma

2.4.1. Since the robber must also start at least 4 columns to the left of $c_r$ (as mentioned above), the result follows.

**Case 2:** the robber starts in row 1, moves horizontally $k_1$ times ($k_1 \geq 0$), and then moves up. Without loss of generality, assume that the robber initially moves right.

(a) Suppose $k_1 = 0$ (so the robber moves up on his first turn). The robber must eventually return to row 1 in order to win and can do so no sooner than his third turn. During this time, each cop moves down to row 1 and at least three columns closer to the robber. At this point, all horizontal edges in row 1 are intact. Thus, if either cop is now within two columns of the robber, then the cops will trap him regardless of his next move. The claim now follows.

(b) Next suppose that $k_1 \geq 1$ and that after moving up, the robber next moves right $k_2$ times (where $k_2 \geq 1$) before moving down. Note that just before the robber moves down, he has moved $k_1 + k_2$ columns to the right, while $c_r$ has moved $k_1 + k_2 + 2$ columns to the left. Thus the robber must have started at least $2k_1 + 2k_2 + 3$ columns to the left of $c_r$ to avoid capture by $c_r$. Additionally, the robber must have started at least three columns to the right of $c_\ell$, since otherwise $c_\ell$ will capture him immediately after he moves up. In total, for the robber to avoid capture, the cops must start at least $2k_1 + 2k_2 + 6$ columns apart; this is at least 10, as claimed.

(c) Finally, suppose that $k_1 \geq 1$ and that after moving up, the robber next moves left $k_2$ times (where $k_2 \geq 1$) before moving down. During his first $k_1+1$ turns, the robber has moved $k_1$ columns to the right; during her first $k_1 + 2$ turns, $c_r$ has moved $k_1+2$ columns to the left. Thus for the robber to avoid capture

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by $c_r$, he must start at least $2k_1 + 3$ columns to her left. Additionally, just before the robber moves down, he has moved $k_2 - k_1$ columns to the left, while $c_\ell$ has moved $k_1 + k_2 + 2$ columns to the right. Thus the robber must start at least $2k_2 + 3$ columns to the right of $c_\ell$. Once again, for the robber to avoid capture, the cops must start at least $2k_1 + 2k_2 + 6$ columns apart, and the claim follows.

$\square$

We are finally ready to determine $c_b(G_{2,n})$.

**Theorem 2.4.3.** $c_b(G_{2,n}) = \left\lceil \dfrac{n+2}{9} \right\rceil$.

*Proof.* We first show that $c_b(G) \leq \left\lceil \frac{n+2}{9} \right\rceil$ by explaining how this many cops can capture the robber. If $n \leq 7$, one cop suffices: the cop begins in column 3 (or in column $n-1$ if $n \leq 3$) and, by Lemma 2.4.1, has a winning strategy. Otherwise, place one cop in column 3 and one cop in column $n-4$. Next, starting from the cop in column 3, repeatedly place a cop 9 columns to the right of the previous cop until there are at most 9 columns between the cop just placed and the cop in column $n-4$. In total, we have placed $\left\lceil \frac{n-7}{9} \right\rceil$ cops in the first $n-4$ columns in addition to the cop in column $n-4$, for a total of $\left\lceil \frac{n-7}{9} \right\rceil + 1$ cops, which simplifies to $\left\lceil \frac{n+2}{9} \right\rceil$. By Lemmas 2.4.1 and 2.4.2, this cop placement ensures that the cops can capture the robber, hence $c_b(G) \leq \left\lceil \frac{n+2}{9} \right\rceil$.

For the reverse inequality, suppose $k$ cops suffice to win the game. Label the cops $c_1, ..., c_k$ and suppose that cop $c_i$ begins in column $m_i$, with $m_i \leq m_j$ whenever $i < j$.

We first claim that if $m_1 \neq m_2$, then $m_1 \leq 3$. Suppose that $4 \leq m_1 < m_2$ and, without loss of generality, that $c_1$ starts on vertex $(m_1, 0)$. If the robber starts on $(0,0)$, then he can isolate himself on $(0,0)$ by moving right to $(1,0)$, down to

$(1, 1)$, left to $(0, 1)$, and finally up to $(0, 0)$; it is straightforward to verify that $c_1$ cannot reach the robber quickly enough to capture him. (Refer to Figure 2.) This proves the claim, and by symmetry it follows that if $m_{k-1} \neq m_k$, then $m_k \geq n - 4$.

If instead $m_1 = m_2$, then the same robber strategy given in the previous paragraph shows that $m_1 \geq 4$; symmetrically, if $m_{k-1} = m_k$, then $m_k \leq n - 5$.

Next, we claim that for all $i \in \{1, ..., k\}$, if either column $m_i$ or column $m_{i+1}$ contains only one cop, then $m_{i+1} \leq m_i + 9$. Suppose $m_i = j$ and $m_{i+1} \geq j + 10$ for some $i$, and suppose by symmetry that column $m_i$ contains only one cop. Without loss of generality, assume $c_i$ starts on vertex $(j, 0)$. The robber can now isolate himself on $(j + 3, 0)$ by starting on $(j + 2, 0)$ and moving right to $(j + 3, 0)$, down to $(j + 3, 1)$, right to $(j + 4, 1)$, up to $(j + 4, 0)$, and left to $(j + 3, 0)$. Note that $c_i$ cannot reach the robber within three moves, and after the robber's third move, $c_i$ no longer occupies the same component as the robber. On the other hand, $c_{i+1}$ starts too far away from the robber to reach him before he has isolated himself. (Refer to Figure 3.)

A similar argument shows that if both columns $m_i$ and $m_{i+1}$ contain two cops, then $m_{i+1} \leq m_i + 10$ (and hence $m_{i+2} \leq m_i + 10$).

To minimize $k$ subject to the constraints established above, we may clearly take $m_1 = 3$ and $m_{i+1} = m_i + 9$ for $1 \leq i \leq k - 1$ (where $m_k$ is reduced to $n - 1$ if needed). This yields $m_k = \min\{n - 1, 3 + 9(k - 1)\}$, which suffices so long as $m_k \geq n - 4$, i.e. $3 + 9(k - 1) \geq n - 4$, or $9k \geq n + 2$. Thus we obtain $k \geq \frac{n+2}{9}$; since $k$ is an integer, in fact $k \geq \lceil \frac{n+2}{9} \rceil$, as claimed. $\qquad\square$

Before proceeding, we remark that Lemma 2.4.2 and an argument along the lines of that used for Theorem 2.4.3 together yield $c_b(P_2 \square C_n) = \lceil \frac{n}{9} \rceil$ for $n \geq 10$; we omit the details.

We next tackle general $m \times n$ grids. As a first step toward this goal, we will
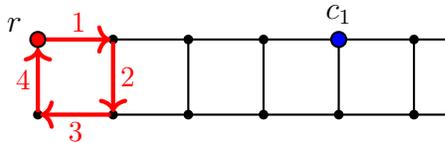
Figure 2. Robber strategy at corner for Theorem 2.4.3.
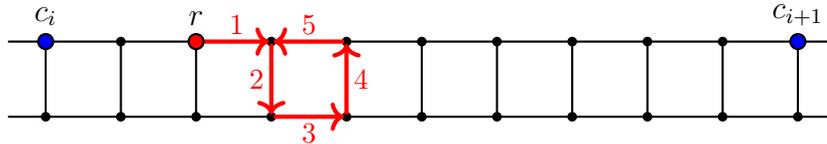


Figure 3. Robber strategy between cops for Theorem 2.4.3.

actually consider $m \times n$ tori, since the analysis is simpler and uses many of the same techniques we will use for grids. We begin by building up sufficient conditions for the cops to win on $T_{m,n}$. Our first lemma actually applies to any graph in which all vertices have even degree, so it may be useful for graphs other than $T_{m,n}$.

**Lemma 2.4.4.** *Let $G$ be a graph in which every vertex has even degree. If at any point any cop can reach the robber's starting vertex, then the cops can capture the robber.*

*Proof.* Consider the game played on $G$, and suppose some cop $c$ reaches the robber's starting vertex $v$. For the remainder of the game, the cop plays as follows:

- If the robber does not move closer to $v$, then the cop moves closer to the robber.

- If the robber moves closer to $v$, then the cop moves closer to $v$.

Note that the cop ensures that she is never further from $v$ than the robber, so the robber can never safely return to $v$. We now show that this strategy does in fact enable the cop to capture the robber.

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The cop's strategy ensures that the robber cannot remain on the same vertex indefinitely, so long as there is some path joining the cop and the robber. Thus, the only way for the robber to avoid capture is to disconnect $G$ and wind up in a component that contains no cops. We claim that the cop guarding $v$ is always in the same component as the robber, hence the robber cannot perpetually escape capture.

The cop's strategy ensures that there is always a path from her current position to $v$, so it suffices to show that the robber is always in the component containing $v$. Each vertex in $G$ initially has even degree, and when the robber passes through a vertex, he deletes two edges incident to that vertex. Thus, at all points in the game, every vertex has even degree except perhaps for $v$ and the robber's current position, $u$. Since the cop's strategy prevents the robber from returning to $v$, we must have $u \neq v$. Thus, the graph has exactly two vertices of odd degree, namely $u$ and $v$. Since every component of the graph must contain an even number of vertices with odd degree, $u$ and $v$ must belong to the same component, as claimed. $\qquad\square$

We next apply Lemma 2.4.4 to establish a simpler sufficient condition for the cops to win on $T_{m,n}$.

**Lemma 2.4.5.** *For the torus $T_{m,n}$ with $m \geq n$, if some cop starts the game within distance 5 of the robber, then that cop can capture the robber.*

*Proof.* Suppose cop $c$ starts the game within distance 5 of the robber. By symmetry we may assume that $c$ begins at vertex $(0,0)$, while the robber begins at some vertex $(i,j)$ with $0 \leq j \leq i$ and $i + j \leq 5$. We consider five cases (see Figure 4). In the cases below, it will be helpful to note that if $c$ can reach some neighbor of the robber's starting vertex (with the edge between the two vertices still intact), then she can either capture the robber or reach his starting vertex.

- **Case 1:** $(i, j) \in \{(0,0), (1,0), (2,0), (3,0), (1,1), (2,1)\}$. If $(i, j) = (0,0)$, then $c$ captures the robber immediately. Otherwise, she moves to the right on her first move; from here it is straightforward to verify that she can either capture the robber or reach his starting vertex, regardless of the robber's strategy.

- **Case 2:** $(i, j) = (2, 2)$. This time, $c$ moves right on her first turn and down on her second. On her next turn she can move either right to $(2, 1)$ or down to $(1, 2)$; at least one of these two vertices must still be adjacent to $(2, 2)$, so she can either capture the robber or reach his starting vertex.

- **Case 3:** $(i, j) \in \{(4, 0), (1, 3)\}$. In this case, $c$ moves right on her first two turns, and again she can either capture the robber or reach his starting vertex. (This is easy to verify by inspection unless the robber began on $(4, 0)$ and moved left, then down on his first two turns; in this case $c$ can reach the robber's starting vertex by moving right, up, right, and down on her next four turns.)

- **Case 4:** $(i, j) = (3, 2)$. This time, $c$ moves right on her first two turns and down on her third. Once again she can either capture the robber or reach his starting vertex by way of either $(3, 1)$ or $(2, 2)$, since the robber has not taken enough turns to delete the edges joining each of these vertices to his starting vertex.

- **Case 5:** $(i, j) \in \{(5, 0), (4, 1)\}$. This time, $c$ moves right on her first three turns and again she can capture the robber or reach his starting vertex, similarly to Case 3.

In any case, $c$ can capture the robber, as claimed. $\qquad\square$

Figure 4. Cases for Lemma 4.

Our next sufficient condition for the cops to win is somewhat technical, but quite powerful.

**Lemma 2.4.6.** *In the game on $T_{m,n}$, suppose the robber begins on some vertex $(i,j)$. If some cop $c_1$ begins the game within distance 7 of $(i-1, j-1)$ and some cop $c_2$ begins the game within distance 7 of $(i+1, j+1)$, or if $c_1$ begins within distance 7 of $(i+1, j-1)$ and $c_2$ begins within distance 7 of $(i-1, j+1)$, then the cops can capture the robber.*

*Proof.* Suppose cops $c_1$ and $c_2$ begin within distance 7 of vertices $(i-1, j-1)$ and $(i+1, j+1)$, respectively; the other case is symmetric. If $c_1$ either begins on $(i-1, j-1)$ itself or begins both strictly to the right of and strictly below $(i-1, j-1)$, then she is within distance 5 of $(i,j)$ and can capture the robber by Lemma 2.4.5. Thus we may suppose that $c_1$ begins either strictly to the left of or strictly above $(i-1, j-1)$; by symmetry we may assume the former. Likewise, we may assume that $c_2$ begins strictly to the right of $(i+1, j+1)$; a similar argument suffices if she begins strictly below.

By Lemma 2.4.4, it suffices to show that at least one cop can reach $(i,j)$. To this end, cop $c_1$ first attempts to travel either up or down to row $j-1$, then right to $(i-1, j-1)$. Once she has reached $(i-1, j-1)$, she attempts to reach $(i,j)$

via either $(i-1, j)$ or $(i, j-1)$, provided that one of these paths remains intact. Similarly, $c_2$ attempts to travel left or right to column $i+1$, then up to $(i+1, j+1)$, and from there to $(i, j)$ via either $(i+1, j)$ or $(i, j+1)$.

We claim that the robber cannot prevent both $c_1$ and $c_2$ from reaching $(i, j)$. There are two ways for the robber to thwart $c_1$: he could prevent $c_1$ from reaching $(i-1, j-1)$, or he could allow $c_1$ to reach $(i-1, j-1)$ but prevent her from then reaching $(i, j)$. Preventing $c_1$ from reaching $(i-1, j-1)$ would require visiting some vertex $(i', j-1)$ with $i' \le i-2$. This would take the robber at least three turns and leave him at least three steps left and up from $v$. Since the robber takes at most six turns before $c_2$ reaches $(i+1, j+1)$, he cannot prevent $c_2$ from reaching $(i+1, j+1)$ with both length-2 paths to $(i, j)$ intact. Consequently, $c_2$ can either reach $(i, j)$ or, if the robber attempts to traverse one of these paths, capture him directly. Thus the robber cannot safely prevent $c_1$ from reaching $(i-1, j-1)$, nor (by symmetry) can he prevent $c_2$ from reaching $(i+1, j+1)$.

By the time $c_1$ and $c_2$ reach $(i-1, j-1)$ and $(i+1, j+1)$ respectively, the robber has taken at most six turns, hence at least one of the two cops can reach $(i, j)$ in two steps; at this point the robber cannot prevent that cop from reaching $(i, j)$, since burning an edge along the relevant path would leave the robber on or adjacent to the cop, resulting in his capture. □

We will also need conditions that guarantee a robber win. We begin with a useful lemma that applies not only to tori and grids, but to all graphs.

**Lemma 2.4.7.** *Fix a graph $G$ and positive integers $k$ and $d$, and let $v$ be a vertex of $G$. Let $d_i$ denote the robber's distance from $v$ after his $i$th move in the original graph $G$ (that is, disregarding any edge deletions that may occur during the game). If the robber can play so that $i + d_i < d$ for $0 \le i \le k-1$, and if no cop begins within distance $d$ of $v$, then the cops cannot capture the robber before his $k$th move.*

*Proof.* Consider an arbitrary cop $c$; it suffices to show that $c$ cannot capture the robber before his $k$th move. Let $u$ denote $c$'s starting vertex, and for $i \in \{0, 1, \ldots, k-1\}$, let $v_i$ denote the robber's position after his $i$th move. In order for $c$ to capture the robber on $v_i$, we must have $\text{dist}_G(u, v_i) \le i+1$, since $c$ can take at most $i+1$ steps before the robber leaves $v_i$. However,

$$\text{dist}_G(u, v) \le \text{dist}_G(u, v_i) + \text{dist}_G(v_i, v),$$

and so

$$\text{dist}_G(u, v_i) \ge \text{dist}_G(u, v) - \text{dist}_G(v_i, v) \ge (d+1) - d_i > (d+1) - (d-i) = i+1,$$

so $c$ cannot capture the robber on $v_i$. It follows that $c$ cannot capture the robber before his $k$th move. $\square$

Lemma 2.4.7 leads to a useful sufficient condition for the robber to win on $G_{m,n}$ or $T_{m,n}$.

**Lemma 2.4.8.** *Fix positive integers $m$ and $n$, and consider the game on either $T_{m,n}$ or $G_{m,n}$. If there is some vertex $v$ of degree 4 such that no cop starts within distance 5 of $v$ and at most one cop starts within distance 9, then the robber can win.*

*Proof.* Suppose no cop starts within distance 5 of $(i,j)$ and at most one cop starts within distance 9. If in fact no cops start within distance 9, then the robber can win by starting on $(i,j)$, then moving right to $(i+1,j)$, up to $(i+1, j-1)$, left to $(i, j-1)$, down to $(i,j)$, left to $(i-1,j)$, down to $(i-1, j+1)$, right to $(i, j+1)$, and finally up to $(i,j)$. (Refer to Figure 5.) Let $d_k$ denote the distance from the robber to $(i,j)$ after the robber's $k$th turn. The robber's strategy ensures that for $1 \le k \le 7$, we have $k + d_k < 9$. Since no cop started the game within distance 9 of

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$(i, j)$, Lemma 2.4.7 shows that no cop can reach the robber before his 8th move, at which point he isolates himself.

Suppose now that no cops begin within distance 5 of $(i, j)$ and exactly one cop, $c$, begins within distance 9. Suppose without loss of generality that $c$ begins at vertex $(k, \ell)$, where $k \leq i$ and $\ell \leq j$. The robber now plays as follows. He begins at $(i, j)$, then moves up to $(i, j-1)$, left to $(i-1, j-1)$, down to $(i-1, j)$, and right to $(i, j)$. At this point, he pauses to assess the situation.

Since the initial distance from $c$ to $(i, j)$ was at least 6, but $c$ has taken only five turns, she cannot have yet captured the robber. Moreover, by the assumptions that $k \leq i$ and $\ell \leq j$, she cannot reach $(i+1, j+1)$ in fewer than three steps, and she cannot reach either $(i+1, j)$ or $(i, j+1)$ in two steps. We claim that $c$ cannot be within three steps of both $(i+1, j)$ and $(i, j+1)$ simultaneously. Suppose otherwise. For the cop to be within three steps of both $(i+1, j)$ and $(i, j+1)$ as well as at least three steps for $(i+1, j+1)$, she must occupy a vertex that is within distance 3 of both $(i+1, j)$ and $(i, j+1)$ in the original graph (before any edge deletions) and within distance 4 of $(i+1, j+1)$. There are only five such vertices: $(i-1, j)$, $(i, j+1)$, $(i-2, j)$, $(i-1, j-1)$, and $(i, j-2)$. If $c$ occupies $(i-1, j)$ or $(i-2, j)$ then she cannot reach $(i+1, j)$ within three steps, since the robber has deleted the edge from $(i-1, j)$ to $(i, j)$; likewise, if she occupies $(i, j-1)$ or $(i, j-2)$, then she cannot reach $(i, j+1)$ within three steps, and if she occupies $(i-1, j-1)$ then she cannot reach either vertex within three steps.

Thus, suppose $c$ cannot reach $(i, j+1)$ within three steps (the other case is similar). The robber now moves right to $(i+1, j)$, down to $(i+1, j+1)$, left to $(i, j+1)$, and finally up to $(i, j)$. The cop clearly cannot capture the robber on $(i+1, j)$. She cannot capture the robber on $(i+1, j+1)$, because she takes only two moves before the robber leaves that vertex. Likewise, she cannot capture the

robber on $(i, j+1)$ because she takes only three steps before the robber returns to $(i, j)$. Hence the robber safely returns to $(i, j)$ and, having done so, isolates himself on $(i, j)$.

Thus $c$ cannot capture the robber. Moreover, since no other cop begins within distance 9 of the robber, an argument similar to that used at the beginning of the proof shows that no other cop can capture the robber either. Thus, the robber wins. $\square$
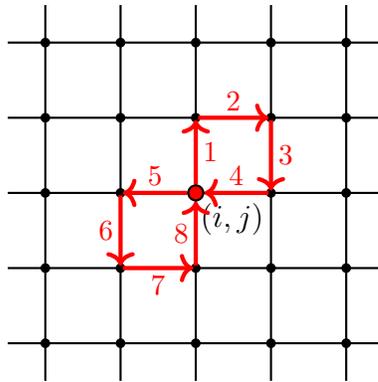


Figure 5. Robber strategy for Lemma 2.4.8.

We are finally ready to establish bounds on $c_b(T_{m,n})$.

**Theorem 2.4.9.** *For all positive integers $m$ and $n$,*

$$\left\lceil \frac{mn}{121} \right\rceil \leq c_b(T_{m,n}) \leq 2 \left\lceil \frac{m}{16} \right\rceil \left\lceil \frac{n}{14} \right\rceil.$$

*Proof.* We begin with the lower bound. Consider an initial cop placement on $T_{m,n}$. We define a weighting function on $V(T_{m,n})$ as follows: a vertex having $k$ cops within distance 5 and another $\ell$ within distance 9 receives weight $k + \ell/2$. If any vertex has weight less than 1, then by Lemma 2.4.8, the robber has a winning strategy. Thus, for the cops to win, every vertex must have weight at least 1, so the sum of the weights of all vertices in $T_{m,n}$ must be at least $mn$. For any vertex $v$ in $T_{m,n}$,

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there are at most 61 vertices within distance 5 of $v$ and at most an additional 120 within distance 9; consequently, each cop's contribution to the total weight is at most $61 + 120/2$, or 121. Thus the total number of cops must be at least $mn/121$, hence $c_b(T_{m,n}) \geq \lceil \frac{mn}{121} \rceil$.

For the upper bound, we give an initial cop placement and claim that regardless of where the robber starts, at least one cop can either capture the robber or reach the robber's starting vertex (at which point she can capture the robber using the strategy outlined in Lemma 2.4.4). For all $0 \leq k < 2\lceil n/16 \rceil$ and $0 \leq \ell < 2\lceil m/16 \rceil$ such that $k + \ell$ is odd, we place a cop at $(7k, 8\ell)$. (Throughout the proof, for any vertex $(i, j)$, we take $i$ modulo $n$ and $j$ modulo $m$ as needed.)

Now consider the $9 \times 8$ block of vertices with upper-left corner $(7k, 8\ell)$ for some $k, \ell$ such that $0 \leq 7k < n$ and $0 \leq 8\ell < m$. Suppose $k + \ell$ is even (the case where $k + \ell$ is odd is symmetric). Cops occupy the lower-left and upper-right vertices of this block, namely $(7k, 8\ell + 8)$ and $(7k + 7, 8\ell)$; denote these cops by $c_1$ and $c_2$, respectively. (Refer to Figure 6.)

We claim that if the robber begins anywhere within this block, then the cops can capture him. Indeed, every vertex in this block satisfies the hypotheses of Lemma 2.4.5 or Lemma 2.4.6, except perhaps for the top-left and lower-right corners, i.e. $(7k, 8\ell)$ and $(7k + 7, 8\ell + 8)$. To show that the cops can capture robbers who begin at these vertices, we consider several cases.

- **Case 1:** the robber starts at $(7k, 8\ell)$ with $k \geq 1$. In this case, there is also a cop $c_3$ at vertex $(7k - 7, 8\ell)$. Now cops $c_2$ and $c_3$ can capture the robber by Lemma 2.4.6.

- **Case 2:** the robber starts at $(7k, 8\ell)$ with $k = 0$, i.e. at $(0, 8\ell)$. Unlike in case 1, this time there need not be a cop at $(-7, 8\ell)$. The cops' strategy ensures that in row $8\ell$, cops appear 14 columns apart. In particular, there

is a cop $c_3$ at vertex $(i, 8\ell)$ for some $i \in \{-7, -6, \ldots, 6\}$. If $i \in \{-7, -6\}$, then cops $c_2$ and $c_3$ can capture the robber by Lemma 2.4.6. If instead $i \in \{-5, -4, \ldots, 5\}$, then the robber starts within distance 5 of $c_3$, who can thus capture him by Lemma 2.4.5. Finally, if $i = 6$, then $c_3$ occupies vertex $(6, 8\ell)$, while some other cop $c_4$ occupies $(-1, 8\ell + 8)$. Now $c_3$ and $c_4$ can capture the robber by Lemma 2.4.6.

- **Case 3:** the robber starts at $(7k + 7, 8\ell + 8)$. The cops' placement ensures that there are cops at $(7k + 14, 8\ell + 8)$ and $(7k + 7, 8\ell + 16)$, so the cops can capture the robber as in Case 1.

Thus the cops can capture the robber if he starts anywhere within the $9 \times 8$ block. Since every vertex in the torus belongs to at least one such block, the cops can always capture the robber.

$\square$

Note that the lower bound on $c_b(T_{m,n})$ in Theorem 2.4.9 is about $mn/121$, while the upper bound is about $mn/112$. With a somewhat more detailed argument, the lower bound can be improved to $mn/120$. However, we suspect that in fact $c_b(T_{m,n}) \sim mn/112$, i.e. that the upper bound is asymptotically tight.

On the grid, the situation is slightly more complex. The edges and corners of the grid contain vertices of low degree, which may allow the robber to isolate himself more quickly than he could in the middle of the grid. Thus when playing on the grid, we must use more cops than when playing on the torus.

**Lemma 2.4.10.** *In the game on $G_{m,n}$, if there exists some vertex $v$ of degree 2 or 3 such that no cop starts within distance 5 of $v$, then the robber can win.*

*Proof.* Suppose no cop starts within distance 5 of some vertex $v$ having degree 2
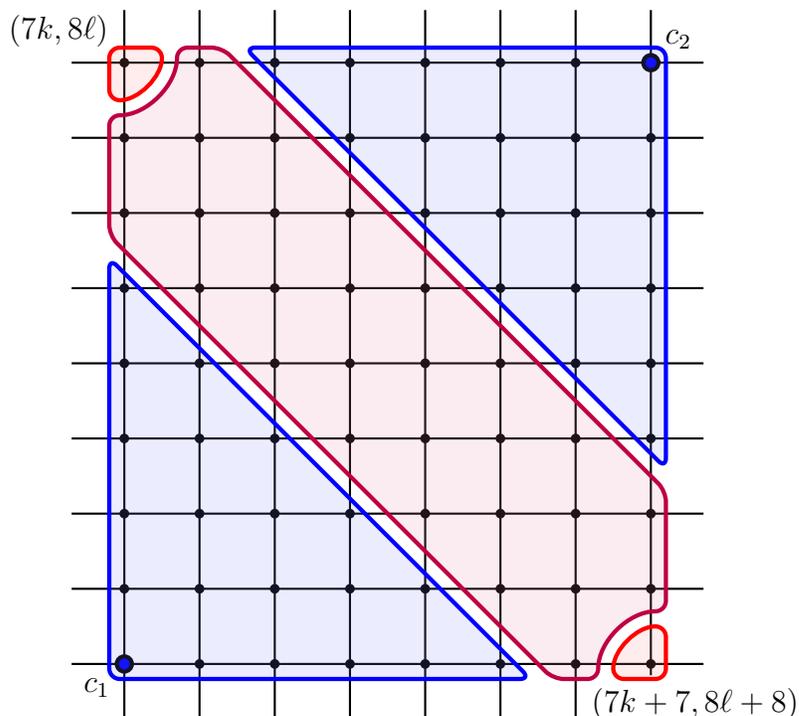
Figure 6. The $9 \times 8$ block with upper-left corner $(7k, 8\ell)$, for $k + \ell$ even. Vertices within the two triangles satisfy the hypotheses of Lemma 2.4.5; vertices within the central region satisfy the hypotheses of Lemma 2.4.6; the remaining two vertices satisfy neither.

or 3. By symmetry, we may suppose $v = (i, 0)$ for some $i \in \{0, 1, \ldots, n-1\}$. We consider two cases.

- **Case 1:** $i \leq 1$. In this case, the robber can isolate himself on $(0, 0)$ by starting on $(0, 0)$ and moving down to $(0, 1)$, right to $(1, 1)$, up to $(1, 0)$, and left to $(0, 0)$. (Refer to Figure 7.) Let $d_k$ denote the robber's distance from $(1, 0)$ after $k$ moves. The robber's strategy ensures that $k + d_k < 4$ for $0 \leq k \leq 3$. Since no cop starts within distance 5 of $v$ it follows that no cop starts within distance 4 of $(1, 0)$, so by Lemma 2.4.7, no cop can reach the robber before his fourth turn. Hence the robber successfully isolates himself and thus wins.

- **Case 2:** $i \geq 2$. In this case, the robber can isolate himself on $(i - 1, 0)$ by starting on $(i - 2, 0)$ and moving right to $(i - 1, 0)$, down to $(i - 1, 1)$, right to $(i, 1)$, up to $(i, 0)$, and left to $(i - 1, 0)$. (Refer to Figure 8.) Letting $d_k$ denote the robber's distance from $v$ after $k$ moves, the robber's strategy ensures that $k + d_k < 5$ for $0 \leq k \leq 4$, so by Lemma 2.4.7, no cop can reach the robber before he has isolated himself.

$\square$
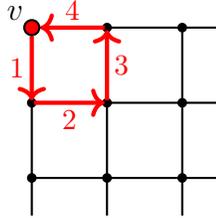


Figure 7. Robber strategy for Case 1 of Lemma 2.4.10.
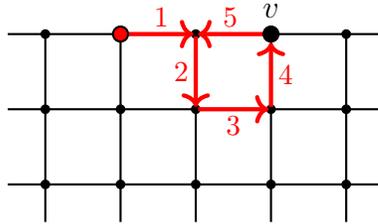


Figure 8. Robber strategy for Case 2 of Lemma 2.4.10.

**Theorem 2.4.11.** *For all positive integers $m$ and $n$,*

$$\left\lceil \frac{mn}{121} \right\rceil \leq c_b(G_{m,n}) \leq 2 \left\lfloor \frac{m}{16} \right\rfloor \left\lfloor \frac{n}{14} \right\rfloor + 3 \left( \left\lfloor \frac{m}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor \right) + 4.$$

*Proof.* The lower bound follows by the same argument as in Theorem 2.4.11. For the upper bound, we give a winning cop strategy. First, we place cops at vertices $(1, 5k + 2)$ and $(n - 2, 5k + 2)$ for all $k$ such that $0 \leq 5k + 2 < m$, along with vertices

$(1, m-1)$ and $(n-2, m-1)$. Similarly, we place cops at vertices $(5\ell+2, 1)$ and $(5\ell+2, m-2)$ for all $\ell$ such that $0 \le 5\ell+2 < n$, along with vertices $(n-1, 1)$ and $(n-1, m-2)$. These cops will prevent the robber from safely visiting the border of the grid; we refer to them as the *border patrol* cops. Next, we place some *central cops* in a manner similar to that used in the proof of Theorem 2.4.9: for all nonnegative integers $k$ and $\ell$ such that $7k < n$, $8\ell < m$, and $k+\ell$ is odd, we place a cop at $(7k, 8\ell)$. Finally, we place *peripheral cops* in columns $0, 10, 20, \ldots$ of row $m-8$ and in columns $5, 15, \ldots$ of row $m-2$. Similarly, we place $\lfloor m/5 \rfloor$ peripheral cops in rows $0, 10, 20, \ldots$ of column $n-2$ and in rows $5, 15, \ldots$ of column $n-8$.

Note that there are at most $2 \lfloor m/5 \rfloor + 2 \lfloor n/5 \rfloor + 4$ border patrol cops, at most $\frac{1}{2} \lfloor m/8 \rfloor \lfloor n/7 \rfloor$ central cops, and $\lfloor m/5 \rfloor + \lfloor n/5 \rfloor$ peripheral cops, so the total number of cops used is at most

$$2 \left\lfloor \frac{m}{16} \right\rfloor \left\lfloor \frac{n}{14} \right\rfloor + 3 \left( \left\lfloor \frac{m}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor \right) + 4$$

as claimed.

To show that the cops can win from this starting position, we first show that the border patrol cops can prevent the robber from safely visiting the border of the grid. In particular, we give a strategy for the border patrol cops placed on row 1; the other border patrol cops use a symmetric strategy. Consider a border patrol cop $c$ who begins on vertex $(k, 1)$. We say that columns $k-2, k-1, \ldots, k+2$ are *assigned* to $c$. Throughout the bulk of the game, $c$ moves left or right within row 1 and within her assigned columns. On her turn, if the robber is not in row 0, then $c$ moves horizontally toward the robber, except that she never moves farther left than column $k-2$ or farther right than column $k+2$. (If $c$ is in the same column as the robber, then she remains in place.)

Suppose the robber begins at vertex $(i, j)$. To show that $c$ prevents the robber from safely entering row 0 of her assigned columns, we consider three cases.

- **Case 1:** $j \geq 2$. In this case, $c$ can make at least two moves before the robber enters row 0. With these moves she can either reach the same column as the robber or the assigned column closest to him, and henceforth it is clear that she can prevent the robber from entering row 0 in her assigned columns.

- **Case 2:** $j = 1$. As before, $c$ clearly prevents the robber from entering row 0 in her assigned columns unless $i \in \{k - 2, k + 2\}$. Suppose without loss of generality that $i = k + 2$. On her first turn, $c$ moves right from $(k, 1)$ to $(k + 1, 1)$. The robber cannot remain in place or move left, lest $c$ capture him. He cannot move right, since column $k + 3$ either does not exist or, if it does exist, is assigned to another border patrol cop, who must currently occupy $(k + 4, 1)$. Thus, he must move up to $(k + 2, 0)$ or down to $(k + 2, 2)$. In the former case, $c$ moves up to $(k + 1, 0)$, and the robber is trapped: he cannot move down, he cannot remain still or move left lest $c$ capture him, and he cannot move right (since either column $k + 3$ does not exist or, if it does, its assigned cop prevents him from entering). In the latter case, the cops can clearly prevent the robber from ever returning to row 1 and hence from ever reaching row 0.

- **Case 3:** Finally, suppose $j = 0$. The robber cannot move down to row 1 on his first turn, since this would result in capture by whichever cop was assigned column $i$. However, if he remains in row 0 on his first turn, then the cops prevent him from safely leaving row 0. Now he cannot remain still indefinitely or he will be captured, but he cannot move indefinitely since he will eventually run out of edges. Thus, the cops eventually capture him.

We may thus suppose that the robber never enters row 0, row $m - 1$, column 0, or column $n - 1$.

Next, we establish analogues of Lemmas 2.4.4, 2.4.5, and 2.4.6 for use on $G_{m,n}$. The proof of Lemma 2.4.4 does not apply on the grid because the grid has vertices of odd degree. However, the border patrol cops prevent the robber from entering any of these vertices. Thus these vertices continue to have odd degree throughout the game and, moreover, belong to the same component at all times. It now follows, as in the proof of Lemma 2.4.4 that the robber must always occupy the same component as his starting vertex; consequently, if some cop can reach the robber's starting vertex, then she can capture him. Thus we may henceforth apply Lemma 2.4.4, from which it follows that we may also apply Lemmas 2.4.5 and 2.4.6 as well (provided in both cases that the robber begins at a vertex of degree 4, but the border patrol cops force him to do so).

To complete the proof, suppose the robber begins at vertex $(i, j)$. If $i = 1$, $j = 1$, $i = n - 1$, or $j = m - 1$, then the border patrol cops can capture the robber as explained above. If $2 \leq i \leq n - 8$ and $2 \leq j \leq m - 9$, then the central cops can capture him as in the proof of Theorem 2.4.9. Otherwise, some peripheral cop begins within distance 5 of the robber and so can capture him by Lemma 2.4.5. $\square$

To finish this section, we determine the bridge-burning cop number of the $n$-dimensional hypercube $Q_n$ – that is, the $n$-dimensional grid with vertices in $\{0, 1\}^n$.

**Theorem 2.4.12.** *For all positive integers $n$, we have $c_b(Q_n) = 1$.*

*Proof.* It suffices to give a strategy for one cop to capture the robber on $Q_n$. As usual, we view the vertex set of $Q_n$ as $\{0, 1\}^n$. We refer to a vertex with a 1 in its $k^{\text{th}}$ coordinate as a vertex in the $k^{th}$ *dimension*. Additionally, we say that the cop or robber *visits* the $k^{\text{th}}$ dimension by starting the game on or moving to a vertex in that dimension, i.e. by changing the $k^{\text{th}}$ coordinate of their position from 0 to 1. Similarly, we will say that the cop or robber *leaves* the $k^{\text{th}}$ dimension by

moving from a vertex inside the dimension to one outside, i.e. by changing the $k^{\text{th}}$ coordinate of their position from 1 to 0.

The cop begins at vertex $(1, 1, \ldots, 1)$. We claim that if, after some cop turn, there exists some $k$ such that the players' positions differ only in coordinate $k$ and the robber hasn't yet visited the $k^{\text{th}}$ dimension, then the cop has a winning strategy.

Indeed, if this situation arises, then all edges incident to vertices in dimension $k$ must be present since the robber hasn't visited the $k^{\text{th}}$ dimension. In addition, the cop must be in the $k^{\text{th}}$ dimension, since the players' positions differ in coordinate $k$. From this point on, the cop always mirrors the robber's move: that is, if the robber changes his $j^{\text{th}}$ coordinate, then the cop changes hers to match. (Of course, if the robber changes his $k^{\text{th}}$ coordinate, then he has moved onto the cop's vertex and thus loses; if the robber remains on his current vertex, then the cop moves onto the robber's vertex and wins, which is possible since all edges incident to vertices in the $k^{\text{th}}$ dimension are still present.) The cop can always do this, since all edges joining vertices in the $k^{\text{th}}$ dimension are still present. Moreover, the cop's strategy prevents the robber from ever visiting the $k^{\text{th}}$ dimension and thus ensures that after all cop turns, the two players' positions agree in all coordinates except the $k^{\text{th}}$. Since the graph is finite, the robber cannot keep moving forever, so the cop eventually captures him.

Hence, we need only show that the cop can always reach a vertex adjacent to the robber and in a dimension that the robber has not yet visited. On her first turn, the cop moves closer to the robber in any coordinate she wishes. Henceforth, the cop plays as follows. If the robber moves away from the cop by changing his $j$th coordinate, then the cop changes her $j$th coordinate in the same way; if the robber sits still or moves closer to the cop, then the cop takes one step closer to

the robber (in any direction she wishes). As long as there is some dimension $k$ that the cop occupies and the robber has never visited, all edges in dimension $k$ are intact and hence the cop can always employ this strategy. It suffices to show that some such dimension always exists up until the point where the cop reaches a vertex adjacent to the robber.

Suppose for the sake of contradiction that the robber can visit all $n$ dimensions while avoiding capture. The cop's strategy ensures that once the players' positions agree in some coordinate, they will continue to do so after every cop turn so long as the robber has not visited all $n$ coordinates. Hence once the robber has visited $n-1$ different dimensions, the players' positions agree in $n-1$ coordinates, meaning that the cop is adjacent to the robber – from which it follows that the cop eventually wins. □

We remark that the argument used to prove Theorem 2.4.12 can be applied more generally to graphs of the form $G_1 \square G_2 \square \ldots \square G_n$, where each $G_i$ is one of $P_2$, $P_3$, and $C_3$. The details are nearly identical to those given above and have been omitted.

## 2.5   Capture Time

In this section, we look not at the cop number, but at a related concept. In the usual model of Cops and Robbers, the *capture time* of a graph $G$, denoted $\text{capt}(G)$, is a measure of how quickly the cops can capture the robber. Formally, $\text{capt}(G)$ is the minimum number of rounds needed for the cops to guarantee a win, provided that there are exactly $c(G)$ cops. Capture time was introduced for by Bonato et al. [2], who showed that $\text{capt}(G) \leq n - 3$ whenever $G$ has cop number 1; this bound was later improved by Gavenčiak [9] to $\text{capt}(G) \leq n - 4$ under the additional condition that $G$ has at least 7 vertices.

We denote capture time in the bridge-burning model by $\text{capt}_b(G)$, and we aim to determine the maximum capture time of an $n$-vertex graph on which a single cop can win. We start with an easy upper bound.

**Theorem 2.5.1.** *For any graph $G$ where one cop can capture the robber,* $\text{capt}_b(G) = O(n^3)$.

*Proof.* In a game on $G$, the robber can move at most $|E(G)|$ times. Between moves, the robber can remain on his current vertex no more than $n$ times provided that the cop is playing optimally, since the cop will move on each turn and will never revisit a vertex while the robber remains in place. Thus, the number of rounds needed for the cop to win is at most $|E(G)| \cdot n$, which is $O(n^3)$. □

One might expect capture times to be lower, in general, in the bridge-burning model of Cops and Robbers than in the ordinary model, since the graph necessarily becomes smaller as the game proceeds. However, for a lower bound on the maximum capture time in the bridge-burning model, we give a graph $G$ with $\text{capt}_b(G) = \Omega(n^2)$ – an order of magnitude larger than the maximum capture time under the usual model!

**Theorem 2.5.2.** *There exists a graph $G$ such that* $\text{capt}_b(G) = \Omega(n^2)$.

*Proof.* Let $k, m$ be positive integers with $m(k-1)$ even. Consider the graph $G_{m,k}$ formed as follows. Begin with a complete graph with vertices $v_1, ... v_k$ and a complete $k$-partite graph with partite sets $S_1, S_2, ..., S_k$, each containing exactly $m$ vertices. For all $i \in \{1, \ldots, k\}$, add an edge between $v_i$ and every vertex in $S_i$. Finally, to each vertex $v_i$, add a pendant neighbor $u_i$.

Vertices $v_1, ..., v_k$ form a dominating clique of $G$, so by Theorem 2.2.2, a single cop can capture the robber. To show that the cop cannot win too quickly, we give a strategy for the robber to avoid capture for a long while. If the cop begins

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anywhere except on some $v_i$, then the robber begins on some $v_j$ not adjacent to the cop; on the robber's first turn, he moves to $u_j$, thereby isolating himself and winning the game. Thus we may suppose the cop begins on some $v_i$.

The robber starts on a vertex in $S_j$ for some $j \neq i$. In addition, the robber fixes some Eulerian cycle in the subgraph of $G$ induced by $S_1 \cup S_2 \cup \cdots \cup S_k$; he can do so because this subgraph is regular of degree $m(k-1)$, which by assumption is even. If the cop moves to $u_i$ or to some vertex in $S_i$, then the robber moves to $v_j$ and, on his next turn, to $u_j$, thereby isolating himself. If the cop moves to $v_\ell$ for some $\ell \neq j$, then the robber remains on his current vertex. Finally, suppose the cop moves to $v_j$. The robber must move, so he moves to the next vertex in his chosen Eulerian cycle. The robber maintains this strategy until he has completed the entire cycle, at which point he remains on his current vertex and awaits his imminent capture.

The robber's strategy ensures that he cannot be captured before burning all edges of the complete multipartite graph induced by $S_1 \cup S_2 \cup \cdots \cup S_k$, of which there are $(mk \cdot m(k-1))/2$. Thus the number of rounds needed for the cop to win is at least $m^2 k(k-1)/2 + 1$. Letting $|V(G)| = n$, we have $n = km + 2k$, so $n = mk + O(k)$ and so

$$\mathrm{capt}_b(G) \geq m^2 k(k-1)/2 + 1 = \Omega(n^2).$$

$\square$

The lower bound in Theorem 2.5.2 differs by an order of magnitude from the upper bound in Theorem 2.5.1; we conjecture that the upper bound gives the correct order of growth.

**Conjecture 2.5.3.** *There exists an n-vertex graph $G$ with $c_b(G) = 1$ and* $\mathrm{capt}_b(G) = \Omega(n^3)$.

## 2.6  Open Problems

We conclude the paper by suggesting a few directions for future research on the bridge-burning game.

- **Characterize the graphs with bridge-burning cop number 1**. A nice structural characterization is known for graphs with cop number 1 under the usual model (see [12], [13]), but we have no such characterization for the bridge-burning model. One principal difficulty in tackling this problem is that in the bridge-burning model, the graph changes as the game progresses, so any structural properties satisfied by the graph at the beginning of the game need not be satisfied throughout the game.

- **Determine the asymptotics of $c_b(T_{m,n})$ and $c_b(G_{m,n})$**. We have shown that both parameters are asymptotically $c \cdot mn$ for some constant $c$ between 112 and 121; could it be that both are asymptotically $mn/112$?

- **Study the game on grids of arbitrary dimension**. Theorem 2.4.12 provides a first step toward this problem, but it is not clear how or if the techniques used therein would extend to grids with larger side lengths.

- **Examine Cartesian products of general trees and/or cycles**. The cop numbers for products of trees and for products of cycles have been completely determined under the usual model of Cops and Robbers (see [10, 11]) as well as for several variants. It would be interesting to see how the situation differs in the bridge-burning model.

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# CHAPTER 3

## The Game of Weak Directed Cops and Robbers

William B. Kinnersley and Eric Peterson

Department of Mathematics

University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

**Abstract**

We introduce a variant of Cops and Robbers played on a directed graph $G$ in which the cops can only move in accordance with each edge's direction, but the robber may move in either direction on any edge. We primarily examine the minimum number of cops needed in this variant to capture a robber on $G$, called the *weak directed cop number of $G$* and denoted $c_w(G)$. In doing so, we first examine a sufficient condition for a directed graph $G$ to be *cop-win*, i.e. to have $c_w(G) = 1$. We then give tight bounds for $c_w$ on several families of Cartesian products of graphs, in particular $\overrightarrow{T_1} \square \overrightarrow{T_2}$ where $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are arborescences, and $\overrightarrow{C_m} \square \overrightarrow{C_n}$ where $\overrightarrow{C_m}$ and $\overrightarrow{C_n}$ are directed cycles. For directed paths $\overrightarrow{P_m}$, $\overrightarrow{P_n}$, and $\overrightarrow{P_r}$, we provide bounds on the weak directed cop number for the three-dimensional directed Cartesian grid $\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}$. We also determine a tight bound for $c_w$ on strongly-connected outerplanar directed graphs and, for each $n \in \mathbb{N}$, give a construction for a strongly-connected planar directed graph satisfying $c_w(G) \geq n$.

## 3.1 Introduction

*Cops and Robbers* is a pursuit-evasion game played on a graph $G$ by two players; one player controls a predetermined number of *cops* (we will use the pronouns she/her for each cop) while the other player controls one *robber* (we will use the pronouns he/his for the robber). Throughout the course of the game, the cops and robber occupy vertices of $G$ and move from vertex to vertex using the edges of $G$. The game begins with all cops choosing their initial vertices and then the robber choosing his initial vertex. The game then proceeds in *rounds*; each round consists of a cop turn and then a robber turn. When it is the cops' turn, each cop chooses either to move to an adjacent vertex or to remain on the vertex she currently occupies. The robber then makes the same decision during his turn. At all times, the cops and robber are fully aware of each others' positions in $G$. The cops win

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the game when one or more cops occupies the same vertex as the robber, in which case we say the cops *capture* the robber. If the robber is able to perpetually avoid capture from the cops, then the robber wins the game.

Aside from the standard variant described above, many other variants of Cops and Robbers have been analyzed; in each variant, the rules of the game are modified slightly. Some examples include: the cops having limited information about the robber's location [5, 7], the cops and robber playing on different edge sets [15], and the robber moving faster than the cops [8, 9, 19]. In each variant, it is typical to look at the minimum number of cops required to capture a robber on a graph $G$. In the standard variant, we call this the *cop number of $G$*, denoted $c(G)$. Additional background on Cops and Robbers can be found in [2]. We note that the above variants operate under the assumption that $G$ is an undirected graph. The standard variant naturally extends to the case where $G$ is a directed graph. This model has been studied on planar directed graphs, oriented graphs, and tournaments [3, 6, 10, 12, 18]. Aside from this, however, very little else is known about the standard game on directed graphs. In particular, this model currently lacks a characterization of *cop-win* graphs, or those where a single cop can win on a directed graph, as well as analysis for $c(G)$ on other common graph classes.

Here, we introduce a variant of Cops and Robbers played on a directed graph $G$ in which, if a cop decides to move from a vertex $u$ to a vertex $v$, then she may do so only if $\overrightarrow{uv} \in E(G)$; however, if the robber elects to move from $u$ to $v$, then he may do so if either $\overrightarrow{uv} \in E(G)$ or $\overrightarrow{vu} \in E(G)$. In other words, each cop may only move along an edge of $G$ according to the direction of the edge; the robber may move along an edge of $G$ regardless of the direction the edge points. The rules of the game otherwise remain otherwise unchanged from the standard variant. We refer to this game as the *weak directed game of Cops and Robbers* and define the

*weak directed cop number of* $G$, denoted $c_w(G)$, to be the minimum number of cops required to win this game on a directed graph $G$. This model is named after (and in part motivated by) the *weak searching game* on directed graphs in which the searchers are "weaker" than the intruder [20].

We hope that studying the weak directed game of Cops and Robbers will yield insights that will be useful in the study of the standard game on directed graphs. It is clear the robber has an advantage over the cops in the weak game in that his movement capabilities are much stronger. As such, a winning cop strategy for the weak directed game works for the standard game on a directed graph as well; thus $c(G) \le c_w(G)$ for any directed graph $G$. On some graphs, these parameters are equal; for example $c\left(\overrightarrow{P_n}\right) = c_w\left(\overrightarrow{P_n}\right) = 1$ for any directed path $\overrightarrow{P_n}$. For other graphs, these parameters can be arbitrarily far apart. For example, $c(G) = 1$ when $G$ is a directed acyclic graph with one source, since a single cop can use a topological ordering of the graph to force the robber onto a sink vertex and win. In the weak game, this is not always the case; we will show in Theorem 3.3.4 that the weak directed cop number is unbounded for the three-dimensional directed Cartesian grid (which is a directed acyclic graph with one source).

In this paper, we examine the weak directed game of Cops and Robbers for general directed graphs and also classes of graphs that are commonly studied in other variants. In Section 3.2, we briefly look at $c_w$ for directed paths, cycles, and complete graphs. We then give a sufficient condition for a directed graph to be cop-win in Theorem 3.2.5, and we show in Theorem 3.2.7 that this condition completely characterizes cop-win oriented graphs. In Section 3.3, we look at Cartesian products of graphs. In Theorem 3.3.1, we show that $c_w\left(\overrightarrow{T_1} \square \overrightarrow{T_2}\right) \le 2$ for any arborescences $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$; in particular, for the two-dimensional directed Cartesian grid $\overrightarrow{P_m} \square \overrightarrow{P_n}$, we have $c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n}\right) \le 2$. In Theorem 3.3.3, we show that

for the two-dimensional directed discrete torus $\overrightarrow{C_m} \square \overrightarrow{C_n}$, we have $c_w\left(\overrightarrow{C_m} \square \overrightarrow{C_n}\right) \leq 4$. In Theorem 3.3.6, we show that for the three-dimensional directed Cartesian grid $\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}$, we have $\log_3 d \leq c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}\right) \leq 2d$ where $d = \min\{m, n, r\}$; in Theorem 3.3.7, we demonstrate that under certain natural restrictions on the cops' strategy, we can establish the asymptotically stronger lower bound $c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}\right) \geq \frac{8}{27}d^{1-\log_3 2}$. In Section 3.4, we examine planar and outerplanar graphs. In Theorem 3.4.1, we determine that $c_w(G) \leq 3$ for any strongly-connected outerplanar directed graph, which is tight. For each $n \in \mathbb{N}$, we then provide in Theorem 3.4.2 a construction of a strongly-connected planar directed graph $G$ that satisfies $c_w(G) \geq n$.

Before proceeding, we give some background definitions and terminology pertaining to directed graphs. An *oriented graph* is a directed graph such that for all $u, v \in V(G)$, if $\overrightarrow{uv} \in E(G)$, then $\overrightarrow{vu} \notin E(G)$. For a directed graph $G$, the *underlying undirected graph* of $G$ is the undirected graph created using the vertex set $V(G)$ and replacing all directed edges in $E(G)$ with undirected edges. The *open in-neighborhood* of a vertex $v$, denoted $N^-(v)$ (respectively the *open out-neighborhood* of $v$, denoted $N^+(v)$), is the set of all $u \in V(G)$ such that $\overrightarrow{uv} \in E(G)$ (respectively, the set of all $u \in V(G)$ such that $\overrightarrow{vu} \in E(G)$). The *closed in-neighborhood* of $v$, denoted $N^-[v]$ (respectively the *closed out-neighborhood* of $v$, denoted $N^+[v]$), is the set $N^-(v) \cup \{v\}$ (respectively, the set $N^+(v) \cup \{v\}$). The *open neighborhood* of $v$, denoted $N(v)$, is the set $N^-(v) \cup N^+(v)$; similarly, the *closed neighborhood* of $v$, denoted $N[v]$, is the set $N^-[v] \cup N^+[v]$.

## 3.2 General Bounds

We begin by looking at elementary directed graphs, namely directed paths and directed cycles.

**Proposition 3.2.1.**

(a) $c_w\left(\overrightarrow{P_n}\right) = 1$ *for all $n$.*

(b) $c_w\left(\overrightarrow{C_n}\right) = 2$ *for all $n \geq 3$.*

*Proof.*

(a) On $\overrightarrow{P_n}$, the cop starts on the source vertex. She is clearly able to capture the robber wherever he starts.

(b) On $\overrightarrow{C_n}$, both cops start on arbitrary vertices. One cop pursues the robber around the cycle while the other cop sits still. Eventually, the robber is trapped.

$\square$

We now provide a trivial upper bound on $c_w$ for any directed graph $G$, which we use to provide an upper bound on $c_w$ for tournaments. We say $S \subseteq V(G)$ is a *dominating set* of $G$ if for each $v \in V(G)$, either $v \in S$, or there exists a vertex $u \in S$ such that $\overrightarrow{uv} \in E(G)$. The cardinality of a minimum size dominating set is the *domination number of $G$*, denoted $\gamma(G)$. It is well-known that the domination number of an undirected graph is an upper bound on the cop number in the standard game [2]; Theorem 3.2.2 is the analogous result for directed graphs in this variant.

**Theorem 3.2.2.** *For every directed graph $G$, we have $c_w(G) \leq \gamma(G)$.*

*Proof.* Suppose there are $\gamma(G)$ cops in play and let $S$ be a minimum size dominating set of $G$ (hence $|S| = \gamma(G)$). We place one cop on each vertex of $S$. It follows that some cop can capture the robber during her first turn regardless where the robber starts. $\square$

**Corollary 3.2.3.** *If $G$ is a tournament on $n$ vertices, then $c_w(G) \leq \lfloor \log_2(n+1) \rfloor$.*

*Proof.* Combining Theorem 3.2.2 from above and Theorem 1 in [11],

$$c_w(G) \leq \gamma(G) \leq \lfloor \log_2(n+1) \rfloor.$$

$\square$

In an undirected graph $G$, a vertex $u \in V(G)$ is a *corner* of $G$ if there exists $v \in N[v]$ such that $N[u] \subseteq N[v]$. It has been shown that an undirected graph $G$ in the standard variant of Cops and Robbers is cop-win if and only if $G$ is *dismantlable*, i.e. there exists some sequence of vertices $u_1, ..., u_m$ such that $u_i$ is a corner of $G \backslash \{u_1, ..., u_{i-1}\}$ for each $i$, and $G \backslash \{u_1, ..., u_m\} = K_1$ (see [16, 17]). Since this is a fundamental result in the literature for Cops and Robbers, it is natural to consider how similar ideas can be applied to the characterization of cop-win directed graphs in the weak game.

We say that $u \in V(G)$ is a *weak corner* of a directed graph $G$ if there exists $v \in N^-[u]$ such that $u \neq v$ and $N[u] \subseteq N^+[v]$. We note that such a corner is named weak due to the variant; in fact, the property of being a weak corner is "stronger" than that of being a corner (in the underlying undirected graph), in the sense that every weak corner is also a corner, but not vice-versa. Here, we also say $v$ *weakly covers* the weak corner $u$. Moreover, a directed graph $G$ has a *weak dismantling* if there exists some sequence of vertices $u_1, ..., u_m$ such that $u_i$ is a weak corner in $G \setminus \{u_1, ..., u_{i-1}\}$ for each $i$, and $G \setminus \{u_1, ..., u_m\} = K_1$. If $G$ has such a weak dismantling, then the ordering given by the sequence $u_1, ..., u_m$ is called a *weak dismantling order*.

**Proposition 3.2.4.** *If $G$ is cop-win in the weak game, then $G$ contains at least one weak corner.*

*Proof.* Let $G$ be a cop-win directed graph, suppose the cop executes a winning strategy, and consider the state of the game immediately prior to the robber's last turn, with the cop on a vertex $v$ and the robber on a vertex $u$. It follows that $\overrightarrow{vu} \in E(G)$ or else the robber could remain on $u$ and the cop wouldn't win during their next move. Furthermore, $N(u) \subseteq N^+[v]$ must hold, or else the robber could move to a vertex that the cop cannot reach during their next move. Hence, $N[u] \subseteq N^+[v]$ must hold, so $u$ is a weak corner.

$\square$

The following result shows that having a weak dismantling is sufficient for a directed graph to be cop-win in the weak game.

**Theorem 3.2.5.** *If a directed graph $G$ has a weak dismantling, then $G$ is cop-win in the weak game.*

*Proof.* Let $G$ be a directed graph on $n$ vertices with a weak dismantling; we proceed by induction on $n$. For the case where $n = 1$, the claim holds trivially. We now assume that for some $n \geq 1$, all directed graphs on $n$ vertices that have weak dismantlings are cop-win, and let $G$ be a directed graph on $n + 1$ vertices with a weak dismantling. Since $G$ has a weak dismantling, $G$ has a weak corner, so let $u$ be the first weak corner of $G$ in the weak dismantling order and let $v$ be a vertex that weakly covers $u$. Upon removal of $u$, it follows that $G \setminus u$ also has a weak dismantling, so it is cop-win by the inductive hypothesis. We now show that $G$ is cop-win.

On the graph $G$, the cop now plays as if the graph is $G \setminus u$ and attempts to execute a winning strategy for $G \setminus u$. If the robber moves to $u$ during his turn, the cop views this as the robber instead being positioned on $v$ in $G \setminus u$, which is valid since $N[u] \subseteq N^+[v]$. This does not upset the cop's strategy; if the robber moves from $u$, he must move to some neighbor of $v$ since $N[u] \subseteq N^+[v]$. Hence, the cop

can view this movement on $G$ as being valid in $G \setminus u$. Moreover, if the cop moves to $v$ as if to capture the robber when playing the game on $G \setminus u$, the cop either captures the robber if he is indeed on $v$, or the cop is now on $v$ and the robber is on $u$. In the latter case, since $v$ weakly covers $u$, the cop captures the robber during the next round. Hence, $G$ is cop-win.

$\square$

It is worth noting that the cop strategy in Theorem 3.2.5 works regardless of whether or not the robber must move according to the directions of the edges of $G$. Hence, this strategy may be applied to the standard variant on directed graphs. In fact, in the standard variant, we can give a stronger sufficient condition for a directed graph to be cop-win.

We say that $u \in V(G)$ is a *directed corner* of a directed graph $G$ if there exists $v \in N^-[u]$ such that $u \neq v$ and $N^+[u] \subseteq N^+[v]$. As before, the notion of being a weak corner is "stronger" than that of being a directed corner, in the sense that every weak corner is also a directed corner, but not vice-versa. A directed graph $G$ has a *directed dismantling* if there exists some sequence of vertices $u_1, ..., u_m$ such that $u_i$ is a directed corner in $G \setminus \{u_1, ..., u_{i-1}\}$ for each $i$, and $G \setminus \{u_1, ..., u_m\} = K_1$. If $G$ has such a directed dismantling, then the ordering given by the sequence $u_1, ..., u_m$ is called a *directed dismantling order*.

Using the same argument as in Theorem 3.2.5, we have the following result.

**Theorem 3.2.6.** *If a directed graph $G$ has a directed dismantling, then $G$ is cop-win in the standard game.*

If $G$ is an oriented graph, the sufficiency condition in Theorem 3.2.5 is also necessary in the weak game. For a directed graph $G$ and a subgraph $H \subseteq G$, we say that a vertex $u \in V(H)$ is *$H$-covered* if there exists some vertex $v \in V(G)$ such that $u \neq v$ and $(N[u] \cap V(H)) \subseteq N^+[v]$; in this case, we also say $v$ *$H$-covers* $u$.

**Theorem 3.2.7.** *An oriented graph $G$ is cop-win in the weak game if and only if there is a weak dismantling of $G$.*

*Proof.* The backward direction follows directly from Theorem 3.2.5. For the forward direction, let $G$ be an oriented graph that has no weak dismantling; we show that $G$ is robber-win.

Since $G$ has no weak dismantling, any attempt to first remove a weak corner from $G$ and then iteratively remove weak corners from what remains eventually results in some nontrivial subgraph of $G$ that has no weak corners. For the sake of contradiction, suppose that $G$ is cop-win and let $v_1, v_2, ..., v_m$ be a sequence of vertices of $G$ such that:

- Each $v_i$ is a weak corner in $G_i = G \setminus \{v_1, ..., v_{i-1}\}$.

- $G \setminus \{v_1, ..., v_m\}$ has no weak corners.

Let $S = \{v_1, .., v_m\}$ and $H = G \setminus S$. Any winning strategy for the cop on $G$ requires the existence of some vertex $v$ that $H$-covers some vertex $u \in V(H)$; otherwise, the robber can evade capture forever without leaving $H$. If $v \in V(H)$, then $u$ is a weak corner in $H$, which is a contradiction. Hence, $v \in S$.

Now let $j$ be the maximum index such that there exists $w \in V(H)$ whereby $v_j$ $H$-covers $w$. By definition, $v_j$ is a weak corner in $G_j$, so there exists some $x \in G_j$ that $G_j$-covers $v_j$. Note that since $\overrightarrow{v_j w} \in E(G)$ and $G$ is oriented, then $\overrightarrow{w v_j} \notin E(G)$ and therefore $x \neq w$. Since $x$ $G_j$-covers $v_j$, it is also true that $x$ $H$-covers $w$. Since $H$ has no weak corners, $x \in S$ and so $x = v_k$ for some $k$. By removing $v_j$, we see that $x \in G_{j+1}$, which implies $k > j$. Since $x$ $H$-covers $w$, this contradicts the choice of $j$.

Hence, there is no vertex in $G \setminus H$ that $H$-covers some $w \in V(H)$. As a result, the cop cannot force the robber to leave $H$, so the robber can win on $G$ by remaining in $H$.
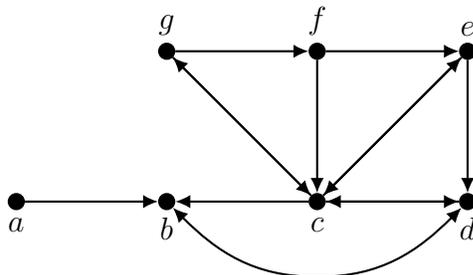
Figure 9. A non-oriented directed graph that is cop-win but not weakly dismant-lable

$\square$

The characterization in Theorem 3.2.7 does not apply to all non-oriented directed graphs. Let $G$ be the directed graph in Figure 9. A single cop can win on $G$ by starting on $a$ and then moving along the walk $a - b - d - c - g - f - c$ to force the robber onto the weak corner $d$, at which point the cop wins. The graph $G$ is not weakly dismantlable, however. In an attempt to iteratively remove weak corners from $G$, we must remove $d$, $e$, $f$, and then $g$ in that order; the remaining graph contains no weak corners and is not $K_1$.

## 3.3    Cartesian Products of Graphs

In this section, we look at several Cartesian products of graphs whose under-lying undirected graphs have been studied in other variants of Cops and Robbers.

We first determine the weak directed cop number for the Cartesian product of two arborescences $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$. An *arborescence* is a directed graph $G$ containing a root vertex $u$ such that, for all $v \in V(G)$, there is exactly one directed path from $u$ to $v$; we may also think of an arborescence as an orientation of a tree where all edges point away from the root. For the Cartesian product of $n$ nontrivial undirected trees, Maamoun and Meyniel showed that the cop number in the standard variant is at most $\lceil \frac{n+1}{2} \rceil$ (see [14]); here we show that $c_w \left( \overrightarrow{T_1} \square \overrightarrow{T_2} \right) \leq 2$.

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**Theorem 3.3.1.** *If $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are arborescences on $m$ and $n$ vertices respectively, then*

$$c_w\left(\overrightarrow{T_1}\square\overrightarrow{T_2}\right) \leq 2, \text{ with equality reached when } m, n \geq 2.$$

*Proof.* If $m = 1$ or $n = 1$, it is clear that $c_w\left(\overrightarrow{T_1}\square\overrightarrow{T_2}\right) = 1$. Let $m, n \geq 2$; we will first show that two cops are necessary in this case.

Suppose only one cop is in play and, without loss of generality, she starts on the source vertex. The robber can start at some sink vertex $(v_1, v_2)$ where $v_1$ is a leaf of $\overrightarrow{T_1}$ and $v_2$ is a leaf of $\overrightarrow{T_2}$, such that the cop cannot move to $(v_1, v_2)$ during her first turn. Let $u_1$ be the parent of $v_1$ and $u_2$ the parent of $v_2$. Once the cop reaches either $(u_1, v_2)$ or $(v_1, u_2)$, a vertex adjacent to the robber, then the robber responds by fleeing to $(v_1, u_2)$ or $(u_1, v_2)$ respectively and wins since the cop can no longer reach the robber.

We now give a strategy for two cops $c_1$ and $c_2$ to capture the robber. Let $r_1$ (respectively $r_2$) be the root vertex of $\overrightarrow{T_1}$ (respectively $\overrightarrow{T_2}$) and let both cops start at $(r_1, r_2)$, the only source vertex of $\overrightarrow{T_1}\square\overrightarrow{T_2}$. We will say that a cop is at *distance* $\langle d_x, d_y \rangle$ from the robber if $d_x$ is the distance from the cop to the robber in $\overrightarrow{T_1}$ and $d_y$ is the distance from the cop to the robber in $\overrightarrow{T_2}$. If the robber in not reachable by the cop in $\overrightarrow{T_1}$ (respectively in $\overrightarrow{T_2}$), we let $d_x = \infty$ (respectively $d_y = \infty$). Furthermore, we say that a cop moves *towards the robber in* $\overrightarrow{T_1}$ (respectively $\overrightarrow{T_2}$) if the cop moves one vertex closer to the robber in $\overrightarrow{T_1}$ (respectively $\overrightarrow{T_2}$), thereby decreasing $d_x$ (respectively $d_y$) by one.

At the start of the game, let the cops be at distance $\langle d_1, d_2 \rangle$. If $d_1 > d_2$, then both cops move together towards the robber in $\overrightarrow{T_1}$ until $d_1 = d_2$ (note this could happen after either a cop or robber turn). During this time, the robber cannot move to a vertex not reachable by the cops in $\overrightarrow{T_1}$ since this requires $d_1 = 0$, which contradicts the assumption that $d_1 > d_2$. Furthermore, since the cops remain

at $r_2$ in $\overrightarrow{T_2}$, the robber cannot move to a vertex not reachable by the cops in $\overrightarrow{T_2}$. Hence, the robber cannot win by moving to a vertex in $\overrightarrow{T_1} \square \overrightarrow{T_2}$ not reachable by the cops throughout this process. Similarly, if $d_1 < d_2$, then both cops move together toward the robber in $\overrightarrow{T_2}$ until $d_1 = d_2$ and the robber cannot move to a vertex not reachable by the cops.

After this point, their strategy is as follows:

- **Case 1:** If the cops are both at distance $\langle d, d \rangle$ for some $d$ after the robber's turn, then $c_1$ moves towards the robber in $\overrightarrow{T_1}$ and $c_2$ moves towards the robber in $\overrightarrow{T_2}$. The cops are now at distances $\langle d - 1, d \rangle$ and $\langle d, d - 1 \rangle$ respectively.

- **Case 2:** If the cops are both at distance $\langle d - 1, d \rangle$ (respectively $\langle d, d - 1 \rangle$) for some $d$ after the robber's turn, then both cops move towards the robber in $\overrightarrow{T_2}$ (respectively in $\overrightarrow{T_1}$). The cops are now both at distance $\langle d - 1, d - 1 \rangle$. If $d = 1$, the robber has been captured.

- **Case 3:** If $c_1$ is at distance $\langle d - 1, d \rangle$ and $c_2$ is at distance $\langle d, d - 1 \rangle$ for some $d$ after the robber's turn, then $c_1$ moves towards the robber in $\overrightarrow{T_2}$ and $c_2$ moves towards the robber in $\overrightarrow{T_1}$. The cops are now both at distance $\langle d - 1, d - 1 \rangle$. If $d = 1$, the robber has been captured.

- **Case 4:** If $c_1$ is at distance $\langle d - 1, d - 1 \rangle$ and $c_2$ is at distance $\langle d, d - 2 \rangle$ for some $d$ after the robber's turn, then both cops move towards the robber in $\overrightarrow{T_1}$. The cops are now at distances $\langle d - 2, d - 1 \rangle$ and $\langle d - 1, d - 2 \rangle$.

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- **Case 5:** If $c_1$ is at distance $\langle d-2, d \rangle$ and $c_2$ is at distance $\langle d-1, d-1 \rangle$ for some $d$ after the robber's turn, then both cops move towards the robber in $\overrightarrow{T_2}$. The cops are now at distances $\langle d-2, d-1 \rangle$ and $\langle d-1, d-2 \rangle$.

We now show that the cops will eventually capture the robber. Consider all possible robber moves with the cops utilizing the strategy outlined above:

- If the cops are both at distance $\langle d, d \rangle$ for some $d$ before the robber's turn, then any robber move ends with the cops at distances covered in Cases 1, 2, or 3.

- If $c_1$ is at distance $\langle d-1, d \rangle$ and $c_2$ is at distance $\langle d, d-1 \rangle$ for some $d$ before the robber's turn, then any robber move ends with the cops at distances covered in Cases 3, 4 or 5.

As a result, the cops have a response for every robber move. Observe that the above strategy ensures that the cops' distances from the robber in $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are always nonincreasing, and only remain the same if the robber moves away from the cops in either tree. Since the graph is finite, however, the robber can only do this for so long. If the robber is unable to move past the cops during the game, he is eventually captured by the cops.

Suppose the robber is able to move past the cops during his turn. This implies that after the robber's previous turn, $d_x = 0$ or $d_y = 0$ for some cop. The cops would then move according to either Case 2 where $d = 1$, Case 3 where $d = 1$, Case 4 where $d = 2$, or Case 5 where $d = 2$. In Case 2 and Case 3, the cops would capture the robber. In Case 4 or Case 5, the cops would be at distance $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ from the robber, so the robber cannot attempt to move past one cop without being captured by the other. $\qquad \square$

**Corollary 3.3.2.** *If $\overrightarrow{P_m}$ and $\overrightarrow{P_n}$ are directed paths on $m$ and $n$ vertices respectively, then $c_w\left(\overrightarrow{P_m}\square\overrightarrow{P_n}\right) \leq 2$, with equality when $m, n \geq 2$.*

Next, we make use of the strategy in Theorem 3.3.1 to examine the weak directed cop number for the two-dimensional directed discrete torus $\overrightarrow{C_m}\square\overrightarrow{C_n}$. In the standard variant of Cops and Robbers on undirected graphs, the cop number of $C_m\square C_n$ is at most 3 (see [13]); here we show that $c_w\left(\overrightarrow{C_m}\square\overrightarrow{C_n}\right) = 4$.

To do so, we view the vertices of $\overrightarrow{C_m}\square\overrightarrow{C_n}$ as ordered pairs $(i, j)$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. For fixed $i$, we say that the vertices of the form $(i, k)$ are in *column* $i$; similarly those of the form $(\ell, i)$ are in *row* $i$. Note that $\overrightarrow{C_m}\square\overrightarrow{C_n}$ has a total of $m$ columns and $n$ rows; a vertex's column is indexed by its first coordinate, while its row is indexed by the second coordinate. We say that vertex $(i, j)$ is *to the left* of $(i', j')$ if $i < i'$ and *to the right* if $i > i'$. Similarly, $(i, j)$ is *above* $(i', j')$ if $j < j'$ and *below* if $j > j'$. When a player moves from vertex $(i, j)$ to $(i + 1, j)$, we say they *move right*; likewise, when they move to $(i - 1, j)$, $(i, j + 1)$, or $(i, j - 1)$ we say that they *move left*, *move down*, or *move up*, respectively.

**Theorem 3.3.3.** *If $m = 3$ or $n = 3$, then $c_w\left(\overrightarrow{C_m}\square\overrightarrow{C_n}\right) = 3$. For all $m, n \geq 4$, we have $c_w\left(\overrightarrow{C_m}\square\overrightarrow{C_n}\right) = 4$.*

*Proof.* Without loss of generality, we first consider $m = 3$ and $n \geq 3$. For the lower bound, assume that $k$ cops are sufficient to win the game. Hence, on the robber's final turn before he is captured, every vertex in the robber's undirected closed neighborhood must be in the closed out-neighborhood of some cop. It is easily seen that this requires at least three cops regardless of the robber's position, leading to $k \geq 3$.

For the upper bound, we give a winning cop strategy for three cops $c_1$, $c_2$, and $c_3$; we assume $c_1$ starts on $(1, 1)$, $c_2$ starts on $(1, 2)$, and $c_3$ starts on $(1, 3)$. Without loss of generality, assume that $c_3$ starts in the same row as the robber.

For the remainder of the game, $c_3$ utilizes the following strategy with the goal of either ending up in the same row as the robber, or in the row above the robber.

- **Case 1:** If the robber is in the same row as $c_3$ at the beginning of the cops' turn, $c_3$ stays put; $c_3$ and the robber are now in the same row.

- **Case 2:** If the robber is in the row below $c_3$ at the beginning of the cops' turn, $c_3$ moves down; $c_3$ and the robber are now in the same row.

- **Case 3:** If the robber is in the row above $c_3$ at the beginning of the cops' turn, $c_3$ moves down; $c_3$ is now in the row above the robber.

As a result of this strategy, $c_3$ remains in column 1 throughout the course of the game. Note that the robber cannot enter column 1 without being captured; if he were to do so, he would either move onto $c_3$ or on the vertex below $c_3$ due to $c_3$'s strategy, at which point $c_3$ could capture him.

The strategy for $c_1$ and $c_2$ is as follows. Both cops move right during each round until they reach some column $i$ such that the robber is located in column $i + 1$. We note that this eventually happens since the robber is not able to move right into column 1 due to $c_3$, and he is clearly unable to move left past $c_1$ and $c_2$ at any point during this time. At this point, if the robber is on $(i+1, 1)$ or $(i+1, 2)$ at the beginning of the cops' turn, he is captured by one of these two cops. If the robber is on $(i+1, 3)$, then $c_1$ moves right to $(i+1, 1)$ and $c_2$ moves down to $(i, 3)$. The robber must now move right to avoid capture, at which point $c_2$ moves right to $(i + 1, 3)$ and the two cops are positioned as before in two distinct rows of the same column. These cops now use a symmetric strategy and continue to force the

robber to move right; he is eventually unable to do so without being captured by $c_3$, whereby the cops win.

We now consider $m, n \geq 4$. For the lower bound, we assume that $k$ cops are sufficient to win the game as before. Hence, on the robber's final turn before he is captured, every vertex in the robber's undirected closed neighborhood must be in the closed out-neighborhood of some cop. It is easily seen that this requires at least four cops regardless of the robber's position, leading to $k \geq 4$.

For the upper bound, we give a winning cop strategy for four cops $c_1$, $c_2$, $c_3$, and $c_4$; assume that all cops start on $(1, 1)$. Cops $c_1$ and $c_2$ implement the strategy outlined in Theorem 3.3.1 at all times. Since $\overrightarrow{C_m}$ and $\overrightarrow{C_n}$ are not trees, this strategy need not terminate; however, the robber can only avoid capture from $c_1$ and $c_2$ if he can move right and/or down infinitely often.

We now provide the strategy for the remaining two cops. We start with $c_3$:

- If $c_3$ is not in the same column as the robber, then $c_3$ moves right.

- If $c_3$ and the robber are in the same column, then $c_3$ stays put.

We provide a similar strategy for $c_4$:

- If $c_4$ is not in the same row as the robber, then $c_4$ moves down.

- If $c_4$ and the robber are in the same row, then $c_4$ stays put.

We now show that the robber is forced to move left, move up, or stay put at least once every $2m + 2n$ rounds of the game due to the movements of $c_3$ and $c_4$. When he does so, his distance from $c_1$ and $c_2$ decreases; consequently, he will eventually be captured.

Suppose that the robber only moves right and down. Note that $c_3$ advances one column closer each time the robber moves down, and $c_4$ advances one row

closer each time the robber moves right. Hence, within $m+n$ rounds of the game, the robber reaches the same column as $c_3$ and/or the same row as $c_4$. From this point onward, if the robber moves right, he moves closer to $c_4$. Similarly, if the robber moves down, he moves closer to $c_3$. Thus, within an additional $m+n$ rounds, the robber must make a move other than right or down to avoid capture. Once this happens, the robber's distance from $c_1$ and $c_2$ decreases. Now $c_3$ and $c_4$ begin their strategy once more and this process repeats at least once every $2m+2n$ rounds. $\qquad\square$

Finally, we look at the three-dimensional directed Cartesian grid $\overrightarrow{P_m}\Box\overrightarrow{P_n}\Box\overrightarrow{P_r}$. In the standard variant of Cops and Robbers on undirected graphs, the cop number for $P_m\Box P_n\Box P_r$ is at most 3 (see [14]); here we show that $\log_3 d \leq c_w\left(\overrightarrow{P_m}\Box\overrightarrow{P_n}\Box\overrightarrow{P_r}\right) \leq 2d$ where $d = \min\{m,n,r\}$.

To do so, we require some additional terminology. On the Cartesian product graph $\overrightarrow{P_m}\Box\overrightarrow{P_n}\Box\overrightarrow{P_r}$, we say that the robber *evades* a cop if the robber occupies a vertex such that at least one coordinate is less than the corresponding coordinate of the cop's vertex; we refer to such a cop as an *evaded* cop. Note that an evaded cop cannot reach the robber's vertex. Similarly, we say that a robber *nearly evades* a cop if the robber occupies a vertex such that at least one coordinate is equal to the corresponding coordinate of the cop's vertex; we refer to such a cop as a *nearly evaded* cop.

**Lemma 3.3.4.** *Consider the weak game played on $\overrightarrow{P_n}\Box\overrightarrow{P_n}\Box\overrightarrow{P_n}$ where all cops start at a distance of at least $2n$ from the sink vertex $(n,n,n)$. The robber has a strategy to start on $(n,n,n)$, evade at least one cop, end up on a vertex in $\left\{\left(\left\lfloor\frac{n}{3}\right\rfloor,n,n\right),\left(n,\left\lfloor\frac{n}{3}\right\rfloor,n\right),\left(n,n,\left\lfloor\frac{n}{3}\right\rfloor\right)\right\}$, and remain at a distance of at least $2\left\lfloor\frac{n}{3}\right\rfloor$ from all remaining cops.*

*Proof.* Under the above conditions, the robber starts at the sink vertex $(n,n,n)$.

Suppose that one of the cops initially has a coordinate greater than $\left\lfloor \frac{n}{3} \right\rfloor$; without loss of generality, assume this cop starts at $(i, j, k)$ where $k > \left\lfloor \frac{n}{3} \right\rfloor$. The robber immediately moves $n - \left\lfloor \frac{n}{3} \right\rfloor$ steps to $\left(n, n, \left\lfloor \frac{n}{3} \right\rfloor\right)$ and in doing so evades this cop. Since all cops started a distance of at least $2n$ from the robber, no cop can have captured the robber during this time, and the remaining cops are now a distance of at least $2n - 2\left(n - \left\lfloor \frac{n}{3} \right\rfloor\right) = 2\left\lfloor \frac{n}{3} \right\rfloor$ from the robber.

Suppose instead that all cops initially start on vertices in $S = \left\{(i, j, k) \mid 1 \le i, j, k \le \left\lfloor \frac{n}{3} \right\rfloor\right\}$. The robber starts on the sink vertex $(n, n, n)$ and waits until some cop leaves $S$; without loss of generality, assume this cop moves to $\left(\left\lfloor \frac{n}{3} \right\rfloor + 1, j, k\right)$ where $1 \le j, k \le \left\lfloor \frac{n}{3} \right\rfloor$. As above, the robber moves $n - \left\lfloor \frac{n}{3} \right\rfloor$ steps to $\left(\left\lfloor \frac{n}{3} \right\rfloor, n, n\right)$ and in doing so evades this cop. Since all vertices in $S$ are a distance at least $2n$ from the robber, the remaining cops are once again a distance of at least $2n - 2\left(n - \left\lfloor \frac{n}{3} \right\rfloor\right) = 2\left\lfloor \frac{n}{3} \right\rfloor$ from the robber.

$\qquad\square$

**Lemma 3.3.5.** *For each positive integer $n$,*

$$c_w\left(\overrightarrow{P_n} \square \overrightarrow{P_n} \square \overrightarrow{P_n}\right) \le c_w\left(\overrightarrow{P_{n+1}} \square \overrightarrow{P_n} \square \overrightarrow{P_n}\right).$$

*Proof.* Let $G = \overrightarrow{P_{n+1}} \square \overrightarrow{P_n} \square \overrightarrow{P_n}$. Let $c_w\left(\overrightarrow{P_n} \square \overrightarrow{P_n} \square \overrightarrow{P_n}\right) = k$, and suppose there are fewer than $k$ cops playing the game on $G$. Hence, there are fewer than $k$ cops starting on the subgraph $H = \overrightarrow{P_n} \square \overrightarrow{P_n} \square \overrightarrow{P_n}$ induced by the vertices in $\{(i, j, k) \mid 1 \le i, j, k \le n\}$. Moreover, a cop located outside $H$ cannot reenter it. The robber therefore has a winning strategy using only vertices in $H$, and therefore wins the game on $G$.

$\qquad\square$

**Theorem 3.3.6.** *Consider the weak game played on $\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}$ and let $d = \min\{m, n, r\}$. If $d$ is sufficiently large, then $\lfloor \log_3 d \rfloor \le c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}\right) \le 2d$.*

*Proof.* We first show the lower bound. For simplicity, consider the case where $m = n = r$ and suppose $n = 3^\ell$ for some $\ell \in \mathbb{Z}^+$; the case for general $n$ follows from monotonicity due to Lemma 3.3.5. Suppose that there are $p$ cops in play, where $p < \log_3 n$. We give a strategy for the robber to evade all $p$ cops; it follows that $c_w\left(\overrightarrow{P_n} \square \overrightarrow{P_n} \square \overrightarrow{P_n}\right) \geq \log_3 n = \log_3 d \geq \lfloor \log_3 d \rfloor$.

Suppose, without loss of generality, that all cops start at the source vertex. The robber uses the strategy outlined in Lemma 3.3.4. Without loss of generality, he evades at least one cop and ends up on $(\frac{n}{3}, n, n)$ at a distance of at least $\frac{2n}{3}$ from all remaining cops. The robber now views $(\frac{n}{3}, n, n)$ as the sink vertex of the graph $\overrightarrow{P_{\frac{n}{3}}} \square \overrightarrow{P_{\frac{n}{3}}} \square \overrightarrow{P_{\frac{n}{3}}}$ formed by the vertices in

$T = \left\{(i, j, k) \mid 1 \leq i \leq \frac{n}{3}, \text{ and } \frac{2n}{3} + 1 \leq j, k \leq n\right\}.$ For each unevaded cop on $(i', j', k') \notin T$, the robber instead considers that cop as being located on her *shadow*, the vertex

$\left(\min\{i', \frac{n}{3}\}, \max\{j', \frac{2n}{3} + 1\}, \max\{k', \frac{2n}{3} + 1\}\right) \in T$; the robber now plays as if on $\overrightarrow{P_{\frac{n}{3}}} \square \overrightarrow{P_{\frac{n}{3}}} \square \overrightarrow{P_{\frac{n}{3}}}$. Note that each cop's shadow is never further from the robber than the cop itself, so if a cop's shadow is unable to capture the robber, then that cop is also unable to capture the robber. At this point, the conditions required for Lemma 3.3.4 are again satisfied.

From here, the robber iteratively repeats the above strategy by considering the remaining cops that have not yet been evaded. Each time the robber executes this strategy, we refer to this as a *phase* of the robber's gameplay. Hence, during phase $q$, the robber takes $\frac{2n}{3^q}$ steps in some direction, evades one cop, and ends up at a vertex that is a distance of at least $\frac{2n}{3^q}$ from all remaining cops. The robber then views this vertex as the sink of a copy of $\overrightarrow{P_{\frac{n}{3^q}}} \square \overrightarrow{P_{\frac{n}{3^q}}} \square \overrightarrow{P_{\frac{n}{3^q}}}$. The robber continues to execute phases until $\frac{n}{3^q} = 3$ and in doing so evades at least $q = \log_3 n - 1$ cops. Since the number of starting cops was $p < \log_3 n$, therefore the robber has evaded

all cops.

For the upper bound, we show a winning cop strategy using $2d$ cops in play. Without loss of generality, suppose $d = r$. We will refer to the set of vertices in $\{(i, j, k) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ for fixed $k$ as the $k^{th}$ *layer* of $\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}$. We start by positioning a pair of cops on the source vertex of each layer of the graph; note that each layer induces the graph $\overrightarrow{P_m} \square \overrightarrow{P_n}$.

Each pair of cops now ignores the third coordinate in their vertex location as well as the vertex location of the robber and plays the game on their respective layer. After each pair of cops executes the strategy outlined in Theorem 3.3.1 where $\overrightarrow{T_1} = \overrightarrow{P_m}$ and $\overrightarrow{T_2} = P_n$, the robber is captured on some layer of the graph. $\square$

Under certain natural restrictions on the cops' strategy, we can obtain a stronger lower bound on $c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}\right)$ than the one presented in Theorem 3.3.6.

**Theorem 3.3.7.** *Consider the weak game played on $\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}$ where each cop starts on $(1, 1, 1)$ and must move each round. Let $d = \min\{m, n, r\}$. If $d$ is sufficiently large, then $c_w\left(\overrightarrow{P_m} \square \overrightarrow{P_n} \square \overrightarrow{P_r}\right) \geq \frac{8}{27} d^{1 - \log_3 2}$.*

*Proof.* For simplicity, consider the case where $m = n = r$; a similar argument suffices for the general case. Suppose there are $k$ cops in play where $k < \frac{8}{27} n^{1 - \log_3 2}$. We give a strategy for the robber to evade all $k$ cops; it follows that $c_w\left(\overrightarrow{P_n} \square \overrightarrow{P_n} \square \overrightarrow{P_n}\right) \geq \frac{8}{27} n^{1 - \log_3 2} = \frac{8}{27} d^{1 - \log_3 2}$.

The robber starts at the sink vertex $(n, n, n)$ and stays put for $2n - 3$ rounds. At this point, all cops are exactly distance $n$ from the robber. By the pigeonhole principle, for each cop in play there exists some coordinate $i$ such that the distance $\ell$ from the cop to the robber in coordinate $i$ satisfies $\ell \leq \lfloor \frac{n}{3} \rfloor$; we will refer to any such cop as an *i-vulnerable* cop. Consequently, there exists some coordinate $i$ such that there are at least $\frac{k}{3}$ $i$-vulnerable cops. For his next $\lfloor \frac{n}{3} \rfloor$ turns, the robber moves forward toward the source in coordinate $i$. As a result, the robber's $i^{th}$

coordinate is now less than or equal to that of all $i$-vulnerable cops, so these cops are all either evaded or nearly evaded. Furthermore, the distance between the robber and the remaining cops after the cops' turn is at least $\lceil \frac{n}{3} \rceil$. The robber now waits until the unevaded cops are exactly a distance of $\lfloor \frac{n}{3} \rfloor$ away before moving again.

From here, the robber iteratively repeats the above strategy considering only those cops that have not yet been evaded or nearly evaded; each time the robber executes this strategy, we will refer to this as a *phase* of the robber's gameplay. Hence, during phase $j$, the robber takes $\lfloor \frac{n}{3^j} \rfloor$ steps toward the cops that have not been evaded or nearly evaded in some coordinate $i$ where at least $\frac{1}{3}$ of these cops are $i$-vulnerable. At this point, the distance between the robber and the remaining cops after the cops' turn is at least $\lceil \frac{n}{3^j} \rceil$, and the robber waits until the remaining cops are exactly a distance of $\lfloor \frac{n}{3^j} \rfloor$ away before starting the next phase.

The robber continues to execute phases as long as he can do so without having taken more than $\frac{n}{2} - 3$ steps from the sink vertex. Under these conditions, let $p$ be the number of phases the robber is able to successfully execute. Note that the robber cannot have been captured during this process since the cops are still at least a distance of 6 away after the $p^{\text{th}}$ phase. Since the robber cannot execute a $(p+1)^{\text{th}}$ phase,

$$\frac{n}{2} - 3 \le \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor + \cdots + \left\lfloor \frac{n}{3^{p+1}} \right\rfloor$$

$$\frac{n}{2} \le \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor + \cdots + \left\lfloor \frac{n}{3^{p+1}} \right\rfloor + 3$$

$$\le \left( \frac{n}{3} + \frac{n}{9} + \cdots + \frac{n}{3^{p+1}} \right) + 3$$

$$= \frac{n}{3} \cdot \left( 1 + \frac{1}{3} + \cdots + \frac{1}{3^p} \right) + 3$$

$$= \frac{n}{3} \cdot \frac{1 - 1/(3^{p+1})}{1 - 1/3} + 3$$

$$= \frac{n}{2} \cdot \left( 1 - \frac{1}{3^{p+1}} \right) + 3.$$

Hence,

$$3 \ge \frac{1}{3^{p+1}} \cdot \frac{n}{2}$$

$$3^{p+2} \ge \frac{n}{2}$$

$$p \ge \log_3 n - \log_3 2 - 2$$

$$\ge \log_3 n - 3$$

Let $k_i$ be the number of cops evaded or nearly evaded during the first $i$ phases of the game; we claim that $k_i \ge k - \left( \frac{2}{3} \right)^p k$ for all $i \ge 1$. During the first round, the robber's strategy ensures that $k_1 \ge \left\lceil \frac{k}{3} \right\rceil \ge \frac{k}{3} = k - \left( \frac{2}{3} \right) k$, so the claim holds for $i = 1$. Assuming the claim holds for some fixed $j \ge 1$, the robber's strategy ensures during the next phase of the game that

$$k_{j+1} \geq k_j + \frac{1}{3}(k - k_j)$$

$$= \frac{2}{3}k_j + \frac{1}{3}k$$

$$\geq \frac{2}{3}\left[k - \left(\frac{2}{3}\right)^j k\right] + \frac{1}{3}k$$

$$= k - \left(\frac{2}{3}\right)^{j+1} k.$$

Thus, during the first $p$ phases of the game, the robber evades or nearly evades $k_p$ cops where $k_p \geq k - \left(\frac{2}{3}\right)^p$. Since the robber is a distance of at least 6 away from the cops, he can now safely move one step towards the source in each direction and thus evades all of these cops. If we let $t$ be the number of cops not yet evaded by the robber, then $t \leq k - k_p \leq \left(\frac{2}{3}\right)^p k \leq \left(\frac{2}{3}\right)^{\log_3 n - 3} k$. Given that $k < \frac{8}{27}n^{1-\log_3 2}$, we have

$$t < \left(\frac{2}{3}\right)^{\log_3 n - 3} \cdot \frac{8}{27}n^{1-\log_3 2}$$

$$= \frac{27}{8}\left(\frac{2^{\log_3 n}}{3^{\log_3 n}}\right) \cdot \frac{8}{27}n^{1-\log_3 2}$$

$$= \frac{3^{\log_3(2^{\log_3 n})}}{n} \cdot n^{1-\log_3 2}$$

$$= \frac{3^{\log_3 n \cdot \log_3 2}}{n} \cdot n^{1-\log_3 2}$$

$$= \frac{n^{\log_3 2}}{n} \cdot n^{1-\log_3 2}$$

$$= 1.$$

Thus, $t = 0$ and the robber has actually already evaded all cops in play.

$\square$

## 3.4   Planar Graphs

In this section, we consider planar graphs. It is well-known that undirected outerplanar graphs have cop number at most 2 and undirected planar graphs have cop number at most 3 (see [1, 4]). We say that a directed graph is *strongly-connected* if, for any two vertices $u$ and $v$, there is both a directed path from $u$ to $v$ and a directed path from $v$ to $u$. Recently, Khatri et al. showed that strongly-connected outerplanar oriented graphs have cop number at most 2 (see [10]) in the standard model of the game. Moreover, Loh and Oh proved there exists a strongly-connected planar directed graph with cop number at least 4 (see [12]). It is still an open question to determine whether the cop number of planar directed graphs can be bounded above by a constant. Here, for each $n \in \mathbb{N}$, we provide a construction of a strongly-connected planar directed graph with weak directed cop number at least $n$.

We first establish an upper bound on the weak directed cop number for outerplanar graphs.

**Theorem 3.4.1.** *If $G$ is a strongly-connected outerplanar directed graph, then $c_w(G) \leq 3$ and this is tight.*

*Proof.* We first provide a strategy for three cops $c_1$, $c_2$, and $c_3$ to win the game. Suppose first that $G$ is 2-connected, in which case we embed $G$ in the plane so that all vertices lie on the outside face with no edge crossings. Choosing an arbitrary vertex and calling it $v_1$, we enumerate the vertices of $G$ as $v_1, ..., v_n$ in clockwise order around the outside face starting at $v_1$.

We initially position $c_1$ at $v_1$, $c_2$ at the other endpoint of any edge containing $v_1$ (call this endpoint $v_j$), and $c_3$ at $v_2$. Suppose without loss of generality that the robber starts at a vertex in the arc $\{v_2, ..., v_{j-1}\}$. The cops move as follows:

Let $v_i$ be the greatest-indexed vertex in $\{v_2, ..., v_{j-1}\}$ such that $v_i$ shares an

edge with $v_1$ (we note such a vertex exists since, at the very least, $v_1$ and $v_2$ share an edge). Initially, $c_1$ and $c_2$ stay put, while $c_3$ moves to $v_i$ (perhaps over the course of many turns) which is possible since $G$ is strongly-connected. Since there is an edge between $v_1$ and $v_j$, and since this embedding of $G$ has no edge crossings, there is no edge between a vertex in $\{v_2, ... v_{j-1}\}$ and any vertex in $\{v_{j+1}, ..., v_n\}$. As a result, with $c_1$ and $c_2$ staying put, the robber cannot leave $\{v_2, ..., v_{j-1}\}$ while the aforementioned movement of $c_3$ is occurring.

After $c_3$ reaches $v_i$, the robber is either in the arc $\{v_2, ..., v_{i-1}\}$ or the arc $\{v_{i+1}, ..., v_{j-1}\}$. Suppose first that the robber is in $\{v_2, ..., v_{i-1}\}$. Since $v_1$ and $v_i$ share an edge, and there are no edge crossings in this embedding of $G$, the robber cannot leave this arc so long as $c_1$ and $c_3$ remain on $v_1$ and $v_i$, respectively. Suppose instead that the robber is in the arc $\{v_{i+1}, ..., v_{j-1}\}$. In this case, $v_1$ shares an edge each with $v_i$ and $v_j$, and, by choice of $v_i$, shares no edge with any vertex internal to this arc. Hence, there is an undirected cycle formed by $v_1$, $v_i$, the undirected path of vertices in this arc, and $v_j$. Since there are no edge crossings in this embedding of $G$ and no chords inside this cycle incident to $v_1$, any edge incident to a vertex internal to this arc must have both endpoints in $\{v_i, ..., v_j\}$. The robber therefore cannot leave this arc so long as $c_3$ and $c_2$ remain on $v_i$ and $v_j$, respectively. We see that in either case, two cops can confine the robber to the arc $\{v_2, ..., v_{i-1}\}$ or the arc $\{v_{i+1}, ..., v_{j-1}\}$ by remaining in place.

We now repeatedly iterate this strategy where two cops stand guard over the arc the robber is in while the third cop moves to an appropriately-chosen vertex inside this arc. It is clear that each time this strategy is executed, the size of the arc that the robber is in decreases. Since $G$ is finite, the robber is eventually captured.

Suppose now that $G$ is not 2-connected. Since $G$ is strongly-connected and

outerplanar, so is each block of $G$. Hence, we may embed each block in the plane so that all vertices lie on the outside face with no edge crossings. Initially, we position the three cops in some block $H$ of $G$ according to the strategy outlined above. Note that the robber is either initially positioned in $H$ or in some other block such that every path from $H$ to that block uses some cut-vertex $u$ in $H$.

The cops now use the strategy outlined above, except that if the robber is not in $H$, then the cops move as if the robber is on the corresponding cut-vertex $u$ as described above. As a result, either the cops capture the robber on $H$, or some cop ends up on some cut-vertex $u$ such that every path from $H$ to the robber's current location must pass through $u$. In the latter case, the remaining cops also move to $u$. Let $H_1$ be the block of $G$ such that $V(H) \cap V(H_1) = \{u\}$ and every path from $u$ to the robber's current location must pass through $H_1$. We now treat $u$ as $v_1$ and the cops execute this same strategy in $H_1$. Note that since this strategy begins with $c_1$ starting on $u$, the robber can never reach $u$, and thereby can never reach $H$.

By iterating this strategy, the robber is never able to reach a block of $G$ that the cops have already visited, so since $G$ is finite, the robber is eventually captured. Thus, for any strongly-connected outerplanar directed graph $G$, it follows that $c_w(G) \leq 3$.

To achieve tightness of this upper bound, we present a strongly-connected outerplanar oriented graph $G$ in which $c_w(G) = 3$ in Figure 10.

First, we provide a winning robber strategy for $G$ with two cops $c_1$ and $c_2$ in play. Since there are only two cops, it follows that there is some $v_i$ where $1 \leq i \leq 4$ such that neither cop starts in $N^-[v_i]$. Without loss of generality, suppose the robber can start on $v_1$ and avoid capture in the first round. The robber now stays put until some cop reaches the unique in-neighbor of $v_1$. At this
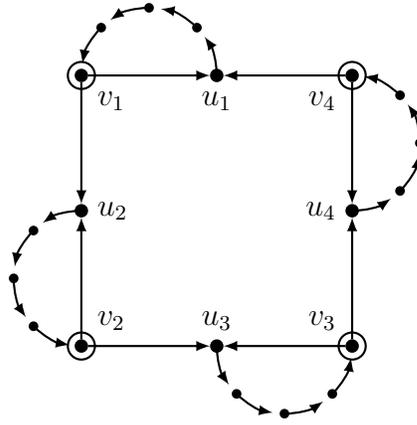
Figure 10. Strongly-connected outerplanar graph $G$ where $c_w(G) = 3$

point, if the other cop is on $u_2$, $v_2$, or within distance 2 of the unique in-neighbor of $v_2$, then the robber moves to $u_1$ and next $v_4$; neither cop can catch the robber while this movement is occurring. If this is not the case, then the robber moves to $u_2$ and next $v_2$; similarly, neither cop can capture the robber while this movement is occurring. The robber perpetually repeats this strategy and therefore wins the game. □

In the next theorem, we use the graph in Figure 10 to construct a strongly-connected directed planar graph with high weak directed cop number (the graph we construct is in fact an oriented graph; note that a doubly-directed edge is the same as an edge pointing in only one direction from the robber's perspective, but the latter of these is more limiting for the cops' movement).

**Theorem 3.4.2.** *For each $n \in \mathbb{N}$, there exists a strongly-connected directed planar graph with $c_w(G) \geq n$.*

*Proof.* Let $n \in \mathbb{N}$. We will construct a graph $G$ with $c_w(G) \geq n$. First, take the strongly-connected outerplanar directed graph presented in Theorem 3.4.1, and extending the directed paths between each $u_i$ and $v_i$ to be of length $25n + 1$; we will refer to this graph as a *cell* of $G$. We then let $G$ be the graph containing

$(5n + 5)^2$ cells sorted into $5n + 5$ rows and $5n + 5$ columns and refer to the cell in row $i$ and column $j$ as $G_{i,j}$. We connect adjacent cells with edges as follows:

- For each $1 \leq i \leq 5n + 5$ and each $1 \leq j \leq 5n + 4$, add an edge directed from $v_3$ in $G_{i,j}$ to $u_2$ in $G_{i,j+1}$.

- For each $1 \leq i \leq 5n + 5$ and each $2 \leq j \leq 5n + 5$, add an edge directed from $v_1$ in $G_{i,j}$ to $u_4$ in $G_{i,j-1}$.

- For each $1 \leq i \leq 5n + 4$ and each $1 \leq j \leq 5n + 5$, add an edge directed from $v_2$ in $G_{i,j}$ to $u_1$ in $G_{i+1,j}$.

- For each $2 \leq i \leq 5n + 5$ and each $1 \leq j \leq 5n + 5$, add an edge directed from $v_4$ in $G_{i,j}$ to $u_3$ in $G_{i-1,j}$.

Note that there is now a directed path from any cell to every other, and since each cell itself is strongly-connected, $G$ is therefore strongly-connected. We also see that $G$ is planar since each cell is planar and the additional edges drawn as described above can be drawn without introducing edge crossings. Since $G$ contains $5n + 5$ rows and columns and there are $n$ cops in play, by the Pigeonhole Principle, there is a set of at least five consecutive rows with no cop and a set of at least five consecutive columns with no cop, both at the start of the game and after every cop turn. We will refer to a row (respectively, column) of cells in $G$ in which the nearest cop is at least 3 rows away (respectively, at least 3 columns away) as an *ideal row* (respectively, *ideal column*). Similarly, we will refer to a row (respectively, column) of cells in $G$ in which the nearest cop is at least 2 rows away (respectively, at least 2 columns away) as a *safe row* (respectively, *safe column*). Hence, at any point in the game, there is an ideal row/column in $G$ whose adjacent row(s)/column(s) are necessarily safe row(s)/column(s). A winning strategy for the robber is as follows:
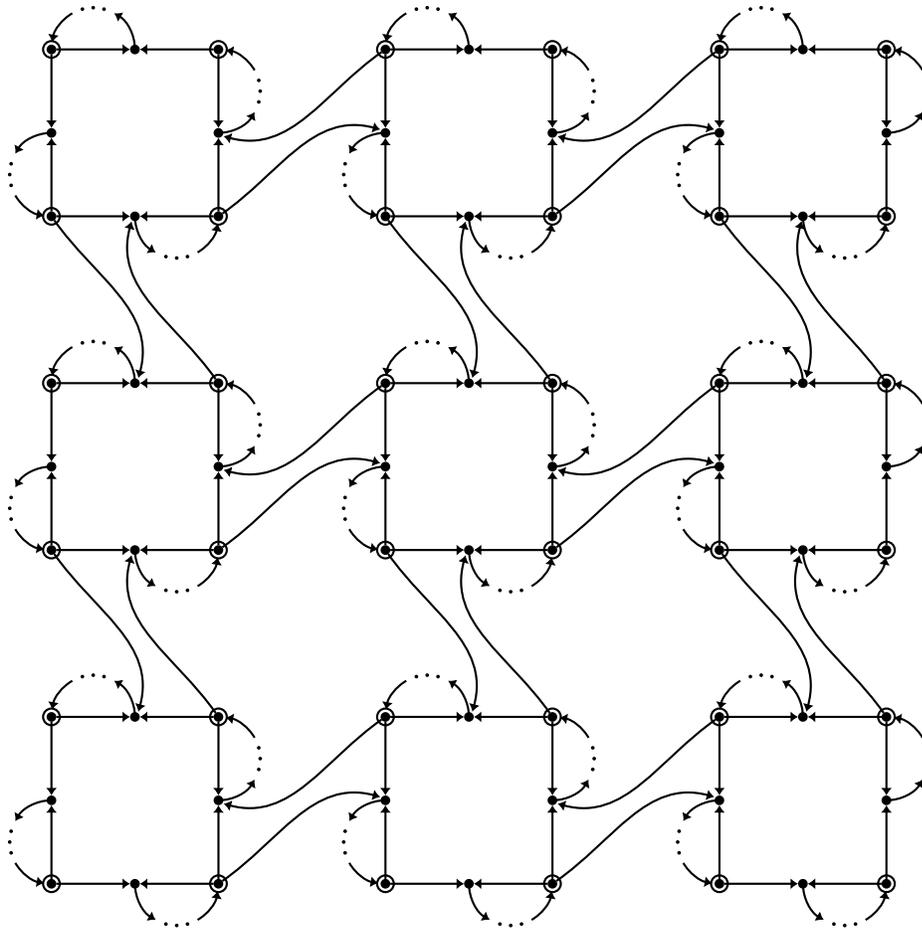
Figure 11. Construction of strongly-connected planar graph in Theorem 3.4.2 with 3 rows and 3 columns

The robber initially starts at any vertex in a cell that's located in an ideal row. Note that the robber can clearly move from one cell in $G$ to any adjacent cell in no more than 5 rounds. He now waits until the row is no longer ideal. Once this happens, he moves along his original row to a cell that is located in an ideal column, which he can do in no more than $5(5n) = 25n$ rounds. Since each cop must traverse at least one directed path of length $25n + 1$ to move a distance of two cells in any direction, no cop can reach the robber's row during this time, so the robber evades capture. Once the robber reaches this column, it may no longer be ideal due to the cops' movement, but it is still safe. The robber now waits until the column is no longer safe. Once this happens, he moves along this column to a cell that is located in an ideal row, which he can do in no more than $5(5n) = 25n$ rounds. As above, he avoids capture from the cops during this time. Once the robber reaches this row, it may no longer be ideal due to the cops' movement, but it is still safe.

The robber can repeat this strategy of alternating between moving to an ideal row and ideal column, and indefinitely avoids capture.

$\square$

For each $k \in \mathbb{N}$, the construction in Theorem 3.4.2 uses at least $(5k + 5)^2 \cdot 4(25k + 1) = \Theta(k^3)$ vertices and has weak directed cop number at least $k$. Hence, this result shows that for each $n \in \mathbb{N}$, there exists a strongly-connected planar directed graph $G$ of size $n$ with weak directed cop number $\Omega(\sqrt[3]{n})$. It would be interesting to determine the maximum weak directed cop number of an $n$-vertex strongly-connected planar directed graph, as it has been shown in the standard variant on directed graphs that every $n$-vertex strongly-connected planar directed graph has cop number at most $O(\sqrt{n})$ (see [12]) in the standard model.

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## 3.5  Open Problems

We conclude with suggestions for future problems to investigate pertaining to the weak game.

- **Characterize cop-win directed graphs for the weak game**. The condition stated in Theorem 3.2.5 has been shown to be sufficient but not necessary for all directed graphs to be cop-win. Since this condition characterizes oriented graphs, perhaps it can be built upon to provide a full characterization for all directed graphs.

- **Investigate general Cartesian products of trees and/or cycles**. The strategies and techniques used in Theorem 3.3.1 and Theorem 3.3.3 may extend to general Cartesian products of trees and/or cycles.

- **Improve the bounds on the three-dimensional directed Cartesian grid**. We are not sure if either the lower or upper bounds in Theorem 3.3.6 are asymptotically tight or not; this requires deeper analysis.

- **Examine directed Cartesian grids of arbitrary dimension**. Perhaps the arguments used in Theorem 3.3.6 or Theorem 3.3.7 can somehow be extended to establish bounds for the weak directed cop number on directed Cartesian grids of arbitrary dimension.

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