

2020

ON THE HAVEL-HAKIMI RESIDUE OF DEGREE SEQUENCES AND ITS RELATION TO THE INDEPENDENCE NUMBER

Benjamin Lantz
University of Rhode Island, benjamin_lantz@uri.edu

Follow this and additional works at: https://digitalcommons.uri.edu/oa_diss

Recommended Citation

Lantz, Benjamin, "ON THE HAVEL-HAKIMI RESIDUE OF DEGREE SEQUENCES AND ITS RELATION TO THE INDEPENDENCE NUMBER" (2020). *Open Access Dissertations*. Paper 1158.
https://digitalcommons.uri.edu/oa_diss/1158

This Dissertation is brought to you for free and open access by DigitalCommons@URI. It has been accepted for inclusion in Open Access Dissertations by an authorized administrator of DigitalCommons@URI. For more information, please contact digitalcommons@etal.uri.edu.

ON THE HAVEL-HAKIMI RESIDUE OF DEGREE SEQUENCES AND ITS
RELATION TO THE INDEPENDENCE NUMBER

BY
BENJAMIN LANTZ

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

UNIVERSITY OF RHODE ISLAND

2020

DOCTOR OF PHILOSOPHY DISSERTATION
OF
BENJAMIN LANTZ

APPROVED:

Dissertation Committee:

Major Professor Michael Barrus

William Kinnersley

Noah Daniels

Nassar H. Zawia

DEAN OF THE GRADUATE SCHOOL

UNIVERSITY OF RHODE ISLAND

2020

ABSTRACT

The Havel-Hakimi residue (or residue) of a graph is the number of zeros left after iteratively applying the Havel-Hakimi algorithm to a degree sequence. Favaron, Mahéo, and Saclé showed that the residue is a lower bound on the independence number. Determining how good of a bound this is remains an open question, including in what cases the bound is realized.

This dissertation looks to help answer when the bound is realized by examining the Maxine heuristic, which reduces a graph G , to an independent set of size $M(G)$. It has been shown that given a graph G , $M(G)$ is bounded between the independence number and the residue of a graph. We find a class of graphs characterized by a list of forbidden subgraphs, an improvement on a list from Barrus and Molnar, such that $M(G)$ is equal to the independence number for all graphs in the class.

Furthermore, to help understand the relationship between the independence number and the residue, the number of reorderings required in the Havel-Hakimi algorithm is found for all regular sequences. It is known that threshold degree sequences, a well known family of degree sequences, have independence number equal to the residue. This dissertation shows that threshold degree sequences require no reorderings and thus begs the question if the number of reorderings is related to the difference in the bound between the independence number and the residue. Then the cases of one reordering and a maximum number of reorderings is determined and analyzed.

ACKNOWLEDGMENTS

I would like to thank all of those who have supported me throughout my academic career. Specifically, I would like to thank my advisor Michael Barrus in helping me pose interesting questions and in being a sounding board when I needed one most. Furthermore I would like to thank my wife Orsi who has sacrificed so much in order for me to complete my program. Her, alongside my dog Echo, have been the best team I could have asked for through this journey.

Contents

ABSTRACT	ii
ACKNOWLEDGMENTS	iii
Contents	iv
List of Figures	vi
Chapter	
1 Introduction	1
1.1 Introduction to graphs	1
1.2 Introduction to degree sequences	3
1.3 The Havel-Hakimi Algorithm	4
List of References	6
2 Graphs in which the Maxine heuristic produces a maximum independent set	8
2.1 Introduction	9
2.2 Preliminary lemmas	11
2.3 The case with independence number equal to 3	13
2.4 The case with independence number greater than 3	17
2.5 Open questions	20
List of References	21
3 Regular sequence reorderings	22
3.1 Introduction	23

	Page
3.2 The next regular sequence in the Havel-Hakimi algorithm when $\gcd(a + 1, n - 1) = 1$	29
3.3 The number of reorderings that appear in the Havel-Hakimi al- gorithm when $\gcd(a + 1, n - 1) = 1$	40
3.4 The next regular sequence in the Havel-Hakimi algorithm when $\gcd(a + 1, n - 1) = d$	43
3.5 The number of reorderings that appear in the Havel-Hakimi al- gorithm when $\gcd(a + 1, n - 1) = d$	53
3.6 Open questions	58
List of References	58
4 Minimum and maximum number of reorderings	60
4.1 Introduction	61
4.2 Zero reorderings	63
4.3 One reordering	64
4.4 Maximum number of reorderings	68
4.5 Independence number and maximum number of reorderings . . .	72
4.6 Open questions	73
List of References	74
BIBLIOGRAPHY	75

List of Figures

Figure		Page
1	A cycle on 6 vertices and its complement	3
2	The two distinct realizations of (22211)	4
3	The Q and N sets with independent set $\{v, u, w\}$	13
4	The Havel-Hakimi algorithm applied to (5^{14})	25
5	A generalized version of the Josephus problem on 14 people with a label of 5	27
6	The Havel-Hakimi algorithm applied to (5^{14}) and the corre- sponding X_{a-j} sets	28
7	The Havel-Hakimi algorithm on (4^{13})	40
8	The Havel-Hakimi algorithm on (10^{20})	42
9	The Havel-Hakimi algorithm applied to (13^{22})	52

CHAPTER 1

Introduction

1.1 Introduction to graphs

Graph theory is the study of a collection of objects and the connections between said objects, represented by a graph. The popularity of graph theory has increased in the last century because of the wide breadth of applications in mathematics and other disciplines, as most any application can be modeled with a graph. Note that the contents of this section can be found in any graph theory textbook; in this dissertation Reinhard Diestel's book on graph theory was used [1]. A graph $G = (V, E)$ is a set of edges E , and vertices V , where an edge is an unordered pair of vertices representing a connection between those vertices. In this dissertation we will only consider simple graphs, that is graphs in which there can be no repeated edges and edges must only exist between distinct vertices (i.e. no loops). Oftentimes results made about simple graphs can be generalized to include graphs that contain multiple edges and loops.

We say that for $u, v \in V$, u and v are adjacent if $[u, v] \in E$ (i.e. that a connection between the vertices exists). We will denote adjacency between u and v with $u \sim v$ and non-adjacency between vertices with $u \not\sim v$. The neighborhood of a vertex $v \in V$ is the set of adjacent vertices to v , denoted $N(v)$. The size of the neighborhood of v , $|N(v)|$ is said to be the degree of that vertex. Given a set of vertices $U \subseteq V$, we have that the neighborhood of U , denoted $N(U)$ is defined to be the set of all vertices of $G \setminus U$ adjacent to vertices in U . We say that a vertex v dominates a set of vertices U if every vertex in U is adjacent to v . Likewise, a vertex v is isolated from a set of vertices U if every vertex in U is non-adjacent to v .

Oftentimes we will only be interested in a certain part of a graph. We will

call the graph $H = (V', E')$ a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Given a specific subset of vertices $U \subseteq V$, we have the induced subgraph of G on U , denoted $G[U]$ is the subgraph consisting of the vertex set U and whose edge set consists of elements $[u, v] \in E$ such that both u and v are vertices of U .

Sometimes it is useful to describe a graph by subgraphs that are absent. Given a graph H , we say that a graph G is H -free (that is G forbids H) if H is not an induced subgraph of G . Moreover, given a family of graphs \mathcal{S} , we say a graph G is \mathcal{S} -free if no graph in \mathcal{S} appears as an induced subgraph of G .

We will give special names to well known graphs that are used consistently through this dissertation. We will call a path on n vertices (consisting of $n - 1$ edges) P_n and a cycle on n vertices, C_n . A graph on n vertices with every possible edge present is known as the complete graph, denoted K_n . We will reference a complete set of vertices of a graph as a clique. A bipartite graph is one in which no odd cycle (C_{2n-1}) is present as a subgraph of the graph, and a complete bipartite graph, denoted $K_{m,n}$ is a graph with two sets of vertices of A and B of size m and n respectively with every possible edge between them, but with no edge present within A or B .

The complement of a graph G , denoted \overline{G} , has the same vertex set as G , but whose edge set is the complement of that of G . So for example, in Figure 1 we can see C_6 and $\overline{C_6}$ both pictured.

The independence number of a graph, denoted $\alpha(G)$ is said to be the size of the largest set of pairwise non-adjacent vertices in G . An open problem in graph theory is to find a computationally easy way to determine $\alpha(G)$ for any graph, or to bound $\alpha(G)$ in an optimal way, as finding $\alpha(G)$ is NP-hard [2]. In fact, this open problem can be translated to fit many different parameters of a graph.

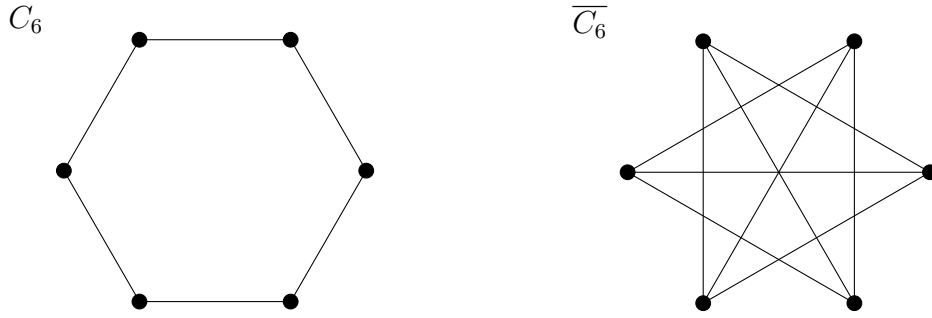


Figure 1: A cycle on 6 vertices and its complement

1.2 Introduction to degree sequences

A list of non-negative integers (d_1, d_2, \dots, d_n) where $d_i \geq d_{i+1}$, is said to be a degree sequence if d_i is the degree of a vertex v_i in a simple graph $G(V, E)$ where $V = \{v_1, \dots, v_n\}$ (i.e. the list of integers represents the degrees of a graph). The graph corresponding to the degree sequence is said to be a realization of the degree sequence. In this case it is said that the list of integers is graphic. In order for an integer sequence to be graphic, the sum of the entries must be even, as each edge must be counted twice when summing up the degrees. This result is often known as the Handshaking Lemma [1].

Starting in the 1950's, there have been many characterizations of graphic sequences. In this dissertation we will reference those built by Erdős and Gallai [3], Havel [4], and Hakimi [5] and focus specifically on the algorithm constructed independently by Havel and Hakimi, detailed in Section 1.3.

Theorem 1 (Erdős-Gallai Inequalities [3]). *A list (d_1, \dots, d_n) of non-negative integers in non-increasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i), \text{ for } 1 \leq k \leq n.$$

A graphic degree sequence can have more than one realization. One example of this is the degree sequence $(2, 2, 2, 1, 1)$ which has two distinct realizations, shown

in Figure 2. Note that for degree sequences with all single digit entries, we will omit the commas, hence $(2, 2, 2, 1, 1)$ will be written as (22211) .

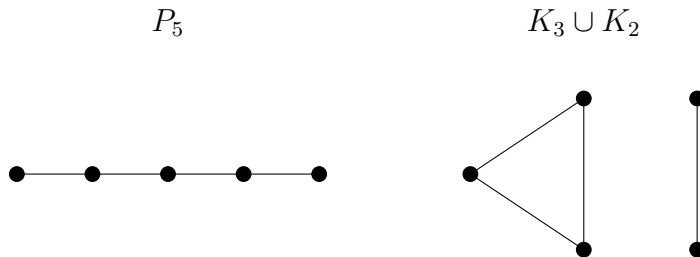


Figure 2: The two distinct realizations of (22211) .

A type of degree sequence that will be studied heavily is a threshold sequence, which is the degree sequence of a threshold graph. There are many characterizations of threshold graphs, but we will define a threshold graph G to be one that can be built by adding a sequence of isolated or dominating vertices to an isolated vertex [6]. Another characterization of threshold sequences involves the Erdős-Gallai inequalities.

Theorem 2 ([7]). *Let $d = (d_1, \dots, d_n)$ be a degree sequence with $m(d) = \max\{i \mid d_i \geq i - 1\}$. Then d is a threshold sequence if and only if the k^{th} Erdős-Gallai inequality is satisfied with equality for all $k \leq m(d)$.*

1.3 The Havel-Hakimi Algorithm

Havel in 1955 [4] and Hakimi in 1962 [5] independently introduced an algorithm used to determine if an integer sequence is graphic or not.

Theorem 3 (The Havel-Hakimi Algorithm[4][5]). *Let $d = (d_1, \dots, d_n)$ with $d_1 > 0$ be a non-negative, non-increasing integer sequence. Then d is graphic if and only if $H(d) = (d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is a graphic integer sequence.*

We will call the reduction of the integer sequence d , seen in Theorem 3, a Havel-Hakimi step. As long as the resulting integer sequence has non-negative

entries with positive first entry, we can iterate the Havel-Hakimi step. We will let the i^{th} application of the Havel-Hakimi step on d be labelled as $H^i(d)$, and the j^{th} entry of said integer sequence, $H_j^i(d)$.

After repeated applications of Havel-Hakimi steps, we will eventually return either an integer sequence with negative integers or a sequence of zeros. Since a degree sequence must contain only non-negative entries, if negative entries appear we must have that the original degree sequence is not graphic. Furthermore, a list of zeros is graphic as it can be realized by a set of isolated vertices of degree zero and once we have a list of zeros the algorithm terminates as the first entry is not positive. Thus we have,

Corollary 1. *Let $d = (d_1, \dots, d_n)$ with $d_1 > 0$ be a non-negative, non-increasing integer sequence. Then d is graphic if and only if the Havel-Hakimi algorithm produces a finite list of zeros.*

Given a graphic integer sequence d , the number of zeroes left after the terminal step of the Havel-Hakimi algorithm is the Havel-Hakimi residue (or residue) of d , denoted $R(d)$. Similarly, if a graph G is realized by a degree sequence d , then we say that the residue of G is the residue of d , $R(G) = R(d)$.

The residue is of interest because of its connection to the independence number of a graph, $\alpha(G)$. In 1988, the conjecture-making computer program Graffiti [8] proposed the following theorem,

Theorem 4 ([9]). *For every graph G , $R(G) \leq \alpha(G)$.*

This result was proven by Favaron et al. in 1991 and improved upon by Griggs and Kleitman [10], Triesch [11], and Jelen [12] in the 1990's. Determining the independence number is NP-hard [2], but since it takes only $O(E)$ steps to determine the residue where E is the number of edges in a graph, it is of interest to know how well $R(G)$ approximates $\alpha(G)$ and when the bound is realized.

The bound has been shown to be arbitrarily weak by considering the example $K_{n,n}$, the complete bipartite graph on $2n$ vertices. The independence number of $K_{n,n}$ is n while the residue is 2. Thus as n gets large, the difference in $\alpha(K_{n,n})$ and $R(K_{n,n})$ gets large.

It is often useful to consider a graphic realization of a degree sequence in relation to the Havel-Hakimi algorithm. Let G be a realization of a degree sequence d , and let $v \in V(G)$. We say that v has the Havel-Hakimi property if v is of maximum degree and its neighbors are vertices of highest possible degree. This vertex is named such because a graphic realization of $H(d)$ is $G \setminus \{v\}$, where a step in the Havel-Hakimi algorithm corresponds to the deletion of a vertex in a realization. Consider the example of (22211) , which has two graphic realizations as seen in Figure 2. In the P_5 realization, the center vertex has the Havel-Hakimi property, but the other two max degree vertices do not, while in the $K_3 \cup K_2$ realization, all max degree vertices have the Havel-Hakimi property. We will see in this dissertation that graphs that have max degree vertices without the Havel-Hakimi property will have the potential to have a difference in the residue and independence number.

List of References

- [1] R. Diestel, *Graph Theory 5th edn.* Berlin: Springer, 2017.
- [2] M. R. Garey and D. S. Johnson, *Computers and intractability.* freeman San Francisco, 1979, vol. 174.
- [3] P. Erdős and T. Gallai, “Graphs with prescribed degrees of vertices,” *Mat. Lapok*, vol. 11, pp. 264–274, 1960.
- [4] V.Havel, “A remark on the existence of finite graphs,” *Časopis Pěst Mat.*, vol. 80, pp. 477–480, 1955.
- [5] S. Hakimi, “On the realizability of a set of integers as degree sequences of the vertices of a graph,” *SIAM Appl. Math.*, vol. 10, pp. 496–506, 1962.

- [6] M. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*. New York: Academic Press, 1980.
- [7] N. Mahadev and U. Peled, *Threshold Graphs and Related Topics*. Amsterdam: Elsevier Science B.V., 1995.
- [8] S. Fajtlowicz, “On conjectures of graffiti, iii,” *Cong. Numer.*, vol. 66, pp. 23–32, 1988.
- [9] O. Favaron, M. Mahéo, and J. F. Saclé, “On the residue of a graph,” *Graph Theory*, vol. 15, no. 1, pp. 39–64, 1991.
- [10] J. Griggs and D. Kleitman, “Independence and the havel-hakimi residue,” *Discrete Mathematics*, vol. 127, no. 1–3, pp. 209–212, 1994.
- [11] E. Triesch, “Degree sequences of graphs and dominance order,” *J. Graph Theory*, vol. 22, no. 1, pp. 89–93, 1996.
- [12] F. Jelen, “ k -independence and k -residue of a graph,” *J. Graph Theory*, vol. 127, pp. 209–212, 1999.

CHAPTER 2

Graphs in which the Maxine heuristic produces a maximum independent set

Benjamin Lantz

Department of Mathematics

University of Rhode Island

Kingston, RI 02881

This manuscript was submitted to *Graphs and Combinatorics* on October 10, 2019.

Keywords: degree sequence, Havel-Hakimi Algorithm, Havel-Hakimi residue, Maxine heuristic, independence number

AMS 2010 Mathematics Subject Classification: 05C35

Abstract. The residue of a graph is the number of zeros left after iteratively applying the Havel-Hakimi algorithm to its degree sequence. Favaron, Mahéo, and Saclé showed that the residue is a lower bound on the independence number. The Maxine heuristic reduces a graph to an independent set of size M . It has been shown that given a graph G , M is bounded between the independence number and the residue of a graph for any application of the Maxine heuristic. We improve upon a forbidden subgraph classification of graphs such that M is equal to the independence number given by Barrus and Molnar in 2015.

2.1 Introduction

We will be considering simple graphs and we will let $N(v)$ represent the neighborhood of a vertex v in a graph, and let $u \sim v$ represent that u and v are adjacent in the graph. For such a graph G and subset of vertices U in the graph, let $G[U]$ be the induced subgraph on the set U . For a set of graphs \mathcal{S} , a graph G is said to be \mathcal{S} -free, if no graph in \mathcal{S} appear as an induced subgraph in G .

Given a degree sequence $d = (d_1, d_2, \dots, d_n)$, an iterative step in the Havel-Hakimi algorithm, developed independently by Havel [1] and Hakimi [2], reduces d to $H(d) = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$. After reordering the terms to be non-increasing, the algorithm iterates until no positive entries are present. The algorithm arose to determine when an integer sequence is graphic: that is a list of integers d is graphic if and only if the Havel Hakimi algorithm terminates in a list of zeros. The number of these zeros is said to be the residue of the degree sequence, and the residue of a graph G , denoted $R(G)$, is the residue of the degree sequence of G . The residue is of interest because of its connection to the independence number of a graph, $\alpha(G)$. In 1988, the conjecture-making computer program Graffiti [3] proposed the following theorem,

Theorem 5 ([4]). *For every graph G , $R(G) \leq \alpha(G)$.*

This result was proven by Favaron et al. in 1991 and improved upon by Griggs and Kleitman [5], Triesch [6], and Jelen [7] in the 1990's. Determining the independence number is NP-hard [8], but since it takes only $O(E)$ steps to determine the residue where E is the number of edges in a graph, it is of interest to know how well $R(G)$ approximates $\alpha(G)$ and when the bound is realized.

To further illustrate the relationship between the residue and the independence number, we can consider the Maxine heuristic, which is the process of iteratively deleting vertices of maximum degree until an independent set of vertices is realized [5]. We will call M the size of the independent set achieved by the Maxine heuristic and note that this is clearly a lower bound on the independence number. Note that the heuristic depends on our choice of deleted vertices and M can vary accordingly. It was shown by Griggs and Kleitman [5] that

Theorem 6 ([5]). *If M is the size of the independent set produced by any application of the Maxine heuristic for a graph G , then $R(G) \leq M \leq \alpha(G)$.*

Thus if $R(G) = \alpha(G)$ for some G , then every application of the Maxine heuristic must achieve a maximum independent set.

A vertex in a graph is said to have the Havel-Hakimi property if it is of maximum degree and its neighbors are of maximal degree, i.e. the deletion of said vertex corresponds to the reduction in the degree sequence by one step of the Havel-Hakimi algorithm. Not every graph has a vertex with this property, but every degree sequence has a realization that has such a vertex [9]. If at each step of the Maxine heuristic, a vertex with the Havel-Hakimi property is deleted, then $R(G) = M$. To find when $M = \alpha(G)$ we will consider graphs with certain conditions.

A vertex v in a non-empty graph G is said to have maximum degree-independence conditions (or MDI conditions) if it is has maximum degree and

is a part of every maximum independent set. Also we will say that a non-empty graph G has maximum degree-independence conditions (or MDI conditions), if there exists a vertex $v \in V(G)$ that has MDI conditions.

In 2016, Barrus and Molnar found that if a vertex v in G has MDI conditions, then G must contain an induced subgraph of C_4 (the cycle on 4 vertices) containing v or an induced subgraph of P_5 (the path on 5 vertices) with v as the center vertex [10]. From this it can be quickly shown that

Theorem 7 ([10]). *The Maxine heuristic always produces a maximum independent set when applied to a $\{C_4, P_5\}$ -free graph.*

2.2 Preliminary lemmas

We will work to strengthen Theorem 7 by examining the case where v with MDI conditions is in an induced copy of C_4 , since C_4 does not have MDI conditions itself. Since we will only strengthen the condition on C_4 , we will assume that all graphs considered have no subgraph isomorphic to P_5 in which the center vertex has MDI conditions. We will call a graph P_5^* -free when referring to the condition that the center vertex must have MDI conditions, as we will not restrict the existence of an induced P_5 in general. We will allude to the aforementioned MDI conditions as the maximum degree condition and independence condition separately. To start, we will prove a few lemmas to reduce our search of induced subgraphs needed to strengthen the C_4 condition.

Lemma 1. *If $v \in V(G)$ has MDI conditions and is a part of more than one maximum independent set, then there is a nonempty induced subgraph of G in which v also has MDI conditions and there is only one maximum independent set.*

Proof. Let v belong to maximum independent sets I_1, I_2, \dots, I_n . Then we can consider the subgraph induced by deleting $\bigcup_{i=2}^n I_i \setminus I_1$. Because v is of maximum

degree in G and G is nonempty, v must have a neighbor in G , and since no vertices adjacent to v were deleted, v still has a neighbor and hence $G \setminus \bigcup_{i=2}^n I_i \setminus I_1$ is nonempty as well. Furthermore, the maximum degree condition is not violated since none of the deleted edges were adjacent to v , and there is exactly one maximum independent set in the induced subgraph. \square

Because of Lemma 1, we will now only consider a graph G with one maximum independent set I including a vertex v such that v has MDI conditions.

Lemma 2. *Let G have MDI conditions and let x be a vertex such that $x \notin N(v) \cup I$ where I is the lone maximum independent set. Then $G \setminus \{x\}$ has MDI conditions as well.*

Proof. Deleting x does not change the degree of v and thus the maximum degree condition is unaffected. Furthermore, since x is not in I , the independent set is unaffected as well. Thus v still has MDI conditions in $G \setminus \{x\}$. \square

If $\alpha(G) = 1$, G must be a complete set of vertices, and if there is only one maximum independent set, then G must be an isolated vertex. Furthermore, if $\alpha(G) = 2$ with unique maximum independent set $\{u, v\}$, then $N(v) = N(u)$ must form a clique and thus every element of $N(v)$ must have strictly larger degree than both u and v . Since we require some element of the maximum independent set to have maximum degree, $N(v)$ must be empty and G must be the graph of two isolated vertices. Hence, if G has MDI conditions and $\alpha(G) \leq 2$, then every application of the Maxine heuristic vacuously produces a set of size α .

Thus we will now assume that the size of I is 3 and that $I = \{u, v, w\}$ where v is the vertex with MDI conditions and $I' = I \setminus \{v\}$. Note that if $x \in N(v)$ where v has MDI conditions and x is not adjacent to any other element in I , the maximum independent set, then we have another maximum independent set

$(I \setminus \{v\}) \cup \{x\}$. Thus, in considering an induced subgraph of G of minimal size with MDI conditions and same independence number of G , we only need to consider the vertices in $N(v) \cup I$ having neighbors in I' . We will then partition $N(v)$ into three sets: Q_u , Q_w , and N . Let Q_u and Q_w be the sets of vertices in $N(v)$ whose neighbors in I' are only u and w respectively. We will call $Q = Q_u \cup Q_w$. Let N be the set of vertices in $N(v)$ that are adjacent to both u and w . Since the independence number of G must be 3 and I is the unique independent set of size 3, we have that Q_u and Q_w must have independence number at most 1; hence Q_u and Q_w are cliques, since otherwise there would exist another independent set of size 3. Similarly, N must have independence number at most 2. Then since G must be P_5^* -free, we have that Q must form a clique as every vertex in Q_u must dominate Q_w and vice versa as otherwise there exists $q_u \in Q_u$ and $q_w \in Q_w$ non-adjacent; hence $\{u, q_u, v, q_w, w\}$ induce P_5 with v as the center vertex. In Figure 3, the sets Q_u , Q_w , and N are diagrammed, with a bold line signifying that all possible adjacencies are present.

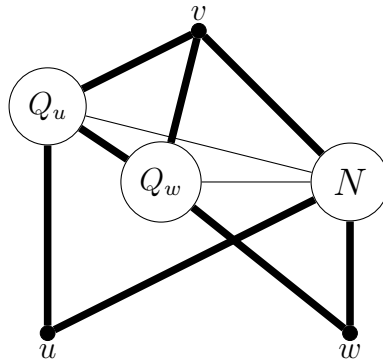


Figure 3: The Q and N sets with independent set $\{v, u, w\}$

2.3 The case with independence number equal to 3

Theorem 8. *Let G have MDI conditions with $\alpha(G) = 3$. Then G has at least one of the following induced subgraphs where Q' is a subset of Q and N' a subset of N :*

1. $|Q'| = 0$, $G[N'] \cong \overline{C_n}$.
2. $|Q'| = 1$, $G[N' \cup Q'] \cong \overline{C_n}$.
3. $|Q'| = 2$, $G[N' \cup Q'] \cong \overline{P_n}$ where the elements of Q are the endpoints of P_n in the complement.

Proof. We will first consider the case where $|Q| = 0$. First note that if $|N| = 0$, then $N(v)$ is empty and G is only the independent set and we have a contradiction as G has MDI conditions and thus must be nonempty. Thus we will assume that N is non-empty. We have that every vertex in N has at least two non-neighbors in N as Q is empty and every vertex in N is also adjacent to u , v , and w , otherwise v would not have maximum degree as $N(v) = N \cup Q$. Since the degree of every vertex in N^c must be at least 2, there must exist a cycle in the complement. Consider a smallest cycle complement, and label its vertices x_0, \dots, x_{m-1} where x_i is non-adjacent to both x_{i+1} and x_{i-1} with indices given modulo m . If there exists an x_i that does not dominate the rest of the cycle complement, then we have a smaller cycle complement which is a contradiction. Thus we have that x_i dominates the rest of the cycle complement for every i and thus we have $G[N'] \cong \overline{C_m}$ where N' is the vertex set of the cycle complement.

We will next consider the case where $|Q| = 1$. We will call q the lone vertex in Q . If $|N| = 0$, then q has larger degree than v , which is a contradiction so we will assume that N is non-empty. Note that every vertex in N has to have at least 2 non-neighbors in $N \cup Q$ otherwise v is not of maximum degree, as every vertex in N is also adjacent to u , v , and w . If q dominates N then $\deg(q) > \deg(v)$ which is a contradiction. Thus there exists a non-neighbor of q in N ; call it x_0 , and call the other guaranteed non-neighbor of x_0 , x_1 . Similarly, x_1 is guaranteed to have another non-neighbor in $N \cup Q$ as $x_1 \in N$ and must have at least two non-neighbors in $N \cup Q$. If this other non-neighbor is q then we have that $\{q, x_0, x_1\}$

induce $\overline{C_3}$ and we are done. Thus we will assume that the other non-neighbor is in N , call it x_2 . Inductively this creates a sequence of non-neighbors in N , $\{x_i\}$, as each x_i must be adjacent to q otherwise we are done as $\overline{C_{i+1}}$ is induced on $\{q, x_1, \dots, x_i\}$. Furthermore each x_i must be adjacent to $\{x_0, \dots, x_{i-2}\}$ otherwise we have an induced copy of $\overline{C_n}$ in N for some n . Since we have a finite graph, this sequence must terminate at x_m for some m , and thus we have that x_m must be non-adjacent to either q or some vertex in $\{x_0, \dots, x_{m-2}\}$ giving the result.

Finally we will show the result if $|Q| \geq 2$. We will proceed by induction on the size of Q . We will now consider the base case where $|Q| = 2$, calling the two vertices q_1, q_2 . Note that $q_1 \sim q_2$ as Q forms a clique. Similar to the case $|Q| = 1$, if N is empty then q_1 has strictly larger degree than v which is a contradiction. Thus we will assume that N is non-empty. Each of the vertices in Q has at least one non-neighbor in N ; if they have the same non-neighbor then those three vertices induce the desired $\overline{P_3}$ and we are done. Thus we will assume that they have different non-neighbors, call them x_1 and x_2 respectively. If $x_1 \approx x_2$, then the four vertices induce the desired $\overline{P_4}$ and we are done, so assume that $x_1 \sim x_2$. Each of these vertices has another non-neighbor in N ; if they share a non-neighbor then the five vertices induce the desired $\overline{P_5}$, so assume that x_1 and x_2 have different non-neighbors call them x_3 and x_4 respectively. Inductively, we have that the pair of vertices x_{2i+1}, x_{2i+2} are the new non-neighbors of x_{2i-1} and x_{2i} . Note that x_{2i+1} and x_{2i+2} must be adjacent to q_1 and q_2 respectively otherwise there is an induced complement of a cycle and we are done. Furthermore x_{2i+1} must be adjacent to each x_j for j odd and at most $2i - 3$ and x_{2i+2} must be adjacent to each x_j for j even and at most $2i - 2$, otherwise we have an induced complement of a cycle in N . Then x_{2i+1} must be adjacent to each x_j for j odd, and x_{2i+2} must be adjacent to each x_j for j even, otherwise we have the desired induced complement of a path.

We thus have that both x_{2i+1} and x_{2i+2} must have another non-neighbor in N . Since we have a finite graph, this process must terminate in either odd indexed or even indexed x 's containing a cycle complement, or there being an edge from some vertex x_{2j+1} to some vertex x_{2k} which yields a path-complement hence yielding the result.

We will now show that if $|Q| > 2$, G has one of the desired induced subgraphs above. We will proceed by induction on $|Q|$, noting that the base case of $|Q| = 2$ is done above. Assume the result is true for $|Q| < k$ and consider the case with $|Q| = k$. We will label the vertices of Q , $\{q_1, q_2, \dots, q_k\}$. Each of these has a non-neighbor in N , call it x_i for each q_i . Note that these are distinct otherwise we have an induced copy of $\overline{P_3}$ with two elements of Q has endpoints in the complement. Furthermore $q_i \sim x_j$ for all $i \neq j$ as otherwise we have an induced $\overline{P_4}$. Then there exists another non-neighbor of x_1 in N , call it y_1 . We have that

- $y_1 \sim q_1$, otherwise $\{q_1, x_1, y_1\}$ induce $\overline{C_3}$.
- $y_1 \sim q_j$ for all $j > 1$ otherwise $\{q_1, x_1, y_1, q_j\}$ induce $\overline{P_4}$.
- $y_1 \sim x_j$ for all $j > 1$, otherwise $\{q_1, x_1, y_1, x_j, q_j\}$ induce $\overline{P_5}$.

We then have that y_1 must have another non-neighbor in N , call it y_2 . Inductively let y_k be the other non-neighbor of y_{k-1} where each y_i for $1 \leq i < k$ dominates all preceding vertices except y_{i-1} . Then we have that

- $y_k \sim q_1$, otherwise $\{q_1, x_1, y_1, \dots, y_k\}$ induce $\overline{C_{k+2}}$.
- $y_k \sim q_j$ for all $j > 1$, otherwise $\{q_1, x_1, y_1, \dots, y_k, q_j\}$ induce $\overline{P_{k+3}}$.
- $y_k \sim x_1$, otherwise $\{x_1, y_1, \dots, y_k\}$ induce $\overline{C_{k+1}}$.
- $y_k \sim x_j$ for all $j > 1$, otherwise $\{q_1, x_1, y_1, \dots, y_k, x_j, q_j\}$ induce $\overline{P_{k+4}}$.

- $y_k \sim y_i$ for all $i < k$ otherwise inductively there is an induced complement of a cycle.

Thus y_k has another non-neighbor in N . Since our graph is finite, this process must terminate and the result holds. □

2.4 The case with independence number greater than 3

We will now extend the result to graphs with independence number greater than 3. In this section we will refer to Q as the set of vertices adjacent to v and exactly one other element of the maximum independent set. Furthermore, we will refer to N as the set of vertices adjacent to v and exactly 2 other vertices of the maximum independent set.

Theorem 9. *Let G have MDI conditions with $\alpha(G) = k$ such that $k > 3$. Then the result from Theorem 8 holds as well. Namely that, G has at least one of the following induced subgraphs where Q' is a subset of Q and N' a subset of N :*

1. $|Q'| = 0$, $G[N'] \cong \overline{C_n}$.
2. $|Q'| = 1$, $G[N' \cup Q'] \cong \overline{C_n}$.
3. $|Q'| = 2$, $G[N' \cup Q'] \cong \overline{P_n}$ where the elements of Q are the endpoints of P_n in the complement.

Proof. We will assume the contrary, that there exists such a graph without the desired induced subgraphs and derive a contradiction.

From Lemma 1 we have that G has one maximum independent set with v a vertex with MDI conditions. First call I the lone independent set, and $I' = I \setminus \{v\}$. Furthermore, we will say that $A \subseteq N(v)$ induces H on $G_{i,j}$ to mean that $H = G[\{v, A, i, j\}]$, where i, j are elements of I' . Then call $Q_i \subseteq N(v)$ the set

of vertices that are adjacent to exactly i members of I' . Then $\{Q_i\}_{i=1}^{k-1}$ partitions $N(v)$, using Lemma 2. Also note that in order for G to have MDI conditions, every vertex in Q_i must have at least i non-neighbors in $N(v)$, otherwise v would not have maximum degree. To show the desired induced subgraphs and derive a contradiction we need to show that a subset $A \subseteq N(v)$ induces \overline{C}_n or \overline{P}_n in $G_{i,j}$ for some $i, j \in I'$. We will first show that Q_{k-1} must be empty.

Let $q \in Q_{k-1}$. We will show that q must have at most one non-neighbor in $N(v) \setminus Q_{k-1}$. Suppose that q has two such non-neighbors: x and y .

First suppose $x \sim y$. If x and y have distinct neighbors in I' , call them u and w respectively, such that $x \approx w$ and $y \approx u$, then $\{q, x, y\}$ induce \overline{P}_3 in $G_{u,w}$. Otherwise, without loss of generality, $(N(x) \cap I') \subseteq (N(y) \cap I')$, and we must have that $N(x) \cap I'$ is non-empty, so it contains an element u , and there exists $w \in I' \setminus N(y)$ since $y \notin Q_{k-1}$. We then have that, again, $\{q, x, y\}$ induce \overline{P}_3 in $G_{u,w}$.

Then suppose that $x \not\sim y$. We must have that x and y cannot each have a neighbor in I' (say a and b respectively) to which the other is non-adjacent, as otherwise $\{x, v, y, a, b\}$ would induce P_5^* . Then x and y share a neighbor in I' , call it u and note that both x, y cannot belong to Q_1 , as Q_1 forms a clique. Thus, without loss of generality, we can say that y has another neighbor, w , in I' , and thus $\{q, x, y\}$ induce \overline{C}_3 in $G_{u,w}$.

We thus have that, for each $q \in Q_{k-1}$, q must have at most one non-neighbor in $N(v) \setminus Q_{k-1}$, and thus must have at least 2 non-neighbors in Q_{k-1} . As in the proof of Theorem 8, we can arrange a smallest cycle complement of non-neighbors and thus we have an induced \overline{C}_n in $G_{u,w}$ where u, w are some two members of I' . This is a contradiction, and thus Q_{k-1} must be empty.

We will then proceed by induction to show that Q_i is empty for $3 \leq i \leq k-1$.

We will assume that Q_i is empty for all $i > \ell$, and we will show that Q_ℓ is empty as well.

Let $q \in Q_\ell$, and assume that q has two non-neighbors in $N(v) \setminus Q_\ell$, call them x and y . If any pair of $\{x, y, q\}$ have distinct neighbors in I' not adjacent to the other vertex, then we have an induced P_5^* , as seen above. Thus we must have that, without loss of generality, $(N(x) \cap I') \subseteq (N(y) \cap I')$, and since q has the most neighbors in I' , $(N(y) \cap I') \subseteq (N(q) \cap I')$. Then we argue, in the same way as in the base case of Q_{k-1} , that q can only have at most one non-neighbor in $N(v) \setminus Q_\ell$. Thus, q has at least $\ell - 1$ non-neighbors in Q_ℓ as Q_i is empty for all $i > \ell$, and as above this means that we have an induced \overline{C}_n , a contradiction. Thus we have that Q_ℓ must be empty. Hence by induction we have that Q_i is empty for all $i > 2$.

Note that $N(v)$ must be non-empty, as we cannot have an edgeless graph, and Q_2 cannot be empty as Q_1 forms a clique, and each element of Q_1 must have at least one non-neighbor in $N(v)$. Then let $q \in Q_2$. If q has two non-neighbors in Q_1 , adjacent to distinct u and w respectively in I' , then q must also be adjacent to u and w , otherwise we have an induced P_5^* . If q has two non-neighbors in Q_1 both adjacent to u , then those vertices induce \overline{P}_3 in $G_{u,w}$ where w is any other vertex in I' . Thus the three vertices induce \overline{P}_3 in $G_{u,w}$ where w is any other vertex in I' . Then assume that q has exactly one non-neighbor in Q_1 , call it x and a non-neighbor in Q_2 , call it y . We must have that q, y share the same neighbors in I' , otherwise we have an induced P_5^* , and likewise the neighbor of x in I' is shared by both q and y . We then have that if $x \not\sim y$, we have that $\{x, y, q\}$ induce \overline{C}_3 . We will thus assume that $x \sim y$.

Then if all $q \in Q_2$ have two non-neighbors in Q_2 , we must have an induced copy of \overline{C}_n in Q_2 . Suppose then that there are two vertices in Q_2 , q and q' that have a non-neighbor in Q_1 , and choose these vertices such that the distance between

them in Q_2^c is as small as possible. Note that there must exist a chain of vertices in Q_2 such that $q \approx q_1 \approx q_2 \approx \dots \approx q'$, such that q_i does not have a non-neighbor in Q_1 . Furthermore, q , q' , and q_i must share the same neighbors in I' , otherwise we have an induced copy of P_5^* . If q and q' have the same non-neighbor in Q_1 , call it x , then $\{x, q, q', q_1, \dots\}$ induce $\overline{C_n}$. If q and q' have different non-neighbors, x and x' in Q_1 , then $\{x, x', q, q', q_1, \dots\}$ induce $\overline{P_n}$. This is a contradiction, and thus for every graph G with conditions and $\alpha(G) = k > 3$, we have the result. \square

For ease, we will call the families of induced subgraphs in Theorem 8, \mathcal{F} . We wanted to improve the C_4 condition introduced by Barrus and Molnar as C_4 itself was not MDI. By construction, each graph in \mathcal{F} is itself MDI alongside P_5 . We then have the immediate corollary,

Corollary 2. *The Maxine heuristic always produces a maximum independent set when applied to a $\{\mathcal{F}, P_5\}$ -free graph.*

2.5 Open questions

Barrus and Molnar used their results to show that if a graph is $\{P_5, 4\text{-pan}, K_{2,3}, K_{2,3}^+, \text{kite}, 2P_3, P_3 + K_3, \text{stool}, \text{co-dominio}\}$ -free, then $R(G) = \alpha(G)$ [10]. It can be expected that this class of graphs can be expanded with the strengthened conditions shown in this paper. We pose the following open questions/problems:

- Can we fully classify the graphs in which the Maxine heuristic produces a maximum independent set?
- What other conditions, other than forbidding MDI conditions, can be considered to guarantee that the Maxine heuristic produces a maximum independent set?
- Can we fully classify the graphs in which the Maxine heuristic produces a graph with an independent set the same size as the residue? Note that graphs

with the strong Havel-Hakimi property introduced in [10] are a subset of these graphs.

- Can we fully classify the graphs in which the residue equals the independence number?

List of References

- [1] V.Havel, “A remark on the existence of finite graphs,” *Časopis Pěst Mat.*, vol. 80, pp. 477–480, 1955.
- [2] S. Hakimi, “On the realizability of a set of integers as degree sequences of the vertices of a graph,” *SIAM Appl. Math.*, vol. 10, pp. 496–506, 1962.
- [3] S. Fajtlowicz, “On conjectures of graffiti, iii,” *Cong. Numer.*, vol. 66, pp. 23–32, 1988.
- [4] O. Favaron, M. Mahéo, and J. F. Saclé, “On the residue of a graph,” *Graph Theory*, vol. 15, no. 1, pp. 39–64, 1991.
- [5] J. Griggs and D. Kleitman, “Independence and the havel-hakimi residue,” *Discrete Mathematics*, vol. 127, no. 1–3, pp. 209–212, 1994.
- [6] E. Triesch, “Degree sequences of graphs and dominance order,” *J. Graph Theory*, vol. 22, no. 1, pp. 89–93, 1996.
- [7] F. Jelen, “ k -independence and k -residue of a graph,” *J. Graph Theory*, vol. 127, pp. 209–212, 1999.
- [8] M. R. Garey and D. S. Johnson, *Computers and intractability*. freeman San Francisco, 1979, vol. 174.
- [9] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs, 5th edn.* Boca Raton: CRC Press, 2011.
- [10] M. Barrus and G. Molnar, “Graphs with the strong havel-hakimi property,” *Graphs and Combinatorics*, vol. 32, pp. 1689–1697, 2016.

CHAPTER 3

Regular sequence reorderings

Benjamin Lantz

Department of Mathematics

University of Rhode Island

Kingston, RI 02881

This manuscript is currently being prepared for submission.

Keywords: degree sequence, Havel-Hakimi Algorithm, Havel-Hakimi residue,
reorderings, regular sequences, independence number

AMS 2010 Mathematics Subject Classification: 05C35

Abstract. Given a degree sequence, the Havel-Hakimi algorithm reduces the sequence to a list of R zeroes, where R is deemed the residue of said sequence. Each step of the algorithm requires that the terms in the degree sequence be listed in non-increasing order, and thus sometimes a reordering of the terms is required. In this manuscript, the number of reorderings of all regular degree sequences is determined.

3.1 Introduction

Given a degree sequence $d = (d_1, d_2, \dots, d_n)$, an iterative step in the Havel-Hakimi algorithm, developed independently by Havel [1] and Hakimi [2], reduces d to $H(d) = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$. After reordering the terms to be non-increasing, the algorithm iterates until no positive entries are present. The algorithm arose to determine when an integer sequence is graphic: that is a list of integers d is graphic if and only if the Havel-Hakimi algorithm terminates in a list of zeros. The number of these zeros is said to be the residue of the degree sequence, and the residue of a graph G , denoted $R(G)$, is the residue of the degree sequence of G . The residue is of interest because of its connection to the independence number of a graph, $\alpha(G)$. In 1988, the conjecture-making computer program Graffiti [3] proposed the following theorem.

Theorem 10 ([4]). *For every graph G , $R(G) \leq \alpha(G)$.*

This result was proven by Favaron et al. in 1991 and improved upon by Griggs and Kleitman [5], Triesch [6], and Jelen [7] in the 1990's. Determining the independence number is NP-hard [8] but since it takes only $O(E)$ steps to determine the residue, where E is the number of edges in a graph, it is of interest to know how well $R(G)$ approximates $\alpha(G)$ and when the bound is realized.

Given two degree sequences, $d = (d_1, \dots, d_n)$ and $e = (e_1, \dots, e_n)$ with $\sum_{i=1}^n d_i = \sum_{i=1}^n e_i$, we say that d majorizes e if $\sum_{i=1}^k d_i \geq \sum_{i=1}^k e_i$ for all $i \in [n]$, de-

noted $d \succeq e$. Majorization of degree sequences with a fixed sum of entries exhibits a partial order. At the top of the majorization partial order are threshold sequences, and at the bottom of the partial order, if we fix the length of the degree sequences, are semi-regular degree sequences (degree sequences of the form $(k^A, (k-1)^{n-A})$) [9]. Semi-regular degree sequences and their relation to the Havel-Hakimi residue have been studied heavily by Nelson and Radcliffe in [10]. Majorization is important when discussing the Havel-Hakimi algorithm because of the following result which was used to help prove Theorem 10.

Theorem 11 ([4]). *Let d and e be degree sequences of the same length and same sum of their entries. If $d \succeq e$, then $R(d) \geq R(e)$.*

Regular degree sequences (which are a special case of semi-regular degree sequences $(k^A, (k-1)^{n-A})$ with $A = n$) are of a simple form and thus are often easier to work with than an arbitrary degree sequence. Because of this and their relation to the majorization partial order and hence Havel-Hakimi residue, this manuscript will focus primarily on regular degree sequences only with the intent that results can be expanded to that of semi-regular degree sequences in general and then arbitrary degree sequences.

During certain iterations of the Havel-Hakimi algorithm, a reordering of the terms must occur in order to ensure that the terms are listed in non-increasing order. Our goal is to track how many reorderings are needed in the Havel-Hakimi algorithm given a certain degree sequence, in particular a regular degree sequence. For example, consider the Havel-Hakimi steps in reducing (5^{14}) to (0^3) . Note in Figure 4 that the term in bold is to be laid off and will decrease the underlined terms by 1, a convention that will be used throughout this dissertation.

We will relate the process of the Havel-Hakimi algorithm conducted on a regular degree sequence to the Josephus Problem, which dates back to the 1st

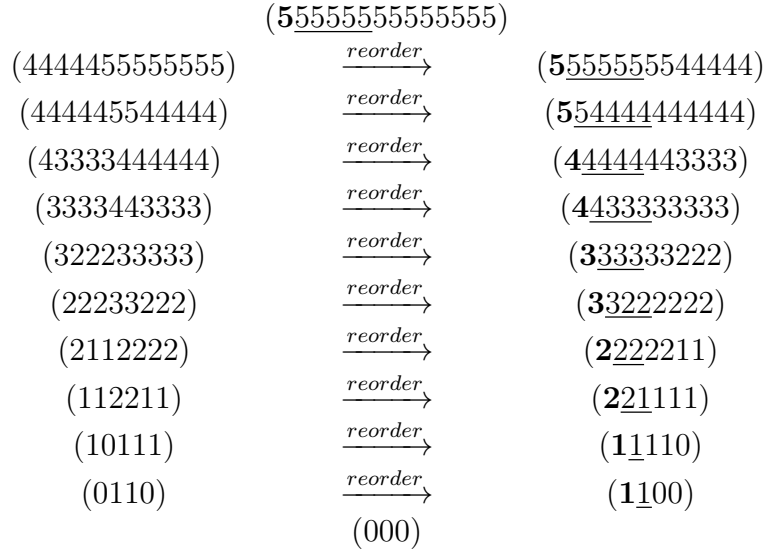


Figure 4: The Havel-Hakimi algorithm applied to (5^{14})

century. The problem is attributed to Flavius Josephus, and states that forty-one Jews escaped a city-seige and were hiding in a cave. They decided to kill themselves to evade capture, and decided to do so in a way that every third man would be killed by his companions until there was one left. The Josephus problem is stated in a way that given n people, and an integer $k < n$, in what position will the last remaining man be situated if the k^{th} person is eliminated each time [11].

Similar to the Josephus problem, we will consider a group of n people sitting in a circle with the same objective as the Josephus problem. We can think that each person has been assigned the same positive number a . The first person is eliminated, and the next a people (as that is the label of the person eliminated) have their number reduced by one. The next person is then eliminated and the process repeats, until all members have a label of 0. Consider the example, seen in Figure 5 of fourteen people sitting in a circle with a label of 5 given to all of them. Unlike the Josephus problem, there may be more than one person left at the end. The number of people left at the end will be the residue of the degree

sequence (a^n) . The cyclic nature of the Josephus problem will be useful as any reordering of a semi-regular degree-sequence can be seen as a cyclic rotation of the degree sequence.

In the Havel-Hakimi algorithm, all terms of maximum degree will be laid off or reduced before any terms of smaller degree as once a term is reduced by one it will not be laid off or reduced again until all other remaining terms are affected. Thus we can think of the laying off/reduction of terms in a graphic regular sequence as cyclically laying off/reducing through the indices $[n]$, in order, until all the terms have been laid off or reduced to zero (akin to the Josephus problem diagrammed in Figure 5).

Managing diagrams as in Figure 4 and Figure 5 is time consuming and impractical for larger ordered degree sequences. We will introduce another diagram that will help identify the number of reorderings in regular degree sequences. In Figure 6, each term of the initial degree sequence corresponds to a column in a table. The columns are headed by the term indices $1, \dots, 14$. We represent and record laid off terms during the algorithm in shaded boxes. Then the appropriate number of unshaded boxes following the laid off term correspond to the sequence terms that the algorithm reduces by 1. The column immediately following these corresponds to the next laid off term in the algorithm.

Consider a regular degree sequence $d = (a^n)$ with $a > 0$, noting that $n \geq a + 1$. Let q and r be non-negative integers such that $n - 1 = q(a + 1) + r - 1$ and $0 \leq r - 1 \leq a$ and hence $n = q(a + 1) + r$. For the rest of this section, we will refer to the terms d_1, \dots, d_n using two systems of indexing. The terms will continue to be indexed by their subscripts from $[n]$ corresponding to their initial location in the degree sequence. When considering the steps in the Havel-Hakimi algorithm, we will not reorder the terms as to preserve their original indices from $\{1, \dots, n\}$ and

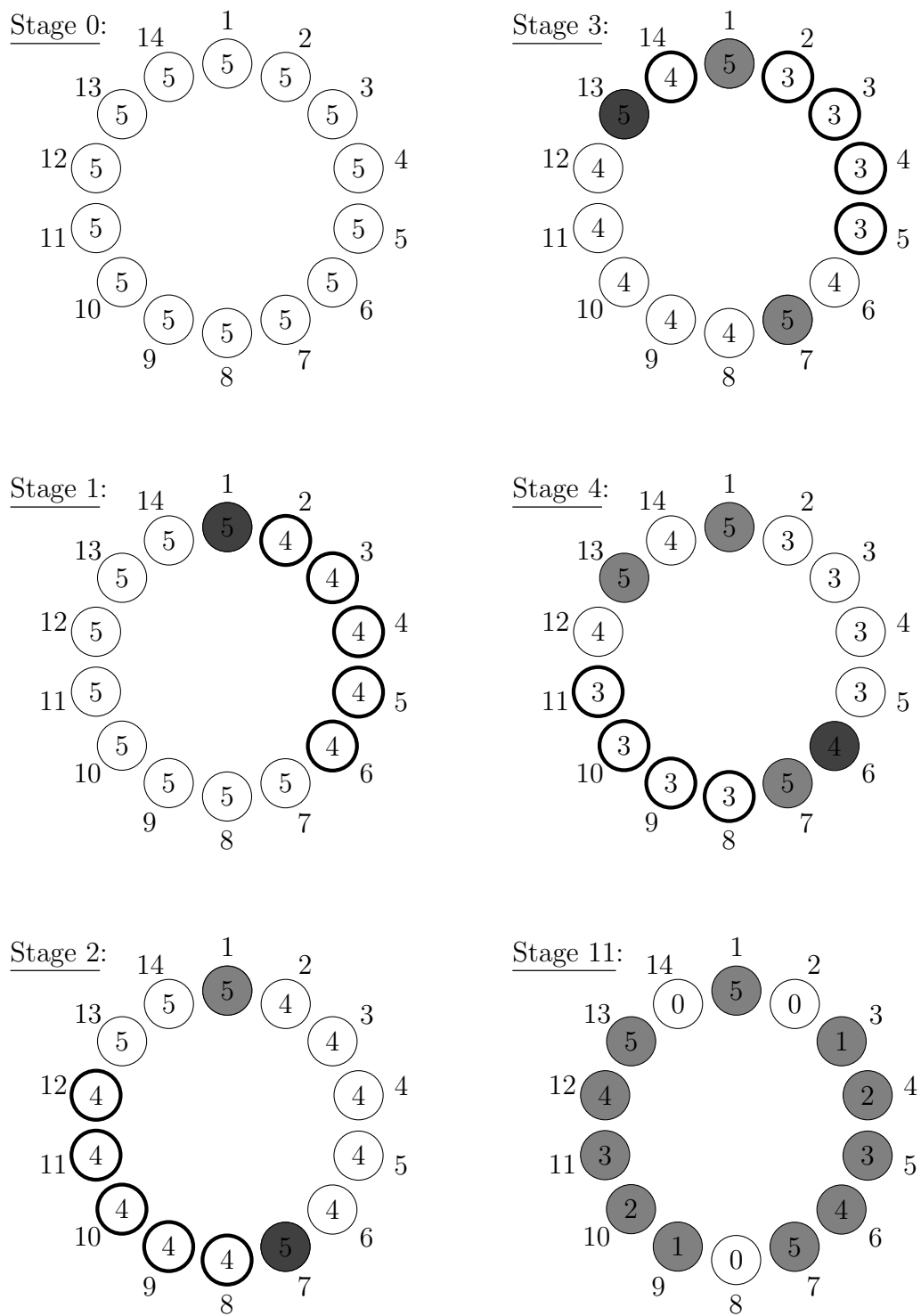


Figure 5: A generalized version of the Josephus problem on 14 people with a label of 5

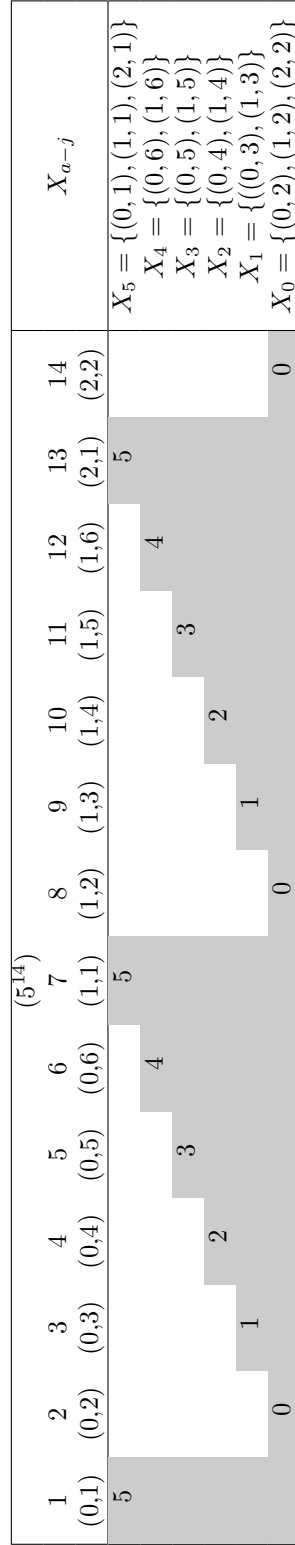


Figure 6: The Havel-Hakimi algorithm applied to (5^{14}) and the corresponding X_{a-j} sets

will retain their index from $[n]$ throughout. For the second indexing system, we will partition the terms d_1, \dots, d_n into q disjoint blocks of $a + 1$ consecutive terms and one block containing the remaining r terms. If $k = q'(a+1) + r'$ with $1 \leq r' \leq a+1$, then we will also refer to the k^{th} term d_k using the ordered pair (q', r') as an index. We will say that the (q', r') -entry has *remainder* r' . The ordered pair indices of $d = (a^n)$ are ordered lexicographically when listed in sequence, and note that in the cyclic reduction interpretation of the Havel-Hakimi algorithm, before a term having index (i, j) is laid off by the Havel-Hakimi algorithm, the terms with indices before (i, j) , and not already laid off, will be 1 less than the (i, j) -term and later terms that are not already laid off. Let X_{a-j} be the set of ordered pair indices of terms having value $a - j$ the moment they are laid off for $0 \leq j \leq a$.

3.2 The next regular sequence in the Havel-Hakimi algorithm when $\gcd(a + 1, n - 1) = 1$

We will show that when performing a step of the Havel-Hakimi algorithm on a regular degree sequence, a reordering always occurs during a Havel-Hakimi step on a semi-regular degree sequence unless the new sequence produced is a regular sequence.

Lemma 3. *Let $d = (B^A, (B - 1)^{n-A})$ be a semi-regular degree sequence. Then a Havel-Hakimi step in the algorithm does not require a reordering if and only if $d_1 = n - 1$ (a dominating term) or if $A = B + 1$ so $d = (B^{B+1}, (B - 1)^{n-B-1})$.*

Proof. We will first show the backwards implication. If d_1 is dominating, then in a Havel-Hakimi step, all terms will be reduced by one, hence preserving the non-increasing order. Thus no reordering is needed.

If $d = (B^{B+1}, (B - 1)^{n-B-1})$, then $d_1 = B$ will be laid off and the next B terms will be reduced. Since the next B terms are all of the remaining terms with degree B , then the subsequent degree sequence will be $((B - 1)^{n-1})$, which does

not require any reordering as it is a regular degree sequence.

Now we will show the forwards implication. If $n - 1 > A > B + 1$, then there will necessarily be a term of degree B which is not reduced and thus a reordering will be required, as shown below where the term in bold is laid off and the underlined entries are reduced by 1.

$$\begin{aligned}
(B^A, (B - 1)^{n-A}) &= (\mathbf{B}, \underline{B, \dots, B}, B^{A-(B+1)}, (B - 1)^{n-A}) \\
&\quad \downarrow \text{Havel-Hakimi step} \\
((B - 1)^B, B^{A-(B+1)}, (B - 1)^{n-A}) & \\
&\quad \downarrow \text{reorder} \\
(B^{A-(B+1)}, (B - 1)^{n-A+B}) &
\end{aligned}$$

If $A < B + 1$, then there will necessarily be a term of degree $B - 1$ which is reduced and thus a reordering will be required.

$$\begin{aligned}
(B^A, (B - 1)^{n-A}) &= (\mathbf{B}, \underline{B^{A-1}, (B - 1)^{B-(A-1)}}, (B - 1)^{n-B-1}) \\
&\quad \downarrow \text{Havel-Hakimi step} \\
((B - 1)^{A-1}, (B - 2)^{B-(A-1)}, (B - 1)^{n-B-1}) & \\
&\quad \downarrow \text{reorder} \\
((B - 1)^{n-B+A-2}, (B - 2)^{B-(A-1)}) &
\end{aligned}$$

Thus if $A \neq B + 1$ and $A \neq n - 1$, then we have a reordering and the result follows. □

As shown in the proof of Lemma 3 a regular degree sequence will only yield semi-regular or regular degree sequences after a Havel-Hakimi step. Thus from

Lemma 3, a reordering will not occur if and only if there is no dominating term (a term with maximum degree) or when an intermediate subsequent degree sequence is regular. Furthermore, if a regular degree sequence has a dominating term then the graph must be complete and hence there are no reorderings. Thus, assuming the regular degree sequence is not of a complete graph, we will count the number of reorderings by counting the number of regular degree sequences that appear in the Havel-Hakimi algorithm. There are exactly $n - R(d)$ steps in the Havel-Hakimi algorithm for a degree sequence d of residue $R(d)$, and hence

$$\text{Number of reorderings} = n - R(d) - (\text{number of regular sequences that appear}).$$

Now we will focus on counting the number of regular sequences that appear in the Havel-Hakimi algorithm. All of the results will involve the division algorithm and modular arithmetic, and we will first need a lemma that relates the greatest common divisor to the division algorithm (that gives $n - 1 = q(a + 1) + (r - 1)$).

Lemma 4. *Given integers m, ℓ, s, t , such that $m = s\ell + t$ and $0 \leq t < \ell$, if $\gcd(m, \ell) = 1$, then $\gcd(t, \ell) = 1$.*

Proof. If t and ℓ have a common factor, then m and ℓ must also have a common factor as we must be able to divide both sides of the equality by that number. Then the contrapositive of this statement yields the result. \square

To count the number of regular sequences that appear in the Havel-Hakimi algorithm, we will consider the value of a term at the moment it is deleted; we will determine the structure of X_{a-j} for all $0 \leq j \leq a$. In Lemma 5 we will show that X_{a-j} has a uniform structure for a fixed j , with the second coordinate of the ordered pair index the same for all entries.

Lemma 5. *Suppose that $\gcd(a + 1, n - 1) = 1$ and $0 \leq j \leq a$. Let $x_{a-j} = 1 + (-j(r - 1) \bmod a + 1)$, then*

$$X_{a-j} = \begin{cases} \{(i, x_{a-j}) \mid i \in 0, \dots, q\} & \text{if } x_{a-j} \leq r \\ \{(i, x_{a-j}) \mid i \in 0, \dots, q - 1\} & \text{if } x_{a-j} > r. \end{cases}$$

Proof. We will prove this result by induction on j . Note that by Lemma 4, since $\gcd(n - 1, a + 1) = 1$, we have $\gcd(r - 1, a + 1) = 1$. Consider first the base case of the induction with $j = 0$ and the first step of the Havel-Hakimi algorithm lays off the $(0, 1)$ -term, so $(0, 1) \in X_a$ where $x_a = 1$. Since this term has degree a , the next a entries will each be reduced by one, and the next term to be laid off will be the $(1, 1)$ -term. Similarly, we have that the $(i, 1)$ -term will be laid off for all $i \in \{0, \dots, q\}$ as we are laying off terms of degree a each time. Thus $X_a = \{(i, 1) \mid i = 0, \dots, q\}$ as the next term after the $(q, 1)$ -term to be laid off must necessarily be one that has already been reduced once before, concluding the base case.

Then we will assume that $j \geq 1$ and $X_{a-\ell}$ is of the desired form for all $\ell < j$. We want to show that every entry in X_{a-j} is of the form (i, x_{a-j}) . We will set $k = x_{a-(j-1)}$.

Note if $k = r$, then $r + (j - 1)(r - 1) \equiv 1 \pmod{a + 1}$ by definition. Thus $r + j(r - 1) - (r - 1) \equiv 1 \pmod{a + 1}$, hence $j(r - 1) \equiv 0 \pmod{a + 1}$. This means that $j \equiv 0 \pmod{a + 1}$ as $\gcd(r - 1, a + 1) = 1$. Since $j \leq a$ and we assume that $j \geq 1$ in the inductive hypothesis, this is a contradiction, hence we will assume that $k \neq r$.

Case 1: $x_{a-(j-1)} = k > 2r$.

If $k > 2r$, then the last entry in $X_{a-(j-1)}$ is $(q - 1, k)$. Consider the ordered

pair indices of the next $a + 1$ terms in d :

$$(q - 1, k + 1), \dots, (q - 1, a + 1), (q, 1), \dots, (q, r), (0, 1), \dots, (0, k - r).$$

Note that if $k = a + 1$, then the indices $(q - 1, k + 1)$ through $(q - 1, a + 1)$ are omitted. We will show that of the terms corresponding to those indices, j have already been laid off before laying off terms with value $a - j$ (that is in $\bigcup_{\ell=0}^{j-1} X_{a-\ell}$), and $a - (j - 1)$ will be reduced by 1. Then since $\gcd(r - 1, a + 1) = 1$, $x_{a-\ell}$ for $\ell < j$ are distinct as, given $\ell_1, \ell_2 < j$, $1 - \ell_1(r - 1) \equiv 1 - \ell_2(r - 1) \pmod{a + 1}$ would require $\ell_1 \equiv \ell_2 \pmod{a + 1}$, hence $\ell_1 = \ell_2$ as $1 \leq \ell < j < a + 1$. Moreover, note that $x_{a-\ell} \not\equiv 1 - (j - 1)(r - 1) - (r - 1) \pmod{a + 1} = k - (r - 1) \pmod{a + 1}$ for any $\ell < j$. Thus the next index $(0, k - (r - 1))$ will be the index of the next term to be laid off, where $k - (r - 1) = x_{a-j}$ as desired.

In referring to the remainders, we will use the notation $[y, z]$ to mean the set of remainders greater than or equal to y and less than or equal to z , noting that all values in $[y, z]$ will be integers. Of the $a + 1$ indices listed above, observe that those with remainder in $[1, r]$ appear twice, those with remainder in $[r + 1, k - r] \cup [k + 1, a + 1]$ appear once, and those with remainder in $[k - r + 1, k]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap [1, r]$
- $M_1 = D \cap ([r + 1, k - r] \cup [k + 1, a + 1])$
- $M_0 = D \cap [k - r + 1, k]$.

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_2$ for some $0 < i < j - 1$. Then $x_{a-i} \not\equiv 1$ as $i \neq 0$, hence $x_{a-i} \in [2, r]$. By the inductive hypothesis $x_{a-(i-1)} \equiv$

$x_{a-i} + (r-1) \pmod{a+1}$, and thus $x_{a-(i-1)} \geq r+1$ and hence not in M_2 . Because the length of $[k-r+1, k]$ is r , $x_{a-(i-c)} \in M_0$ for minimal positive integer c such that $x_{a-(i-c)} = x_{a-i} + c(r-1) \leq a+1$. Furthermore $x_{a-(i-c+1)} \notin M_0$ as then $x_{a-(i-c)} = k-r+1$, hence $x_{a-(i-c+1)} = k$ which cannot happen as $x_{a-(j-1)} = k$ and $i-c+1 < j-1$. A symmetric result can be said for an arbitrary element x_{a-i} in M_0 for $i \in [2, j-2]$. Note that for two arbitrary elements x_{a-i_1} and x_{a-i_2} in M_2 , their corresponding values $x_{a-(i_1-c_1)}$ and $x_{a-(i_2-c_2)}$ in M_2 described above must be unique by the inductive hypothesis as each $x_{a-\ell}$ value is unique for all $\ell < j$. Thus for $i \in [1, j-2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that the number of already laid off terms of the $a+1$ indices listed above is

$$\begin{aligned}
(0 \cdot |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\
&= |M_1| + |M_2| + |M_0| \\
&= |D| = j
\end{aligned}$$

Then, of the next $a+1$ entries, j have already been laid off and $a-(j-1)$ will be reduced yielding $(0, k-r+1)$ as the index of the next term to be laid off (as it has not already been laid off), and hence $(0, k-(r-1)) \in X_{a-j}$.

Then in the following $(a+1)$ entries, j are already laid off and $a-j$ will be reduced giving $(1, k-(r-1)) \in X_{a-j}$. Similarly we can show that $X_{a-j} = \{(i, k-(r-1)) \mid i \in 0, \dots, q-1\}$ as $k-r+1 > 2r-r+1 > r$. The result is satisfied for this case.

Case 2: $x_{a-(j-1)} = k = 2r$.

If $k = 2r$, then the last entry in $X_{a-(j-1)}$ is $(q-1, k)$. Consider the ordered

pair indices of the next $a + 1$ terms in d :

$$(q - 1, k + 1), \dots, (q - 1, a + 1), (q, 1) \dots, (q, r), (0, 1) \dots, (0, k - r)$$

. Of the $a + 1$ indices listed above, observe that those with remainder in $[1, k - r]$ appear twice, those with remainder in $[k + 1, a + 1]$ appear once, and those with remainder in $[k - r + 1, k]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap [1, r]$
- $M_1 = D \cap [k + 1, a + 1]$
- $M_0 = D \cap [k - r + 1, k]$.

The same argument works for this case as Case 1, as the only difference is the size of M_1 which does not affect the argument.

Case 3: $r < k < 2r$ where $k = x_{a-(j-1)}$.

If $r < k < 2r$, then the last entry in $X_{a-(j-1)}$ is $(q - 1, k)$. Consider the ordered pair indices of the next $a + 1$ terms in d :

$$(q - 1, k + 1), \dots, (q - 1, a + 1), (q, 1) \dots, (q, r), (0, 1) \dots, (0, k - r).$$

Of the $a + 1$ indices listed above, observe that, those with remainder in $[1, k - r]$ appear twice, those with remainder in $[k - r + 1, r] \cup [k + 1, a + 1]$ appear once, and those with remainder in $[r + 1, k]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap [1, k - r]$

- $M_1 = D \cap ([k - r + 1, r] \cup [k + 1, a + 1])$
- $M_0 = D \cap [r + 1, k]$.

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_2$ for some $0 < i < j - 1$. Then $x_{a-i} \neq 1$ as $i \neq 0$, hence $x_{a-i} \in [2, r]$. By the inductive hypothesis $x_{a-(i-1)} \equiv x_{a-i} + (r - 1) \pmod{a + 1}$, and thus $x_{a-(i-1)}$ is in $[r + 1, k]$ and thus in M_0 . Because the length of $[r + 1, k]$ is r , $x_{a-i-2} \notin M_0$. A symmetric result can be said for an arbitrary element x_{a-i} in M_0 for $i \in [2, j - 2]$. Thus for $i \in [1, j - 2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that number of already laid off terms of the next $a + 1$ entries is

$$\begin{aligned}
(0 \cdot |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\
&= |M_1| + |M_2| + |M_0| \\
&= |D| = j
\end{aligned}$$

Then, of the next $a + 1$ entries, j have already been laid off and $a - (j - 1)$ will be reduced yielding $(0, k - r + 1)$ as the index of the next term to be laid off (as it has not already been laid off), and hence $(0, k - (r - 1)) \in X_{a-j}$.

Then in the following $(a + 1)$ entries, j are already laid off and $a - j$ will be reduced giving $(1, k - (r - 1)) \in X_{a-j}$. Similarly we can show that $X_{a-j} = \{(i, k - (r - 1)) \mid i \in 0, \dots, q\}$ as $k - r + 1 < 2r - r + 1 \leq 2r - r + 1 - 1 = r$. The result is satisfied for this case.

Case 4: $x_{a-(j-1)} = k < r$.

If $k < r$, then the last entry in $X_{a-(j-1)}$ is (q, k) . Consider the ordered pair indices of the next $a + 1$ terms in d :

$$(q, k + 1), \dots, (q, r), (0, 1) \dots, (0, a + 1 - (r - k)).$$

Of the $a + 1$ indices listed above, observe that those with remainder in $[k + 1, r]$ appear twice, those with remainder in $[1, k] \cup [r + 1, (a + 1) - (r - k)]$ appear once, and those with remainder in $[(a + 1) - (r - k) + 1, a + 1]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap [k + 1, r]$
- $M_1 = D \cap ([1, k] \cup [r + 1, (a + 1) - (r - k)])$
- $M_0 = D \cap [k - r + 1, k]$.

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_0$ for some $0 < i < j - 1$, except that $x_{a-i} \neq (a + 1) - (r - k) + 1$ as $x_{a-(i-1)} = k$. However this cannot happen as $x_{a-(j-1)} = k$ and $i - 1 < j - 1$. By the inductive hypothesis $x_{a-(i-1)} \equiv x_{a-i} + (r - 1) \pmod{a + 1}$, and thus $x_{a-(i-1)}$ is in $[k + 1, r - 1]$ and thus in M_2 . Because the length of $[k + 1, r]$ is $r - k$, $x_{a-i-2} \notin M_2$. A symmetric result can be said for an arbitrary element x_{a-i} in M_2 for $x_{a-i} \in [k + 1, r - 1]$, noting that $x_{a-i} \neq r$, as then $x_{a-i+1} = 1$ but $x_a = 1$ and $i + 1 > 0$. Thus for $i \in [1, j - 2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that number of already laid off terms of the next $a + 1$ entries is

$$\begin{aligned}
(0 \cdots |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\
&= |M_1| + |M_2| + |M_0| \\
&= |D| = j
\end{aligned}$$

Then, of the next $a + 1$ entries, j have already been laid off and $a - (j - 1)$ will be reduced yielding $(0, k - r + 1 \pmod{a + 1}) = (0, (a + 1) - (r - k) + 1)$ as the index of the next term to be laid off (as it has not already been laid off), and hence $(0, k - (r - 1) \pmod{a + 1}) = (0, (a + 1) - (r - k) + 1) \in X_{a-j}$.

Then in the following $(a + 1)$ entries, j are already laid off and $a - j$ will be reduced giving $(1, (a + 1) - (r - k) + 1) \in X_{a-j}$. Similarly we can show that either $X_{a-j} = \{(i, (a + 1) - (r - k) + 1) \mid i \in 0, \dots, q\}$ or $X_{a-j} = \{(i, (a + 1) - (r - k) + 1) \mid i \in 0, \dots, q - 1\}$ depending on if $(a + 1) - (r - k) + 1$ is less than or equal to r or greater than r . The result is satisfied for this case. □

Theorem 12. *Given a degree sequence (a^n) with $\gcd(a + 1, n - 1) = 1$. Let $k = -(r - 1)^{-1} \pmod{a + 1}$ and $\ell = (a + 1)^{-1} \pmod{n - 1}$. The next regular degree sequence that appears in the Havel-Hakimi algorithm will be (b^m) where $b = a - k$ and $m = n - \ell$.*

Proof. We will order the terms as described in the proof above, and use the notation for X_{a-j} and x_{a-j} . We will have a new regular degree sequence only after laying off the $(q - 1, r + 1)$ -term, as then the index of the next term to be laid off will be $(0, 2)$ (because $r + 1 - (r - 1) \equiv 2 \pmod{a + 1}$). Since $(0, 1)$ is the index of the first term to be laid off, we have that $(0, 2)$ is the index of the first remaining term at this point and thus the degree of all remaining terms are equal to b where

$$x_{a-(b-1)} \equiv 1 - (b - 1)(r - 1) \equiv r + 1 \pmod{a + 1}.$$

Thus we have that $1 - (b-1)(r-1) \equiv (r-1)+2 \pmod{a+1}$ and hence $-b(r-1) \equiv 1 \pmod{a+1}$. Hence $a-b \equiv a - (r-1)^{-1} \pmod{a+1}$ as desired, noting that this exists as $\gcd(r-1, a+1) = 1$.

We will enumerate all terms by their placement in the lexicographic order, that is $(0,1) = 0, (0,2) = 1, \dots, (q,r) = n-1$. Note that the difference in the lexicographic order between consecutive elements in X_{a-j} is $a+1$ for all j , and there are $a+2$ terms between the last element of X_{a-j} and the first element of $X_{a-(j+1)}$ (consecutively laid off terms of different degrees) if 0 is preceded by $n-1$ in order.

Since we lay off the $(0,1)$ -term first for every degree sequence, we will then consider only the distance in regards to the remaining $n-1$ terms $1, \dots, n-1$. Note that doing this preserves the difference between consecutive terms within X_{a-j} but changes the number of terms between consecutively laid off terms of different degrees to $a+1$ as $(0,1) = 0$ was included in the counting previously. Thus we have that the p^{th} term to be laid off will have index $(p-1)(a+1) \pmod{n-1}$.

As above, we have a new regular degree sequence only after laying off the term with index $(q-1, r+1) = n-a-1 = n-(a+1)$. Thus we need that $(p-1)(a+1) \equiv n-(a+1) \pmod{n-1}$. We then have that $p(a+1) \equiv n \equiv 1 \pmod{n-1}$, hence $p \equiv (a+1)^{-1} \pmod{n-1}$. Note that this exists since $\gcd(a+1, n-1) = 1$. Thus $m \equiv n - (a+1)^{-1} \pmod{n-1}$ as desired. \square

To see the results of Theorem 12, we will examine the degree sequence (4^{13}) . In this example we have $n = 13$ and $a = 4$, hence $n-1 = 12$ and $a+1 = 5$. Also because $12 = 2(5) + 2$, we have $r-1 = 2$. Since $\gcd(12, 5) = 1$, Theorem 12 applies. We have $-(r-1)^{-1} \pmod{a+1} = -(2)^{-1} \pmod{5} = 2$, and $(a+1)^{-1} \pmod{n-1} = 5^{-1} \pmod{12} = 5$. Thus by the theorem, the next regular sequence

to appear in the Havel-Hakimi algorithm is $((4 - 2)^{13-5}) = (2^8)$. In Figure 7 we can see since the $(0, 1)$ -term is laid off when the degree is 2, that indeed the next regular sequence that appears in the Havel-Hakimi algorithm has degree 2.

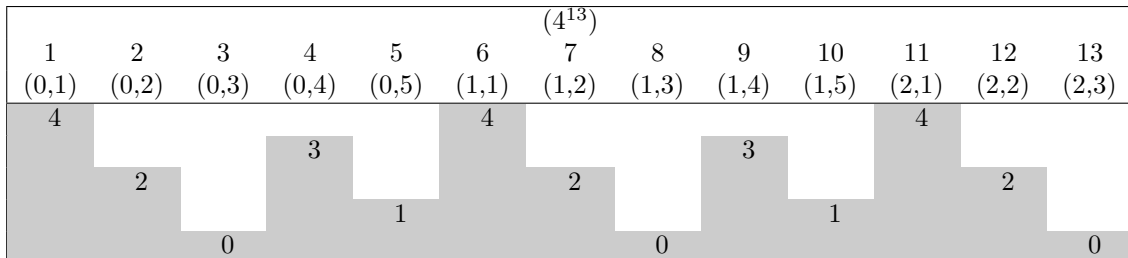


Figure 7: The Havel-Hakimi algorithm on (4^{13})

3.3 The number of reorderings that appear in the Havel-Hakimi algorithm when $\gcd(a + 1, n - 1) = 1$

Next we will attempt to find all regular degree sequences that appear in the Havel-Hakimi algorithm in the case where $\gcd(n - 1, a + 1) = 1$. In Figure 8 we have an example of a degree sequence, (10^{20}) , with multiple regular sequences appearing in the Havel-Hakimi algorithm. In this example we have $a + 1 = 11$ and $n - 1 = 19$ hence $\gcd(a + 1, n - 1) = 1$, meaning Theorem 12 applies. We have that $19 = 11(1) + 8$, hence $r - 1 = 8$. The inverse of $-8 \pmod{11}$ (i.e., $-(r - 1)^{-1} \pmod{a + 1}$) is $4 = k$, and the inverse of $11 \pmod{19}$ (i.e., $(a + 1)^{-1} \pmod{n - 1}$) is $7 = \ell$. Thus the next regular degree sequence to appear is $((10 - 4)^{20-7}) = (6^{13})$. Furthermore, we can apply Theorem 12 to (6^{13}) again as $\gcd(6 + 1, 13 - 1) = 1$. We have that $12 = 7(1) + 5$ and hence $r - 1 = 5$. The inverse of $-5 \pmod{7}$ (i.e., $-(r - 1)^{-1} \pmod{a + 1}$) is $4 = k$, and the inverse of $7 \pmod{12}$ (i.e., $(a + 1)^{-1} \pmod{n - 1}$) is $7 = \ell$ (the same values as before). Thus the next regular sequence following (6^{13}) is $((6 - 4)^{13-7}) = (2^5)$. In Theorem 13 we will show that as long as $a + 1$ and $n - 1$ are large enough, we can iteratively apply Theorem 12 with the

same values of k and ℓ . After a certain point (in the preceding example at (2^5)) we can still apply Theorem 12 but the values of k and ℓ will change.

Now let's show that this iterative process will yield the same values of k and ℓ for large enough $(a + 1)$ and $(n - 1)$.

Theorem 13. *Let $\ell = (a + 1)^{-1} \pmod{n - 1}$ and $k = -(r - 1)^{-1} \pmod{a + 1}$ and let j be the maximum value such that $a > jk$ and $n > j\ell$. Then in the first $j\ell$ steps of the Havel-Hakimi algorithm, the regular sequences that appear are $((a - ik)^{n - i\ell})$ for $i \in [j]$.*

Proof. We will proceed by induction on i , noting that the base case was considered in Theorem 12, and so we will now assume the result for i and show the result for $i + 1$.

Since $k = -(r - 1)^{-1} \pmod{a + 1}$, there is a $v \in \mathbb{N}$ such that $k(r - 1) = v(a + 1) - 1$. First we will show that $\ell(a + 1) = k(n - 1) + 1$ and $\ell = kq + v$. We have

$$\begin{aligned} k(n - 1) + 1 &= k(q(a + 1) + r - 1) + 1 \\ &= kq(a + 1) + k(r - 1) + 1 \\ &= kq(a + 1) + v(a + 1) - 1 + 1 \\ &= (a + 1)(kq + v). \end{aligned}$$

Thus $(a + 1)(kq + v) \equiv 1 \pmod{n - 1}$, and then since inverses are unique $kq + v = \ell$, and thus $\ell(a + 1) = k(n - 1) + 1$.

Next we will show that the remainder of $n - 1 - i\ell \pmod{a + 1 - ik}$ is $r - 1 - iv$. We have

$$\begin{aligned} n - 1 - i\ell &= q(a + 1) + r - 1 - i\ell \\ &= q(a + 1) + r - 1 - i(kq + v) \\ &= q(a + 1 - ik) + (r - 1 - iv). \end{aligned}$$

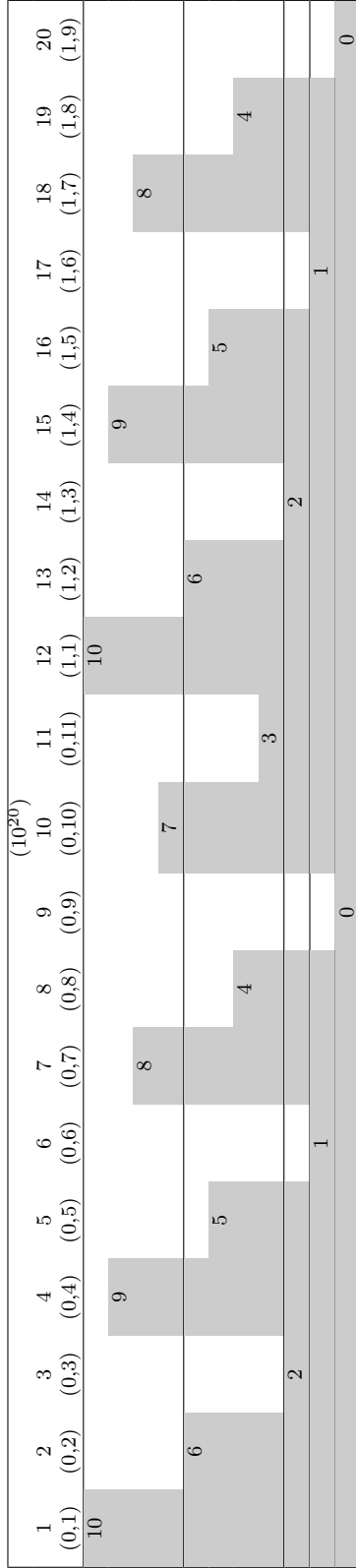


Figure 8: The Havel-Hakimi algorithm on (10^{20})

Then we will show that $\ell = (a + 1 - ik)^{-1} \pmod{n - 1 - i\ell}$. We have

$$\begin{aligned}\ell(a + 1 - ik) &= \ell(a + 1) - \ell ik \\ &= k(n - 1) + 1 - \ell ik \\ &= k(n - 1 - i\ell) + 1.\end{aligned}$$

Hence $\ell(a + 1 - ik) \equiv 1 \pmod{n - 1 - i\ell}$ and $\ell = (a + 1 - ik)^{-1} \pmod{n - 1 - i\ell}$.

Similarly, we will show that $k = -(r - 1 - iv)^{-1} \pmod{a + 1 - ik}$. We have

$$\begin{aligned}k(r - 1 - iv) &= k(r - 1) - kiv \\ &= v(a + 1) - 1 - kiv \\ &= v(a + 1 - ik) - 1\end{aligned}$$

Hence $k(r - 1 - iv) \equiv -1 \pmod{a + 1 - ik}$ and $k = -(r - 1 - iv)^{-1} \pmod{a + 1 - ik}$.

Since these inverses exist, we have $\gcd(a + 1 - ik, n - 1 - i\ell) = 1$, and thus from the result above we have that the next regular sequence that appears is $((a - ik - k)^{n - i\ell - \ell}) = ((a - (i + 1)k)^{n - (i + 1)\ell})$ and the result follows by induction. \square

3.4 The next regular sequence in the Havel-Hakimi algorithm when $\gcd(a + 1, n - 1) = d$

Now we will consider the case when $\gcd(a + 1, n - 1) \neq 1$.

Lemma 6. *Let $\gcd(a + 1, n - 1) \neq 1$ with $(a + 1)\ell \equiv 0 \pmod{n - 1}$ where $0 < \ell < n - 1$. By Lemma 4, $\gcd(r - 1, a + 1) \neq 1$ and thus there exists $0 < \ell < a + 1$ minimal such that $(r - 1)k \equiv 0 \pmod{a + 1}$. We have the following*

(a) For $j \in [0, k - 1]$,

$$X_{a-j} = \begin{cases} \{(i, 1 - j(r - 1)) \mid & i \in 0, \dots, q - 1 \\ & \text{if } 1 - j(r - 1) \pmod{a + 1} > r \end{cases}$$

$$\begin{cases} \{(i, 1 - j(r - 1)) \mid & i \in 0, \dots, q \\ & \text{if } 1 - j(r - 1) \pmod{a + 1} \leq r. \end{cases}$$

(b) For $j = k$, $X_{a-j} = \{(i, 2) \mid i \in 1, \dots, q\}$.

(c) For $j \in [k + 1, 2k - 1]$,

$$X_{a-j} = \begin{cases} \{(i, 1 - j(r - 1) + 1) \mid & i \in 0 \dots, q - 1 \\ & \text{if } 1 - j(r - 1) + 1 \pmod{a + 1} > r\} \\ \{(i, 1 - j(r - 1) + 1) \mid & i \in 0, \dots, q \\ & \text{if } 1 - j(r - 1) + 1 \pmod{a + 1} \leq r\}. \end{cases}$$

Proof. We will first show that the result for (a) follows from the proof of Lemma 5 using induction on j . The only possible issue is if $x_{a-(j-1)} = r$ for some j . That is $1 - (j - 1)(r - 1) \equiv r \equiv (r - 1) + 1 \pmod{a + 1}$, hence $-j(r - 1) \equiv 0 \pmod{a + 1}$ and $j(r - 1) \equiv 0 \pmod{a + 1}$. The value of k is minimal such that $k(r - 1) \equiv 0 \pmod{a + 1}$ and since $j < k$ this cannot happen. Thus the result follows the proof of Lemma 5.

To show (b), we will consider $x_{a-(k-1)} = 1 + (-(k - 1)(r - 1) \pmod{a + 1}) \equiv r \pmod{a + 1}$. Then considering the next $a + 1$ values,

$$(0, 0), \dots, (0, a + 1),$$

$k - 1$ are laid off and $(a + 1) - (k - 1)$ will be reduced. Thus the next available entry will be laid off. Since the $(1, 1)$ -term has already been laid off, the $(1, 2)$ -term is the next candidate. We need to show that this term has not already been laid off. If it had already been laid off we would have $x_{a-i} = 1 - i(r - 1) \equiv 2 \pmod{a + 1}$ for some $i \in [0, k - 1]$. This would yield $-i(r - 1) \equiv 1 \pmod{a + 1}$, and hence $-i \equiv (r - 1)^{-1} \pmod{a + 1}$. However, since $\gcd(r - 1, a + 1) \neq 1$, this inverse does not exist. Thus $(1, 2) \in X_{a-k}$. Using the same argument as in Lemma 5, this gives the correct form for X_{a-k} .

Finally we will show (c). We will again follow the proof of the previous lemma; however, because X_{a-k} does not include the term $(0, 2)$, special care needs to be taken. We will again proceed by induction on j , noting that we can use $j = k$ as

the base case (proven above in part (b)) as $x_{a-k} = 1 - k(r-1) + 1 \equiv 2 \pmod{a+1}$. We will assume the result for values less than j and show it is true for j . We will consider the 4 cases as in Lemma 5 based on the relationship between $p = x_{a-(j-1)}$ and r , noting that $p \neq r$ because $x_{a-(k-1)} = r$ and j is assumed to be at least $k+1$.

Case 1: $x_{a-(j-1)} = p > 2r$.

If $p > 2r$, then the last entry in $X_{a-(j-1)}$ is $(q-1, p)$. Consider the ordered pair indices of the next $a+1$ terms in d :

$$(q-1, p+1), \dots, (q-1, a+1), (q, 1), \dots, (q, p), (0, 1), \dots, (0, p-r).$$

We will show that of the terms corresponding to those indices, j have already been laid off before laying off terms with value $a-j$ (that is in $\bigcup_{\ell=0}^{j-1} X_{a-\ell}$), and $a-(j-1)$ will be reduced by 1. We then want to show that the next term with index $(0, p-(r-1))$ has not already been laid off. If it were, then we would have $1 - j(r-1) + 1 \equiv 1 - i(r-1) \pmod{a+1}$ for some $i \in [0, p-1]$ or $1 - j(r-1) + 1 \equiv 1 - i(r-1) + 1 \pmod{a+1}$ for some $i \in [p, j-1]$. This would then mean that either $(j-i)(r-1) \equiv 1 \pmod{a+1}$ which cannot happen as $r-1$ does not have an inverse modulo $a+1$ or $(j-i)(r-1) \equiv 0 \pmod{a+1}$ which cannot happen as $j-i < k$ and k is minimal such that $k(r-1) \equiv 0 \pmod{a+1}$. Thus the next term with index $(0, p-(r-1))$ will be the next term to be laid off, where $p-(r-1) = x_{a-j}$ as desired.

Of the $a+1$ indices listed above, observe that, those with remainder in $[1, r] \setminus \{2\}$ appear twice, those with remainder in $\{2\} \cup [r+1, p-r] \cup [p+1, a+1]$ appear once, and those with remainder in $[p-r+1, p]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j-1\}$, and we will partition D into

- $M_2 = D \cap ([1, r] \setminus \{2\})$

- $M_1 = D \cap (\{2\} \cup [r + 1, p - r] \cup [p + 1, a + 1])$
- $M_0 = D \cap [p - r + 1, p]$.

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_2$ for some $0 < i < j - 1$. Note that $x_{a-i} \equiv x_{a-(i-1)} - (r - 1) \pmod{a + 1}$ for all $i \in [0, j]$ except for when $i = k$ and $x_{a-k} \in M_1$. Then $x_{a-i} \neq 1$ as $i \neq 0$, hence $x_{a-i} \in [3, r]$. By the inductive hypothesis $x_{a-(i-1)} = x_{a-i} + (r - 1) \pmod{a + 1}$ (except for when $i = k$ however since $x_{a-i} \neq 2$ this will not happen) and thus $x_{a-(i-1)} \geq r + 1$ and hence not in M_2 . Because the length of the interval $[p - r + 1, p]$ is r , $x_{a-i-c} \in M_0$ for a minimal value c , where $x_{a-(i-c)} = x_{a-i} + c(r - 1) \leq a + 1$ (not modulo $a + 1$) and furthermore $x_{a-(i-c+1)} \notin M_0$ as then $x_{a-(i-c)} = p - r + 1$, hence $x_{a-(i-c+1)} = p$ which cannot happen as $x_{a-(j-1)} = p$ and $i - c + 1 < j - 1$. A symmetric result can be said for an arbitrary element x_{a-i} in M_0 for $i \in [2, j - 2]$. Thus for $i \in [1, j - 2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that the number of already laid off terms of the next $a + 1$ entries is

$$\begin{aligned}
(0 \cdot |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\
&= |M_1| + |M_2| + |M_0| \\
&= |D| = j.
\end{aligned}$$

Then, of the next $a + 1$ entries, j have already been laid off and $a - (j - 1)$ will be reduced yielding $p - r + 1$ as the remainder of the next term to be laid off (as it has not already been laid off), and hence $(0, p - (r - 1)) \in X_{a-j}$.

Then in the following $(a + 1)$ entries, j are already laid off and $a - j$ will be reduced giving $(1, p - (r - 1)) \in X_{a-j}$. Similarly we can show that $X_{a-j} = \{(i, p - (r - 1)) \mid i \in 0, \dots, q - 1\}$ as $p - r + 1 > 2r - r + 1 > r$. The result

is satisfied for this case.

Case 2: $x_{a-(j-1)} = p = 2r$.

If $p = 2r$, then the last entry in $X_{a-(j-1)}$ is $(q-1, p)$. Consider the ordered pair indices of the next $a+1$ terms in d :

$$(q-1, p+1), \dots, (q-1, a+1), (q, 1) \dots, (q, r), (0, 1) \dots, (0, p-r).$$

Of the $a+1$ indices listed above, observe that those with remainder in $[1, p-r] \setminus \{2\}$ appear twice, those with remainder in $\{2\} \cup [p+1, a+1]$ appear once, and those with remainder in $[p-r+1, p]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j-1\}$, and we will partition D into

- $M_2 = D \cap ([1, p-r] \setminus \{2\})$
- $M_1 = D \cap (\{2\} \cup [p+1, a+1])$
- $M_0 = D \cap [p-r+1, p]$

The same argument works for this case as in Case 1, as the only difference is the size of M_1 which does not affect the argument.

Case 3: $r < p < 2r$ where $p = x_{a-(j-1)}$.

If $r < p < 2r$, then the last entry in $X_{a-(j-1)}$ is $(q-1, p)$. Consider the ordered pair indices of the next $a+1$ terms in d :

$$(q-1, p+1), \dots, (q-1, a+1), (q, 1) \dots, (q, r), (0, 1) \dots, (0, p-r).$$

Of the $a+1$ indices listed above, observe that those with remainder in $[1, p-r] \setminus \{2\}$ appear twice, those with remainder in $\{2\} \cup [p-r+1, r] \cup [p+1, a+1]$ appear once,

and those with remainder in $[r + 1, p]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap ([1, p - r] \setminus \{2\})$
- $M_1 = D \cap (\{2\} \cup [p - r + 1, r] \cup [p + 1, a + 1])$
- $M_0 = D \cap [r + 1, p]$.

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_2$ for some $0 < i < j - 1$. Then $x_{a-i} \neq 1$ as $i \neq 0$, hence $x_{a-i} \in [3, r]$. By the inductive hypothesis $x_{a-(i-1)} \equiv x_{a-i} + (r - 1) \pmod{a + 1}$, and thus $x_{a-(i-1)}$ is in $[r + 1, p]$ and hence in M_0 . Because the length of the interval $[r + 1, p]$ is $p - r$, $x_{a-i-2} \notin M_0$. A symmetric result can be said for an arbitrary element x_{a-i} in M_0 for $i \in [2, j - 2]$. Thus for $i \in [1, j - 2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that number of already laid off terms of the next $a + 1$ entries is

$$\begin{aligned}
(0 \cdot |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\
&= |M_1| + |M_2| + |M_0| \\
&= |D| = j.
\end{aligned}$$

Then, of the next $a + 1$ entries, j have already been laid off and $a - (j - 1)$ will be reduced yielding $p - r + 1$ as the remainder of the next term to be laid off (as it has not already been laid off as shown in Case 1), and hence $(0, p - (r - 1)) \in X_{a-j}$.

Then in the following $(a + 1)$ entries, j are already laid off and $a - j$ will be reduced giving $(1, p - (r - 1)) \in X_{a-j}$. Similarly we can show that $X_{a-j} = \{(i, p - (r - 1)) \mid i \in 0, \dots, q\}$ as $p - r + 1 < 2r - r + 1 \leq 2r - r + 1 - 1 = r$.

The result is satisfied for this case.

Case 4: $x_{a-(j-1)} = p < r$.

If $p < r$, then the last entry in $X_{a-(j-1)}$ is (q, p) . Consider the ordered pair indices of the next $a + 1$ terms in d :

$$(q, p + 1), \dots, (q, r), (0, 1) \dots, (0, a + 1 - (r - p)).$$

Of the $a + 1$ indices listed above, observe that, those with remainder in $[p + 1, r]$ appear twice, those with remainder in $[1, p] \cup [r + 1, (a + 1) - (r - p)] \setminus \{2\}$ appear once, and those with remainder in $\{2\} \cup [(a + 1) - (r - k) + 1, a + 1]$ do not appear. We will call the remainders of the terms already laid off $D = \{x_{a-\ell} \mid 0 \leq \ell \leq j - 1\}$, and we will partition D into

- $M_2 = D \cap [p + 1, r]$
- $M_1 = D \cap ([1, p] \cup [r + 1, (a + 1) - (r - p)] \setminus \{2\})$
- $M_0 = D \cap (\{2\} \cup [(a + 1) - (r - k) + 1, a + 1])$

We will now show that there is a bijective correspondence between M_2 and M_0 . Suppose that there exists an $x_{a-i} \in M_0$ for some $0 < i < j - 1$, except that $x_{a-i} \neq (a + 1) - (r - p) + 1$ as $x_{a-(i-1)}$ would then be equal to p which cannot happen as $x_{a-(j-1)} = p$ and $i - 1 < j - 1$. By the inductive hypothesis $x_{a-(i-1)} = x_{a-i} + (r - 1) \pmod{a + 1}$, and thus $x_{a-(i-1)}$ is in $[p + 1, r - 1]$ and thus in M_2 . Furthermore, note that if $x_{a-i} = 2 = x_{a-k} \in M_0$, we have that $x_{a-(k-1)} = 2 + (r - 1) - 1 = r \in M_2$. Because the length of the interval $[p + 1, r]$ is $r - p$, $x_{a-i-2} \notin M_2$. A symmetric result can be said for an arbitrary element x_{a-i} in M_2 for $x_{a-i} \in [p + 1, r - 1]$, noting that $x_{a-i} \neq r$, as then $x_{a-i+1} = 1$ but $x_a = 1$

and $i + 1 > 0$. Thus for $i \in [1, j - 2]$, there is a bijective correspondence between M_0 and M_2 . Then since $x_a \in M_2$ and $x_{a-(j-1)} \in M_0$, we have that $|M_2| = |M_0|$. Thus we have that number of already laid off terms of the next $a + 1$ entries is

$$\begin{aligned} (0 \cdots |M_0|) + (1 \cdot |M_1|) + (2 \cdot |M_2|) &= |M_1| + |M_2| + |M_2| \\ &= |M_1| + |M_2| + |M_0| \\ &= |D| = j. \end{aligned}$$

Then, of the next $a + 1$ entries, j have already been laid off and $a - (j - 1)$ will be reduced yielding $(a + 1) - (r - p) + 1$ as the remainder of the next term to be laid off (as it has not already been laid off as shown in Case 1), and hence $(0, (a + 1) - (r - p) + 1) \in X_{a-j}$.

Then in the following $(a + 1)$ entries, j are already laid off and $a - j$ will be reduced giving $(1, (a + 1) - (r - p) + 1) \in X_{a-j}$. Similarly we can show that $X_{a-j} = \{(i, p - (r - 1) \mid i \in 0, \dots, q\}$ or $X_{a-j} = \{(i, p - (r - 1) \mid i \in 0, \dots, q - 1\}$ depending on if $(a + 1) - (r - p) + 1$ is less than or equal to r or greater than r . The result is satisfied for this case.

Therefore, all cases are satisfied and the result follows by the principle of mathematical induction.

□

Theorem 14. *Let $\gcd(a + 1, n - 1) \neq 1$ with $(a + 1)\ell \equiv 0 \pmod{n - 1}$ where $0 < \ell < n - 1$ is minimal. By Lemma 4 $\gcd(r - 1, a + 1) \neq 1$ and thus there exists $0 < k < a + 1$ such that $(r - 1)k \equiv 0 \pmod{a + 1}$. The next regular degree sequence that appears in the Havel-Hakimi algorithm will be (b^m) where $b = a - 2k$ and $m = n - 2\ell$.*

Before proving Theorem 14, let's consider the example (13^{22}) shown in Fig-

ure 9. In this example we have $a+1 = 14$ and $n-1 = 21$, hence $\gcd(a+1, n-1) = 7$. Since $14 \cdot 3 \equiv 0 \pmod{21}$ and $7 \cdot 2 \equiv 0 \pmod{14}$, we have that $\ell = 3$ and $k = 2$. Thus by Theorem 14, the next regular sequence that appears is $((13 - 2(2))^{22-2(3)}) = (9^{16})$. We can see in Figure 9 that the last term of degree 10 laid off, reduces all remaining terms in that row, yielding a new regular degree sequence of degree 9. Similarly, as we will see in Theorem 15, this process repeats for 2 more steps, yielding regular degree sequences $((9 - 2(2))^{16-2(3)}) = (5^{10})$, and $((5 - 2(2))^{10-2(3)}) = (1^4)$ as both of the new regular degree sequences appeal to Theorem 14.

Proof. We will order the terms as described in the proof of Lemma 6, and use the notation for X_{a-j} and x_{a-j} . We will have a new regular degree sequence only after laying off the $(q-1, r+1)$ -term, as then the next term to be laid off will have index $(0, 2)$ (since $r+1 - (r-1) = 2$). Because the $(0, 1)$ -term is the first term to be laid off, we have that $(0, 2)$ is the index of the first remaining term at this point (as the $(0, 2)$ -term was not laid off in X_{a-k} even though $x_{a-k} = 2$) and thus the degree of all remaining terms are equal to $a-b$ where $x_{a-(b-1)} \equiv r+1 \pmod{a+1}$. We have that if $b = 2k$, then

$$\begin{aligned}
x_{a-(b-1)} = x_{a-(2k-1)} &\equiv 1 - (2k-1)(r-1) + 1 \pmod{a+1} \\
&\equiv 2 - 2k(r-1) + (r-1) \pmod{a+1} \\
&\equiv 2 - (r-1) \pmod{a+1} \\
&\equiv r+1 \pmod{a+1}.
\end{aligned}$$

The previous Lemma showed that all of the x_{a-j} values are unique for $j \in [0, 2k-1]$, and thus $b = a - 2k$ as desired.

We will enumerate all terms by their placement in the lexicographic order, that is $(0, 1) = 0, (0, 2) = 1, \dots, (q, r) = n - 1$. Note that the difference between

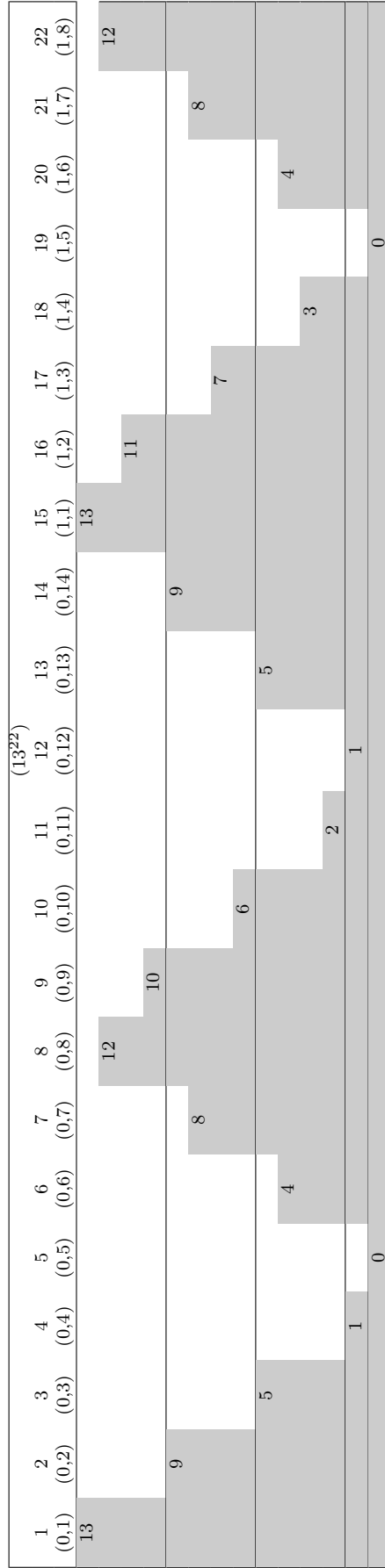


Figure 9: The Havel-Hakimi algorithm applied to (13^{22})

consecutive elements in X_{a-j} is $a+1$ for all j , and there are $a+2$ terms between the last element of X_{a-j} and the first element of $X_{a-(j+1)}$ (consecutively laid off terms of different degrees), except that of $X_{a-(k-1)}$ and X_{a-k} in which the difference is $a+3$.

Since the $(0,1)$ -term is laid off first for every degree sequence, we will then consider only the difference in regards to the remaining $n-1$ terms $1, \dots, n-1$. Note that doing this preserves the difference between consecutive elements within X_{a-j} but changes the number of terms between consecutively laid off terms of different degrees to $a+1$ as $(0,1) = 0$ was included in the counting previously and $a+2$ for $X_{a-(k-1)}$ and X_{a-k} . Thus we have that the p^{th} term to be laid off will have index $(p-2)(a+1) + (a+2) \equiv (p-1)(a+1) + 1 \pmod{n-1}$.

As above, we have a new regular degree sequence only after laying off the term with index $(q-1, r+1) = n-a-1 = n-(a+1)$. Thus we need that $(p-1)(a+1) + 1 \equiv n-(a+1) \pmod{n-1}$. We then have that $p(a+1) + 1 \equiv n \equiv 1 \pmod{n-1}$ and $p(a+1) \equiv n-1 \equiv 0 \pmod{n-1}$, hence $p = t\ell \pmod{n-1}$ for some $t \in \mathbb{Z}$.

First consider the case where $p = \ell$ (that is $t = 1$). We have $x_{a-(k-1)} = r$ and we can find how many terms must be laid off (p') before laying off the term with index $(q, r) = n-1$. From Lemma 6, we have the same differences discussed above (without the exception) and hence $(p')(a+1) \equiv n-1$ hence $p' = \ell$. We then have that $p \neq \ell$, and is thus equal to 2ℓ . Thus $m \equiv n-2\ell$ as desired. \square

3.5 The number of reorderings that appear in the Havel-Hakimi algorithm when $\gcd(a+1, n-1) = d$

Now that we know what the next regular sequence will be in the Havel-Hakimi algorithm, we will now show all regular sequences that appear in the algorithm. Using this information we will then be able to predict the number of reorderings

required in the Havel-Hakimi algorithm.

Lemma 7. *Let $\gcd(a + 1, n - 1) = d$ with $k, \ell \in \mathbb{N}$ minimal such that $k(r - 1) \equiv 0 \pmod{a + 1}$ and $\ell(a + 1) \equiv 0 \pmod{n - 1}$ and let $s, t \in \mathbb{N}$ such that $\ell(a + 1) = s(n - 1)$ and $k(r - 1) = t(a + 1)$. Then $\ell = \frac{n-1}{d}$, $s = \frac{a+1}{d}$, $k = \frac{a+1}{d}$, and $t = \frac{r-1}{d}$.*

Proof. We have that $\frac{n-1}{d}(a + 1) \equiv 0 \pmod{n - 1}$ as $\frac{a+1}{d} \in \mathbb{N}$, and thus we need to show that $\frac{n-1}{d}$ is minimal. Suppose that $\ell < \frac{n-1}{d}$. Then $\ell(a + 1) < \frac{n-1}{d}(a + 1) = \frac{a+1}{d}(n - 1)$, hence $s < \frac{a+1}{d}$. Furthermore, we must have that $\frac{s(n-1)}{a+1} = \frac{s(\frac{n-1}{d})}{(\frac{a+1}{d})} \in \mathbb{N}$. Since $\gcd(\frac{n-1}{d}, \frac{a+1}{d}) = 1$, we have that $\frac{a+1}{d}$ divides s , and thus $s = (\frac{a+1}{d})\alpha$ for some $\alpha \in \mathbb{N}$. However since $s < \frac{a+1}{d}$, this is a contradiction and thus $\ell = \frac{n-1}{d}$, hence $s = \frac{a+1}{d}$.

A symmetric argument holds for showing the result for k and t . □

Theorem 15. *Let $\gcd(a + 1, n - 1) = d$. The only regular sequences that appear in the first $2\left(\frac{d-1}{2}\right)\ell$ steps of the Havel-Hakimi algorithm are those of the form $((a - 2ik)^{n-2i\ell})$ where $i \in [1, \frac{d-1}{2}]$.*

Proof. We will proceed by induction on i , noting that the base case was taken care of in the result above, that being that the next regular sequence that appears in the Havel Hakimi algorithm is $((a - 2k)^{n-2\ell})$. Then assume that the only regular sequences that appear in the first $2i\ell$ steps of the Havel-Hakimi algorithm are those of the correct form and show the result for $i + 1$. We then have that after $2i\ell$ steps we have the regular degree sequence $((a - 2ik)^{n-2i\ell})$, and we will show that the next regular degree sequence to appear is $((a - 2(i + 1)k)^{n-2(i+1)\ell})$.

By the division algorithm we have integers q', r' such that $n - 1 - 2i\ell = q'(a + 1 - 2ik) + r' - 1$ with $r' - 1 < a + 1 - 2ik$. We will first show that $r' - 1 = r - 1 - 2it$. Since $n - 1 = q(a + 1) + r - 1$, we have that

$$\begin{aligned}
n - 1 - 2i\ell &= q(a + 1) + r - 1 - 2i\ell \\
&= q(a + 1 - 2ik + 2ik) + r - 1 - 2i\ell \\
&= q(a + 1 - 2ik) + 2ikq + r - 1 - 2i\ell
\end{aligned}$$

and thus $r' - 1 \equiv 2ikq + r - 1 - 2i\ell \pmod{a + 1 - 2ik}$.

Then, using Lemma 7,

$$\begin{aligned}
2ikq + r - 1 - 2i\ell &= 2iq\left(\frac{a+1}{d}\right) + r - 1 - 2i\left(\frac{n-1}{d}\right) \\
&= \frac{1}{d}(2iq(a+1) + (r-1)d - 2i(n-1)) \\
&= \frac{1}{d}(2iq(a+1) + (r-1)d - 2i(q(a+1) + (r-1))) \\
&= \frac{1}{d}(2iq(a+1) + (r-1)d - 2iq(a+1) - 2i(r-1)) \\
&= \frac{1}{d}((r-1)d - 2i(r-1)) \\
&= (r-1) - 2i\left(\frac{r-1}{d}\right) \\
&= (r-1) - 2it.
\end{aligned}$$

Now we will show that $\gcd(a + 1 - 2ik, n - 1 - 2i\ell) = d - 2i$. We have

$$\begin{aligned}
\gcd(a + 1 - 2ik, n - 1 - 2i\ell) &= \gcd\left(a + 1 - 2i\left(\frac{a+1}{d}\right), n - 1 - 2i\left(\frac{n-1}{d}\right)\right) \\
&= \gcd\left(\frac{(a+1)(d-2i)}{d}, \frac{(n-1)(d-2i)}{d}\right) \\
&= d - 2i
\end{aligned}$$

as $\gcd\left(\frac{a+1}{d}, \frac{n-1}{d}\right) = 1$. Furthermore, we then have that $\frac{a+1-2ik}{d-2i} = a + 1$ and $\frac{n-1-2i\ell}{d-2i} = n - 1$.

Next we will show that the value of minimal ℓ' such that $\ell'(a + 1 - 2ik) \equiv 0 \pmod{n - 2i\ell}$ is equal to ℓ , and the minimal k' such that $k'(r' - 1) \equiv 0 \pmod{a + 1 - 2ik}$

$2ik$) is equal to k . To do this, first we will show that $\ell(a+1-2ik) = s(n-1-2i\ell)$, and then we will show that ℓ is minimal.

$$\begin{aligned}
\ell(a+1-2ik) &= \ell(a+1) - 2ik\ell \\
&= \frac{n-1}{d}(a+1) - 2i\left(\frac{a+1}{d}\right)\ell \\
&= \frac{a+1}{d}(n-1-2i\ell) \\
&= s(n-1-2i\ell)
\end{aligned}$$

We will now show that ℓ is minimal. Assume there exists an ℓ' such that $\ell'(a+1-2ik) = s'(n-1-2i\ell')$ with $\ell' < \ell$ and we will show a contradiction. Then

$$\begin{aligned}
\ell'(a+1-2ik) &< \ell(a+1-2ik) \\
&= \frac{n-1}{d}(a+1-2i\left(\frac{a+1}{d}\right)) \\
&= \frac{(n-1)(a+1)}{d} - \frac{2i\left(\frac{a+1}{d}\right)(n-1)}{d} \\
&= \left(\frac{a+1}{d}\right)\left((n-1) - \frac{2i(a+1)}{d}\right) \\
&= \left(\frac{a+1}{d}\right)(n-1-2i\ell).
\end{aligned}$$

Hence $s' < s = \frac{a+1}{d}$. Furthermore, we must have that

$$\frac{s'(n-1-2i\ell')}{a+1-2ik} = \frac{s'\left(\frac{n-1-2i\ell'}{d-2i}\right)}{\left(\frac{a+1-2ik}{d-2i}\right)} \in \mathbb{N}$$

Since $\gcd\left(\frac{n-1-2i}{d-2i}, \frac{a+1-2ik}{d-2i}\right) = 1$, then $\frac{a+1-2ik}{d-2i} = a+1$ divides s' , and thus $s' = (a+1)\alpha$ for some $\alpha \in \mathbb{N}$. However since $s' = (a+1)\alpha > \frac{a+1}{d} = s$, this is a contradiction and thus $\ell' = \ell = \frac{n-1}{d}$.

Next we will show that $k(r-1-2it) = t(a+1-2ik)$, and then we will show that k is minimal.

$$\begin{aligned}
k(r-1-2it) &= k(r-1) - 2ikt \\
&= \frac{a+1}{d}(r-1) - 2i\left(\frac{r-1}{d}\right)k \\
&= \left(\frac{r-1}{d}\right)(a+1-2ik) \\
&= t(a+1-2ik)
\end{aligned}$$

Next we will show that k is minimal. Assume there exists a k' such that $k'(r-1-2it) = t'(a+1-2ik)$ with $t' < t$ and we will show a contradiction. Then

$$\begin{aligned}
k'(r-1-2it) &< k(r-1-2it) \\
&= \frac{a+1}{d}(r-1-2i\left(\frac{r-1}{d}\right)) \\
&= \frac{(a+1)(r-1)}{d} - \frac{2i\left(\frac{r-1}{d}\right)(a+1)}{d} \\
&= \left(\frac{r-1}{d}\right)\left((a+1) - \frac{2i(a+1)}{d}\right) \\
&= \left(\frac{r-1}{d}\right)(a+1-2ik).
\end{aligned}$$

Hence $t' < t = \frac{r-1}{d}$. Furthermore, we must have that

$$\frac{t'(a+1-2ik)}{r-1-2it} = \frac{t'\left(\frac{a+1-2ik}{d-2i}\right)}{\left(\frac{r-1-2it}{d-2i}\right)} \in \mathbb{N}$$

Since $\gcd\left(\frac{a+1-2i}{d-2i}, \frac{r-1-2it}{d-2i}\right) = 1$, then $\frac{r-1-2it}{d-2i} = r-1$ divides t' , and thus $t' = (r-1)\alpha$ for some $\alpha \in \mathbb{N}$. However since $t' = (r-1)\alpha > \frac{r-1}{d} = t$, this is a contradiction and thus $k' = k = \frac{a+1}{d}$.

We then have for $i < \frac{d-1}{2}$, the next regular sequence is $((a-2(i+1)k)^{n-2(i+1)\ell})$, and the result follows by induction. □

Then note in the proof above we have that the $\frac{d-1}{2}$ th regular sequence to appear in the Havel-Hakimi algorithm is necessarily $((a - (d - 1)k)^{n - (d-1)\ell})$ with $\gcd(a + 1 - (d - 1)k, n - 1 - (d - 1)\ell) = d - 2(\frac{d-1}{2}) = 1$. Thus to find the rest of the regular degree sequences that appear in the algorithm, we can apply Theorem 13.

3.6 Open questions

The goal of understanding the number of reorderings is to understand the difference between the independence number and the residue. This leads to some open questions and possibilities of further research.

- Is there a closed formula for the number of reorderings of all regular degree sequences?
- What are the number of reorderings of semi-regular degree sequences and how does this relate to that of regular degree sequences?
- What are the number of reorderings of an arbitrary degree sequence?
- Does the number of reorderings directly correspond to the difference in the residue and independence number?
- In what ways is the number of reorderings of regular degree sequences related to the original Josephus problem solution and can this generalization be investigated further and expanded?

List of References

- [1] V.Havel, “A remark on the existence of finite graphs,” *Časopis Pěst Mat.*, vol. 80, pp. 477–480, 1955.
- [2] S. Hakimi, “On the realizability of a set of integers as degree sequences of the vertices of a graph,” *SIAM Appl. Math.*, vol. 10, pp. 496–506, 1962.
- [3] S. Fajtlowicz, “On conjectures of graffiti, iii,” *Cong. Numer.*, vol. 66, pp. 23–32, 1988.

- [4] O. Favaron, M. Mahéo, and J. F. Saclé, “On the residue of a graph,” *Graph Theory*, vol. 15, no. 1, pp. 39–64, 1991.
- [5] J. Griggs and D. Kleitman, “Independence and the havel-hakimi residue,” *Discrete Mathematics*, vol. 127, no. 1–3, pp. 209–212, 1994.
- [6] E. Triesch, “Degree sequences of graphs and dominance order,” *J. Graph Theory*, vol. 22, no. 1, pp. 89–93, 1996.
- [7] F. Jelen, “ k -independence and k -residue of a graph,” *J. Graph Theory*, vol. 127, pp. 209–212, 1999.
- [8] M. R. Garey and D. S. Johnson, *Computers and intractability*. freeman San Francisco, 1979, vol. 174.
- [9] S. R. Arikati and U. N. Peled, “Degree sequences and majorization,” 1994.
- [10] P. Nelson and A. J. Radcliffe, “Semi-regular graphs of minimum independence number,” *Discrete Mathematics*, vol. 275, pp. 237–263, 2004.
- [11] L. Halbeisen and N. Hungerbühler, “The josephus problem,” *Journal de Théorie des Nombres de Bordeaux*, vol. 9, no. 2, pp. 303–318, 1997.

CHAPTER 4

Minimum and maximum number of reorderings

Benjamin Lantz

Department of Mathematics

University of Rhode Island

Kingston, RI 02881

This manuscript is currently being prepared for submission.

Keywords: degree sequence, Havel-Hakimi Algorithm, Havel-Hakimi residue, reorderings, threshold sequences, independence number

AMS 2010 Mathematics Subject Classification: 05C35

Abstract. Given a degree sequence, the Havel-Hakimi algorithm reduces the sequence, one term at a time, to a list of R zeroes, where R is deemed the residue of said sequence. Each step of the algorithm requires that the terms in the degree sequence be listed in non-increasing order, and thus sometimes a reordering of the terms is required. In this manuscript, the cases of zero reorderings, one reordering, and a maximal amount of reorderings are considered. A characterization of degree sequences of each is identified and certain graphic realizations of those degree sequences are analyzed.

4.1 Introduction

Whether or not an integer sequence has a graphic realization or not can be determined by the Erdős-Gallai inequalities:

Theorem 16. (*Erdős-Gallai Inequalities [1]*) *A list (d_1, \dots, d_n) of non-negative integers in non-increasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i), \text{ for } 1 \leq k \leq n.$$

Another way to determine whether or not an integer sequence is graphic or not is the Havel-Hakimi algorithm. Given a degree sequence $d = (d_1, d_2, \dots, d_n)$, an iterative step in the Havel-Hakimi algorithm, developed independently by Havel [2] and Hakimi [3], reduces d to $H(d) = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$. After reordering the terms to be non-increasing, the algorithm iterates until no positive entries are present. Given a degree sequence $H(d)$, we call d a parent in the Havel-Hakimi algorithm, noting that parents are not unique. The algorithm arose to determine when an integer sequence is graphic: that is a list of integers d is graphic if and only if the Havel-Hakimi algorithm terminates in a list of zeros. The number of these zeros is said to be the residue of the degree sequence, and the

residue of a graph G , denoted $R(G)$, is the residue of the degree sequence of G . The residue is of interest because of its connection to the independence number of a graph, $\alpha(G)$. In 1988, the conjecture-making computer program Graffiti [4] proposed the following theorem,

Theorem 17. [5] *For every graph G , $R(G) \leq \alpha(G)$.*

This result was proven by Favaron et al. in 1991 and improved upon by Griggs and Kleitman [6], Triesch [7], and Jelen [8] in the 1990's. Determining the independence number is NP-hard, but since it takes only $O(E)$ steps to determine the residue where E is the number of edges in a graph, it is of interest to know how well $R(G)$ approximates $\alpha(G)$ and when the bound is realized.

Given two degree sequences, $d = (d_1, \dots, d_n)$ and $e = (e_1, \dots, e_n)$ with $\sum_{i=1}^n d_i = \sum_{i=1}^n e_i$, we say that d majorizes e if $\sum_{i=1}^k d_i \geq \sum_{i=1}^k e_i$ for all $i \in [n]$, denoted $d \succeq e$. Majorization of degree sequences with a fixed sum of entries exhibits a partial order. At the top of the majorization partial order are threshold sequences, and at the bottom of the partial order for a fixed length of degree sequence are semi-regular degree sequences (degree sequences of the form $(k^A, (k-1)^{n-A})$).

Theorem 18. [5] *Let d and e be degree sequences of the same length and same sum of their entries. If $d \succeq e$, then $R(d) \geq R(e)$.*

In order to understand any significance of the number of reorderings of a degree sequence during the Havel-Hakimi algorithm, we need to analyze the cases in which the minimum number and maximum number of reorderings are attained. The minimum number of reorderings possible (zero) is attained by threshold sequences. Threshold sequences have a number of characterizations; one that we will focus on is in relation to the Erdős-Gallai inequalities,

Theorem 19. [9] *Let $d = (d_1, \dots, d_n)$ be a degree sequence with $m(d) = \max\{i \mid$*

$d_i \geq i - 1\}$. Then d is a threshold sequence if and only if the k^{th} Erdős-Gallai inequality is satisfied with equality for all $k \leq m(d)$.

After understanding the case of zero reorderings, it is logical to then consider the case of one reordering and to analyze what happens to the difference in the independence number and residue of the degree sequence.

Finally, on the other end of the spectrum, we can consider the case of a maximum number of reorderings. Again, this case can be analyzed to help determine the difference in the bound of the residue and independence number.

4.2 Zero reorderings

Theorem 20. *A degree sequence has no reorderings in the Havel-Hakimi algorithm if and only if the degree sequence is threshold.*

Proof. We will first consider the forward implication and let $d = (d_1, \dots, d_n)$, with $n \geq 1$, be a degree sequence requiring no reorderings in the Havel-Hakimi algorithm and proceed by induction on n . If $n = 1$, there is only one possible degree sequence, namely (0) , which trivially has no reorderings in the Havel-Hakimi algorithm, and is trivially threshold. We will then assume that any degree sequence of length less than n , where n is at least 2, that has no reorderings must be threshold. Since d has no reorderings, $H(d)$ must also not have any reorderings, and by the inductive hypothesis, $H(d)$ must be threshold. From Section 1.2, a threshold degree sequence is the degree sequence resulting from a graph that can be constructed by adding a sequence of isolated vertices or dominating vertices to an empty graph. Thus $H(d)_1$ must be the degree of the last term appended in this sequence corresponding to a dominating vertex, hence $H(d) = (H(d)_1, \dots, H(d)_{H(d)_1+1}, 0^{n-H(d)_1-1})$. Furthermore, $d_1 > H(d)_1$ (as there is no reordering in the first step of the Havel-Hakimi algorithm) and thus $d = (d_1, H(d)_1 + 1, \dots, H(d)_{H(d)_1+1} + 1, 1^a, 0^b)$ for $a, b \geq 0$. Thus we can

build a realization of d by adding a sequence of dominating/isolated vertices to an empty graph in such a way as to construct a realization of $H(d)$ (as $H(d)$ is threshold), then adding a dominating vertex and finally b isolated vertices. The resulting graph (and hence d) is threshold, and the result follows by induction.

We will now consider the reverse implication and again proceed by induction on n . If $n = 1$, there is only one degree sequence (0) which is threshold and trivially has no reorderings in the Havel-Hakimi algorithm. Let $n \geq 2$ and assume that any degree sequence of length less than n that is threshold must have no reorderings in the Havel-Hakimi algorithm. Let d be a threshold sequence. Since d is threshold, either $d_1 = n - 1$, or $d = (d_1, \dots, d_{d_1+1}, 0^a)$, since d_1 is the degree of the last dominating vertex added. In either case, the Havel-Hakimi algorithm decreases all non-zero entries by 1 and thus $H(d)$ does not need to be reordered. Since threshold graphs are closed under vertex deletions, the resulting sequence $H(d)$ is threshold and thus there are no further reorderings in the Havel-Hakimi algorithm by the inductive hypothesis. Thus the result follows by induction. \square

4.3 One reordering

Now we will examine the case in which there is only one reordering in the Havel-Hakimi algorithm. We say that v has the Havel-Hakimi property if v is of maximum degree and its neighbors are vertices of highest possible degree. This vertex is named such because a graphic realization of $H(d)$ is $G \setminus \{v\}$, where a step in the Havel-Hakimi algorithm corresponds to the deletion of a vertex in a realization.

Theorem 21. *A degree sequence has one reordering in the Havel-Hakimi algorithm if and only if it can be built from the following degree sequences:*

- (i) $(2^a, 0^a)$ where $a \geq 0$

(ii) $(n - 3, 1^{n-1}, 0^a)$ where $n \geq 4$ and $a \geq 0$

(iii) $(\ell + 1, n - \ell - 1, 1^{n-2}, 0^a)$ where $n \geq 4$, $\frac{n-2}{2} \leq \ell \leq n - 2$, and $a \geq 0$

(iv) $(b + \ell - 1, n - \ell - 1, d_3, \dots, d_b, 1^{n-b}, 0^a)$ where $b \geq 3$, $n \geq 4$, $\frac{n-2}{2} \leq \ell \leq n - 2$, $a \geq 0$, and $(d_3 - 2, \dots, d_b - 2)$ is a threshold degree sequence.

by appending to the degree sequence a sequence of terms corresponding to vertices with the Havel-Hakimi property that have degree strictly greater than that of the previous max degree term appended.

Proof. Let $e = (e_1, \dots, e_m)$ be a degree sequence with one reordering in the Havel-Hakimi algorithm, and let that reordering happen at the j^{th} stage of the Havel-Hakimi algorithm. Note that appending or removing any 0 terms to the degree sequence will not affect the number of reorderings. Thus the number of reorderings of e is equal to that of $e' = (e_1, \dots, e_m, 0^a)$ for any $a \geq 0$. Thus in building (i), (ii), (iii), and (iv) we will let $a = 0$ and assume that the last entry of each degree sequence is strictly positive.

We will let $d = (d_1, \dots, d_n) = H^{j-1}(e)$, and thus $H(d)$ is a threshold degree sequence. In order for the creation of $H(d)$ to need a reordering, $d_{d_1+1} = d_{d_1+2}$. Thus d is of the form $d = (d_1, \dots, d_b, d_{b+1}^p, d_{b+p+1}, \dots, d_n)$, where $d_{b+1} = d_{d_1+1} = d_{d_1+2}$, $b \geq 0$, and $p \geq 2$. Here $b + 1$ is the first index i such that $d_i = d_{d_1+1}$, and p is the multiplicity of this term in d . Thus if $n > b + p$, then $d_{b+p} > d_{b+p+1}$. We will first show that d is of the form of one of the four degree sequences listed in the theorem.

Case 1.1: $d_n > 1$ and $H(d)_1 = d_2 - 1$.

Because $d_n > 1$, we have that $H(d)_i > 0$ for all i . Since $H(d)$ is threshold, it satisfies the k^{th} Erdős-Gallai inequality (Theorem 16) with equality for all k satisfying $H(d)_k \geq k - 1$ by Theorem 19. Since $d_2 - 1 \geq 0$, we have

$$d_2 - 1 = 1(1 - 0) + \sum_{i=2}^{n-1} \min(H(d)_i, 1) = \sum_{i=2}^{n-1} 1 = n - 2.$$

Thus we have that $d_2 = n - 1$, and hence $d_1 = n - 1$, which would mean that there is no reordering in the Havel-Hakimi algorithm which is a contradiction.

Case 1.2: $d_n > 1$ and $H(d)_1 = d_{b+1}$.

In this case we must have that $d_2 - 1 = d_{b+1} - 1$, and hence $d_2 = d_{b+1}$. Because $d_n > 1$, we have that $H(d)_i > 0$ for all i . Since $H(d)$ is threshold, it satisfies the k^{th} Erdős-Gallai inequality with equality for all k satisfying $H(d)_k \geq k - 1$. Since $d_2 - 1 \geq 0$, we have

$$d_{b+1} = d_2 = 1(1 - 0) + \sum_{i=2}^{n-1} \min(H(d)_i, 1) = \sum_{i=2}^{n-1} 1 = n - 2.$$

If $d_1 = n - 1$, then there is no reordering which is a contradiction, thus $d_1 = n - 2$. Then since $d_1 = d_2 = d_{b+1} = d_{d_1+2} = d_n$, we have $d = ((n - 2)^n)$, noting that n must be even. Then $H(d) = ((n - 2), (n - 3)^{n-2})$ which requires a reordering and $H(H(d)) = ((n - 4)^{n-2})$ which does not require a reordering. Furthermore, $H(H(H(d)))$ will require a reordering for $n \geq 6$ as the degree sequence is the same form as d . Thus in this case, there is a reordering every other step of the Havel-Hakimi algorithm, and thus the only time there is only one reordering is if $n - 2 = 2$, thus $d = (2^4)$, yielding (i).

Case 2: $d_n = 1$.

If $H(d)_{n-1} > 0$, the first Erdős-Gallai inequality must hold with equality and therefore must have the same value for $d_2 = d_{b+1} = n - 2$ yielding the same result as in Case 1. Thus we will assume that $H(d)_{n-1} = 0$. This then means that $d_{b+1} = 1$, as some terms need to be reduced to 0. Thus d is of the form $d = (d_1, \dots, d_b, 1^{n-b})$.

Note that if $b = 0$ we have $d = (1^n)$ which will require one reordering only when $n = 4$. Also if $b = 1$, we have that $d = (\ell, 1^{n-1})$ with $\ell \geq 1$, and hence $H(d) = (1^{n-1-\ell}, 0^\ell)$ which has another reordering in the next step of the Havel-Hakimi algorithm unless $n - 1 - \ell = 2$, hence $\ell = n - 3$, giving the case that $d = (n - 3, 1^{n-1})$, yielding (ii).

If $b = 2$, $d_1 = \ell + 1$, where ℓ is the number of zeros in $H(d)$ and thus if $H(d)$ is threshold, the first Erdős-Gallai equality gives $d_2 - 1 = n - \ell - 2$, yielding $d = (\ell + 1, n - \ell - 1, 1^{n-2})$, where since $d_1 \geq d_2$ we find that $\frac{n-2}{2} \leq \ell \leq n - 2$. Then $H(d) = (n - \ell - 2, 1^{n-\ell-2}, 0^\ell)$, and $H^2(d) = (0^{\ell+1})$, and thus there is only one reordering in this case, yielding (iii).

Now let $b > 2$. Then we have that $H(d) = (d_2 - 1, \dots, d_b - 1, 1^{n-b-\ell}, 0^\ell)$ for some $\ell > 0$ and $d_1 = b + \ell - 1$. This then means that $H(d)_1 = d_2 - 1 \geq 1$. Since $H(d)$ is threshold, it satisfies the Erdős-Gallai inequalities with equality as long as $H(d)_k \geq k - 1$. Since $d_2 - 1 \geq 0$, we have

$$d_2 - 1 = 1(1 - 0) + \sum_{i=2}^{n-1} \min(H(d)_i, 1) = \sum_{i=2}^{n-1-\ell} 1 + \sum_{i=n-\ell}^{n-1} 0 = n - 2 - \ell.$$

Thus we have that $d_2 = n - \ell - 1$, and thus $d = (b + \ell - 1, n - \ell - 1, d_3, \dots, d_b, 1^{n-b})$, then we have that $H(d) = (n - \ell - 2, d_3 - 1, \dots, d_b - 1, 1^{n-b-\ell}, 0^\ell)$, and $H^2(d) = (d_3 - 2, \dots, d_b - 2, 0^{n-b})$. Since $H(d)$, and thus $H^2(d)$ must be threshold, then we have that $(d_3 - 2, \dots, d_b - 2)$ must be a threshold degree sequence, yielding (iv).

Then from these four types of degree sequences we can build larger degree sequences by appending either vertices with the Havel-Hakimi property or isolated vertices to a realization of the degree sequence. Adding an isolated vertex to a degree sequence with one reordering will not change the number of reorderings as

the associated term will always be at the end of the degree sequence. Adding a vertex with the Havel-Hakimi property to a degree sequence with one reordering will also not increase the number of reorderings because the vertex is of maximum degree and all neighbors have degree strictly greater than those of non-neighbors. Thus, when the associated term is laid off during a step of the Havel-Hakimi algorithm, all terms associated with the neighbors of the vertex with the Havel-Hakimi property will be reduced by one and, hence no reordering is required. Conversely, consider a degree sequence, d , with one reordering whose first step in the Havel-Hakimi algorithm does not require a reordering and is not one of the degree sequences listed in the statement of the theorem. We will show that a realization of d can be built by adding a vertex with the Havel-Hakimi property to a realization of $H(d)$. The first step in the Havel-Hakimi algorithm does not have a reordering, so the terms reduced were all strictly larger than the other terms. So given a realization of $H(d)$, a realization of d can be achieved by adding a vertex with the Havel-Hakimi property to the realization of $H(d)$. Inductively, this shows that every degree sequence requiring one reordering can be built in this way, proving the result.

□

4.4 Maximum number of reorderings

On the other end of the spectrum we will investigate the degree sequences with a maximum number of reorderings. The maximum number of steps in the Havel-Hakimi algorithm increases as n increases and thus the maximum number of potential reorderings increases as n increases; thus we will fix a value of n . Furthermore, as the residue increases, the maximum number of steps in the Havel-Hakimi algorithm decreases, and so the maximum number of potential reorderings will occur when the residue is smallest. However, note that when the residue is

one, the degree sequence is threshold and therefore there are no reorderings. So, in order to look at the maximum number of potential reorderings, it is appropriate to look at those with residue 2.

Theorem 22. *Given the length of a degree sequence $n \geq 4$, and the residue of that degree sequence $R = 2$, the degree sequence with the maximum number of reorderings is $((n-2)^2, (n-3)^{n-2})$, $((n-3)^n)$ or (2222) (in the case where $n = 4$) and the maximum number of reorderings is $n - 3$.*

Proof. A parent of a degree sequence, d , in the Havel-Hakimi algorithm, is a degree sequence in which one step of the Havel-Hakimi algorithm produces d . We will build the necessary degree sequences from (0^2) inductively and derive the possible parents in the Havel-Hakimi algorithm that require a reordering, noting that the terminal step of the algorithm necessarily does not have a reordering as (0^2) is regular. Moreover, the step producing any regular degree sequence in the Havel-Hakimi algorithm also necessarily does not require a reordering (as the resulting terms are all of the same degree). Then since there are $n - 2$ steps in Havel-Hakimi algorithm, to reduce to (0^2) and the last step does not require a reordering, the maximum number of reorderings is at most $n - 3$ and we will show equality to yield the result.

The only parents of (0^2) are (110) and (211) . In order to find possible parents, we must append to the degree sequence a term of degree greater than or equal to that of the maximum degree as otherwise that term would not be laid off in the Havel-Hakimi algorithm. Furthermore, a term appended to the degree sequence having the same maximum degree must be adjacent to only vertices of strictly smaller degree in the corresponding graph, as otherwise a term of larger degree is created which would not be a viable parent in the algorithm. Furthermore, if a term corresponding to a dominating vertex in a realization is

appended to the degree sequence, the first step of the Havel-Hakimi algorithm would necessarily not have a reordering. Thus we will not consider parents produced by appending a term corresponding to a dominating vertex in a realization, as the maximum number of reorderings would then be strictly less than $n - 3$.

Case 1: For (211), we can only append to the degree sequence a term of degree 2. This term must be adjacent to only the two vertices of degree 1 in the corresponding graph (as otherwise a term corresponding to a dominating term in a realization would be created, hence no reordering), yielding (2222). However, note that the step in the Havel-Hakimi that produces (2222) would not require a reordering, hence unless $n = 4$, (211) cannot be encountered in the Havel-Hakimi algorithm of a degree sequence of maximum ordering.

Case 2: For (110), we can add a term of degree 1 or 2. If we add a term of degree 1, it must be adjacent to the isolated vertex in the corresponding graph, yielding (1111) as a parent. However, the step in the Havel-Hakimi algorithm that produces (1111) would not require a reordering, hence unless $n = 4$, (1111) cannot be encountered in the Havel-Hakimi algorithm.

A term of degree 2 can be added in two different ways, namely (2220) (adding to the two 1 degree terms, or (2211) (adding to one degree 1 term and the degree zero term). In the case of (2220), a reordering is not required, so a sequence requiring a maximum number of reorderings would not encounter this sequence. The case of (2211) does require a reordering and can have reorderings in previous steps of the algorithm.

Thus for the length of the degree sequence $n > 4$, if there is a reordering

in every step of the Havel-Hakimi algorithm (except necessarily for the terminal step) we have that (2211) must be encountered in the algorithm. We will proceed by induction on n , noting that the base case of $n \leq 4$ was considered above. We will then assume that in order to have a maximum number of reorderings in the algorithm, $((k-3)^2, (k-4)^{k-3})$ must be encountered or $((k-3)^k)$ is the degree sequence. Then we will show that $((k-2)^2, (k-3)^{k-2})$ must also be encountered or that $((k-2)^{k+1})$ is the degree sequence. Note that the result of a Havel-Hakimi step on a degree sequence with maximum number of reorderings equal to $n-3$, will yield a degree sequence with a maximum number of reorderings as every step of the algorithm (except the terminal one) will require a reordering.

Similar to Case 2 above, we could append to the degree sequence either a term of degree $k-2$ or $k-3$ (noting that adding a term of degree $k-1$ would correspond to a dominating vertex in a realization). If we add a term of degree $k-3$, it must necessarily be adjacent to all of the vertices of degree $k-4$ in the corresponding graph, yielding $((k-3)^k)$ as a parent. However, the Havel-Hakimi step producing that degree sequence would not require a reordering and thus unless $n = k$, $((k-3)^k)$ cannot be encountered in the algorithm.

A term of degree $k-2$ can be appended to the degree sequence in two different ways, namely $((k-2)^3, (k-3)^{k-4}, (k-4))$ (adding to the two $k-3$ degree terms), or $((k-2)^2, (k-3)^{k-2})$ (adding to only one $k-3$ degree term). In the case of $((k-2)^3, (k-3)^{k-4}, (k-4))$, a reordering is not required in the next Havel-Hakimi step and so we will not consider it as a maximum number of $n-3$ reorderings is not possible. The case of $((k-2)^2, (k-3)^{k-2})$ does require a reordering and can have reorderings in previous steps of the algorithm. The result then follows by the principle of mathematical induction. \square

4.5 Independence number and maximum number of reorderings

We will now look at realizations of the graphs exhibited in Theorem 22. One realization of $((n-3)^n)$ is $\overline{C_n}$, where $\alpha(\overline{C_n}) = 2$. Furthermore, since every vertex in a realization must be adjacent to all but 2 vertices, the independence number of a realization can only be three for $n \geq 6$, realized by $\overline{K_3 \cup C_{n-3}}$. For $((n-2)^2, (n-3)^{n-2})$, one realization is two dominating vertices attached to $\overline{C_{n-2}}$ with independence number 2 and another is two dominating vertices attached to $\overline{K_3 \cup C_{n-5}}$ for $n \geq 8$ with independence number 3. Similar to $((n-3)^n)$, the largest independence number possible is 3. Note that in Chapter 2, we also saw $\overline{C_n}$ in the construction of the graphs with Maxine number different than that of the independence number as seen in Theorem 8, all of which have an independence number of 3. The three minimal subgraphs exhibited in this theorem are

- (i) Three dominating vertices attached to $\overline{C_{n-3}}$, which has a degree sequence of $((n-3)^n)$.
- (ii) Two dominating vertices attached to $\overline{C_{n-3}}$ and one vertex of degree $n-2$ adjacent to all but one vertex in $\overline{C_{n-3}}$ which has a degree sequence of $((n-3)^{n-2}, (n-4)^2)$.
- (iii) One dominating vertex attached to $\overline{P_{n-3}}$ and two vertices of degree $n-2$ adjacent to all but one of the endpoints in $\overline{P_{n-3}}$ which has a degree sequence of $((n-3)^{n-2}, (n-4)^2)$.

In (i), this is the same degree sequence found in Theorem 22, and thus the degree sequence exhibits a maximum number of reorderings. In (ii) and (iii), after one application of the Havel-Hakimi algorithm, which does not require a reordering, produces $((n-4)^{n-1})$, which is of the same form as $((n-3)^n)$ with one less term. Thus when applying the Havel-Hakimi algorithm, there are one

less than the maximum number of reorderings possible. These minimal subgraphs all have a discrepancy in the Maxine number and independence number, and are building blocks for a discrepancy in the residue and independence number.

Theorem 22 can be extended to sequences with residue greater than 2 as well.

Corollary 3. *Given the length of a degree sequence n , and the residue of that degree sequence R , a degree sequence with the maximum number of reorderings (given residue R) is $((n - R)^2, (n - R - 1)^{n-R-2}, 0^R)$ or $((n - R - 1)^{n-R}, 0^R)$ and the maximum number of reorderings is $n - R - 1$.*

Proof. Following the proof of Theorem 22, every step of the Havel-Hakimi algorithm on $((n - R)^2, (n - R - 1)^{n-R-2}, 0^R)$ or $((n - R - 1)^{n-R}, 0^R)$ (except the terminal one) requires a reordering as appended zeroes to the degree sequence do not affect the existence of reorderings, giving the result. \square

Although the above result gives the existence of one family of degree sequences with maximum possible reorderings, there are other degree sequences with residue $R > 2$ that exhibit $n - R - 1$ reorderings. However, characterizing these degree sequences is difficult. One example of another degree sequence with maximum possible reorderings is (5^{14}) which has been diagrammed in the introduction of Chapter 3.

4.6 Open questions

The goal of understanding the number of reorderings of a degree sequence is to understand the difference between the residue and the independence number of a degree sequence which is minimum independence number across all realizations of the degree sequence. This leads to the following open questions and potential future research:

- What is the independence number of a degree sequence with exactly one reordering?
- Can we characterize the degree sequences with a maximum number of reorderings given a degree sequence with residue greater than 2, and how does the independence number of these degree sequences relate to the residue?
- Is there a correspondence between the number of reorderings and the difference in the residue and the independence number?

List of References

- [1] P. Erdős and T. Gallai, “Graphs with prescribed degrees of vertices,” *Mat. Lapok*, vol. 11, pp. 264–274, 1960.
- [2] V. Havel, “A remark on the existence of finite graphs,” *Časopis Pěst Mat.*, vol. 80, pp. 477–480, 1955.
- [3] S. Hakimi, “On the realizability of a set of integers as degree sequences of the vertices of a graph,” *SIAM Appl. Math.*, vol. 10, pp. 496–506, 1962.
- [4] S. Fajtlowicz, “On conjectures of graffiti, iii,” *Cong. Numer.*, vol. 66, pp. 23–32, 1988.
- [5] O. Favaron, M. Mahéo, and J. F. Saclé, “On the residue of a graph,” *Graph Theory*, vol. 15, no. 1, pp. 39–64, 1991.
- [6] J. Griggs and D. Kleitman, “Independence and the havel-hakimi residue,” *Discrete Mathematics*, vol. 127, no. 1–3, pp. 209–212, 1994.
- [7] E. Triesch, “Degree sequences of graphs and dominance order,” *J. Graph Theory*, vol. 22, no. 1, pp. 89–93, 1996.
- [8] F. Jelen, “ k -independence and k -residue of a graph,” *J. Graph Theory*, vol. 127, pp. 209–212, 1999.
- [9] N. Mahadev and U. Peled, *Threshold Graphs and Related Topics*. Amsterdam: Elsevier Science B.V., 1995.

BIBLIOGRAPHY

- Arikati, S. R. and Peled, U. N., “Degree sequences and majorization,” 1994.
- Barrus, M. and Molnar, G., “Graphs with the strong havel-hakimi property,” *Graphs and Combinatorics*, vol. 32, pp. 1689–1697, 2016.
- Chartrand, G., Lesniak, L., and Zhang, P., *Graphs & Digraphs, 5th edn.* Boca Raton: CRC Press, 2011.
- Diestel, R., *Graph Theory 5th edn.* Berlin: Springer, 2017.
- Erdős, P. and Gallai, T., “Graphs with prescribed degrees of vertices,” *Mat. Lapok*, vol. 11, pp. 264–274, 1960.
- Fajtlowicz, S., “On conjectures of graffiti, iii,” *Cong. Numer.*, vol. 66, pp. 23–32, 1988.
- Favaron, O., Mahéo, M., and Saclé, J. F., “On the residue of a graph,” *Graph Theory*, vol. 15, no. 1, pp. 39–64, 1991.
- Garey, M. R. and Johnson, D. S., *Computers and intractability.* freeman San Francisco, 1979, vol. 174.
- Golumbic, M., *Algorithmic Graph Theory and Perfect Graphs.* New York: Academic Press, 1980.
- Griggs, J. and Kleitman, D., “Independence and the havel-hakimi residue,” *Discrete Mathematics*, vol. 127, no. 1–3, pp. 209–212, 1994.
- Hakimi, S., “On the realizability of a set of integers as degree sequences of the vertices of a graph,” *SIAM Appl. Math.*, vol. 10, pp. 496–506, 1962.
- Halbeisen, L. and Hungerbühler, N., “The josephus problem,” *Journal de Théorie des Nombres de Bordeaux*, vol. 9, no. 2, pp. 303–318, 1997.
- Jelen, F., “ k -independence and k -residue of a graph,” *J. Graph Theory*, vol. 127, pp. 209–212, 1999.
- Mahadev, N. and Peled, U., *Threshold Graphs and Related Topics.* Amsterdam: Elsevier Science B.V., 1995.
- Nelson, P. and Radcliffe, A. J., “Semi-regular graphs of minimum independence number,” *Discrete Mathematics*, vol. 275, pp. 237–263, 2004.
- Triesch, E., “Degree sequences of graphs and dominance order,” *J. Graph Theory*, vol. 22, no. 1, pp. 89–93, 1996.

V.Havel, "A remark on the existence of finite graphs," *Časopis Pěst Mat.*, vol. 80, pp. 477–480, 1955.