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Global Dynamics of Some Discrete Dynamical Systems in Mathematical Biology

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GLOBAL DYNAMICS OF SOME DISCRETE DYNAMICAL SYSTEMS IN MATHEMATICAL BIOLOGY

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ABSTRACT

This thesis will be presented in manuscript format. The first chapter will introduce preliminary definitions and theorems of difference equations that will be utilized in chapters 2, 3, and 4.

The second chapter will investigate the global behavior of two difference equations with exponential nonlinearities

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots$$

where the parameters b, c are positive real numbers and $p \in (0, 1)$ and

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots$$

where the parameters a, b are positive numbers. The initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. The two equations are well known mathematical models in biology, which behavior was studied by other authors and resulted in partial global dynamics behavior. In this manuscript, we complete the results of other authors and give the global dynamics of both equations. In order to obtain our results we will prove several results on global attractivity and boundedness and unboundedness for general second order difference equations

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

which are of interest on their own.

The third chapter will investigate the global behavior of the cooperative system

$$x_{t+1} = \min\{r_{11}x_t + r_{12}y_t, K_1\}, y_{t+1} = \min\{r_{21}x_t + r_{22}y_t, K_2\}, t = 0, 1, \ldots$$

where the initial conditions x_0, y_0 are arbitrary nonnegative numbers. This system models a population comprised of two subpopulations on different patches of land.

The model considers the minimum between the maximum carrying capacity of each patch (K_1 or K_2 resp.) and the linear combination of the population from patch i from the last time step with those who migrated to patch i for i=1,2. We break the behavior of the system into several cases based on whether the linear combination of the population or maximum carrying capacity is greater. We are able to conclude that either one fixed point will be a global attractor of the interior region of $\mathbb{R}^2_+$ or there will exist a line of fixed points with the stable manifolds as the basins of attractions. We then extend some of these results to the n–dimensional case using similar techniques. We investigate the global behavior of the general cooperative system

$$x^i_{t+1} = \min\{r_{i1}x^1_t + r_{i2}x^2_t + \ldots + r_{ii}x^i_t + \ldots + r_{in}x^n_t, K_i\},$$

for $i = 1, 2, \ldots, n$, and t=0,1,... where the initial conditions of x^i_0 are arbitrary nonnegative numbers for i=1,2, ..., n. We are able to conclude in some cases that one fixed point will be a global attractor of the interior region of $\mathbb{R}^n_+$.

Finally, in the fourth chapter we will prove general results regarding the global stability of monotone systems without minimal period two solutions on a rectangular region $\mathcal{R}$. We will illustrate the general results in two examples of well known systems used in mathematical biology. The first of the systems that will be investigated is a modified Leslie-Gower system of the form

$$x_{n+1} = \alpha x_n + (1-\alpha)\frac{cx_n}{a + cx_n + y_n} \text{ and } y_{n+1} = \beta y_n + (1-\beta)\frac{dy_n}{b + x_n + dy_n}, n = 0, 1, \ldots,$$

where the parameters a, b, c, d are positive numbers, α and β are positive values less than 1, and the initial conditions x_0, y_0 are arbitrary nonnegative numbers. In most cases for different values of a, b, c, and d, there will either be one, two, three, or four equilibrium solutions present with at most one an interior equilibrium point. In the case when $c = d = 1$ and $a = b$, there will exist an infinite number of interior equilibrium points in which case we will find the basin of attraction for each of the equilibrium points.

The second system that will be investigated is a version of a Lotka-Volterra model of the form

$$x_{n+1} = \frac{x_n(A - y_n)}{K_1 + x_n} \quad \text{and} \quad y_{n+1} = \frac{y_n(A - x_n)}{K_2 + y_n}, \quad n = 0, 1, 2, ...,$$

where the parameters of A, K_1, and K_2 are all positive and the initial conditions x_0, y_0 are arbitrary nonnegative numbers, which is a semi implicit discretization of the continuous version. In most cases, there will be between one and three equilibrium points with solutions converging to one of the points. In one case when $A > K_1 = K_2$, however, there will exist an infinite number of equilibrium points. In this case for each equilibrium point, there will be a stable manifold as its basin of attraction.

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I am fortunate to have made lifelong friends at URI. From day one, my colleagues in the Department of Mathematics have been like family, supporting one another academically and emotionally. I feel like I can count on my peers for anything. Everything from help grinding out homework problems to hearing out any personal concerns, I know they have my back. I hope to have many more exciting adventures with them in the future.

Finally, I want to thank my loving family, who has shown unwavering support over the course of my education. They have given me the strength to always persevere and be the best version of myself. I am very fortunate to have such a wonderful family, and I cannot picture going through this process without them. I would like to thank my sister Laura, Mom and Dad, as well as Greg Laudani and my entire extended family for their love and support.

PREFACE

This thesis will be in manuscript format. Chapter 1 will introduce basic definitions and theorems of difference equations that will be used throughout the thesis. Chapters 2, 3, and 4 will be three separate manuscripts. The first manuscript in chapter 2 was accepted to the *Journal of Computational Analysis and Applications* on November 7, 2018 and published in issue 4, volume 28, 2020. The second manuscript in chapter 3 was submitted on February 12, 2019 to the *International Journal of Difference Equations*. The third manuscript in chapter 4 is currently being prepared for submission.

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CHAPTER 1

Introduction

This thesis will primarily focus on difference equations and mathematical biology. In this chapter, we will outline some of the introductory theory of difference equations.

First we will consider the general system of difference equations of the form

$$\begin{cases} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{cases}, \quad n = 0, 1, \ldots \tag{1}$$

Definition 1 *A point* $(\bar{x}, \bar{y})$ *is said to be an equilibrium point or fixed point if* $f(\bar{x}, \bar{y}) = \bar{x}$ *and* $g(\bar{x}, \bar{y}) = \bar{y}$.

The following theory is used for local stability analysis. For an equilibrium point in the System (1), we define the stability in the following way.

Definition 2 *(a) An equilibrium point* $(\bar{x}, \bar{y})$ *is stable if for any* $\epsilon > 0$ *there is* $\delta > 0$ *such that for each initial point* (x_0, y_0) *for which* $\|(x_0, y_0) - (\bar{x} - \bar{y})\| < \delta$, *the iterates* (x_n, y_n) *of* (x_0, y_0) *satisfy* $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \epsilon$ *for all* $n > 0$.

(b) An equilibrium point $(\bar{x}, \bar{y})$ *is unstable if it is not stable.*

(c) An equilibrium point $(\bar{x}, \bar{y})$ *is asymptotically stable if there exists* $r > 0$ *such that* $(x_n, y_n) \to (\bar{x}, \bar{y})$ *as* $n \to \infty$ *for all* (x_0, y_0) *that satisfy* $\|(x_0, y_0) - (\bar{x}, \bar{x})\| < r$.

(d) A periodic point (x_p, y_p) *of period* m *is stable if* (x_p, y_p) *is a stable point of* F^m *where* F *is the map of the equilibrium point.*

We define the Jacobian matrix and linearization of the map F of the System (1) to be the following. This will then help us to define hyperbolic and non-hyperbolic.

Definition 3 *(a) Let $(\bar{x}, \bar{y})$ be a fixed point of the map $F = (f, g)$ where f and G are continuously differentiable functions at $(\bar{x}, \bar{y})$. The Jacobian matrix of F at $(\bar{x}, \bar{y})$ is the matrix*

$$J_F(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \\ \frac{\partial g}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial g}{\partial y}(\bar{x}, \bar{y}) \end{bmatrix}.$$

The linear map $J_F(\bar{x}, \bar{y}) : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J_F(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\bar{x}, \bar{y})x + \frac{\partial f}{\partial y}(\bar{x}, \bar{y})y \\ \frac{\partial g}{\partial x}(\bar{x}, \bar{y})x + \frac{\partial g}{\partial y}(\bar{x}, \bar{y})y \end{bmatrix}$$

is called the linearization of the map F at the fixed point $(\bar{x}, \bar{y})$.

(b) An equilibrium point $(\bar{x}, \bar{y})$ of the map F is said to be hyperbolic if the linearization of F is hyperbolic, that is if the Jacobian matrix $J_F(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y})$ has no eigenvalues on the unit circle. If $J_F(\bar{x}, \bar{y})$ has at least one eigenvalue on the unit circle, then it is a non-hyperbolic equilibrium point.

Based on the eigenvalues of the Jacobian matrix we can make conclusions of the stability of an equilibrium point.

Theorem 1 *Let $F = (f, g)$ be a continuously differentiable function defined on an open set W in $\mathbb{R}^2$, and let $(\bar{x}, \bar{y})$ in W be a fixed point of F.*

(a) If all the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x}, \bar{y})$ is asymptotically stable.

(b) If at least one of the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x}, \bar{y})$ is unstable.

The following theorem can be used to check the local stability of an equilibrium point.

Theorem 2 *(a) An equilibrium point $(\bar{x}, \bar{y})$ is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - tr\ J_F(\bar{x}, \bar{y})\lambda + det\ J_F(\bar{x}, \bar{y}) = 0 \tag{2}$$

lies inside the unit circle, that is, if

$$|tr\ J_F(\bar{x}, \bar{y})| < 1 + det\ J_F(\bar{x}, \bar{y}) < 2.$$

(b) An equilibrium point $(\bar{x}, \bar{y})$ is locally a repeller if and only if every solution of (2) lies outside the unit circle, that is, if

$$|tr\ J_F(\bar{x}, \bar{y})| < |1 + det\ J_F(\bar{x}, \bar{y})| \quad and \quad |\ det\ J_F(\bar{x}, \bar{y})| > 1.$$

(c) An equilibrium point $(\bar{x}, \bar{y})$ is locally a saddle point if and only if the solutions of (2) has one root that lies inside the unit circle and one root that lies outside the unit circle, that is, if

$$|tr\ J_F(\bar{x}, \bar{y})| > |1 + det\ J_F(\bar{x}, \bar{y})| \quad and \quad trJ_F(\bar{x}, \bar{y})^2 - 4det\ J_F(\bar{x}, \bar{y}) > 0.$$

(d) An equilibrium point $(\bar{x}, \bar{y})$ is non-hyperbolic if and only if the solutions of (2) has at least one root that lies on the unit circle, that is, if

$$|tr\ J_F(\bar{x}, \bar{y})| = |1 + det\ J_F(\bar{x}, \bar{y})|\ or\ \ det\ J_F(\bar{x}, \bar{y}) = 1\ and\ |\ tr\ J_F(\bar{x}, \bar{y})| \leq 2.$$

Definition 4 *A set M is said to be invariant under the map $F = (f, g)$ if $F(M) \subset M$.*

We can also consider a general second order difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}).$$

We can formalize a theorem used to perform the local stability analysis.

Theorem 3 *(a) An equilibrium point $\bar{x}$ is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - P\lambda - Q = 0 \tag{3}$$

where

$$P = \frac{\partial f}{\partial x}(\bar{x}, \bar{x}) \quad \text{and} \quad Q = \frac{\partial f}{\partial y}(\bar{x}, \bar{x}),$$

lies inside the unit circle, that is, if

$$|P| < 1 - Q < 2.$$

(b) An equilibrium point $\bar{x}$ is a local repeller if and only if every solution of the characteristic equation (3) *lies outside the unit circle, that is, if*

$$|P| < |1 - Q| \quad \text{and} \quad |Q| > 1.$$

(c) An equilibrium point $\bar{x}$ is a saddle point if and only if the characteristic equation (3) *has one root that lies inside the unit circle and one root that lies outside the unit circle, that is, if*

$$|P| > |1 - Q| \quad \text{and} \quad P^2 + 4Q > 0.$$

(d) As equilibrium point $\bar{x}$ is non-hyperbolic if and only if the characteristic equation (3) *has at least one root that lies on the unit circle, that is, if*

$$|P| = |1 - Q| \quad \text{or} \quad Q = -1 \quad \text{and} \quad |P| \leq 2.$$

Finally we will give the formal definition of a periodic solution.

Definition 5 *(a) A solution $\{x_n\}$ is said to be periodic with period p if*

$$x_{n+p} = x_n \text{ for all } n \geq -1.$$

(b) A solution $\{x_n\}$ is said to be periodic with prime period p, or a p-cycle if it is periodic with period p and p is the least positive integer for which part (a) holds.

All other necessary definitions and theorems for the manuscripts will be self contained within chapters 2, 3, and 4.

List of References

[1] M.R.S. Kulenović and G. Ladas. Dynamics of Second Order Rational Difference Equations. Chapman & Hall/CRC, Boca Raton, FL, 2002.

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CHAPTER 2

Global Dynamics and Bifurcations of Two Second Order Difference Equations in Mathematical Biology

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Abstract. We investigate the global behavior of two difference equations with exponential nonlinearities

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots$$

where the parameters b, c are positive real numbers and $p \in (0, 1)$, and

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots$$

where the parameters a, b are positive numbers. The initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. The two equations are well known mathematical models in biology, which behavior was studied by other authors and resulted in partial global dynamics behavior. In this paper, we complete the results of other authors and give the global dynamics of both equations. In order to obtain our results we will prove several results on global attractivity and boundedness and unboundedness for general second order difference equations

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

which are of interest on their own.

2.1 Introduction and Preliminaries

We investigate the global behavior of the system of difference equations

$$x_{n+1} = be^{-cx_n} + py_n, \quad y_{n+1} = x_n, \quad n = 0, 1, \ldots$$

where the parameters b and c are positive real numbers, $p \in (0, 1)$, and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. This system can be rewritten in the form of the second order difference equation

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots \tag{4}$$

In [5], the authors originally studied this model to describe the synchrony of ovulation cycles of the Glaucous-winged Gulls. The model assumed that there is an infinite breeding season as well as the number of gulls available to breed is infinite. The value of c is a positive number representing the colony density. The parameter b is the number of birds per day ready to begin ovulating. The parameter p is the probability that a bird will begin to ovulate and $1 - e^{-cx_n}$ is the probability of delaying ovulation. In making the model, the authors assumed that the delay only occurs for birds entering the system, not birds switching between different segments of the cycle. Note the authors state that the bifurcation of two-cycle solutions is the same as ovulation synchrony with the value of c increasing. In [5], they used the local bifurcation theory to come to the conclusion that there exists a unique equilibrium such that for sufficiently small values of c, the equilibrium branch is locally asymptotically stable. Additionally, for large enough values of c, there exists a two-cycle branch that will be locally asymptotically stable. In this paper we will improve these results by making them global. Using the results of Camouzis and Ladas, see [2] and [6], we are able to find the global dynamics of (4), which was not completed in [5]. We will show that Equation (4) exhibits global period doubling bifurcation described by Theorem 5.1 in [11], which shows that global dynamics of Equation (4) changes from global asymptotic stability of the unique equilibrium solution to the global asymptotic stability of the minimal period-two solution within its basin of attraction, as the parameter passes through the critical value.

By using a similar method, we investigate the dynamics of

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots \tag{5}$$

where the parameters a, b are positive real numbers and the the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. As it was mentioned in [8], Equation (5) could be considered as a mathematical model in biology where a represent the constant immigration and b represent the population growth rate. In this paper, we find a simpler equivalent condition to $\frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}}e^{\frac{a+\sqrt{a^2+4a}}{2}} < b$ in [8] for the existence of a minimal period-two solution. We split the results into the two cases of $b \geq e^a$ and $b < e^a$. While using a similar method as in [9] to establish the existence of a period-two solution when $b < e^a$, we are able to find the global dynamics of Equation (5). By using new results for general second order difference equations we will prove the existence of unbounded solutions for the case when $b \geq e^a$. Similar as for Equation (4) we will show that Equation (5) exhibits global period doubling bifurcation described by Theorem 5.1 in [11]. In addition, we give the precise description of the basins of attractions of all attractors of both Equations (4) and (5).

The rest of the paper is organized as follows. In the rest of this section we introduce some known results about monotone systems in the plane needed for the proofs of the main results as well as some new results about the existence of unbounded solutions. Section 2 gives the global dynamics of Equation (4) and Section 3 gives the global dynamics of Equation (5).

The next result, which is combination of two theorems from [2] and [6], is important for the global dynamics of general second order difference equation.

Theorem 4 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is either non-increasing in the first variable and non-decreasing in the second variable or non-decreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f\left(x_n, x_{n-1}\right), \quad x_{-1}, x_0 \in I,\ n = 0, 1, \ldots \tag{6}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

We now give some basic notions about monotone maps in the plane.

A partial ordering $\preceq$ on $\mathbb{R}^2$ where $x, y \in \mathbb{R}^2$ is said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction. *Basin of attraction* of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next few results, from [12, 11], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [14, 13].

Theorem 5 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ.*

Suppose that the following statements are true.

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 7 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 6 *For the curve $\mathcal{C}$ of Theorem 5 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period-two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 5 reduces just to $|\lambda| < 1$. This follows from a change of variables [14] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 7 *(A) Assume the hypotheses of Theorem 5, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 5. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_{-} := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \tag{7}$$

$$\mathcal{W}_{+} := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{8}$$

such that the following statements are true.

(i) $\mathcal{W}_{-}$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_{-}$.

(ii) $\mathcal{W}_{+}$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_{+}$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_{-}$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_{+}$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \exists \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and },\ \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map.

If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 8 *In addition to the hypotheses of part (B) of Theorem 7, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 5 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (6) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (6) is a strictly competitive map on $I \times I$, see [11].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (6) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [12].

Remark 2 The characteristic equation of Equation (6) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{9}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 5-8 depends on the existence or nonexistence of minimal period-two solutions.

We now present theorems relating to the existence of unbounded solutions of Equation (6). The original result was obtained in [4]. Here we give an improved version of Theorem 2.1 in [4] taking out the extraneous conditions of requiring a continuity of f and the existence of an equilibrium solution. Additionally, we have extended the results in [4] to obtain a theorem in which the function f is nondecreasing in both arguments.

Theorem 9 *Assume that the function $f : I \times I \to I$ is nonincreasing in the the first variable and nondecreasing in the second variable, where I is an interval of real numbers that may be infinite. Assume there exists numbers $L, U \in I$ such that $L < U$ which satisfy*

$$f(U, L) \leq L \tag{10}$$

and

$$f(L, U) \geq U, \tag{11}$$

where at least one inequality is strict. If $x_{-1} \leq L$ and $x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \leq L \quad \text{and} \quad x_{2n} \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (6) has no minimal period-two solution then,

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

Similarly, if $x_{-1} \geq U$ *and* $x_0 \leq L$*, then the corresponding solution* $\{x_n\}_{n=-1}^{\infty}$ *satisfies*

$$x_{2n-1} \geq U \quad \textit{and} \quad x_{2n} \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f *is continuous and Equation* (6) *has no minimal period-two solution then,*

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \textit{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

Proof. Assume that $x_{-1} \leq L$ and $x_0 \geq U$. Then by using the monotonicity of f (nonincreasing in the first variable and nondecreasing in the second variable) and conditions (10) and (11) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(U, L) \leq L$$

and

$$x_2 = f(x_1, x_0) \geq f(L, U) \geq U.$$

By using induction it follows that $x_{2n-1} \leq L$ and $x_{2n} \geq U$ for all $n = 0, 1, \ldots$ where at least one inequality is strict. In view of Theorem 4 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is a continuous function and there is no minimal period-two solution. We will consider a few cases based on the properties of the interval I. First suppose there exist $a \in \mathbb{R}$ such that $I = [a, \infty)$ and $a < L$. Then $\{x_{2n-1}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in $[a, L]$. If $\{x_{2n}\}_{n=0}^{\infty}$ converges, this would create a contradiction as there would exist a minimal period-two solution. Therefore,

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Next suppose that for some $b \in \mathbb{R}$, both $I = (-\infty, b]$ and $U < b$. Here $\{x_{2n}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in the interval of $[U, b]$. So $\{x_{2n-1}\}_{n=0}^{\infty}$ cannot converge as there is no minimal period-two solution resulting in

$$\lim_{n\to\infty} x_{2n-1} = -\infty.$$

If $I = (-\infty, \infty)$, then similar to the two cases above, at most one subsequence can converge as there is no minimal period-two solution. So either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the option of both occurring. Finally, we will prove that I cannot be $I = [a, b]$ where $a, b \in \mathbb{R}$. Suppose that $I = [a, b]$ such that $a < L < U < b$ and $a, b \in \mathbb{R}$. Since $x_n \in [a, b]$ for all n, both subsequences would be convergent. As $\lim_{n\to\infty} x_{2n-1} = p < \lim_{n\to\infty} x_{2n} = q$ for some $p, q \in \mathbb{R}$, there exists a period-two solution, which is a contradiction. The case when $x_{-1} \geq U$ and $x_0 \leq L$ will follow similarly to the proof used here.

Many examples of the use of Theorem 9 are provided in [4].

Theorem 10 *Assume that $f : I \times I \to I$ is a function which is nondecreasing in both variables, where I is an interval of real numbers that may be infinite. Assume there exists numbers $L, U \in I$ such that $L < U$ where*

$$f(L, L) \leq L \tag{12}$$

and

$$f(U, U) \geq U \tag{13}$$

are satisfied, where at least one inequality is strict. If $x_{-1}, x_0 \leq L$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (6) satisfies

$$x_n \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (6) has no minimal period-two solution, then either x_n converges to an equilibrium point or

$$\lim_{n\to\infty} x_{2n-1} = -\infty \quad \textit{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

If $x_{-1}, x_0 \geq U$, *then the corresponding solution* $\{x_n\}_{n=-1}^{\infty}$ *satisfies*

$$x_n \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f *is continuous and Equation* (6) *has no period-two solution, then either* x_n *converges to an equilibrium point or*

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \textit{and/or} \quad \lim_{n\to\infty} x_{2n} = \infty.$$

Proof. Assume that $x_{-1}, x_0 \leq L$. Then by using the monotonicity of f (both variables are nondecreasing) and conditions (12) and (13) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(L, L) \leq L \quad \text{and} \quad x_2 = f(x_1, x_0) \leq f(L, L) \leq L.$$

By using induction it follows that $x_{2n-1}, x_{2n} \leq L$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 4 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. We can assume that f is continuous and that there is no minimal period-two solution. We can choose the value of L such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = [a, \infty)$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are less than L, then $x_n \in [a, L]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = [a, \infty)$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = (-\infty, a]$ or $I = (-\infty, \infty)$. Now $x_n \in (-\infty, L]$ for all n. At least one subsequence must be decreasing as the subsequences do not converge to an equilibrium point.

Furthermore since there is no period-two solution, the subsequences cannot be bounded below resulting in either

$$\lim_{n\to\infty} x_{2n} = -\infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the possibility of both options occurring.

Now assume that $x_{-1}, x_0 \geq U$. Then by using the monotonicity of f and conditions (12) and (13) we obtain

$$x_1 = f(x_0, x_{-1}) \geq f(U, U) \geq U$$

and

$$x_2 = f(x_1, x_0) \geq f(U, U) \geq U.$$

By using induction it follows that $x_{2n-1}, x_{2n} \geq U$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 4 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is continuous and that there is no minimal period-two solution. We can choose the value of U such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = (-\infty, b]$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are greater than U, then $x_n \in [U, b]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = (-\infty, b]$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = [a, \infty]$ or $I = (-\infty, \infty)$. Now $x_n \in [U, \infty)$ for all n. At least one subsequence must be increasing as the subsequences do not converge to an equilibrium point.

Furthermore since there is no period-two solution, the subsequences cannot be bounded above resulting in either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = \infty.$$

with the option of both occurring.

Now we give few examples which illustrate possible scenarios of Theorem 10.

Example 1 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2, \quad n = 1, 2, \ldots$$

where $x_{-1}, x_0 \in \mathbb{R}^+$, and $x_n \geq 0$ for $n = 1, 2, \ldots$. Here $f(u, v) = u^2 + v^2$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_+ = 1/2$. The linearized difference equation is $z_{n+1} = 2\overline{x}z_n + 2\overline{x}z_{n-1}$ and the characteristic equation is $\lambda^2 = 2\overline{x}\lambda + 2\overline{x}$. The zero equilibrium $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_+$, $\lambda^2 = \lambda + 1$, so that $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$. As $\frac{1+\sqrt{5}}{2} > 1$ and $\frac{1-\sqrt{5}}{2} \in (-1, 0)$, then $\overline{x}_+$ is a saddle point. There is no minimal period-two solution as

$$\phi = \psi^2 + \phi^2 \quad \text{and} \quad \psi = \phi^2 + \psi^2$$

implies $\phi = \psi$. Now we want to find a $L < U$ that satisfies the conditions (12) and (13). Condition (12) $f(L, L) \leq L$ implies $2L^2 \leq L$, which simplifies to $L \leq 1/2$. As well, $f(U, U) \geq U$ if $2U^2 \geq U$, which simplifies to $U \geq 1/2$. We can choose at least one of these inequalities to be strict. From Theorem 10, we can conclude that every solution with $x_1, x_0 \leq L$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/2 < U$ are arbitrary this conclusion holds for every case where $x_{-1}, x_0 \leq L$ or $x_{-1}, x_0 \geq U$. These results do not give conclusions when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this case one may use theory of monotone maps as in [3].

Example 2 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2 + a, \quad n = 1, 2, \ldots$$

where $a > 1/8$, $x_n \geq 0$, and $x_{-1}, x_0 \in \mathbb{R}$. Here $f(u,v) = u^2 + v^2 + a$ is increasing in both variables. There is no equilibrium points as the discriminant of the equilibrium equation $1 - 8a < 0$ and no minimal period-two solution exists as

$$\phi = \psi^2 + \phi^2 + a \quad \text{and} \quad \psi = \phi^2 + \psi^2 + a$$

implies $\phi = \psi$. We can find U that satisfies the conditions (12) and (13) of Theorem 10. As $f(U,U) \geq U$ simplifies to $2U^2 + a \geq U$, which always holds, every solution will be eventually increasing and tends to ∞.

Example 3 Consider the difference equation

$$x_{n+1} = x_n^5 + x_{n-1}^5, \quad n = 1, 2, \ldots$$

where $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u,v) = u^5 + v^5$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_{\pm} = \pm 1/\sqrt[4]{2}$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = 5\overline{x}^4\lambda + 5\overline{x}^4$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_{\pm}$, $\lambda^2 = 5/2\lambda + 5/2$, so that $\lambda_{1,2} = \frac{5 \pm \sqrt{65}}{4}$. As $\frac{5+\sqrt{65}}{4} > 1$ and $\frac{5-\sqrt{65}}{4} \in (-1, 0)$, then the equilibrium points $\overline{x}_{\pm}$ are saddle points. There is no minimal period-two solution as

$$\phi = \psi^5 + \phi^5 \quad \text{and} \quad \psi = \phi^5 + \psi^5$$

implies $\phi = \psi$.

Now we want to find $L < U$ that satisfies the conditions of Theorem 10. Clearly $f(L,L) \leq L$ if $2L^5 \leq L$, which simplifies to $L \leq 1/\sqrt[4]{2}$ if $L > 0$ and to $L \leq -1/\sqrt[4]{2}$ if $L < 0$. As well, $f(U,U) \geq U$ if $2U^5 \geq U$, which simplifies to

$U \geq 1/\sqrt[4]{2}$. We can choose at least one of these inequalities to be strict. From Theorem 10, we can conclude that every solution with $x_1, x_0 \leq L, L > 0$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/\sqrt[4]{2} < U$ are arbitrary we conclude that

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{when} \quad \bar{x}_- < x_{-1}, x_0 < \bar{x}_+, \\ \infty & \text{when} \quad x_{-1}, x_0 > \bar{x}_+, \\ -\infty & \text{when} \quad x_{-1}, x_0 < \bar{x}_-. \end{cases}$$

Theorem 10 does not apply when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this cases one can use the results from [3].

Example 4 Consider the difference equation

$$x_{n+1} = \frac{ax_n^2}{1+x_n^2} + \frac{bx_{n-1}^2}{1+x_{n-1}^2}, \quad n = 1, 2, \ldots$$

where $a, b > 0$ and $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u, v) = \frac{au^2}{1+u^2} + \frac{bv^2}{1+v^2}$ is increasing in both variables. One equilibrium point is $\overline{x}_0 = 0$. The non-zero equilibrium point satisfies the quadratic equation $1 + \overline{x}^2 - (a + b)\overline{x} = 0$ which has real solutions if $(a + b)^2 - 4 \geq 0$. If $a + b < 2$, then there only exist $\overline{x}_0$, if $a + b = 2$, then there exists $\overline{x}_0$ and $\overline{x}$, and if $a + b > 2$, then there exist three equilibrium points $\overline{x}_0 < \overline{x}_- < \overline{x}_+$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = \frac{2a\overline{x}}{(1+\overline{x}^2)^2}\lambda + \frac{2b\overline{x}}{(1+\overline{x}^2)^2}$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and thus, $\overline{x}_0$ is locally asymptotically stable. The conditions for local stability of the equilibrium points $\bar{x}_\pm$ are quite involved and can be found in [1]. In particular $\overline{x}_-$ will either be a saddle point, repeller, or non-hyperbolic depending on whether $2a(a + b) + (a - b)\sqrt{(a + b)^2 - 4}$ is greater than, less than, or equal to 0, and the equilibrium point $\overline{x}_+$ is either locally asymptotically stable or non-hyperbolic when it exists.

Now we want to find a $L < U$ that satisfies the conditions (12) and (13) of Theorem 10. First $f(L, L) \leq L$ if $\frac{(a+b)L^2}{1+L^2} \leq L$, which simplifies to $0 \leq 1 + L^2 - (a+$

$b)L$. This will occur when $L < L_-$ or $L > L_+$ where we can set $L_- = \overline{x}_-$ and $L_+ = \overline{x}_+$. As well, $f(U,U) \geq U$ if $\frac{(a+b)U^2}{1+U^2} \geq U$, which simplifies to $0 \geq 1 + U^2 - (a+b)U$. This occurs when $U_- < U < U_+$ where we can set $U_- = \overline{x}_-$ and $U_+ = \overline{x}_+$. For both L and U to exist, we need $L < L_-$ to satisfy $L < U$. From Theorem 10, we can conclude that every solution with $x_1, x_0 \leq L$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ converges to $\overline{x}_+$. Note that in the region where L and U exist, no minimal period-two solutions exists. All the period-two solutions are located in the region which is the union of the second and the fourth quadrant with respect to $\overline{x}_-$.

2.2 Global Dynamics of Equation (4)

In this section we present the global dynamics of Equation (4).

2.2.1 Local stability results

We begin by observing that the function $f(u,v) = be^{-cu} + pv$ is decreasing in the first variable and increasing in the second variable and so by Theorem 4, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (4) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (4) has a unique positive equilibrium point $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$ where $0 < \overline{x} < \frac{b}{1-p}$. Note that $\frac{\partial f}{\partial u}(\overline{x},\overline{x}) = -cbe^{-c\overline{x}} = -c(1-p)\overline{x}$ and $\frac{\partial f}{\partial v}(\overline{x},\overline{x}) = p$. The characteristic equation of Equation (4) is

$$\lambda^2 + (1-p)c\overline{x}\lambda - p = 0.$$

Applying local stability test [10] we obtain

Lemma 1 *Equation* (4) *has a unique positive equilibrium solution* $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$.

i) If $\overline{x} < \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is locally asymptotically stable.*

ii) If $\overline{x} > \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is a saddle point.*

iii) If $\overline{x} = \frac{1}{c}$, then the equilibrium point $\overline{x}$ is non-hyperbolic of the stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p$).

Proof.

i) Equilibrium point $\overline{x}$ is locally asymptotically stable if

$$|(1-p)c\overline{x}| < 1 - p < 2.$$

As $p \in (0,1)$ then $1 - p < 2$ holds. As $(1-p)c\overline{x} > 0$, then $\overline{x}$ is stable if

$$(1-p)c\overline{x} < 1 - p \Leftrightarrow c\overline{x} < 1 \Leftrightarrow \overline{x} < \frac{1}{c}.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{1}{c}$

ii) If $|(1-p)c\overline{x}| > |1-p|$, then the equilibrium point $\overline{x}$ is a saddle point. As $(1-p)c\overline{x}$ is positive, we obtain

$$(1-p)c\overline{x} > 1 - p \Leftrightarrow c\overline{x} > 1 \Leftrightarrow \overline{x} > \frac{1}{c}.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{1}{c}$.

iii) The equilibrium point $\overline{x}$ is non-hyperbolic if

$$|(1-p)c\overline{x}| = |1-p|.$$

We see that $c\overline{x} = 1 \Leftrightarrow \overline{x} = \frac{1}{c}$. The characteristic equation at the equilibrium becomes

$$\lambda^2 + (1-p)\lambda - p = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p$.

2.2.2 Periodic solutions

In this section we present results about existence and uniqueness of the minimal period-two solution of Equation (4).

Theorem 11 *If $\overline{x} > \frac{1}{c}$, then Equation (4) has a unique minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots (\phi \neq \psi, \phi > 0 \text{ and } \psi > 0).$$

Proof. Let $\{\phi, \psi\}$ be a minimal period-two solution of Equation (4), where ϕ and ψ are distinct positive real numbers. Then we have

$$\phi = be^{-c\psi} + p\phi, \quad \psi = be^{-c\phi} + p\psi, \tag{14}$$

where $\phi \neq \psi$. This implies

$$\psi = \frac{be^{-c\phi}}{1-p}, \quad \phi = be^{\frac{-cbe^{-c\phi}}{1-p}} + p\phi.$$

Let $F(\phi) = be^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)\phi$. The equilibrium point $\overline{x} = \frac{b}{1-p}e^{-c\overline{x}}$ will be a zero of F as

$$F(\overline{x}) = be^{\frac{-cbe^{-c\overline{x}}}{1-p}} + (p-1)\overline{x} = be^{-c\overline{x}} + (p-1)\overline{x} = 0.$$

Note that $F(0) = be^{\frac{-cb}{1-p}} > 0$ since $b > 0$. Additionally, as ϕ approaches ∞, then $F(\phi)$ approaches $-\infty$. Notice graphically, the the function F begins above the x-axis and ends approaching $-\infty$. As the function F crosses the x-axis at least once at $\overline{x}$, then F must cross the x-axis at least three times when $F'(\overline{x}) > 0$. This will result in the existence of a minimal period-two solution. We want to prove the values of parameters that $F'(\overline{x}) > 0$ holds true. Observe that the derivative of F is

$$F'(\phi) = \frac{b^2c^2}{1-p}e^{-c\phi}e^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)$$

so that when $\overline{x}$ is substituted $F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1)$. Then $F'(\overline{x}) > 0$ when $\overline{x} > \frac{1}{c}$ as

$$F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1) > 0 \Leftrightarrow c^2\overline{x} > \frac{1-p}{b}e^{c\overline{x}} \Leftrightarrow c^2\overline{x} > \frac{1}{\overline{x}} \Leftrightarrow \overline{x} > \frac{1}{c}.$$

Thus when $\overline{x} > \frac{1}{c}$, there will be a minimal period-two solution.

Next we want to prove that the period-two solution is unique. Rewritting (14) we obtain

$$\phi e^{c\psi} = \frac{b}{1-p} = \psi e^{c\phi} \Leftrightarrow \phi e^{-c\phi} = \psi e^{-c\psi}.$$

Let $g(x) = xe^{-cx}$. As $g'(x) = e^{-cx}(1-cx)$, then the global maximum of g is attatined at $x = \frac{1}{c}$. For each y value there will be two corresponding x values when $g(x) < g(\frac{1}{c}) = \frac{1}{ce}$. This will happen when

$$xe^{-cx} < \frac{1}{ce} \Leftrightarrow e^{cx} - ecx > 0.$$

Let $G(x) = e^{cx} - ecx$ and notice that $G(0) = 1$. The derivative of G will be $G'(x) = c(e^{cx} - e)$. Notice $G'(x) \leq 0$ when $e^{cx} \leq e$ such that $x \leq \frac{1}{c}$, and $G'(x) > 0$ when $x > \frac{1}{c}$. Thus, $G(x) > 0$ on $[0, \frac{1}{c}) \cup (\frac{1}{c}, \infty)$ where $G(\frac{1}{c}) = 0$ is a global minimum. Thus when the period-two solution exists, it is unique.

2.2.3 Global stability results

In view of Theorem 4 every bounded solution of Equation (4) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 2 *The solutions of Equation* (4) *are bounded.*

Proof. Equation (4) implies

$$x_{n+1} = be^{-cx_n} + px_{n-1} \leq b + px_{n-1}, \quad n = 0, 1, \ldots.$$

Consider the difference equation of

$$u_{n+1} = b + pu_{n-1}, \quad n = 0, 1, \ldots. \tag{15}$$

The solution of Equation (15) is $u_n = \frac{b}{1-p} + C_1(\sqrt{p})^n + C_2(-\sqrt{p})^n$. As $n \to \infty$, then $u_n \to \frac{b}{1-p}$. In view of the difference inequality result, see [7] $x_n \leq u_n \leq \frac{b}{1-p} + \epsilon = \mathcal{U}$ for $n = 0, 1, \ldots$ and some $\epsilon > 0$ when $x_0 \leq u_0$.

Theorem 12 *(i) If $\bar{x} > \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is a saddle point and the minimal period-two solution $\{\phi, \psi\}, \phi < \psi$ is globally asymptotically stable within the basin of attraction $\mathcal{B}(\phi, \psi) = [0, \infty)^2 \setminus \mathcal{W}^s(\bar{x}, \bar{x})$, where $\mathcal{W}^s(\bar{x}, \bar{x})$ is the global stable manifold of $(\bar{x}, \bar{x})$.*

(ii) If $\bar{x} \leq \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is globally asymptotically stable.

Proof. Using Theorem 4 every bounded solution of Equation (4) converges to an equilibrium solution or period-two solution. By Lemma 2, every solution of Equation (4) is bounded so that all solutions converge to either an equilibrium solution or to the unique period-two solution $\{\phi, \psi\}, \phi < \psi$. When $\overline{x} > \frac{1}{c}$, then $\overline{x}$ is a saddle point, by Lemma 1 part (ii), and has both the global stable $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable $\mathcal{W}^u(\overline{x}, \overline{x})$ manifolds, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ), see Theorems 5-7. In this case in view of Theorem 4 global attractivity of period-two solution implies its local stability since the convergence is monotonic.

When $\overline{x} \leq \frac{1}{c}$, the equilibrium solution is locally and so globally asymptotically stable by Lemma 1 part (i) and part (iii) .

Remark 3 For instance, case i) of Theorem 12 holds when $b = 1, p = .5, c = 2$, case ii) holds when $b = 1, p = .5, c = 1$ and when $b = 1, p = (e-1)/e, c = 1$.

2.3 Global Dynamics of Equation (5)

In this section we present global dynamics of Equation (5).

2.3.1 Local stability results

First, notice that the function $f(u,v) = a + bve^{-u}$ is decreasing in the first variable and increasing in the second variable. By Theorem 4, for all solutions $\{x_n\}_{n=-1}^{\infty}$ of Equation (5) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (5) has a unique positive equilibrium point $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$ where $0 < a < \overline{x}$. Note that $\frac{\partial f}{\partial u}(\overline{x},\overline{x}) = -b\overline{x}e^{-\overline{x}}$ and $\frac{\partial f}{\partial v}(\overline{x},\overline{x}) = be^{-\overline{x}}$. The characteristic equation of Equation (5) is

$$\lambda^2 + b\overline{x}e^{-\overline{x}}\lambda - be^{-\overline{x}} = 0.$$

Lemma 3 *Equation* (5) *has a unique positive equilibrium solution of* $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$.

i) If $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, *then the equilibrium solution* $\overline{x}$ *is locally asymptotically stable.*

ii) If $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, *then the equilibrium solution* $\overline{x}$ *is a saddle point.*

iii) If $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, *then the equilibrium solution* $\overline{x}$ *is non-hyperbolic of stable type (with eigenvalues* $\lambda_1 = -1$ *and* $\lambda_2 = be^{-\overline{x}}$*).*

Proof.

i) The equilibrium point $\overline{x}$ is locally asymptotically stable if

$$\left|b\overline{x}e^{-\overline{x}}\right| < 1 - be^{-\overline{x}} < 2.$$

As $be^{-\overline{x}} > 0$, then $1 - be^{-\overline{x}} < 2$ holds true. So rearranging the other inequality we obtain

$$b\overline{x}e^{-\overline{x}} < 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) < 1 \Leftrightarrow \overline{x}+1 < \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} < \frac{e^{\overline{x}}}{b} - 1.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{e^{\overline{x}}}{b} - 1$. As $\overline{x} = a + b\overline{x}e^{-\overline{x}}$ we have

$$e^{\overline{x}} = \frac{b\overline{x}}{\overline{x}-a}. \tag{16}$$

Then we can equivalently write the condition to be locally asymptotically stable as

$$\overline{x} < \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\overline{x}}{\overline{x}-a} - 1$$
$$\Leftrightarrow \overline{x}^2 - \overline{x}a - a < 0 \Leftrightarrow \overline{x} < \frac{a+\sqrt{a^2+4a}}{2}.$$

ii) If

$$\left|b\overline{x}e^{-\overline{x}}\right| > \left|1 - be^{-\overline{x}}\right|,$$

then the equilibrium solution $\overline{x}$ is a saddle point. Note that $be^{-\overline{x}} < 1$ since

$$be^{-\overline{x}} < 1 \Leftrightarrow \frac{\overline{x}-a}{\overline{x}} < 1 \Leftrightarrow \frac{-a}{\overline{x}} < 0$$

always holds as $a > 0$. The condition for $\overline{x}$ to be a saddle point yields

$$b\overline{x}e^{-\overline{x}} > 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) > 1 \Leftrightarrow \overline{x}+1 > \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} > \frac{e^{\overline{x}}}{b} - 1.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{e^{\overline{x}}}{b} - 1$. By using (16), the inequality can then equivalently be written as

$$\overline{x} > \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a > 0 \Leftrightarrow \overline{x} > \frac{a+\sqrt{a^2+4a}}{2}.$$

iii) The equilibrium point $\overline{x}$ is non-hyperbolic point if

$$\left|b\overline{x}e^{-\overline{x}}\right| = \left|1 - be^{-\overline{x}}\right|.$$

We see that

$$b\overline{x}e^{-\overline{x}} = 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) = 1 \Leftrightarrow \overline{x}+1 = \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} = \frac{e^{\overline{x}}}{b} - 1.$$

In view of (16) this can be rewritten as

$$\overline{x} = \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a = 0 \Leftrightarrow \overline{x} = \frac{a+\sqrt{a^2+4a}}{2}.$$

The characteristic equation at the equilibrium point will become

$$\lambda^2 + (1 - be^{-\overline{x}})\lambda - be^{-\overline{x}} = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = be^{-\overline{x}} \in (0, 1)$.

2.3.2 Periodic solutions

In this section we present results about existence and uniqueness of minimal period-two solutions of Equation (5).

Theorem 13 *Assume that* $b < e^a$. *If* $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, *then Equation* (5) *has minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots \quad (\phi \neq \psi \text{ and } \phi > 0, \psi > 0)\,.$$

Proof. We want to find for which values of $\overline{x}$ there exists a minimal period-two solution (ϕ, ψ) where ϕ and ψ are distinct positive real numbers. A period-two solution satisfies

$$\phi = a + b\phi e^{-\psi}, \quad \psi = a + b\psi e^{-\phi}, \tag{17}$$

where ϕ and ψ are distinct real numbers. Rewritting ψ and then substituting into ϕ we obtain

$$\psi = \frac{a}{1 - be^{-\phi}}, \quad \phi = a + b\phi e^{-\frac{a}{1-be^{-\phi}}}. \tag{18}$$

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

Now

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

is positive as a and b are positive constants. As ϕ approaches ∞, then F approaches $-\infty$ assuming that $b < e^a$. When $F'(\overline{x}) > 0$ then F will cross the $x-$axis at least three times resulting in a minimal period-two solution. Thus, we want to prove when $F'(\overline{x}) > 0$ holds. Taking the derivative of F we have

$$F'(\phi) = (be^{-\frac{a}{1-be^{-\phi}}} - 1) + \frac{\phi ab^2 e^{-\phi}}{(1-be^{-\phi})^2} e^{-\frac{a}{1-be^{-\phi}}}$$

so that $F'(\overline{x}) = \frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a}$. Then $F'(\overline{x}) > 0$ hold true when

$$\begin{aligned}
\frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a} &> 0 \Leftrightarrow \\
\overline{x}^4 b^2 e^{-2\overline{x}} &> a^2 \Leftrightarrow \\
\overline{x}^2 b e^{-\overline{x}} &> a \Leftrightarrow \\
\overline{x}^2 b &> \frac{a\overline{x}b}{\overline{x} - a} \Leftrightarrow \\
\overline{x}(\overline{x} - a) &> a \Leftrightarrow \\
\overline{x}^2 - \overline{x}a - a &> 0.
\end{aligned}$$

Thus, when $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, there will be a minimal period-two solution.

When $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then

$$\frac{a}{1 - be^{-\overline{x}}} > \frac{a + \sqrt{a^2 + 4a}}{2} \Leftrightarrow \frac{2a}{a + \sqrt{a^2 + 4a}} > 1 - be^{-\overline{x}}$$

$$\Leftrightarrow \frac{-a + \sqrt{a^2 + 4a}}{a + \sqrt{a^2 + 4a}} e^{\overline{x}} < b \Leftrightarrow \frac{-a + \sqrt{a^2 + 4a}}{a + \sqrt{a^2 + 4a}} e^{\frac{a+\sqrt{a^2+4a}}{2}} < b.$$

Next we want to prove that the minimal period-two solution is unique. By rewriting (17) we find that

$$\phi(1 - be^{-\psi}) = a = \psi(1 - be^{-\phi}) \Leftrightarrow \frac{\phi}{1 - be^{-\phi}} = \frac{\psi}{1 - be^{-\psi}}.$$

Let $g(x) = \frac{x}{1-be^{-x}}$. Using $g'(x) = \frac{1-be^{-x}(x+1)}{(1-be^{-x})^2}$ to find the critical points we get that $1 - be^{-x}(x + 1) = 0 \Leftrightarrow e^x = b(x + 1)$. There exists a unique value of m where $\frac{1}{m+1} = be^{-m}$ for which this holds. Using the first-derivative theorem we can check that m is a local minima. Note it suffices to check the numerator of $g'(m - 1)$ as the denominator is always positive. Using the fact that $\frac{1}{m+1} = be^{-m}$

$$1 - be^{-(m-1)}m < 0 \Leftrightarrow \frac{1}{m} < be^{-(m-1)} \Leftrightarrow \frac{1}{m} < \frac{e}{m+1} \Leftrightarrow \frac{m+1}{m} < e.$$

This proves that $g'(m-1) < 0$. Next using the same method taking the numerator of $g'(m+1)$ we see that

$$\begin{aligned} 1 - be^{-(m+1)}(m+2) > 0 &\Leftrightarrow \\ \frac{1}{m+2} > be^{-(m+1)} &\Leftrightarrow \\ \frac{1}{m+2} > \frac{e^{-1}}{m+1} &\Leftrightarrow \\ \frac{m+1}{m+2} > e^{-1}&. \end{aligned}$$

This proves that $g'(m+1) > 0$. As the derivative changes from negative to positive around the critical point, it will be a local minima. Note that $g(a) > 0$ and as x approaches ∞, $g(x)$ approaches ∞. Since m is the only critical point, each y value will have two x values with the exception at m. This results in the fact that there can only be one period-two solution.

Proposition 1 *If $b \geq e^a$, there are no minimal period-two solutions.*

Proof. Assume that $\{\phi, \psi\}$ is a period-two solution. Then $\{\phi, \psi\}$ satisfies (17) and so it satisfies (18) as well.

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

We see that

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

which is a positive value as a and b are positive constants. As ϕ approaches ∞, then F approaches ∞ as $b \geq e^a$. As the function begins above the x-axis at a and approaches ∞, F will cross the x-axis an even number of times. Since $F(\overline{x}) = 0$ is one of the points that lie on the x-axis and the only equilibrium point, there cannot be a minimal period-two solution.

The result of proposition 1 has been verified through Mathematica simulations as well.

2.3.3 Global stability results

By Theorem 4 every bounded solution of Equation (5) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 4 *The solutions of Equation* (5) *are bounded if* $b < e^a$ *.*

Proof. By Equation (5),

$$x_{n+1} = a + bx_{n-1}e^{-x_n} \leq a + bx_{n-1}, \quad n = 0, 1, ...$$

Consider the difference equation of

$$u_{n+1} = a + bu_{n-1}, \quad n = 0, 1, ... \tag{19}$$

Suppose that $b < e^a$. The solution of Equation (19) is $u_n = \frac{a}{1-b} + C_1(\sqrt{b})^n + C_2(-\sqrt{b})^n$. As $n \to \infty$, then $u_n \to \frac{a}{1-b}$. In view of difference inequality result, see [7] $x_n \leq u_n \leq \frac{a}{1-b} + \epsilon = \mathcal{U}$ for $n = 0, 1, ...$ when $x_0 \leq u_0$, where $\epsilon > 0$.

Theorem 14 *Consider Equation* (5).

(i) If $b < e^a$ *and* $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, *then there exists a period-two solution that is locally asymptotically stable and the equilibrium point,* $\overline{x}$, *that is is a saddle point. The unique period-two solution attracts all solutions which start off the global stable manifold of* $\mathcal{W}^s(E(\overline{x}, \overline{x}))$.

(ii) If $b < e^a$ *and* $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, *then the equilibrium solution,* $\overline{x}$, *is globally asymptotically stable.*

(iii) If $b < e^a$ *and* $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, *then the equilibrium solution,* $\overline{x}$, *is non-hyperbolic of the stable type and is global attractor.*

Proof.

(i) Using Theorem 4 every bounded solution of Equation (5) converges to an equilibrium solution or period-two solution. By Lemma 4, when $b < e^a$ every solution of Equation (5) is bounded such that all solutions will converge to either an equilibrium solution or period-two solution. If $b < e^a$ and $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ will be a saddle point by Lemma 3 part (ii), and there will be a minimal period-two solution by Theorem 13. In view of Theorems 5-7 there exist the global stable manifold $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable manifold $\mathcal{W}^u(\overline{x}, \overline{x})$, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ).

(ii) When $b < e^a$ and $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ is locally asymptotically stable by Lemma 3 part (i). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

(iii) Moreover, when $b < e^a$ and $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, $\overline{x}$ will be non-hyperbolic of the stable type by Lemma 3 part (iii). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

Theorem 15 *If $b \geq e^a$, then Equation* (5) *has unbounded solutions.*

Proof. We will use Theorem 9 to prove this theorem. The conditions of (10) and (11) of Theorem 9 become

$$f(U, L) = a + bLe^{-U} \leq L \quad \text{and} \quad f(L, U) = a + bUe^{-L} \geq U.$$

These inequalities can be reduced to

$$a \leq L(1 - be^{-U}) \quad \text{and} \quad a \geq U(1 - be^{-L}).$$

Any value of L and U such that $\frac{U}{1-be^{-U}} \leq \frac{L}{1-be^{-L}}$ will satisfy the theorem. Let $G(x) = \frac{x}{1-be^{-x}}$. There is a vertical asymptote at $1 - be^{-x} = 0$ that is at $x = \ln(b)$. In the interval $(\ln(b), \infty)$ we can find L and U that satisfies these inequalities. As $b \geq e^a$, then $\ln(b) \geq a$ so that $(\ln(b), \infty)$ is part of the domain of difference equation (5). An example of where this holds is when $L = a + \epsilon$. Using the fact that $b \geq e^a$ and ϵ is small, then $b \geq e^{a+\epsilon}$. By condition (11) the inequality holds true as

$$a + bUe^{-(a+\epsilon)} \geq U \Leftrightarrow e^{a+\epsilon} \leq \frac{bU}{U - a}.$$

We will use condition (10) and $b \geq e^a$ to find the criteria for U based on our L. Thus,

$$a+b(a+\epsilon)e^{-U} \leq (a+\epsilon) \Leftrightarrow e^U \geq \frac{b(a+\epsilon)}{\epsilon} \Leftrightarrow e^U \geq \frac{e^a(a+\epsilon)}{\epsilon} \Leftrightarrow U \geq a+\ln\left(\frac{a+\epsilon}{\epsilon}\right).$$

Let U be such that $U > a + \ln\left(\frac{a+\epsilon}{\epsilon}\right)$. It holds that $U \geq L$. Overall, as f is continuous and there is no minimal period-two solution by Proposition 1, using Theorem (9) some solutions will approach ∞.

Remark 4 For instance, case $i)$ of Theorem 14 holds when $a = 1, b = 2$, case $ii)$ holds when $a = 4, b = 2$ and case $iii)$ holds when $a = 2, b = \frac{\sqrt{3}-1}{\sqrt{3}+1}e^{1+\sqrt{3}}$, and the conditions of Theorem 15 holds when $a = .5, b = 2$.

In conclusion, Equations (4) and (5) exhibit the global period doubling bifurcation described by Theorem 5.1 in [11]. Checking the conditions of Theorem 5.1 in [11] is exactly the content of Lemmas 1-3 and Theorems 10-12.

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CHAPTER 3

Global Dynamics of a Cooperative System with Ceiling Density Dependence

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Abstract We will investigate the global behavior of the cooperative system

$$x_{t+1} = \min\{r_{11}x_t + r_{12}y_t, K_1\}, y_{t+1} = \min\{r_{21}x_t + r_{22}y_t, K_2\}, t = 0, 1, \ldots$$

where the initial conditions x_0, y_0 are arbitrary nonnegative numbers. This system models a population comprised of two subpopulations on different patches of land. The model, introduced in [4], considers the minimum between the maximum carrying capacity of each patch (K_1 or K_2 resp.) and the linear combination of the population from patch i from the last time step with those who migrated to patch i for i=1,2. We break the behavior of the system into several cases based on whether the linear combination of the population or maximum carrying capacity is greater. We are able to conclude that either one fixed point will be a global attractor of the interior region of $\mathbb{R}^2_+$ or there will exist a line of fixed points with the stable manifolds as the basins of attractions. We then extend some of these results to the n–dimensional case, first introduced in [2], using similar techniques. We investigate the global behavior of general cooperative system

$$x^i_{t+1} = \min\{r_{i1}x^1_t + r_{i2}x^2_t + \ldots + r_{ii}x^i_t + \ldots + r_{in}x^n_t, K_i\},$$

for i = 1, 2, ..., n, and t=0,1,... where the initial conditions of x^i_0 are arbitrary nonnegative numbers for i=1,2, ..., n. We are able to conclude in some cases that one fixed point will be a global attractor of the interior region of $\mathbb{R}^n_+$.

3.1 Introduction

We investigate the global behavior of the cooperative system

$$x_{t+1} = \min\{r_{11}x_t + r_{12}y_t, K_1\}, \quad y_{t+1} = \min\{r_{21}x_t + r_{22}y_t, K_2\}, \quad t = 0, 1, \ldots, \tag{20}$$

where the initial conditions x_0, y_0 are arbitrary nonnegative numbers. In the original paper [4], Rebarber et al. define the variables and constants based on a biological model. The model studies a metapopulation consisting of two subpopulations who live on separate patches of land (patch 1 and patch 2), however do no live entirely separate as migration is allowed. The sizes of the subpopulations fluctuate each breeding season due to migration, death, birth, as well as other factors. Additionally, each subpopulation has a maximum known as a carrying capacity as population growth is restricted by the size of the land, food available, etc. In (20) the density of the subpopulations is represented by x_t (the number of females in patch 1) and y_t (the number of females in patch 2) at a time step $t \in \mathbb{N}$. The maximum carrying capacity is represented by K_1, $K_2 > 0$ for each patch respectively. The constants of $r_{11} \geq 0$, $r_{22} \geq 0$, $r_{12} > 0$, and $r_{21} > 0$ are the probabilities given as

$$r_{11} = (1 - \mu_1)(1 + b_1 f_1)(1 - m_1) \quad \text{and} \quad r_{22} = (1 - \mu_2)(1 + b_2 f_2)(1 - m_2),$$
$$r_{12} = (1 + b_2 f_2) m_2 \alpha \quad \text{and} \quad r_{21} = (1 + b_1 f_1) m_1 \alpha$$

where b_1 and b_2 are the probabilities that a female will give birth, f_1 and f_2 are the probabilities that the baby is female, μ_1 and μ_2 are the probabilities of death, α is the probability of a successful migration, and m_1 and m_2 are the probabilities of migration for each patch respectively. The linear models in the system are

$$x_{t+1} = r_{11} x_t + r_{12} y_t, \quad y_{t+1} = r_{21} x_t + r_{22} y_t,$$

where x_{t+1} and y_{t+1} represent the current populations at time step $t+1$. Here $r_{11} x_t$ is the population of patch 1 that remained from time step t, $r_{12} y_t$ is the population that migrated to patch 1, $r_{22} y_t$ is the population of patch 2 that remained from time step t, and $r_{21} x_t$ is the population that migrated to patch 2. Thus, system (20) will compute the minimum of the maximum carrying capacity and the linear combination of the females remaining in the population from the last time step with those who migrated to the patch. It is assumed that each subpopulation will increase until it reaches the maximum carrying capacity, also known as the ceiling density dependence.

In this paper we will present the global dynamics of the model for all cases of the parameters. In particular, we obtain the basins of attraction in the two cases of an infinite number of fixed points, which was not covered in [4]. For the four cases of a finite number of fixed points, we will use different, simpler techniques of monotone discrete dynamical systems to find the basins of attraction of all fixed points and so to give an alternative proof of the results in [4].

The cooperative system (20) can be generalized to study a metapopulation consisting of n subpopulations who live on n different patches of land. This system was originally studied in [2]. Such a system is as follows

$$x^i_{t+1} = \min\{r_{i1}x^1_t + r_{i2}x^2_t + \ldots + r_{ii}x^i_t + \ldots + r_{in}x^n_t, K_i\} \tag{21}$$

for $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots$. The initial conditions x^i_0 are arbitrary nonnegative numbers for $i = 1, 2, \ldots, n$. For each patch i, the density of females in each subpopulation is represented by x^i_t. Additionally, the maximum carrying capacity of each patch will be represented by $K_i > 0$. In (21) the constant $r_{ii} \geq 0$ is the probability of females in patch i surviving from the last time step (and remaining in the patch) as well as the probability of having a female baby born. The constants $r_{ik} > 0$ for $k = 1, 2, \ldots, i-1, i+1, \ldots, n$ are the probabilities of a female migrating to patch i from patch k including the possibility of a newborn baby female migrating during the time step. As with the 2–patch model, we assume that each population will grow linearly until reaching the carrying capacity. We will present the global dynamics of some of the cases using similar techniques to the 2–dimensional system (20). We are able to give global results for three cases, in which one fixed point will be a global attractor of the interior region of $\mathbb{R}^n_+$ using simpler, different techniques than originally used in [2]. The remaining case will be left as conjecture.

The next section of this paper contains some basic results on the basins of attraction of the equilibrium solutions of monotone systems and order preserving maps. Section 3 contains the main results of the global dynamics of system (20) in all 6 cases, and section 4 contains the main results of the generalized n–dimensional system (21). It is important to mention that no local or global dynamics of system (20) was obtained in [4] in cases 5 and 6. In fact the global dynamics in these two cases is very interesting and has been observed in number of monotone systems, see [1, 9, 10].

3.2 Some Basic Results for Order Preserving Maps

In this section we give some basic attractivity results for order preserving maps and systems from [9, 10], which will be used in the rest of the paper. See also [5, 6, 12].

We will begin with some definitions and vocabulary. A first order system of difference equations

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots \tag{22}$$

where $\mathcal{S} \subset \mathbb{R}^2$ has nonempty interior, $(f, g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is called *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y. If both f and g are nondecreasing in x and y, the system (22) is called *cooperative*. Competitive and cooperative maps are defined similarly. *Strongly cooperative* systems of difference equations or strongly competitive maps are those for which the functions f and g are coordinate–wise strictly monotone.

Let $\preceq$ be a partial order on $\mathbb{R}^n$ with a nonnegative cone P. For x, $y \in \mathbb{R}^n$, the *ordered interval* $[\![x, y]\!]$ is the set of all z such that $x \preceq z \preceq y$. We say that $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of a set P. A map T on a subset of $\mathbb{R}^n$ is *order preserving* if $T(x) \preceq T(y)$ whenever $x \preceq y$, *strictly order preserving* if $T(x) \prec T(y)$ whenever $x \prec y$, and *strongly order preserving* if $T(x) \ll T(y)$ whenever $x \prec y$. We say that $\mathcal{B}(x)$ is the *basin of attraction* of a fixed point x if $T^n(y) \to x$.

Furthermore, we define the *south–east* partial order as $\preceq_{se}$ on $\mathbb{R}^2$ where $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *north–east* partial order as $\preceq_{ne}$ on $\mathbb{R}^2$ where $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$.

Let $T : R \to R$ be a map with a fixed point $\overline{x}$ and let R' be an invariant subset of R that contains $\overline{x}$. We say that $\overline{x}$ is stable (asymptotically stable) relative to R' if $\overline{x}$ is a stable (asymptotically stable) fixed point of the restriction of T to R'.

The next result is stated for order–preserving maps on $\mathbb{R}^n$ and is given here for completeness. See [5] for a more general version valid in ordered Banach spaces.

Theorem 16 *For a nonempty set $R \subset \mathbb{R}^n$ and $\preceq$ a partial order on $\mathbb{R}^n$, let $T : R \to R$ be an order preserving map, and let $a, b \in R$ be such that $a \prec b$ and $[\![a, b]\!] \subset R$. If $a \preceq T(a)$ and $T(b) \preceq b$, then $[\![a, b]\!]$ is invariant and*

i. *There exists a fixed point of T in $[\![a, b]\!]$.*

ii. *If T is strongly order preserving, then there exists a fixed point in $[\![a, b]\!]$ which is stable relative to $[\![a, b]\!]$.*

iii. *If there is only one fixed point in $[\![a, b]\!]$, then it is a global attractor in $[\![a, b]\!]$ and therefore asymptotically stable relative to $[\![a, b]\!]$.*

We say that $\{x_n\}_{n\in\mathbb{Z}}$ is an entire orbit of a map $T : A \to A$, $A \subset \mathbb{R}^n$ if $x_{n+1} = T(x_n)$ for all $n \in \mathbb{Z}$. This orbit is said to join u_1 to u_2 if $x_n \to u_1$ as $n \to -\infty$ and $x_n \to u_2$ as $n \to \infty$. The following result of the order interval trichotomy of Dancer and Hess is for strictly order preserving maps [3, 5]. The result is stated for a partial order $\preceq$ in $\mathbb{R}^n$, but it also holds in Banach spaces.

Theorem 17 *Let $u_1 \preceq u_2$ be distinct fixed points of a strictly order preserving map $T : A \to A$, where $A \subset \mathbb{R}^n$, and let $I = [\![u_1, u_2]\!] \subset A$. Then at least one of the following holds.*

(a) *T has a fixed point in I distinct from u_1 and u_2.*

(b) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_1 to u_2 and satisfying $x_n \preceq x_{n+1}$.*

(c) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_2 to u_1 and satisfying $x_{n+1} \preceq x_n$.*

Corollary 1 ([3]) *If a and b are stable fixed points, then there exists a third fixed point in $[a, b]$.*

The following result is a direct consequence of Theorem 17, see [9, 10].

Corollary 2 *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1,\ u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

The following theorem was proved by Kulenović and Merino [10] for competitive or cooperative systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or nonhyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. This result is useful for determining basins of attraction of fixed points of competitive or cooperative maps.

Theorem 18 *Let T be a competitive (resp. cooperative) map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int(\mathcal{Q}_1(\bar{x}) \cup$*

$\mathcal{Q}_3(\overline{\mathrm{x}})$) *is nonempty (i.e. "*$\overline{\mathrm{x}}$ *is not the NW or SE vertex of* $\mathcal{R}$*"), and* T *is strongly competitive (resp. cooperative) on* Δ*. Suppose that the following statements are true.*

a. *The map* T *has a* C^1 *extension to a neighborhood of* $\overline{\mathrm{x}}$*.*

b. *The Jacobian* $J_T(\overline{\mathrm{x}})$ *of* T *at* $\overline{\mathrm{x}}$ *has real eigenvalues* λ*,* μ *such that* $0 < |\lambda| < \mu$*, where* $|\lambda| < 1$*, and the eigenspace* E^λ *associated with* λ *is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ *through* $\overline{\mathrm{x}}$ *that is invariant and a subset of the basin of attraction of* $\overline{\mathrm{x}}$*, such that* $\mathcal{C}$ *is tangential to the eigenspace* E^λ *at* $\overline{\mathrm{x}}$*, and* $\mathcal{C}$ *is the graph of a strictly increasing (resp. decreasing) continuous function of the first coordinate on an interval. Any endpoints of* $\mathcal{C}$ *in the interior of* $\mathcal{R}$ *are either fixed points or minimal period two points. In the latter case, the set of endpoints of* $\mathcal{C}$ *is a minimal period two orbit of* T*.*

3.3 The 2–Patch System

We will prove the global dynamics of system (20) for two patches of land. Let

$$J = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

be the matrix consisting of the constants of system (20), and let $\mathcal{T}$ be the cooperative map associated with system (20),

$$\mathcal{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \min\{r_{11}x + r_{12}y, K_1\} \\ \min\{r_{21}x + r_{22}y, K_2\} \end{bmatrix} : \mathbb{R}^2_+ \to \mathbb{R}^2_+.$$

Theorem 19 *The following results for system* (20) *hold:*

1. *Suppose that* $\rho(J) < 1$ *and*

$$K_1 \geq r_{11}K_1 + r_{12}K_2, \quad K_2 \geq r_{21}K_1 + r_{22}K_2, \tag{23}$$

where at least one of the inequalities is strict. The fixed point $(0,0)$ *is a global attractor.*

2. *Suppose that* $\rho(J) > 1$, $1 \notin \sigma(J)$, *and*

$$K_1 \leq r_{11}K_1 + r_{12}K_2, \quad K_2 \leq r_{21}K_1 + r_{22}K_2, \tag{24}$$

where at least one of the inequalities is strict. The fixed point $(0,0)$ *is unstable while the fixed point* (K_1, K_2) *is a global attractor of the interior region of* $\mathbb{R}^2_+$ *with the basin of attraction* $\mathcal{B}(K_1, K_2)$.

3. *Suppose that* $\rho(J) > 1$, $1 \notin \sigma(J)$, $r_{22} < 1$, *and*

$$K_1 \leq r_{11}K_1 + r_{12}K_2, \quad K_2 > r_{21}K_1 + r_{22}K_2. \tag{25}$$

Then the fixed point $(0,0)$ *is unstable while the fixed point* $(K_1, K_1r_{21}/(1-r_{22}))$ *is a global attractor of the interior region of* $\mathbb{R}^2_+$ *where the basin of attraction is* $\mathcal{B}(K_1, K_1r_{21}/(1-r_{22}))$.

4. *Suppose that* $\rho(J) > 1$, $1 \notin \sigma(J)$, $r_{11} < 1$, *and*

$$K_1 > r_{11}K_1 + r_{12}K_2, \quad K_2 \leq r_{21}K_1 + r_{22}K_2. \tag{26}$$

Then the fixed point of $(0,0)$ *is unstable while the fixed point of* $(K_2r_{12}/(1-r_{11}), K_2)$ *is a global attractor of the interior region of* $\mathbb{R}^2_+$ *where the basin of attraction is* $\mathcal{B}(K_2r_{12}/(1-r_{11}), K_2)$.

5. *Suppose that* $\rho(J) \geq 1$, $1 \in \sigma(J)$, $r_{22} < 1$, *and*

$$K_1 \leq r_{11}K_1 + r_{12}K_2, \quad K_2 \geq r_{21}K_1 + r_{22}K_2. \tag{27}$$

System (20) has an infinite number of fixed points, $E_x = \{(x, xr_{21}/(1-r_{22})) | 0 \leq x \leq K_1\}$, *which has the stable manifolds* $\mathcal{W}^s(E_x)$ *as its basins of attraction.*

6. *Suppose that $\rho(J) \geq 1$, $1 \in \sigma(J)$, $r_{11} < 1$ and*

$$K_1 \geq r_{11}K_1 + r_{12}K_2, \quad K_2 \leq r_{21}K_1 + r_{22}K_2. \tag{28}$$

System (20) has an infinite number of fixed points, $E_y = \{(yr_{12}/(1 - r_{11}), y) | 0 \leq y \leq K_2\}$, which has the stable manifolds $\mathcal{W}^s(E_y)$ as its basins of attraction.

Proof.

1. Clearly $(0,0)$ is always a fixed point. Using the knowledge of inequalities (23), the cooperative map $\mathcal{T}$ can be rewritten as

$$\mathcal{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} r_{11}x + r_{12}y \\ r_{21}x + r_{22}y \end{bmatrix}.$$

Note that no other fixed points will exist as

$$J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

would have to hold true for some fixed point (x, y). This implies that

$$\begin{aligned} x = r_{11}x + r_{12}y \text{ and } y = r_{21}x + r_{22}y &\Leftrightarrow \\ \frac{(1 - r_{11})x}{r_{12}} = \frac{r_{21}x}{(1 - r_{22})} &\Leftrightarrow \\ 1 + r_{11}r_{22} - r_{12}r_{21} = r_{11} + r_{22} &\Leftrightarrow \\ 1 + \text{ det } (J) = \text{ tr } (J). & \end{aligned}$$

This cannot be the case as $\rho(J) < 1$. The map $\mathcal{T}$ has an invariant interval $[\![(0,0), (K_1, K_2)]\!]$. This can be seen as the map is defined on $\mathbb{R}^2_+$ and the system (20) has a maximum carrying capacity (K_1, K_2). As $(0,0)$ is the unique fixed point of the invariant interval and $\rho(J) < 1$, then

$$\lim_{t\to\infty} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This indeed holds by Theorem 16 so $(0,0)$ is a global attractor.

2. The inequalities in (24) can be rewritten as

$$J \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \geq \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

Therefore as $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ is the minimum, we can conclude that

$$\mathcal{T}\left(\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}\right) = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

This results in the fact that (K_1, K_2) is a fixed point in addition to $(0,0)$. Another fixed point will only exist if either $x = K_1$ and $y \neq K_2$ or $x \neq K_1$ and $y = K_2$. Without loss of generality suppose that $x = K_1$ and $y \neq K_2$. The map $\mathcal{T}$ gives us that

$$y = r_{21}x + r_{22}y \quad \Leftrightarrow \quad y = r_{21}K_1 + r_{22}y \quad \Leftrightarrow \quad y = \frac{r_{21}K_1}{1 - r_{22}}.$$

However, this creates a contradiction as the inequality of (24) can be rewritten as

$$K_2 \leq \frac{r_{21}K_1}{1 - r_{22}}.$$

Therefore, there are no other fixed points. As $\mathcal{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \leq \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2_+$, $\mathcal{T}$ has an invariant interval $[\![(0,0), (K_1, K_2)]\!]$. The fixed point, $(0,0)$ is unstable as $\rho(J) > 1$ (see [7]). By Corollary 2, (K_1, K_2) is a global attractor for the interior of $[\![(0,0), (K_1, K_2)]\!]$, that is

$$\text{int } [\![(0,0), (K_1, K_2)]\!] \subseteq \mathcal{B}((K_1, K_2)).$$

3. Through rearranging one of the inequalities of (25) we have

$$K_1 \leq r_{11}K_1 + r_{12}K_2 \quad \text{and} \quad K_2 > \frac{r_{21}K_1}{1 - r_{22}} \tag{29}$$

We will have the fixed point of $(K_1, K_1 r_{21}/(1 - r_{22}))$ when

$$\mathcal{T}\left(\begin{bmatrix} K_1 \\ \frac{r_{21}K_1}{1 - r_{22}} \end{bmatrix}\right) = \begin{bmatrix} K_1 \\ \frac{r_{21}K_1}{1 - r_{22}} \end{bmatrix}$$

holds true. That is if

$$K_1 \leq r_{11}K_1 + r_{12}\frac{r_{21}K_1}{1-r_{22}} \quad \text{and} \quad K_2 \geq r_{21}K_1 + r_{22}\frac{r_{21}K_1}{1-r_{22}}. \tag{30}$$

By the inequalities of (29) and the fact that

$$r_{21}K_1 + r_{22}\frac{r_{21}K_1}{1-r_{22}} = \frac{r_{21}K_1}{1-r_{22}}$$

the second inequality of (30) indeed holds. The first inequality of (30) can be reduced to

$$\begin{aligned} 1 \leq r_{11} + \frac{r_{12}r_{21}}{1-r_{22}} &\Leftrightarrow \\ 1 - r_{22} \leq r_{11} - r_{11}r_{22} + r_{12}r_{21} &\Leftrightarrow \\ 1 \leq \operatorname{tr}(J) - \det(J) &\Leftrightarrow \\ 1 \leq \lambda_1 + \lambda_2 - \lambda_1\lambda_2&. \end{aligned}$$

Therefore this inequality holds as $\rho(J) > 1$ and furthermore, $(K_1, K_1r_{21}/(1-r_{22}))$ is a fixed point. By using the same argument as in cases 1 and 2 we conclude that there are no other fixed points. By using the fact $[\![a, b]\!] = \{x : a \leq x \leq b\}$ is an invariant set for $\mathcal{T}$ when a and b are fixed points of a monotone map $\mathcal{T}$, then $[\![E_0, E_+]\!]$ is an invariant interval where $E_0 = (0, 0)$ and $E_+ = (K_1, K_1r_{21}/(1-r_{22}))$. As $\rho(J) > 1$, then E_0 is unstable. By Corollary 2, E_+ is an attractor for the interior of $[\![E_0, E_+]\!]$, that is

$$\text{int } [\![E_0, E_+]\!] \subseteq \mathcal{B}(E_+). \tag{31}$$

Since (K_1, K_2) is the maximum carrying capacity for the population and E_+ is a fixed point, using the knowledge that $[\![a, b]\!]$ is an invariant set for monotone map $\mathcal{T}$ when a is fixed point and b is an end point, then $[\![E_+, (K_1, K_2)]\!]$ is an invariant set. By Theorem 16 as E_+ is the only fixed point in the region, then E_+ is an attractor for the interior of $[\![E_+, (K_1, K_2)]\!]$,

$$\text{int } [\![E_+, (K_1, K_2)]\!] \subseteq \mathcal{B}(E_+). \tag{32}$$

For all $(x,y) \notin [\![E_0, E_+]\!] \cup [\![E_+, (K_1, K_2)]\!]$ there exists $(x_L, y_L) \in [\![E_0, E_+]\!]$ and $(x_U, y_U) \in [\![E_+, (K_1, K_2)]\!]$ such that

$$(x_L, y_L) \leq (x, y) \leq (x_U, y_U).$$

Using the fact that $\mathcal{T}$ is monotone,

$$\mathcal{T}^n(x_L, y_L) \leq \mathcal{T}^n(x, y) \leq \mathcal{T}^n(x_U, y_U) \text{ for } n \in \mathbb{N}. \tag{33}$$

In view of (31) and (32), $\lim_{n\to\infty} \mathcal{T}^n(x_L, y_L) = E_+$ and $\lim_{n\to\infty} \mathcal{T}^n(x_U, y_U) = E_+$. Using (33) we can conclude that $\lim_{n\to\infty} T^n(x, y) = E_+$. Thus, E_+ is a global attractor of the interior of $[\![E_0, (K_1, K_2)]\!]$.

4. Through rearranging the inequalities of (26) we have

$$K_1 > \frac{r_{12}K_2}{1 - r_{11}} \quad \text{and} \quad K_2 \leq r_{21}K_1 + r_{22}K_2. \tag{34}$$

We will have the fixed point of $(K_2 r_{12}/(1 - r_{11}), K_2)$ when

$$\mathcal{T}\left(\begin{bmatrix} \frac{r_{12}K_2}{1-r_{11}} \\ K_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{r_{12}K_2}{1-r_{11}} \\ K_2 \end{bmatrix}$$

holds true. This will happen if

$$K_1 \geq r_{11}\frac{r_{12}K_2}{1 - r_{11}} + r_{12}K_2 \quad \text{and} \quad K_2 \leq r_{21}\frac{r_{12}K_2}{1 - r_{11}} + r_{22}K_2. \tag{35}$$

The first inequality of (35) is valid using (34) and the knowledge that

$$r_{11}\frac{r_{12}K_2}{1 - r_{11}} + r_{12}K_2 = \frac{r_{12}K_2}{(1 - r_{11})}.$$

The second inequality of (35) can be reduced to

$$\begin{aligned} 1 &\leq \frac{r_{21}r_{12}}{1 - r_{11}} + r_{22} \Leftrightarrow \\ 1 - r_{11} &\leq r_{21}r_{12} + r_{22} - r_{22}r_{11} \Leftrightarrow \\ 1 &\leq \operatorname{tr}(J) - \det(J) \quad \Leftrightarrow \\ 1 &\leq \lambda_1 + \lambda_2 - \lambda_1\lambda_2. \end{aligned}$$

Therefore this inequality holds as $\rho(J) > 1$, and furthermore, $(K_2 r_{12}/(1 - r_{11}), K_2)$ is a fixed point. The remainder of the proof is analogous to case 3 where we use the fixed point of $E = ((K_2 r_{12}/(1 - r_{11}), K_2)$ instead of $E_+ = (K_1, K_1 r_{21}/(1 - r_{22}))$.

5. If we have that

$$\mathcal{T}\left(\begin{bmatrix} x \\ \frac{x r_{21}}{(1-r_{22})} \end{bmatrix}\right) = \begin{bmatrix} x \\ \frac{x r_{21}}{(1-r_{22})} \end{bmatrix} \text{ for } 0 \leq x \leq K_1,$$

then $\{(x, x r_{21}/(1 - r_{22}))|0 \leq x \leq K_1\}$ are fixed points. As a reminder, in case 3 we proved that $(K_1, K_1 r_{21}/(1 - r_{22}))$ is a fixed point. When $x \neq K_1$, there will exist infinite fixed points whenever

$$x = r_{11} x + \frac{r_{12} r_{21} x}{(1 - r_{22})} \quad \text{and} \quad \frac{x r_{21}}{(1 - r_{22})} = r_{21} x + r_{22} \frac{x r_{21}}{(1 - r_{22})}. \tag{36}$$

The second equality of (36) clearly holds true. The first equality of (36) can be rewritten as

$$\begin{aligned} 1 &= r_{11} + \frac{r_{12} r_{21}}{(1 - r_{22})} \Leftrightarrow \\ 1 - r_{22} &= r_{21} r_{12} + r_{11} - r_{22} r_{11} \Leftrightarrow \\ 1 &\leq \operatorname{tr}(J) - \det(J) \end{aligned}$$

By rearranging the inequalities from (27) we have that

$$K_1 \leq \frac{r_{12}}{1 - r_{11}} K_2.$$

Since K_1 and K_2 are nonnegative and $r_{12} > 0$, then $r_{11} < 1$. The fact that $r_{11}, r_{22} < 1$ implies that $0 < \operatorname{tr}(J) < 2$. As $\operatorname{tr}(J) = \lambda_1 + \lambda_2$, then $0 < \lambda_1 + \lambda_2 < 2$ as well. We can conclude that as $\lambda_1 = 1$ (by $1 \in \sigma(J)$), then $\lambda_2 < 1$, and moreover, $\rho(J) = 1$. This gives that $1 \leq \operatorname{tr}(J) - \det(J)$, and furthermore, there will be infinite fixed points of the form $\{(x, x r_{21}/(1 - r_{22}))|0 \leq x \leq K_1\}$.

Next as $\mathcal{T}$ is defined on $\mathbb{R}_+^2$ and has a maximum of (K_1, K_2), then $\mathcal{T}$ is defined on the rectangular region $\mathcal{R} = [\![(0,0),(K_1,K_2)]\!] \subset \mathbb{R}^2$. Let $\overline{x}$ be a fixed point such that

$$\overline{x} \in \{(x, x \cdot r_{21}/(1 - r_{22}))|0 \leq x \leq K_1\}.$$

Then $\overline{x}$ will be in $\Delta := \mathcal{R} \cap \text{int}(\mathcal{Q}_1(\overline{x}) \cap \mathcal{Q}_3(\overline{x}))$ as all of the fixed points are in $\mathcal{R}$ and in the first quadrant of $\mathbb{R}^2$. As (20) is linear in $\mathcal{R}$, then the map is strictly monotone on $\mathcal{R}$, and moreover strongly cooperative. The map $\mathcal{T}$ has a C^1 extension to a neighborhood of $\overline{x}$ as the map is linear. Since $\lambda_1 = 1$ and $\lambda_2 < 1$, by Theorem 18 there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{x}$ that is invariant and a subset of the basin of attraction of $\overline{x}$ such that $\mathcal{C}$ is the graph of a strictly decreasing continuous function of the first coordinate on an interval, and any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period two points. This holds for every fixed point in $\{(x, xr_{21}/(1 - r_{22}))|0 \leq x \leq K_1\}$ as $\overline{x}$ was arbitrary. Thus, each fixed point of the form $\{(x, xr_{21}/(1 - r_{22}))|0 \leq x \leq K_1\}$ will have a stable manifold. By using Theorem 3.4 from [1] we can prove that the fixed point depends continuously on the initial point.

6. If we have that

$$\mathcal{T}\left(\begin{bmatrix} \frac{yr_{12}}{(1-r_{11})} \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{yr_{12}}{(1-r_{11})} \\ y \end{bmatrix} \text{ for } 0 \leq y \leq K_2,$$

then $\{(yr_{12}/(1 - r_{11}), y)|0 \leq y \leq K_2\}$ are fixed points. In case 4 we proved that $(K_2 r_{12}/(1-r_{11}), K_2)$ is a fixed point. When $y \neq K_2$ there will be infinite fixed points if

$$\frac{yr_{12}}{(1 - r_{11})} = \frac{yr_{11}r_{12}}{(1 - r_{11})} + r_{12}y \quad \text{and} \quad y = \frac{yr_{12}r_{21}}{(1 - r_{11})} + r_{22}y. \tag{37}$$

Note that the first equality of (37) clearly holds true. The second equality of (37) can be rewritten as

$$1 = \frac{r_{12}r_{21}}{(1-r_{11})} + r_{22} \Leftrightarrow$$
$$1 - r_{11} = r_{21}r_{12} + r_{22} - r_{22}r_{11} \Leftrightarrow$$
$$1 \leq \operatorname{tr}(J) - \det(J).$$

By rearranging one of the inequalities from (28) we get

$$K_2 \leq \frac{r_{21}}{1-r_{22}} K_1.$$

Since K_1 and K_2 are nonnegative and $r_{21} > 0$, then $r_{22} < 1$. The fact that $r_{11}, r_{22} < 1$ implies that $0 < \operatorname{tr}(J) < 2$. As $\operatorname{tr}(J) = \lambda_1 + \lambda_2$, then $0 < \lambda_1 + \lambda_2 < 2$ as well. We can conclude that as $\lambda_1 = 1$ (by $1 \in \sigma(J)$), $\lambda_2 < 1$, and moreover, $\rho(J) = 1$. Therefore $1 \leq \operatorname{tr}(J) - \det(J)$ holds, and furthermore, there will be infinite fixed points of the form $\{(yr_{12}/(1-r_{11}), y) | 0 \leq y \leq K_2\}$. The remainder of the proof is analogous to case 5, and thus concludes the proof.

3.4 The n–Patch System

We will prove the global dynamics in some cases of system (21) for n patches of land. Let M be the matrix comprised of constants of (21):

$$M = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn}. \end{bmatrix}$$

Define $\mathcal{T}$ to be the cooperative map associated with system (21)

$$\mathcal{T}\left(\begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}\right) = \begin{bmatrix} \min\{r_{11}x^1 + r_{12}x^2 + \ldots + \ldots + r_{1n}x^n, K_1\} \\ \vdots \\ \min\{r_{n1}x^1 + r_{n2}x^2 + \ldots + \ldots + r_{nn}x^n, K_n\} \end{bmatrix} : \mathbb{R}^n_+ \to \mathbb{R}^n_+.$$

Additionally, let $\bar{x}$ be an equilibrium point of the system (21). By Lemmas 3.2, 3.3, and 3.4 in [2] we know that

Lemma 5 *1. If* $r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \leq K_i$ *for each* $i = 1, 2, \ldots, n$ *with at least one strict inequality, then* $\rho(M) < 1$.

2. If $r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \geq K_i$ *for each* $i = 1, 2, \ldots, n$ *with at least one strict inequality, then* $\rho(M) > 1$.

We state the results of Theorem 3.5 in [2].

Lemma 6 *1. If* $\rho(M) < 1$, *then* $(0, \ldots, 0)$ *is the only fixed point.*

2. If $\rho(M) > 1$ *and* $r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \geq K_i$ *for each* $i = 1, 2, \ldots, n$ *with at least one strict inequality, then the only fixed points are* $(0, \ldots, 0)$ *and* $(K_1, \ldots, K_n)$.

3. If $\rho(M) > 1$ *and both* $r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \leq K_i$ *and* $r_{j1}K_1 + r_{j2}K_2 + \ldots + r_{jj}K_j + \ldots + r_{jn}K_j \geq K_j$ *for* i, j *in* $1, 2, \ldots, n$ *with some of the inequalities strict, then the only fixed points are* $(0, \ldots, 0)$ *and there exist one nonzero fixed point,* E_+, *with some patches at capacity and some below.*

We can now formulate a theorem about the global stability in some cases and provide new proofs based on the theory of monotone maps.

Theorem 20 *The following results for system* (21) *holds:*

1. Suppose that

$$r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \leq K_i \tag{38}$$

for each $i = 1, 2, \ldots, n$ *with at least one strict inequality. Then* $\rho(M) < 1$ *and the fixed point* $(0, \ldots, 0)$ *is a global attractor.*

2. *Suppose that* $1 \notin \sigma(M)$

$$r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \geq K_i \tag{39}$$

for each $i = 1, 2, \ldots, n$ *with at least one strict inequality. Then* $\rho(M) > 1$ *and the fixed point* $(0, \ldots, 0)$ *is unstable, while the fixed point* $(K_1, \ldots, K_n)$ *is a global attractor of the interior region of* $\mathbb{R}^n_+$.

3. *Suppose that* $1 \notin \sigma(M)$

$$r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \leq K_i$$
$$r_{j1}K_1 + r_{j2}K_2 + \ldots + r_{jj}K_j + \ldots + r_{jn}K_j \geq K_j$$

for i *and* j *in* $1, 2, \ldots, n$ *with some of the inequalities strict. Then* $\rho(M) > 1$ *and the fixed point* $(0, \ldots, 0)$ *is unstable while the nonzero fixed point,* E_+*, is a global attractor of the interior region of* $\mathbb{R}^n_+$.

Proof.

1. The point $(0, \ldots, 0)$ will always be a fixed point of (21). We can rewrite the inequalities of (38) as

$$\begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix} \leq M \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}.$$

We can rewrite the cooperative map as

$$\mathcal{T}\left(\begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}\right) = \begin{bmatrix} r_{11}x^1 + r_{12}x^2 + \ldots + \ldots + r_{1n}x^n \\ \vdots \\ r_{n1}x^1 + r_{n2}x^2 + \ldots + \ldots + r_{nn}x^n \end{bmatrix}.$$

The map $\mathcal{T}$ will have an invariant interval $[\![(0, \ldots, 0), (K_1, \ldots, K_n)]\!]$ as the map is defined on the region of $\mathbb{R}^n_+$ and has a maximum carrying capacity

$(K_1, \ldots, K_n)$. As $(0, \ldots, 0)$ is the unique fixed point of the invariant interval and $\rho(M) < 1$, then

$$\lim_{t\to\infty} \begin{bmatrix} x_t^1 \\ \vdots \\ x_t^n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Indeed by Theorem 16, $(0, \ldots, 0)$ is a global attractor.

2. As a result of the inequalities of (39),

$$M \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix} \geq \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}.$$

Thus in addition to $(0, \ldots, 0)$, $(K_1, \ldots, K_n)$ is a fixed point of (21) under the conditions of (39) as we can conclude

$$T\left(\begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}\right) = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}.$$

The map $\mathcal{T}$ has an invariant interval $[\![(0, \ldots, 0), (K_1, \ldots, K_n)]\!]$ since the map is defined on $\mathbb{R}_+^n$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \leq \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}$. The fixed point $(0, \ldots, 0)$ is unstable as $\rho(M) > 1$ (see [7]). By Corollary 2, $(K_1, \ldots, K_n)$ is the global attractor for the interior of $[\![(0, \ldots, 0), (K_1, \ldots, K_n)]\!]$, that is

$$\text{int } [\![(0, \ldots, 0), (K_1, \ldots, K_n)]\!] \subseteq \mathcal{B}((K_1, \ldots, K_n)).$$

3. In addition to the fixed point $E_0 = (0, \ldots, 0)$ there exists one more fixed point E_+ by Lemma 6. As $\mathcal{T}$ is a monotonic map and both E_0 and E_+ are fixed points, then $[\![E_0, E_+]\!]$ is an invariant interval. The fixed point E_0 is

unstable as $\rho(M) > 1$. By Corollary 2, E_+ is an attractor for the interior region of $[\![E_0, E_+]\!]$, that is

$$\text{int } [\![E_0, E_+]\!] \subseteq \mathcal{B}(E_+). \tag{40}$$

Again as $\mathcal{T}$ is a monotonic map and E_+ is a fixed point while $(K_1, \ldots, K_n)$ is the maximum carrying capacity, $[\![E_+, (K_1, \ldots, K_n)]\!]$ is an invariant interval. By Theorem 16 since E_+ is the only fixed point in the interval, then E_+ is an attractor for the interior of $[\![E_+, (K_1, \ldots, K_n)]\!]$, that is

$$\text{int } [\![E_+, (K_1, \ldots, K_n)]\!] \subseteq \mathcal{B}(E_+). \tag{41}$$

It remains to prove that E_+ is the attractor for the interior of the invariant interval $[\![E_0, (K_1, \ldots, K_n)]\!]$. From above we know that for all $(x_1, \ldots, x_n) \notin [\![E_0, E_+]\!] \cup [\![E_+, (K_1, \ldots, K_n)]\!]$ there exists $(x_1^L, \ldots, x_n^L) \in [\![E_0, E_+]\!]$ and $(x_1^U, \ldots, x_n^U) \in [\![E_+, (K_1, \ldots, K_n)]\!]$ such that

$$(x_1^L, \ldots, x_n^L) \leq (x_1, \ldots, x_n) \leq (x_1^U, \ldots, x_n^U).$$

As $\mathcal{T}$ is monotone,

$$\mathcal{T}^k((x_1^L, \ldots, x_n^L)) \leq \mathcal{T}^k((x_1, \ldots, x_n)) \leq \mathcal{T}^k((x_1^U, \ldots, x_n^U)) \text{ for } k \in \mathbb{N}. \tag{42}$$

Using (40) and (41), we can conclude that $\lim_{k\to\infty} \mathcal{T}^k((x_1^L, \ldots, x_n^L)) = E_+$ and $\lim_{k\to\infty} \mathcal{T}^k((x_1^U, \ldots, x_n^U)) = E_+$. From the inequalities of (42) we conclude that $\lim_{k\to\infty} \mathcal{T}^k(x_1, \ldots, x_n) = E_+$. Therefore E_+ is a global attractor of the interior of $[\![E_0, (K_1, \ldots, K_n)]\!]$ and thus concludes the proof.

As at this point in time the stable manifold theory does not extend to $\mathbb{R}_+^n$, for $n > 2$. We will leave cases 5 and 6 from the 2–dimensional case as conjectures.

Conjecture 1 *If $\rho(M) = 1$, $1 \in \sigma(M)$, and*

$$r_{i1}K_1 + r_{i2}K_2 + \ldots + r_{ii}K_i + \ldots + r_{in}K_n \leq K_i,$$

$$r_{j1}K_1 + r_{j2}K_2 + \ldots + r_{jj}K_j + \ldots + r_{jn}K_j \geq K_j$$

for $i, j \in \{1, 2, \ldots, n\}$ with no strict inequalities, then there will be infinite fixed points with the corresponding stable manifold as its basins of attraction.

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CHAPTER 4

Global Dynamics of Two Modified Lotka-Volterra Models

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Abstract. In this paper we will prove general results regarding the global stability of monotone systems without minimal period two solutions on a rectangular region $\mathcal{R}$. We will illustrate the general results in two examples of well known systems used in mathematical biology. The first of the systems that will be investigated is a modified Leslie-Gower system of the form

$$x_{n+1} = \alpha x_n + (1-\alpha)\frac{cx_n}{a + cx_n + y_n}$$
$$y_{n+1} = \beta y_n + (1-\beta)\frac{dy_n}{b + x_n + dy_n}, n = 0, 1, \ldots,$$

where the parameters a, b, c, d are positive numbers, α and β are positive values less than 1, and the initial conditions x_0, y_0 are arbitrary nonnegative numbers [32]. In most cases for different values of a, b, c, and d, there will either be one, two, three, or four equilibrium solutions present with at most one an interior equilibrium point. In the case when $c = d = 1$ and $a = b$, there will exist an infinite number of interior equilibrium points in which case we will find the basin of attraction for each of the equilibrium points.

The second system that will be investigated is a version of a Lotka-Volterra model of the form

$$x_{n+1} = \frac{x_n(A - y_n)}{K_1 + x_n} \quad \text{and} \quad y_{n+1} = \frac{y_n(A - x_n)}{K_2 + y_n}, \quad n = 0, 1, 2, ...,$$

where the parameters of A, K_1, and K_2 are all positive and the initial conditions x_0, y_0 are arbitrary nonnegative numbers, which is a semi implicit discretization of the continuous version [1]. In most cases, there will be between one and three equilibrium points with solutions converging to one of the points. In one case when $A > K_1 = K_2$, however, there will exist an infinite number of equilibrium points. In this case for each equilibrium point, there will be a stable manifold as its basin of attraction.

4.1 Introduction and Preliminaries

In this paper we will give global dynamic results for monotone systems with no minimal period two solutions on a rectangular region $\mathcal{R}$. These results will be found using the theory of global invariant manifolds developed by Kulenović and Merino in [28, 29, 30, 31].

We will illustrate the general results with two examples of systems that have rational functions as transition functions. The first monotone system that will be considered is

$$x_{n+1} = \alpha x_n + (1-\alpha)\frac{cx_n}{a + cx_n + y_n}, \quad y_{n+1} = \beta y_n + (1-\beta)\frac{dy_n}{b + x_n + dy_n}, n = 0, 1, ..., \tag{43}$$

where the parameters a, b, c and d are positive numbers and both α and β are positive numbers less than 1. The initial conditions x_0, y_0 are arbitrary nonnegative numbers. This system was originally outlined in [32] by Pakes and Maller as a way to model the application of plant growth. In particular, the system came as a result of the study of the subterranean clover and its various strains found in the south-west of Western Australia. The motivation around this study of the clover and competing strains can be found in [37] by Rossiter. Rossiter et al. [34] and Pakes and Maller [32] formally derived the model from the experimental data. Their aim was to explore a binary mixture of two strains and observe the competition between desirable and undesirable strains of the clover to see which would endure. One trait that was considered is the hardness of the seeds, where seeds that soften begin to grow while hard seeds become part of the seed pool for the following year. The hardness has been studied by Taylor, Rossiter, and Palmer in [36] as well as by others. It was found that in some strains the seeds soften at a faster rate as the years pass while for other strains the rate remains steady. The other quality observed is whether the seed has burrs or became a free seed, which will effect the rate of softening. It is assumed that the seed becomes free within a year once it softens.

System (43) is a modified Leslie-Gower system. To understand the general Leslie-Gower system we must first consider the system of uncoupled Beverton-Holt equations:

$$x_{n+1} = \frac{cx_n}{1+x_n}, \quad y_{n+1} = \frac{dy_n}{1+y_n}, n = 0, 1, \ldots \tag{44}$$

where $c, d > 0$ and the initial conditions x_0 and y_0 are non negative. This system has been studied by many authors such as Kulenović and Clark in [5] and H. L. Smith [40]. The system (44) has an explicit solution of the form

$$x_n = \begin{cases} \frac{1}{1/(a-1)+(1/x_0-1/(a-1))1/a^n} \text{ if } a\neq 1 \\ \frac{1}{n+1/x_0}, \text{ if } a=1 \end{cases} n = 0, 1, \ldots$$

$$y_n = \begin{cases} \frac{1}{1/(b-1)+(1/y_0-1/(b-1))1/b^n} \text{ if } b\neq 1 \\ \frac{1}{n+1/y_0}, \text{ if } b=1 \end{cases} n = 0, 1, \ldots$$

The following theorem summarizes the well known results regarding system (44).

Theorem 21 *The following statements are true for system* (44)*.*

(1) All solutions (x_n, y_n) are component-wise monotonic (x_n and y_n are increasing or decreasing sequences). Both axes are invariant sets.

(2) If $a \leq 1, b \leq 1$, then $E_0(0,0)$ is the only equilibrium and it is globally asymptotically stable.

(3) If $a \leq 1, b > 1$, then the equilibrium point $E_y(0, b-1)$ is a global attractor of all positive solutions with $x_0 > 0, y_0 \geq 0$. The basin of attraction of E_0 is the x-axis.

(4) If $a > 1, b \leq 1$, then the equilibrium point $E_x(a-1, 0)$ is a global attractor of all positive solutions with $y_0 > 0, x_0 \geq 0$. The basin of attraction of E_0 is the y-axis.

(5) *If $a > 1, b > 1$, then the equilibrium point $E_+(a-1, b-1)$ is a global attractor of all solutions with $x_0, y_0 > 0$. The basin of attraction of E_x (resp. E_y) is x-axis (resp. y-axis) without E_0.*

The Beverton-Holt equations, system (44), can be modified to create a coupled system known as the Leslie-Gower model. This model is the two-species competition model of the form

$$x_{n+1} = \frac{cx_n}{1 + a_{11}x_n + a_{12}y_n}, \quad y_{n+1} = \frac{dy_n}{1 + a_{21}x_n + a_{22}y_n}, n = 0, 1, \ldots, \tag{45}$$

where $c, d, a_{ij} \geq 0$ and the initial conditions x_0, y_0 are arbitrary nonnegative numbers, such that solution is defined for every n. As it was shown in [27] system (45) is semi implicit discretization of the classical Lotka-Volterra system of differential equations. The system (45) is a well known system that has been studied by numerous authors [6, 7, 28, 29]. Note that the terms a_{12} and a_{21} are the constants added to couple the system as they represent the interspecific competition. System (43) is a modified version of this Leslie-Gower model where the linear factors αx_n and βy_n represent the stockings for two species in competition, see [11, 12, 13]. In this paper for system (43), we begin by finding the local stability results as well as proving both the $(\mathcal{O}^+)$ condition and boundedness of solutions. Then, we use the global dynamic results to prove that solutions will converge to one of the equilibrium points in most cases. In one case, however, when $c = d = 1$ and $a = b$ there will exist an infinite number of interior equilibrium solutions. We can conclude that there is a stable manifold which is the basin of attraction for each of the infinite equilibrium points.

The second system that will be considered is

$$x_{n+1} = \frac{x_n(A - y_n)}{K_1 + x_n}, \quad y_{n+1} = \frac{y_n(A - x_n)}{K_2 + y_n}, \quad n = 0, 1, ..., \tag{46}$$

where the parameters of A, K_1, and K_2 are positive and the initial conditions x_0, y_0 are arbitrary nonnegative numbers. As we will show in Section 4, system (46) is another semi implicit discretization of the Lotka-Volterra differential equation model. We will give a local stability analysis of system (46) in which we are able to find all the eigenvalues of the Jacobian matrix for each equilibrium point. Additionally, we prove boundedness in order to use the global dynamics results. System (46) has between one and three equilibrium solutions, where we will prove that solutions will converge to one of the equilibrium points. In addition and of particular interest to us is one case when $K_1 = K_2 < A$. Here there will exist an infinite number of equilibrium points. In this case, there is a stable manifold for each equilibrium point as its basin of attraction.

In this paper, we will first give some basic definitions and results of monotone systems needed throughout the paper. In the second section, we will prove some general global dynamic results regarding monotone systems without minimal period two solutions. In the third section, we will prove the global dynamics of all cases of system (43), and in the fourth section, we will prove the global dynamic results of all cases of system (46).

The theory of monotone maps will be used to help prove global dynamic results for system (43) and (46). We will begin by giving some basic definitions and information regarding monotone maps in the plane.

A first order system of difference equations

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots \tag{47}$$

where $\mathcal{S} \subset \mathbb{R}^2$ has nonempty interior, $(f, g) : \mathcal{S} \to \mathcal{S}$, and both f, g are continuous functions is called *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y. If both f and g are nondecreasing in x and y, system (47) is called *cooperative*. Competitive and cooperative maps are defined similarly. *Strongly cooperative* systems of difference equations or strongly competitive maps are those for which the functions f and g are coordinate–wise strictly monotone.

Given a partial ordering $\preceq$ on $\mathbb{R}^2$ two points x, y are said to be related if $x \preceq y$ or $y \preceq x$, and is said to be a strictly related if $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality is defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

We define a map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ to be a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map, T, is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and furthermore is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. This implies that being related is invariant under iteration for a strongly monotone map.

A *North-East ordering* (NE) is an ordering for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering is defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*. The examples provided in this paper will be competitive and therefore follow a South-East ordering.

Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $(O+)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. The basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p.

The fixed point $(\bar{x}, \bar{y})$ is said to by non-hyperbolic if the Jacobian matrix has at least one eigenvalue on the unit circle ($|\lambda| = 1$). If the other eigenvalue is inside the unit circle ($|\lambda| < 1$) the fixed point is non-hyperbolic of stable type, and if the other other eigenvalue is outside of the unit circle ($|\lambda| > 1$) the fixed point is non-hyperbolic of unstable type. If both eigenvalues lie on the unit circle, the fixed point is non-hyperbolic of resonance type of either (1,1), (1,-1), (-1,1), or (-1,-1) depending on the values of the eigenvalues.

The local stable manifold $W^s_{loc}(x, y)$ and unstable manifold of $W^u_{loc}(x, y)$ of an equilibrium point $(\bar{x}, \bar{y})$ are defined as the sets

$$W^s_{loc} = \{(x, y) : T^n(x, y) \in U \text{ for all } n \geq 0, \text{ and } T^n(x, y) \to (\bar{x}, \bar{y}) \text{ as } n \to \infty\} \text{ and}$$

$$W^u_{loc} = \{(x, y) : T^{-n}(x, y) \in U \text{ for all } n \geq 0, \text{ and } T^{-n}(x, y) \to (\bar{x}, \bar{y}) \text{ as } n \to \infty\}$$

where U is a neighborhood of the equilibrium point and T is the map. The global stable manifold W^s and the global unstable manifold W^u are then defined as the sets

$$W^s(E) = \bigcup_{k=1}^{\infty} T^{-k}(W^s_{loc}(E)) \text{ and } W^u(E) = \bigcup_{k=1}^{\infty} T^{-k}(W^u_{loc}(E)).$$

The following theorem was proved by deMottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [40].

Theorem 22 *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of*

T. If instead (O−) holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.

The next results, from [30, 29], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [40, 41].

Theorem 23 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

In the theorem below criteria is given for the curve $\mathcal{C}$ to have endpoints on the boundary of the region $\mathcal{R}$, that is $\partial\mathcal{R}$.

Theorem 24 *For the curve $\mathcal{C}$ of Theorem 23 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{x}) > 0$, *and* $T(x) = \overline{x}$ *has no solutions* $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{x}) < 0$, *and* $T(x) = \overline{x}$ *has no solutions* $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 23 reduces just to $|\lambda| < 1$. This follows from a change of variables [42] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 25 *(A) Assume the hypotheses of Theorem 23, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 23. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \text{and } \mathcal{W}_+ := \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{48}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ *is invariant, and* $\mathrm{dist}(T^n(x), Q_2(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ *is invariant, and* $\mathrm{dist}(T^n(x), Q_4(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{x}$ *is an interior point of* $\mathcal{R}$ *and T is C^2 and strongly competitive in a neighborhood of* $\overline{x}$*, then T has no periodic points in the boundary of* $Q_1(\overline{x}) \cup Q_3(\overline{x})$ *except for* $\overline{x}$*, and the following statements are true.*

(iii) For every $x \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_2(\overline{x})$ *for* $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_4(\overline{x})$ *for* $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x,\ \text{and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

The following result gives a description of the stable and unstable sets of a saddle point of a competitive map.

Theorem 26 *In addition to the hypotheses of part (B) of Theorem 25, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 23 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Theorem 27 *Let T be a monotone map on a closed and bounded rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Suppose that T has a unique fixed point $\bar{x}$ in $\mathcal{R}$. Then $\bar{x}$ is a global attractor of T on $\mathcal{R}$.*

The next result is stated for order-preserving maps on $\mathbb{R}^n$. See [16] for a more general version that is valid in ordered Banach spaces.

Corollary 3 *If the non-negative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

The following result gives conditions for the existence of the boundary curves of the basin of attraction. The complete results proved by Kulenović and Merino can be found in [31].

Theorem 28 *Let* $p, q \in \mathbb{R}^2$ *be such that* $p \preceq_{se} q$, *and* $\mathcal{R} \subset \mathbb{R}^2$ *such that* $int([\![p, q]\!]_{se}) \subset \mathcal{R} \subset [\![p, q]\!]_{se}$. *Let* T *be a competitive map defined on* $\mathcal{R}$ *that is strongly competitive on* $int(\mathcal{R})$. *If there exist* $r \in \{p, q\}$, *and* $x, y \in int\,(\mathcal{R})$ *such that* $T^n(x) \to r$ *and* $T^n(y) \not\to r$, *then there exists a curve* $\mathcal{C}$ *in* $\mathcal{R}$ *which is strongly north-east linearly ordered and whose endpoints are in* $\partial\mathcal{R}$ *such that the connected components* $\mathcal{A}$ *and* $\mathcal{B}$ *of* $int(\mathcal{R})\backslash\mathcal{C}$ *chosen so that* $x \in \mathcal{A}$, *satisfy* $T^n(z) \to r$ *for* $z \in \mathcal{A}$, *and* $T^n(w) \not\to r$ *for* $w \in \mathcal{B} \cup \mathcal{C}$. *If the point* r *is in* $\mathcal{R}$, *then* r *is a fixed point of* T.

Theorem 29 *Let* $\mathcal{R} = (a_1, a_2) \times (b_1, b_2)$, *and let* $T : \mathcal{R} \to \mathcal{R}$ *be a strongly competitive map with a unique fixed point* $\bar{x} \in \mathcal{R}$, *and such that* T *is twice continuously differentiable in a neighbourhood of* $\bar{x}$. *Assume further that at the point* $\bar{x}$ *the map* T *has associated characteristic values* μ *and* ν *satisfying* $1 < \mu$ *and* $-\mu < \nu < \mu$, *with* $\nu \neq 0$, *and that no standard basis vector is an eigenvector associated to one of the characteristic values.*

Then there exists curves $\mathcal{C}_1$, $\mathcal{C}_2$ *in* $\mathcal{R}$ *and there exist* p_1, $p_2 \in \partial\mathcal{R}$ *with* $p_1 \ll_{se} \bar{x} \ll_{se} p_2$ *such that*

i. *For* $l = 1, 2, \mathcal{C}_l$ *is invariant, north-east strongly linearly ordered, such that* $\bar{x} \in \mathcal{C}_l$ *and* $\mathcal{C}_l \subset \mathcal{Q}_3(\bar{x}) \cup \mathcal{Q}_1(\bar{x})$; *the endpoints* q_l, r_l *of* $\mathcal{C}_l$, *where* $q_l \preceq_{ne} r_l$, *belong to the boundary of* $\mathcal{R}$. *For* $l, j \in 1, 2$ *with* $l \neq j$, $\mathcal{C}_l$ *is a subset of the closure of one of the components of* $\mathcal{R}\backslash\mathcal{C}_j$. *Both* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are tangential at* $\bar{x}$ *to the eigenspace associated with* ν.

ii. *For* $l = 1, 2$, *let* $\mathcal{B}_l$ *be the component of* $\mathcal{R}\backslash\mathcal{C}_l$ *whose closure contains* p_l. *Then* $\mathcal{B}_l$ *is invariant. Also, for* $x \in \mathcal{B}_1$, $T^n(x)$ *accumulates on* $\mathcal{Q}_2(p_1) \cap \partial\mathcal{R}$, *and for* $x \in \mathcal{B}_2$, $T^n(x)$ *accumulates on* $\mathcal{Q}_4(p_2) \cap \partial\mathcal{R}$.

We will use the results from [29, 30, 31] to prove the general global dynamics results of competitive maps in the plane.

4.2 Global Dynamic Results

We will prove some global dynamic results of the general monotone system (47).

Theorem 30 *Consider the map T generated by system* (47) *on a rectangular region $\mathcal{R}$ where the fixed point $E_0 = (0,0)$ is on the bottom left corner of $\mathcal{R}$. Suppose that T is a strongly competitive map with no minimal period two solutions on $\mathcal{R}$. Furthermore, we will assume that conditions a and b of Theorem 23 holds for any saddle fixed point.*

(a) Assume the map T has the fixed points of $E_x = (x,0)$ which is a saddle point, $E_y = (0,y)$ which is locally asymptotically stable, and $E_0 = (0,0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Then every solution which begins off of the x-axis converges to E_y, and every solution which begins on the x-axis without E_0 converges to E_x.

(b) Assume the map T has the fixed point $E_y = (0,y)$ which is a saddle point, the fixed point $E_x = (x,0)$ which is locally asymptotically stable point, and the fixed point $E_0 = (0,0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Then every solution which begins off the y-axis converges to E_x, and every solution which begins on y-axis without E_0 converges to E_y.

(c) Assume the map T has the fixed points $E_x = (x,0)$ and $E_y = (0,y)$ which are both saddle points, the fixed point $E_+ = (x_+, y_+), x_+ > 0, y_+ > 0$ which is a locally asymptotically stable point, and the fixed point $E_0 = (0,0)$ which is a repeller. For the fixed points, $E_y \preceq_{se} E_0 \preceq_{se} E_x$ and $E_y \preceq_{se} E_+ \preceq_{se} E_x$. Then every solution which begins off the x and y axes converges to E_+. Every solution which begins on the x-axis without E_0 converges to E_x and every solution which begins on the y-axis without E_0 converges to E_y.

(d) *Assume the map T has the fixed points $E_x = (x, 0)$ and $E_y = (0, y)$ which are both locally asymptotically stable points, the fixed point $E_+ = (x_+, y_+), x_+ > 0, y_+ > 0$ which is a saddle point, and the fixed point $E_0 = (0, 0)$ which is a repeller. For the fixed points, $E_y \preceq_{se} E_0 \preceq_{se} E_x$ as well as $E_y \preceq_{se} E_+ \preceq_{se} E_x$. Then there will exist the continuous non-decreasing curve of the stable manifold $W^s(E_+)$ with an endpoint at E_0. Every solution which begins to the right the stable manifold $\mathcal{W}^s(E_+)$ converges to E_x and every solution which begins to the left of the stable manifold $W^s(E_+)$ converges to E_y. Every solution which begins on the stable manifold $\mathcal{W}^s(E_+)$ converges to E_+.*

(e) *Assume the map T has the fixed point $E_y = (0, y)$ which is locally asymptotically stable and the fixed point $E_0 = (0, 0)$ which is a saddle point where $E_y \preceq_{se} E_0$. Then every solution which begins off the x-axis converges to E_y and every solution which begins on the x-axis converges to E_0.*

(f) *Assume the map T has the fixed point $E_x = (x, 0)$ which is locally asymptotically stable and the fixed point $E_0 = (0, 0)$ which is a saddle point where $E_0 \preceq_{se} E_x$. Then every solution which begins off the y-axis converges to E_x and every solution which begins on the y-axis converges to E_0.*

(g) *Assume the map T has one fixed point $E_0 = (0, 0)$ which is locally asymptotically stable. Then every solution converges to E_0.*

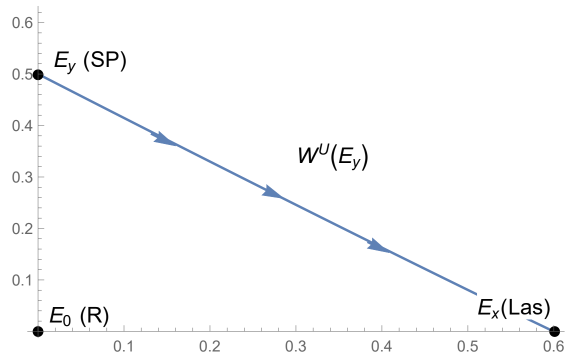

Figure 1. Graph of Case (b)

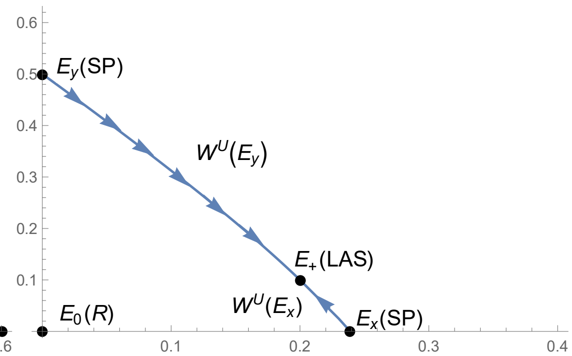

Figure 2. Graph of Case (c)

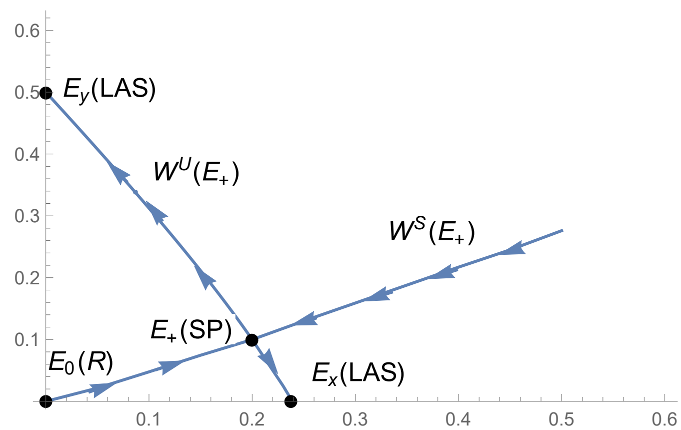

Figure 3. Graph of Case (d)

Proof.

(a) As E_x is a saddle point, there exists a global stable manifold $W^s(E_x)$, and global unstable manifold $W^u(E_x)$, by Theorems 23, 24, 25, and 26. As there are no interior fixed points or minimal period-two solutions, the endpoints of both the stable and unstable manifolds will be on the boundary of $\mathcal{R}$. For the unstable manifold, $W^u(E_x)$ the endpoint will be E_y and for the stable manifold, $W^s(E_x)$ the endpoint will be the x-axis. Any point on the stable manifold, which in this case is the x-axis will converge to E_x. As E_y is locally asymptotically stable, points on the y-axis will converge to E_y. We will consider the global dynamics in two cases based on the location of the initial point $B = (x_0, y_0) \in \text{ int } \mathcal{R}$.

For the first case, suppose the initial point $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and below the unstable manifold $W^u(E_x)$ of E_x. There will exists two projections of B onto the unstable manifold, $W^u(E_x)$, that is $P_x = (x, y_0)$ and $P_y = (x_0, y)$ such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

Taking the limit we obtain

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x),$$

which implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_y.$$

This yields that $\lim_{n\to\infty} T^n(B) = E_y$. Here we use that the unstable manifold $W^u(E_x)$ has an endpoint at E_y, $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_y$.

Next suppose the initial point $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and above the unstable manifold $W^U(E_x)$ of E_x. There will exists two projections of B onto the unstable manifold, $W^u(E_x)$, that is $P_x = (x, y_0)$ and $P_y = (x_0, y)$ such that $P_x \preceq_{se} B \preceq_{se} P_y$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y).$$

Taking the limit we obtain

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y),$$

which implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_y.$$

This yields that $\lim_{n\to\infty} T^n(B) = E_y$. Here we use that the unstable manifold $W^u(E_x)$ has an endpoint at E_y, $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_y$.

Finally suppose that $B = (x_0, y_0) \in \text{int}\mathcal{R}\backslash\mathcal{R}_0$. There exists a projection $P_y = (0, y_0)$ of B onto the y-axis and a projection $P_x = (x_0, 0)$ of B onto the x-axis such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity this implies that

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

As the x-axis is the stable manifold of E_x, then $\lim_{n\to\infty} T^n(P_x) = E_x$. Furthermore, as E_y is locally asymptotically stable, $\lim_{n\to\infty} T^n(P_y) = E_y$. So when the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x)$$

implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

We can conclude that as $n \to \infty$ then $T^n(B) \to \mathcal{R}_0$. Once $T^n(B)$ enters the rectangular region $\mathcal{R}_0$, the global behavior will follow from the previous cases.

(b) This proof is analogous to the proof of case (a). The difference is that now we consider the stable and unstable manifolds of E_y instead of E_x as E_y is a saddle point and E_x is locally asymptotically stable.

(c) By Theorems 23, 24, 25, and 26 as E_x and E_y are saddle points, there exist the global stable manifolds, $W^s(E_x)$ and $W^s(E_y)$, and global unstable manifolds, $W^u(E_x)$ and $W^u(E_y)$. As E_+ is the interior fixed point, it will be the endpoint of $W^u(E_x)$ and $W^u(E_y)$. The y-axis will be the stable manifold $W^s(E_y)$ of E_y. Thus for any initial point that begins on the y-axis will converge to E_y. The x-axis will be the stable manifold $W^s(E_x)$ of the E_x. So we can conclude that any point that begins on the x-axis will converge to E_x. We will consider the global dynamics in a few cases based on the location of the initial point $B = (x_0, y_0) \in \text{int } \mathcal{R}$.

First suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and below both the unstable manifolds $W^u(E_x)$ and $W^u(E_y)$ of E_x and E_y respectively. There will exists two projections $P_x = (x, y_0)$ and $P_y = (x_0, y)$ of B, which will either be on the unstable manifold of E_x ($W^u(E_x)$) or E_y ($W^u(E_y)$) depending on the initial location of the point B. As the proof holds regardless of whether the projections are onto $W^u(E_x)$, $W^u(E_y)$, or both, without loss of generality we can suppose that P_x is on $W^u(E_x)$ and P_y is on $W^u(E_y)$ such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

In view of $\lim_{n\to\infty} T^n(P_x) = E_+$ and $\lim_{n\to\infty} T^n(P_y) = E_+$ we obtain

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x)$$

which implies that

$$E_+ \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_+,$$

and so $\lim_{n\to\infty} T^n(B) = E_+$.

Next suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and above both the unstable manifolds $W^u(E_x)$ and $W^u(E_y)$ of E_x and E_y respectively. There will exists two projections $P_x = (x, y_0)$ and $P_y = (x_0, y)$ of B, which will either be on the unstable manifold of E_x ($W^u(E_x)$) or E_y ($W^u(E_y)$) depending on the initial location of the point B. Without loss of generality suppose that P_x is on $W^u(E_x)$ and P_y is on $W^u(E_y)$ such that $P_x \preceq_{se} B \preceq_{se} P_y$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y).$$

In view of $\lim_{n\to\infty} T^n(P_x) = E_+$ and $\lim_{n\to\infty} T^n(P_y) = E_+$ we obtain

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y)$$

which implies that

$$E_+ \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_+,$$

and so $\lim_{n\to\infty} T^n(B) = E_+$.

Finally suppose that $B = (x_0, y_0) \in$ int $\mathcal{R}\backslash\mathcal{R}_0$. There exists a projection $P_y = (0, y_0)$ of B onto the y-axis and a projection $P_x = (x_0, 0)$ of B onto the x-axis such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity this implies that

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

As the x-axis is the stable manifold of E_x, then $\lim_{n\to\infty} T^n(P_x) = E_x$. Furthermore, as the y-axis is the stable manifold of E_y, $\lim_{n\to\infty} T^n(P_y) = E_y$. So when the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x)$$

implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

We can conclude that as $n \to \infty$ then $T^n(B) \to \mathcal{R}_0$. Once $T^n(B)$ enters the rectangular region $\mathcal{R}_0$, the global behavior will follow from the previous cases.

(d) As E_+ is a saddle point, there exists the global stable manifold $W^s(E_+)$, and the global unstable manifold $W^u(E_+)$, by Theorems 23, 24, 25, and 26. As there are no other interior fixed points besides E_+, the endpoints of $W^u(E_+)$ will be E_x and E_y on the boundary of the region. The endpoint of $W^s(E_+)$ will be E_0. As E_y is locally asymptotically stable, solutions on the y-axis will converge to E_y and as E_x is locally asymptotically stable, solutions on the x-axis will converge to E_x. Any point that begins on the stable manifold $W^s(E_+)$ of E_+ will converge to E_+. We will describe the global dynamics in a few cases based on the location of the initial point $B = (x_0, y_0) \in \text{ int } \mathcal{R}$.

First suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and is both to the left of the stable manifold $W^s(E_+)$ and below the unstable manifold $W^u(E_+)$ of E_+. There will exist a projection $P_y = (x_0, y)$ of B onto the unstable manifold $W^u(E_+)$ of E_+ as well as another projection $P_x = (x, y_0)$ of B such that $P_y \preceq_{se} B \preceq_{se} P_x$. The projection P_x will either be on the unstable manifold $W^u(E_+)$ or on the stable manifold $W^s(E_+)$ depending on the initial point B. We will first suppose that the projection P_x is on the unstable manifold $W^u(E_+)$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

Once the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x),$$

which implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_y$$

as both P_x and P_y are on the unstable manifold so that $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_y$. We can conclude that $\lim_{n\to\infty} T^n(B) = E_y$. Next suppose that the projection P_x is on the stable manifold $W^s(E_+)$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x)$$

still holds. Once the limit of the inequalities is taken we have

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x).$$

As P_y is on the unstable manifold, $\lim_{n\to\infty} T^n(P_y) = E_y$, and as P_x is on the stable manifold, then $\lim_{n\to\infty} T^n(P_x) = E_+$. This implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_+.$$

We can conclude that $\lim_{n\to\infty} T^n(B) = E_y$ as B does not begin on the stable manifold of E_+.

For the next case, suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and is both to the left of the stable manifold $W^s(E_+)$ and above the unstable manifold $W^u(E_+)$ of E_+. There will exist a projection $P_x = (x, y_0)$ of B onto the unstable manifold $W^u(E_+)$ of E_+ as well as another projection $P_y = (x_0, y)$ of B such that $P_x \preceq_{se} B \preceq_{se} P_y$. The projection P_y will either be on the unstable manifold $W^u(E_+)$ or on the stable manifold $W^s(E_+)$ depending on the initial point B. We will first suppose that the projection P_y is on the unstable manifold $W^u(E_+)$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y).$$

Once the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y),$$

which implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_y$$

as both P_x and P_y are on the unstable manifold so that $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_y$. We can conclude that $\lim_{n\to\infty} T^n(B) = E_y$. Next suppose that the projection P_y is on the stable manifold $W^s(E_+)$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y)$$

still holds. Once the limit of the inequalities is taken we have

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y).$$

As P_x is on the unstable manifold, $\lim_{n\to\infty} T^n(P_x) = E_y$, and as P_y is on the stable manifold, then $\lim_{n\to\infty} T^n(P_y) = E_+$. This implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_+.$$

We can conclude that $\lim_{n\to\infty} T^n(B) = E_y$ as B does not begin on the stable manifold of E_+.

Now suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and is both to the right of the stable manifold $W^s(E_+)$ and above the unstable manifold $W^u(E_+)$ of E_+. There will exist a projection $P_y = (x_0, y)$ of B onto the unstable manifold $W^u(E_+)$ of E_+ as well as another projection $P_x = (x, y_0)$ of B such that $P_x \preceq_{se} B \preceq_{se} P_y$. The projection P_x will either be on the unstable manifold $W^u(E_+)$ or on the stable manifold $W^s(E_+)$ depending on the initial point B. We will first suppose that the projection P_x is on the unstable manifold $W^u(E_+)$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y).$$

Once the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y),$$

which implies that

$$E_x \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x$$

as both P_x and P_y are on the unstable manifold so that $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_x$. We can conclude that $\lim_{n\to\infty} T^n(B) = E_x$. Next suppose that the projection P_x is on the stable manifold $W^s(E_+)$. By monotonicity,

$$T^n(P_x) \preceq_{se} T^n(B) \preceq_{se} T^n(P_y)$$

still holds. Once the limit of the inequalities is taken we have

$$\lim_{n\to\infty} T^n(P_x) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_y).$$

As P_y is on the unstable manifold, $\lim_{n\to\infty} T^n(P_y) = E_x$, and as P_x is on the stable manifold, then $\lim_{n\to\infty} T^n(P_x) = E_+$. This implies that

$$E_+ \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

We can conclude that $\lim_{n\to\infty} T^n(B) = E_x$ as B does not begin on the stable manifold of E_+.

For the next case, suppose that $B = (x_0, y_0)$ is inside the rectangular region $\mathcal{R}_0 = [\![E_y, E_x]\!]$ and is both to the right of the stable manifold $W^s(E_+)$ and below the unstable manifold $W^u(E_+)$ of E_+. There will exist a projection $P_x = (x, y_0)$ of B onto the unstable manifold $W^u(E_+)$ of E_+ as well as another projection $P_y = (x_0, y)$ of B such that $P_y \preceq_{se} B \preceq_{se} P_x$. The projection P_y will either be on the unstable manifold $W^u(E_+)$ or on the stable manifold $W^s(E_+)$ depending on the initial point B. We will first suppose that the projection P_y is on the unstable manifold $W^u(E_+)$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

Once the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x),$$

which implies that

$$E_x \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x$$

as both P_x and P_y are on the unstable manifold so that $\lim_{n\to\infty} T^n(P_x) = \lim_{n\to\infty} T^n(P_y) = E_x$. We can conclude that $\lim_{n\to\infty} T^n(B) = E_x$. Next suppose that the projection P_y is on the stable manifold $W^s(E_+)$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x)$$

still holds. Once the limit of the inequalities is taken we have

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x).$$

As P_x is on the unstable manifold, $\lim_{n\to\infty} T^n(P_x) = E_x$, and as P_y is on the stable manifold, then $\lim_{n\to\infty} T^n(P_y) = E_+$. This implies that

$$E_+ \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

We can conclude that $\lim_{n\to\infty} T^n(B) = E_x$ as B does not begin on the stable manifold of E_+.

Finally, suppose that $B = (x_0, y_0) \in \text{ int}\mathcal{R}\backslash\mathcal{R}_0$. There exists a projection $P_y = (0, y_0)$ of B onto the y-axis and a projection $P_x = (x_0, 0)$ of B onto the x-axis such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity this implies that

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

As E_x is locally asymptotically stable, then $\lim_{n\to\infty} T^n(P_x) = E_x$. Furthermore, as E_y is locally asymptotically stable, $\lim_{n\to\infty} T^n(P_y) = E_y$. So when the limit of the inequalities is taken

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x)$$

implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

We can conclude that as $n \to \infty$, then $T^n(B) \to \mathcal{R}_0$. Once $T^n(B)$ enters the rectangular region, $\mathcal{R}_0$, the global behavior will follow from one of the previous cases.

(e) As E_0 is a saddle point, there exists a global stable manifold $W^s(E_0)$, and global unstable manifold $W^u(E_0)$, by Theorems 23, 24, 25, and 26. As there are no interior fixed points or minimal period-two solutions, the endpoints of both the stable and unstable manifolds will be on the boundary of $\mathcal{R}$. The unstable manifold $W^u(E_0)$ will be the y-axis with the endpoint of E_y. So if a point begins on the y-axis, it will converge to E_y. The stable manifold $W^s(E_0)$ will be the x-axis. Thus, if a point begins on the x-axis, it will converge to E_0.

Suppose that $B = (x_0, y_0) \in$ int $\mathcal{R}$. There will exists two projections $P_x = (x_0, 0)$ and $P_y = (0, y_0)$ of B onto the x and y axis such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity,

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

As E_y is locally asymptotically stable, $\lim_{n\to\infty} T^n(P_y) = E_y$. Additionally, as P_x is on the stable manifold of E_0, $\lim_{n\to\infty} T^n(P_x) = E_0$. Taking the limit of the inequalities

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x),$$

which implies

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_0.$$

As B does not begin on the stable manifold and by the monotone system theory [30] as the stable manifold is unique, we can conclude that $\lim_{n\to\infty} T^n(B) = E_y$.

(f) This proof is analogous to the proof of case (e). The difference is that we consider the existence of E_x instead of E_y, where E_x will be locally asymptotically stable.

(g) Let the map T contain one fixed point $E_0 = (0, 0)$ that is locally asymptotically stable. As E_0 is the only fixed point on the rectangular region $\mathcal{R}$ and there are no minimal period two points, all solutions must converge to E_0 by Theorem 27, that is $\lim_{n\to\infty} T^n(B) = E_0$ for an initial point $B = (x_0, y_0)$.

Theorem 31 *Consider the map T generated by system (47) on a rectangular region $\mathcal{R}$ where the fixed point $E_0 = (0, 0)$ is on the bottom left corner of $\mathcal{R}$. Suppose that T is a strongly competitive map with no minimal period two solutions on $\mathcal{R}$. Furthermore, we will assume that conditions a and b of Theorem 23 holds for any saddle fixed point.*

(a) Assume that the map T has the fixed points $E_y = (0, y)$ which is non-hyperbolic of the stable type, $E_x = (x, 0)$ which is locally asymptotically stable, and $E_0 = (0, 0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Every solution which begins off the y-axis converges to E_x and every solution which begins on the y-axis without E_0 converges to E_y.

(b) Assume that the map T has the fixed points $E_y = (0, y)$ which is locally asymptotically stable, $E_x = (x, 0)$ which is non-hyperbolic of the stable type, and $E_0 = (0, 0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Every solution which begins off the x-axis converges to E_y and every solution which begins on the x-axis without E_0 converges to E_x.

(c) Assume that the map T has the fixed points $E_y = (0, y)$ which is non-hyperbolic of the stable type, $E_x = (x, 0)$ which is a saddle point, and $E_0 = (0, 0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Every solution

which begins off the x-axis converges to E_y and every solution which begins on the x-axis without E_0 converges to E_x.

(d) *Assume that the map T has the fixed points $E_y = (0, y)$ which is a saddle point, $E_x = (x, 0)$ which is non-hyperbolic of the stable type, and $E_0 = (0, 0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Every solution which begins off the y-axis converges to E_x and every solution which begins on the y-axis without E_0 converges to E_y.*

(e) *Assume that the map T has the fixed points $E_y = (0, y)$ and $E_x(x, 0)$ which are both non-hyperbolic of the stable type and $E_0 = (0, 0)$ which is a repeller where $E_y \preceq_{se} E_0 \preceq_{se} E_x$. Every solution on the x-axis without E_0 will converge to E_x and every solution on the y-axis without E_0 will converge to E_y. Every solution which begins off the x and y axis will converge to exactly one of E_x or E_y.*

(f) *Assume that the map T has the fixed points $E_y = (0, y)$ which is locally asymptotically stable and $E_0 = (0, 0)$ which is non-hyperbolic of the unstable type where $E_y \preceq_{se} E_0$. Then there will exists two curves, C_1 and C_2, $C_2 \preceq_{se} C_1$ that are continuous and non-decreasing with an endpoint at E_0. If the curves C_1 and C_2 coincide with each other or $C_2 \notin \mathcal{R}$, every solution which begins off the x-axis will converge to E_y. Every solution which begins on the x-axis will converge to E_0. If there exists both $C_1, C_2 \in \mathcal{R}$ then every solution to the left of C_2 will converge to E_y and every solution to the right of C_2 will converge to E_0.*

(g) *Assume that the map T has the fixed points $E_x = (x, 0)$ which is locally asymptotically stable and $E_0 = (0, 0)$ which is non-hyperbolic of the unstable type where $E_0 \preceq_{se} E_x$.Then there will exists two curves, C_1 and C_2, $C_2 \preceq_{se}$*

C_1, that are continuous and non-decreasing with an endpoint at E_0. If the curves C_1 and C_2 coincide with each other or $C_2 \notin \mathcal{R}$, every solution which begins off the y-axis will converge to E_x. Every solution which begins on the y-axis will converge to E_0. If there exists both $C_1, C_2 \in \mathcal{R}$ then every solution to the left of C_1 will converge to E_0 and every solution to the right of C_1 will converge to E_x.

(h) Assume that the map T has one fixed point $E_0 = (0,0)$ which is non-hyperbolic. Then every solution converges to E_0.

(i) Assume the map T has the fixed points $E_x = (x,0)$ and $E_y = (0,y)$ both of which are non hyperbolic of the stable type, $E_0 = (0,0)$ which is a repeller, and an infinite number of equilibrium points which are non-hyperbolic of stable type. Suppose the map T is strongly competitive and has a C^1 extension. For each of the infinite equilibrium points suppose that conditions a and b of Theorem 23 are satisfied. Then for each of the infinite equilibrium points, there is a stable manifold as its basin of attraction. Each stable manifold will have an end point at E_0 and they are graphs of continuous and non-decreasing functions. The points will depend continuously on the initial point (x_0, y_0).

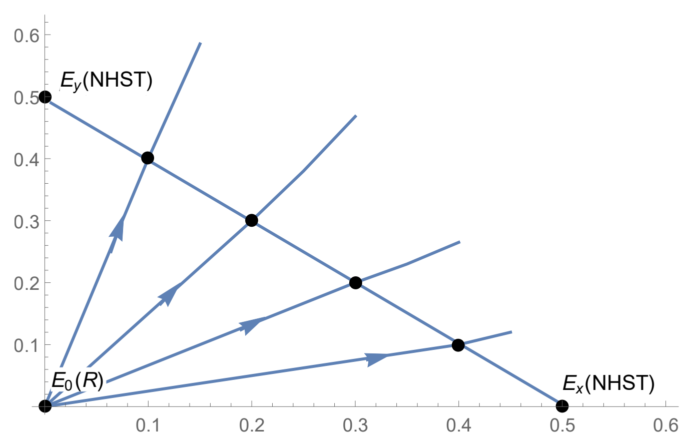

Figure 4. Graph of Case (k)

Proof.

(a) As E_y is non-hyperbolic of the stable type, there exists a stable manifold $W^s(E_y)$ by Theorems 23 and 24, which in this case will be the y-axis. Any point on the y-axis will converge to E_y. As E_x is locally asymptotically stable, an initial point on the x-axis will converge to E_x. Suppose there exist an initial point $B = (x_0, y_0) \in \text{ int } \mathcal{R}$. There will exist two projections of B, $P_x = (x_0, 0)$ onto the x-axis and $P_y = (0, y_0)$ onto the y-axis such that $P_y \preceq_{se} B \preceq_{se} P_x$. By monotonicity this implies

$$T^n(P_y) \preceq_{se} T^n(B) \preceq_{se} T^n(P_x).$$

Furthermore, taking the limits we have

$$\lim_{n\to\infty} T^n(P_y) \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} \lim_{n\to\infty} T^n(P_x),$$

which implies that

$$E_y \preceq_{se} \lim_{n\to\infty} T^n(B) \preceq_{se} E_x.$$

This step was obtained using the fact that E_x is locally asymptotically stable so $\lim_{n\to\infty} T^n(P_x) = E_x$ and the y-axis is the stable manifold $W^s(E_y)$ so that $\lim_{n\to\infty} T^n(P_y) = E_y$. We can conclude that $\lim_{n\to\infty} T^n(B) = E_x$ as the stable manifold of E_y is unique and B does not begin on it.

(b) This case is analogous to case (a) where in this case we use the stable manifold of E_x, and E_y is now locally asymptotically stable.

(c) This proof is analogous to case (a) of Theorem 30 where instead of claiming E_y is locally asymptotically stable, E_y is now non-hyperbolic of the stable type. We can instead use the fact that the y-axis is the stable manifold $W^s(E_y)$ of E_y to proceed with proof using the same technique.

(d) This proof is analogous to the proof of case (b) of Theorem 30 where instead of claiming E_x is locally asymptotically stable, E_x is now non-hyperbolic of the stable type. We can instead use the fact that the x-axis is the stable manifold $W^s(E_x)$ of E_x to proceed with proof using the same technique.

(e) As E_x and E_y are non-hyperbolic of the stable type, there exist stable manifold $W^s(E_x)$, which is the x-axis, and $W^s(E_y)$, which is the y axis respectively in this case by Theorems 23 and 24. Any point on the y-axis without E_0 will converge to E_y and any point on the x-axis without E_0 will converge to the E_x. For both E_x and E_y there will exist a center manifold. This manifold can be used to show all solutions in the interior of $\mathcal{R}$ will either converge to E_x or E_y.

(f) By Theorem 29, there exist two curves C_1 and C_2, where $C_2 \preceq_{se} C_1$ that are continuous and non-decreasing with an endpoint at E_0. The curve C_1 is the boundary of the basin of attraction of a point at infinity, and the curve C_2 is the boundary of the basin of attraction of E_y. This proof is analogous to case (e) of Theorem 30 when the two curves C_1 and C_2 coincide or $C_2 \notin \mathcal{R}$ where instead of claiming E_0 is a saddle point, E_0 is now non-hyperbolic of the unstable type. We can instead use the curve C_1, that is a stable manifold $W^s(E_0)$ of E_0 to proceed with proof using the same technique.

If both $C_1, C_2 \in \mathcal{R}$, then, any points on C_1 and C_2 converge to E_0. Suppose there exists a point $B_0 = (x_0, y_0) \in \mathcal{R}$ and to the right of C_2. There there will exist two projections of B, P_2 onto the curve C_2 and P_1 onto the curve C_1 such that $P_2 \preceq_{se} B_0 \preceq_{se} P_1$. By monotonicity this implies

$$T^n(P_2) \preceq_{se} T^n(B_0) \preceq_{se} T^n(P_1),$$

which once the limits are taken becomes

$$\lim_{n\to\infty} T^n(P_2) \preceq_{se} \lim_{n\to\infty} T^n(B_0) \preceq_{se} \lim_{n\to\infty} T^n(P_1),$$

and furthermore

$$E_0 \preceq_{se} T^n(B_0) \preceq_{se} E_0.$$

The last inequalities were obtained using the fact that P_1 is on C_1 so that $\lim_{n\to\infty} T^n(P_1) = E_0$ and P_2 is on C_2 so that $\lim_{n\to\infty} T^n(P_2) = E_0$. Therefore, $\lim_{n\to\infty} T^n(B_0) = E_0$.

Next suppose that there exists a point $B_0 = (x_0, y_0) \in \mathcal{R}$ and to the left of C_2. Then B_0 is in the region of the basin of attraction to E_y. As B_0 does not begin on C_1 or C_2, then the point will converge to E_y.

(g) This proof is analogous to case (f) of Theorem 30 when the two curves C_1 and C_2 coincide or if $C_2 \notin \mathcal{R}$ where instead of claiming E_0 is a saddle point, E_0 is now non-hyperbolic of the unstable type. We can instead use the curve C_1, that is the stable manifold $W^s(E_0)$ of E_0 to proceed with proof using the same technique. When the two curves C_1 and C_2 do not coincide, the proof will be analogous to case (f) given above.

(h) Assume that the map T has one fixed point of $E_0 = (0, 0)$ which is non-hyperbolic. As E_0 is the only fixed point on $\mathcal{R}$ and there are no minimal period two all solutions must converge to E_0 by Theorem 27, that is $\lim_{n\to\infty} T^n(B) = E_0$ for an initial point $B = (x_0, y_0)$.

(i) For the fixed points suppose both $E_x = (x, 0)$ and $E_y = (0, y)$ are non hyperbolic of the stable type, $E_0 = (0, 0)$ is a repeller or singular point, and there exist an infinite number of interior equilibrium points on the rectangular region $\mathcal{R}$. As the map T is strongly competitive, has a C^1 extension, and for each interior equilibrium point conditions a and b of Theorem 23 are satisfied, then by Theorem 23 there is a stable manifold as its basin of attraction for each of the equilibrium points. Each stable manifold will have an endpoint at E_0 and they are graphs of continuous and non-decreasing functions where the points depend continuously on the initial point (x_0, y_0).

Remark 5 *Theorems 30 and 31 can be generalized. Instead of considering* $E_0 = (0,0)$*,* $E_x = (x,0)$*, and* $E_y = (0,y)$*, you can instead consider a rectangular region* $\mathcal{R}$ *where* E_0 *is a fixed point on the bottom left corner of the boundary,* E_x *is a fixed point on the bottom boundary, and* E_y *is a fixed point on the left boundary of* $\mathcal{R}$ *where the points are not necessarily on the axes.*

4.3 Example 1

We will investigate the global dynamics of system (43): where the parameters a, b, c and d are positive numbers and $0 < \alpha, \beta < 1$.

4.3.1 Local Stability Results

To begin let us find the local stability results of system (43). Additionally, we will prove that the $(\mathcal{O}^+)$ condition is satisfied as well as the fact that system (43) is bounded, which will help in proving the global results.

Lemma 7 *The following holds true for system* (43) *where* $\alpha, \beta \in (0,1)$*:*

(a) $E_0 = (0,0)$ *is always an equilibrium point.*

(b) If $d > b$*, then* $E_y = \left(0, \frac{d-b}{d}\right)$ *is an equilibrium point.*

(c) If $c > a$*, then* $E_x = \left(\frac{c-a}{c}, 0\right)$ *is an equilibrium point.*

(d) If $cd > 1$ *and both* $d(1-c+a) < b$ *and* $c(1+b-d) < a$ *hold or* $cd < 1$ *and both* $d(1-c+a) > b$ *and* $c(1+b-d) > a$ *hold, then* $E_+ = \left(\frac{d(1-c+a)-b}{1-cd}, \frac{c(1+b-d)-a}{1-cd}\right)$ *is an equilibrium point.*

Proof. The equilibrium points satisfy

$$\overline{x} = \alpha\overline{x} + (1-\alpha)\frac{c\overline{x}}{a+c\overline{x}+\overline{y}} \quad \text{and} \quad \overline{y} = \beta\overline{y} + (1-\beta)\frac{d\overline{y}}{b+\overline{x}+d\overline{y}}.$$

Clearly, one of the equilibrium points is always $E_0 = (0,0)$. Suppose that $\overline{x} = 0$ and $\overline{y} \neq 0$. Then $\overline{y} = \frac{d-b}{d}$, and so we get the equilibrium point $E_y = (0, \frac{d-b}{d})$ when $d > b$. Next suppose that $\overline{x} \neq 0$ and $\overline{y} = 0$. It follows that $\overline{x} = \frac{c-a}{c}$, which shows that there is the equilibrium point of $E_x = (\frac{c-a}{c}, 0)$ providing that $c > a$. Finally, assume that $\overline{x}, \overline{y} \neq 0$. Straightforward calculation yields that $\overline{x} = \frac{d(1-c+a)-b}{1-cd}$ and $\overline{y} = \frac{c(1+b-d)-a}{1-cd}$. Therefore, the interior point of $E_+ = (\frac{d(1-c+a)-b}{1-cd}, \frac{c(1+b-d)-a}{1-cd})$ exist when both $\frac{d(1-c+a)-b}{1-cd} > 0$ and $\frac{c(1+b-d)-a}{1-cd} > 0$ holds.

To find the local stability of each equilibrium point we find the Jacobian matrix. The map corresponding to system (43) is $\mathcal{T}(u,v) = (f(u,v), g(u,v))$ where $f(u,v) = \alpha u + (1-\alpha)\frac{cu}{a+cu+v}$ and $g(u,v) = \beta v + (1-\beta)\frac{dv}{b+u+dv}$. The Jacobian matrix of $\mathcal{T}$ is

$$J(u,v) = \begin{bmatrix} \alpha + (1-\alpha)\frac{(a+cu+v)c-c^2u}{(a+cu+v)^2} & (\alpha-1)\frac{cu}{(a+cu+v)^2} \\ (\beta-1)\frac{dv}{(b+u+dv)^2} & \beta + (1-\beta)\frac{(b+u+dv)d-d^2v}{(b+u+dv)^2} \end{bmatrix}.$$

First, the Jacobian matrix evaluated at the equilibrium point $E_0 = (0,0)$ gives

$$J(0,0) = \begin{bmatrix} \frac{c+\alpha(a-c)}{a} & 0 \\ 0 & \frac{d+\beta(b-d)}{b} \end{bmatrix},$$

with the eigenvalues of $\lambda_1 = \frac{c+\alpha(a-c)}{a}$ and $\lambda_2 = \frac{d+\beta(b-d)}{b}$.

Lemma 8 *The equilibrium point $E_0 = (0,0)$ is*

(a) locally asymptotically stable if $a > c$ and $b > d$.

(b) repeller if $a < c$ and $b < d$.

(c) saddle point if $a < c$ and $b > d$ or $a > c$ and $b < d$.

(d) non-hyperbolic if $a = c$ or $b = d$.

Proof. The results follow from the eigenvalues. When both eigenvalues lie within the unit circle, the equilibrium point will be locally asymptotically stable. Note that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ when

$$-a < c + \alpha(a-c) < a \quad \Leftrightarrow \quad -(a+c) < \alpha(a-c) < (a-c),$$
$$-b < d + \beta(b-d) < b \quad \Leftrightarrow \quad -(b+d) < \beta(b-d) < (b-d).$$

This will hold true when $a > c$ and $b > d$. When both eigenvalues lie outside the unit circle, the equilibrium point will be a repeller. So $|\lambda_1| > 1$ and $|\lambda_2| > 1$ when

$$-a > c + \alpha(a-c) > a \quad \Leftrightarrow \quad -(a+c) > \alpha(a-c) > a-c,$$
$$-b > d + \beta(b-d) > b \quad \Leftrightarrow \quad -(b+d) > \beta(b-d) > b-d.$$

This will hold true when $a < c$ and $b < d$. Note that in this case $\lambda_1 > 1$ and $\lambda_2 > 1$, as $\alpha, \beta < 1$. It cannot hold that $-(a+c) > \alpha(a-c)$ and $-(b+d) > \beta(b-d)$. When one eigenvalue lies outside the unit circle and the other within the unit circle the equilibrium point will be a saddle point. Based on the previous calculations this will occur when $a < c$ and $b > d$ or $a > c$ and $b < d$. Finally, when at least one eigenvalue lies on the unit circle, the equilibrium point is non-hyperbolic. So $|\lambda_1| = 1$ or $|\lambda_2| = 1$ respectively when

$$-a = c + \alpha(a-c) = a \quad \Leftrightarrow \quad -(a+c) = \alpha(a-c) = (a-c),$$
$$-b = d + \beta(b-d) = b \quad \Leftrightarrow \quad -(b+d) = \beta(b-d) = (b-d).$$

This will hold true when $a = c$ or $b = d$. Note that as $a, b, c, d > 0$ then $\lambda_1 = 1$ or $\lambda_2 = 1$. It cannot happen that either is equal to -1 as $-(a+c) = \alpha(a-c)$ and $-(b+d) = \beta(b-d)$ cannot hold true.

Next we evaluate the Jacobian matrix at the point of $E_y = (0, \frac{d-b}{d})$:

$$J\left(0, \frac{d-b}{d}\right) = \begin{bmatrix} \frac{cd+\alpha(-b+d+ad-cd)}{-b+d+ad} & 0 \\ -(1-\beta)\frac{(d-b)}{d^2} & \frac{b+\beta(d-b)}{d} \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = \frac{cd+\alpha(-b+d+ad-cd)}{-b+d+ad}$ and $\lambda_2 = \frac{b+\beta(d-b)}{d}$.

Lemma 9 *When $d > b$ the equilibrium point of $E_y = \left(0, \frac{d-b}{d}\right)$ is*

(a) locally asymptotically stable if $b < d(a + 1 - c)$.

(b) saddle point if $b > d(a + 1 - c)$.

(c) non-hyperbolic of the stable type if $b = d(a + 1 - c)$.

Proof. The results follow from the eigenvalues. As $d > b$, then we can conclude $|\lambda_2| < 1$ will always hold true as

$$-d < b + \beta(d - b) < d \Leftrightarrow -(d + b) < \beta(d - b) < (d - b) \Leftrightarrow d > b.$$

When both eigenvalues lie within the unit circle, the equilibrium point will be locally asymptotically stable. Note that $|\lambda_1| < 1$ when

$$-(-b + d + ad) < cd + \alpha(-b + d + ad - cd) < (-b + d + ad) \Leftrightarrow$$
$$-(-b + d + ad + cd) < \alpha(-b + d + ad - cd) < (-b + d + ad - cd).$$

This will hold true when $d(a+1-c) > b$. When one eigenvalue lies outside the unit circle and the other within the unit circle the equilibrium point will be a saddle point. Note that $|\lambda_1| > 1$ when

$$-(-b + d + ad) > cd + \alpha(-b + d + ad - cd) > (-b + d + ad) \Leftrightarrow$$
$$-(-b + d + ad + cd) > \alpha(-b + d + ad - cd) > (-b + d + ad - cd).$$

This will hold true when $d(a + 1 - c) < b$. Note again that as $\alpha < 1$, then $-(-b + d + ad + cd) > \alpha(-b + d + ad - cd)$ cannot hold resulting in the fact that $\lambda_2 > 1$. Finally, when at least one eigenvalue (in this case $|\lambda_2| = 1$) lies on the unit circle, the equilibrium point is non-hyperbolic. This will happen when

$$-(-b + d + ad) = cd + \alpha(-b + d + ad - cd) = (-b + d + ad) \Leftrightarrow$$
$$-(-b + d + ad + cd) = \alpha(-b + d + ad - cd) = (-b + d + ad - cd).$$

This holds true when $d(a+1-c) = b$. As $a, b, c, d > 0$, then $-(-b+d+ad+cd) = \alpha(-b+d+ad-cd)$ does not hold true resulting in the fact that λ_2 cannot be -1.

Next we evaluate Jacobian matrix at the equilibrium point $E_x = (\frac{c-a}{c}, 0)$.

$$J\left(\frac{c-a}{c}, 0\right) = \begin{bmatrix} \frac{a+\alpha(c-a)}{c} & -(1-\alpha)\frac{(c-a)}{c^2} \\ 0 & \frac{cd+\beta(bc+c-a-cd)}{bc+c-a} \end{bmatrix}.$$

It follows that the eigenvalues will be $\lambda_1 = \frac{a+\alpha(c-a)}{c}$ and $\lambda_2 = \frac{cd+\beta(bc+c-a-cd)}{bc+c-a}$.

Lemma 10 *When $c > a$, the equilibrium point of $E_x = \left(\frac{c-a}{c}, 0\right)$ will be*

(a) locally asymptotically stable if $a < c(b+1-d)$.

(b) saddle point if $a > c(b+1-d)$.

(c) non-hyperbolic of the stable type if $a = c(b+1-d)$.

Proof. The results follow from the eigenvalues. As a reminder, $c > a$ so that $|\lambda_1| < 1$ will always be true. Indeed this result holds as

$$-c < a + \alpha(c-a) < c \Leftrightarrow -(c+a) < \beta(c-a) < (c-a) \Leftrightarrow c > a.$$

When both eigenvalues lie within the unit circle, the equilibrium point will be locally asymptotically stable. Note that $|\lambda_2| < 1$ when

$$-(bc+c-a) < cd + \beta(bc+c-a-cd) < (bc+c-a) \Leftrightarrow$$
$$-(bc+c-a+cd) < \beta(bc+c-a-cd) < (bc+c-a-cd).$$

The inequalities hold true when $c(b+1-d) > a$. When one eigenvalue lies outside the unit circle and the other within the unit circle the equilibrium point will be a saddle point. Note that $|\lambda_2| > 1$ when

$$-(bc+c-a) > cd + \beta(bc+c-a-cd) > (bc+c-a) \Leftrightarrow$$
$$-(bc+c-a+cd) > \beta(bc+c-a-cd) > (bc+c-a-cd).$$

This holds when $c(b+1-d) < a$. Note that as $\beta < 1$, then $-(bc+c-a+cd) > \beta(bc+c-a-cd)$ cannot hold. So $\lambda_2 > 1$ in this case. Finally, when at least one eigenvalue (in this case $|\lambda_2| = 1$) lies on the unit circle, the equilibrium point is non-hyperbolic. This will happen when

$$-(bc+c-a) = cd + \beta(bc+c-a-cd) = (bc+c-a) \Leftrightarrow$$
$$-(bc+c-a+cd) = \beta(bc+c-a-cd) = (bc+c-a-cd).$$

This holds when $c(b+1-d) = a$. Note that as $a, b, c, d > 0$, then $-(bc+c-a+cd) = \beta(bc+c-a-cd)$ cannot hold so $\lambda_2 = 1$ in this case.

Next we prove that system (43) is bounded and furthermore, is decreasing when $c \leq a$ and $d \leq b$.

Lemma 11 *The solutions of system* (43) *are bounded. In addition,* $\{x_n\}$ *is decreasing when* $c \leq a$ *and* $\{y_n\}$ *is decreasing when* $d \leq b$.

Proof. Note that

$$x_{n+1} = \alpha x_n + (1-\alpha)\frac{cx_n}{a + cx_n + y_n} \leq \alpha x_n + (1-\alpha) \quad (49)$$

$$y_{n+1} = \beta y_n + (1-\beta)\frac{dy_n}{b + x_n + dy_n} \leq \beta y_n + (1-\beta). \quad (50)$$

Let $x_n \leq u_n$ and $y_n \leq v_n$ where $u_{n+1} = \alpha u_n + 1 - \alpha$ and $v_{n+1} = \beta v_n + 1 - \beta$. Assume the initial conditions of $x_0 \leq u_0$ and $y_0 \leq v_0$ hold. When iterated u_n and v_n become

$$u_n = (u_0 - 1)\alpha^n + 1 \leq (u_0 - 1) + 1 = u_0$$
$$v_n = (v_0 - 1)\beta^n + 1 \leq (v_0 - 1) + 1 = v_0.$$

Therefore, we have that $x_n \leq u_n \leq u_0$ and $y_n \leq v_n \leq v_0$. Next, we want to see when $x_{n+1} \leq x_n$ and when $y_{n+1} \leq y_n$ holds. By reexamining (49) we see that

$$x_{n+1} = \alpha x_n + (1-\alpha)\frac{cx_n}{a + cx_n + y_n} \leq \alpha x_n + (1-\alpha)\frac{cx_n}{a} = (\alpha + (1-\alpha)\frac{c}{a})x_n.$$

Thus $x_{n+1} \leq x_n$ when $(\alpha + (1-\alpha)\frac{c}{a}) \leq 1$. This will happen when $\frac{c}{a} \leq 1$ that is $c \leq a$. Similarly, by rewriting (50) we see that

$$y_{n+1} = \beta y_n + (1-\beta)\frac{dy_n}{b + x_n + dy_n} \leq (\beta - (1-\beta)\frac{d}{b})y_n.$$

Then $y_{n+1} \leq y_n$ when $(\beta - (1-\beta)\frac{d}{b}) \leq 1$. This will hold true when $d \leq b$.

Next, we will use an alternative method to prove the local stability in two cases for the equilibrium point $E_+ = \left(\frac{d(1-c+a)-b}{1-cd}, \frac{c(1+b-d)-a}{1-cd}\right)$.

Lemma 12 *Let $b < d(1+a-c)$, $a < c(1+b-d)$, $c > a$, $d > b$, and $cd < 1$. Furthermore, suppose that E_x and E_y are locally asymptotically stable while E_0 is a repeller. Then $E_y \preceq_{se} E_+ \preceq_{se} E_x$ and E_+ will either be non-hyperbolic of the unstable type or a saddle point.*

Proof. First note that

$$\begin{aligned}
\frac{d-b}{d} > \frac{c(1+b-d)-a}{1-cd} \Leftrightarrow \\
(d-b)(1-cd) > d(c(1+b-d)-a) \Leftrightarrow \\
d - cd^2 - b + bcd > dc + bcd - cd^2 - ad \Leftrightarrow \\
d(1+a-c) > b,
\end{aligned}$$

and

$$\begin{aligned}
\frac{c-a}{c} > \frac{d(1-c+a)-b}{1-cd} \Leftrightarrow \\
(c-a)(1-cd) > c(d(1-c+a)-b) \Leftrightarrow \\
c - c^2 d - a + acd > cd - c^2 d + acd - cb \Leftrightarrow \\
c(1+b-d) > a,
\end{aligned}$$

which implies that $E_y \preceq_{se} E_+ \preceq_{se} E_x$. By Corollary 3, $\text{int}[\![E_y, E_+]\!]$ is a subset of the basin of attraction of either E_y or E_+. Since the equilibrium point E_y is locally asymptotically stable, the interior of $[\![E_y, E_+]\!]$ is a subset of the basin of attraction of E_y. This means that E_+ cannot be locally asymptotically stable. By Theorem 29, there exists an invariant, north-east strongly linearly ordered curve $\mathcal{C}_x$ that is the boundary of the basin of attraction for E_x with an endpoint of E_0 that passes through E_+, and an invariant, north-east strongly linearly ordered curve $\mathcal{C}_y$ that is the boundary of the basin of attraction for E_y with an endpoint of E_0 that passes through E_+ where $\mathcal{C}_x$ may coincide with $\mathcal{C}_y$. Any points on the curves $\mathcal{C}_x$ and $\mathcal{C}_y$ will be attracted to E_+ as they are on the boundary, E_0 is a repeller, and by the theorem cannot cross over to the other boundary. If $\mathcal{C}_y$ and $\mathcal{C}_x$ do not coincide, we can suppose that there exists a point $B_0 = (x, y)$ that is in the region between $\mathcal{C}_x$ and $\mathcal{C}_y$. Additionally, there exists the points of $B_1 \in \mathcal{C}_y$ and $B_2 \in \mathcal{C}_x$ such that $B_1 \preceq_{se} B_0 \preceq_{se} B_2$. This implies that $T^n(B_1) \preceq_{se} T^n(B_0) \preceq_{se} T^n(B_2)$, and furthermore

$$\lim_{n\to\infty} T^n(B_1) \preceq_{se} \lim_{n\to\infty} T^n(B_0) \preceq_{se} \lim_{n\to\infty} T^n(B_2).$$

However, as $\lim_{n\to\infty} T^n(B_1) = \lim_{n\to\infty} T^n(B_2) = E_+$, we conclude that $\lim_{n\to\infty} T^n(B_0) = E_+$. As E_+ attracts some points, E_+ cannot be a repeller. Thus, E_+ is either a saddle point or a non-hyperbolic point. Based on *Mathematica* calculations the eigenvalues of E_+ cannot be 1, however, one eigenvalue can potentially be -1. The other eigenvalue will always be greater than 1. If E_+ exists, it will be non-hyperbolic of the unstable type or a saddle point.

The following lemma will give results regarding the spectral radius based on the slopes of the tangent lines at E_+. Let $\rho(J)$ be the spectral radius of $J(E_+)$.

Lemma 13 *Suppose the tangent lines to $f(x, y) = x$ and $g(x, y) = y$ at E_+ are not parallel to one of the axes. Let m_1 and m_2 be the slopes of the tangent lines respectively. The following holds true:*

(i) If $m_1 - m_2 > 0$, then $\rho(J) > 1$.

(ii) If $m_1 - m_2 = 0$, then $\rho(J) = 1$.

(iii) If $m_1 - m_2 < 0$, then $\rho(J) < 1$.

Proof. For the proof let the equilibrium E_+ be represented as $(\bar{x}, \bar{y})$. Note the Jacobian matrix of $J(E_+)$ can be rewritten as

$$J(E_+) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}.$$

The Taylor expansion of the map $T(x, y)$ is

$$(x, y) = T(x, y) = (\bar{x}, \bar{y}) + J \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + \ldots.$$

We will consider the linear part to find the slope of the tangent lines. Rewritten this becomes

$$x - \bar{x} = f_x(\bar{x}, \bar{y})(x - \bar{x}) + f_y(\bar{x}, \bar{y})(y - \bar{y})$$
$$y - \bar{y} = g_x(\bar{x}, \bar{y})(x - \bar{x}) + g_y(\bar{x}, \bar{y})(y - \bar{y}).$$

Let $\Delta x = x - \bar{x}$ and $\Delta y = y - \bar{y}$. Substituting this in we have

$$\Delta x = f_x(\bar{x}, \bar{y})\Delta x + f_y(\bar{x}, \bar{y})\Delta y$$
$$\Delta y = g_x(\bar{x}, \bar{y})\Delta x + g_y(\bar{x}, \bar{y})\Delta y.$$

From the two equations we have that

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{1 - f_x}{f_y}$$
$$m_2 = \frac{\Delta y}{\Delta x} = \frac{g_x}{1 - g_y}.$$

As the map is competitive, then $f_x, g_y > 0$, $f_y, g_x < 0$, as well as $m_1, m_2 < 0$. This implies that $f_x, g_y < 1$. The term $m_1 - m_2$ is equivalent to

$$m_1 - m_2 = \frac{1 - f_x}{f_y} - \frac{g_x}{1 - g_y} = \frac{1 - g_y - f_x + f_x g_y - g_x f_y}{f_y(1 - g_y)} = \frac{p(1)}{f_y(1 - g_y)}.$$

Note the characteristic polynomial is $p(\lambda) = \lambda^2 - (f_x + g_y)\lambda + (f_x g_y - f_y g_x)$, and $p(1) = 1-(f_x+g_y)+f_x g_y - g_x f_y$. As $f_y(1-g_y) < 0$, then $m_1 - m_2$ will either be less than, greater than, or equal to zero based on $p(1)$. The characteristic polynomial at 1 is equivalent to $p(1) = 1 - \mathrm{tr}(J) + \det(J)$. Then we have that $p(1) > 0$ when $\rho(J) < 1$, $p(1) < 0$ when $\rho(J) > 1$, and $p(1) = 0$ when $\rho(J) = 1$.

For system (43), $x = f(x, y)$ means that

$$x = \alpha x + (1-\alpha)\frac{cx}{a + cx + y} \Leftrightarrow y = -cx + (c - a)$$

and $y = g(x, y)$ means that

$$y = \beta y + (1-\beta)\frac{dy}{b + x + dy} \Leftrightarrow y = -\frac{1}{d}x + (1 - \frac{b}{d}).$$

The criteria of $m_1 - m_2$ is equivalent here to $-c - \frac{-1}{d} = -c + \frac{1}{d}$. This results that $m_1 - m_2 < 0$ when $1 < cd$, $m_1 - m_2 > 0$ when $1 > cd$, and $m_1 - m_2 = 0$ when $1 = cd$.

Lemma 14 *Let $b > d(1 + a - c)$, $a > c(1 + b - d)$, $c > a$, $d > b$, and $cd > 1$. Furthermore, suppose that E_x and E_y are both saddle points while E_0 is a repeller. Then $E_y \preceq_{se} E_+ \preceq_{se} E_x$ and E_+ will be locally asymptotically stable.*

Proof. First note that

$$\begin{aligned} \frac{d-b}{d} > \frac{a - c(1 + b - d)}{cd - 1} &\Leftrightarrow \\ (d - b)(cd - 1) > d(a - c(1 + b - d)) &\Leftrightarrow \\ cd^2 - d - bcd + b > da - cd - bcd + cd^2 &\Leftrightarrow \\ d(1 + a - c) < b&, \end{aligned}$$

and

$$\begin{aligned}
\frac{c-a}{c} > \frac{b-d(1-c+a)}{cd-1} &\Leftrightarrow \\
(c-a)(cd-1) > c(b-d(1-c+a)) &\Leftrightarrow \\
c^2d - c - acd + a > cb - cd + c^2d - acd &\Leftrightarrow \\
c(1+b-d) < a&,
\end{aligned}$$

which implies that $E_y \preceq_{se} E_+ \preceq_{se} E_x$. As E_x and E_y are both saddle points, there exists global stable manifolds, $W^s(E_x)$ and $W^s(E_y)$, and global unstable manifolds, $W^u(E_x)$ and $W^u(E_y)$, by Theorems 23, 24, 25, and 26. The endpoint of the unstable manifolds of $W^u(E_x)$ and $W^u(E_y)$ will be E_+. The stable manifold of $W^s(E_y)$ will be the y-axis and the stable manifold of $W^s(E_x)$ will be the x-axis. So E_+ is clearly not a repeller. Suppose that E_+ is a saddle point and let $\mathcal{R}_{sp} = [\![E_y, E_+]\!]$. As E_+ is a saddle point, by Theorems 23-26 there exists an unstable manifold $W^u(E_+)$. However, this causes a contradiction as the curve conflicts with the unstable manifold $W^u(E_y)$. So E_+ cannot be a saddle point. By Lemma (13) $\rho(J) < 1$ and therefore, E_+ will not be non-hyperbolic and furthermore, must be locally asymptotically stable.

Next we will prove that the $(\mathcal{O}^+)$ condition holds. The $(\mathcal{O}^+)$ condition tells us that there will be no minimal period two solutions.

Lemma 15 *System* (43) *satisfies the* $(\mathcal{O}^+)$ *condition.*

Proof. It suffices to prove that if $\mathcal{T}(x_1, y_1) \leq \mathcal{T}(x_2, y_2)$, then $x_1 \leq x_2$ and $y_1 \leq y_2$, where $\mathcal{T}(x, y)$ is a map associated with system (43). The condition $\mathcal{T}(x_1, y_1) \leq \mathcal{T}(x_2, y_2)$ is equivalent to

$$\begin{aligned}
\alpha x_1 + (1-\alpha)\frac{cx_1}{a + cx_1 + y_1} &\leq \alpha x_2 + (1-\alpha)\frac{cx_2}{a + cx_2 + y_2} \\
\beta y_1 + (1-\beta)\frac{dy_1}{b + x_1 + dy_1} &\leq \beta y_2 + (1-\beta)\frac{dy_2}{b + x_2 + dy_2},
\end{aligned}$$

which can be reduced to

$$\alpha(x_1 - x_2)(a + cx_1 + y_1)(a + cx_2 + y_2) \leq (1 - \alpha)(ac(x_2 - x_1) + c(x_2y_1 - x_1y_2)) \quad (51)$$

$$\beta(y_1 - y_2)(b + x_1 + dy_1)(b + x_2 + dy_2) \leq (1 - \beta)(bd(y_2 - y_1) + d(x_1y_2 - x_2y_1)). \quad (52)$$

We know that either $x_1 \leq x_2$ or $x_1 > x_2$. Suppose that $x_1 > x_2$. By (51), as $\alpha(x_1 - x_2)(a + cx_1 + y_1)(a + cx_2 + y_2) > 0$ then $ac(x_2 - x_1) + c(x_2y_1 - x_1y_2) > 0$. This implies that $c(x_2y_1 - x_1y_2) > ac(x_1 - x_2) > 0$. As $c > 0$, then $x_2y_1 - x_1y_2 > 0$. Since $x_1 > x_2$, then $y_1 > y_2$. Using the fact that $y_1 > y_2$ and (52), $(1 - \beta)(bd(y_2 - y_1) + d(x_1y_2 - x_2y_1)) > 0$. So this implies that $(bd(y_2 - y_1) + d(x_1y_2 - x_2y_1)) > 0$. By reducing this we see that $(x_1y_2 - x_2y_1) > b(y_1 - y_2) > 0$. This is a contradiction as we stated $x_2y_1 - x_1y_2 > 0$ so that $x_1y_2 - x_2y_1 > 0$ cannot hold true. Therefore, $x_1 \leq x_2$ must hold.

Next we know either $y_1 \leq y_2$ or $y_1 > y_2$. Suppose that $y_1 > y_2$. By (52) as $\beta(y_1-y_2)(b+x_1+dy_1)(b+x_2+dy_2) > 0$, then $(1-\beta)(bd(y_2-y_1)+d(x_1y_2-x_2y_1)) > 0$. As $\beta < 1$, then $(bd(y_2-y_1)+d(x_1y_2-x_2y_1)) > 0$. With some reduction this implies $(x_1y_2-x_2y_1)) > b(y_1-y_2) > 0$. Therefore, $(x_1y_2-x_2y_1) > 0$ and moreover $x_1 > x_2$. Using (51) and the fact that $x_1 > x_2$, then $c(x_2y_1 - x_1y_2) > ac(x_1 - x_2) > 0$. This implies that $x_2y_1 - x_1y_2 > 0$. However, this is a contradiction as we already stated that $x_1y_2 - x_2y_1 > 0$. Therefore, $y_1 \leq y_2$.

4.3.2 Global Stability Results

In this section we will compile the local stability results and use both Theorems 30 and 31 to give conclusions regarding the global dynamics of system (43).

Lemma 16 *The following hold:*

(a) If $c > a > 0$, $d > b > 0$, $b < d(1 + a - c)$, *and* $a < c(1 + b - d)$, *then* $cd < 1$.

(b) *If* $c > a > 0$, $d > b > 0$, $b > d(1 + a - c)$, *and* $a > c(1 + b - d)$, *then* $cd > 1$.

Proof.

(a) Rewriting the two inequalities give us

$$d(c - a) < d - b \text{ and } c(d - b) < c - a.$$

This implies that $cd(c - a) < c - a$. As $c > a$ this inequality can be reduced to $cd < 1$.

(b) Rewriting the two inequalities give us

$$.d(c - a) > d - b \text{ and } c(d - b) > c - a$$

This implies that $cd(c - a) > c - a$. As $c > a$ this inequality can be reduced to $cd > 1$.

Lemma 17 *The equilibrium point* E_+ *will not exist when*

(a) $c > a > 0$, $d > b > 0$, $b > d(1 + a - c)$, *and* $a < c(1 + b - d)$.

(b) $c > a > 0$, $d > b > 0$, $b < d(1 + a - c)$, *and* $a > c(1 + b - d)$.

(c) $c > a > 0$, $d > b > 0$, *and* $a = c(1 + b - d)$.

(d) $c > a > 0$, $d > b > 0$, *and* $b = d(1 + a - c)$.

(e) $c > a > 0$ *and* $b > d > 0$.

(f) $a > c > 0$ *and* $d > b > 0$.

(g) $0 < c < a$ *and* $0 < d < b$.

(h) $a = c$.

(i) $d = b$.

Proof.

(a) Clearly E_+ cannot exist by Lemma 7 as either $d(1-c+a) < b$ and $c(1+b-d) < a$ or $d(1-c+a) > b$ and $c(1+b-d) > a$ must hold.

(b) For the same reasons as case (a) clearly E_+ cannot exist.

(c) As $a = c(1+b-d)$, then E_+ can be reduced to

$$\left(\frac{d(1-c+a)-b}{1-cd}, 0\right).$$

However, $\frac{d(1-c+a)-b}{1-cd} = \frac{c-a}{c}$. Note that this can be reduced to $cd(1-c+a) - bc = (c-a)(1-cd)$, and then further to $a = c(1+b-d)$. This results in $E_+ = E_x$.

(d) As $b = d(1+a-c)$, then E_+ can be reduced to

$$\left(0, \frac{c(1+b-d)-a}{1-cd}\right).$$

However, $\frac{c(1+b-d)-a}{1-cd} = \frac{d-b}{d}$. Note that this can be reduced to $dc(1+b-d) - ad = (d-b)(1-cd)$, and then further to $b = d(1+a-c)$. We can conclude $E_+ = E_y$.

(e) First suppose that $cd > 1$. Then

$$\frac{c(1+b-d)-a}{1-cd} > 0$$

holds when $c(1+b-d) < a$. This can be rewritten as $c(b-d) < a-c$, which is not true in this case as $c > a$ and $b > d$. Next suppose that $cd < 1$. Then,

$$\frac{d(1+a-c)-b}{1-cd} > 0$$

when $d(1+a-c) > b$. Note that the inequality can be rewritten as $d(a-c) > b-d$, which is false and therefore, E_+ does not exist.

(f) Suppose that $cd > 1$. Then

$$\frac{d(1+a-c)-b}{1-cd} > 0$$

when $d(1+a-c) < b$. This inequality does not hold true as $a > c$ and $d > b$ since the inequality can be rewritten as $d(a-c) < b-d$. Next suppose that $cd < 1$. Then,

$$\frac{c(1+b-d)-a}{1-cd} > 0$$

when $c(1+b-d)-a > 0$. This inequality can be rewritten as $c(b-d) > a-c$, and so will not hold. Therefore, E_+ does not exist.

(g) Let $cd > 1$. This implies that

$$d(1+a-c)-b < 0 \Leftrightarrow d(a-c) < b-d,$$

and

$$c(1+b-d)-a < 0 \Leftrightarrow c(b-d) < a-c.$$

This will provide a contradiction as this inequalities imply $cd(a-c) < c(b-d) < a-c$. However, this can be reduced to $cd < 1$ since $a > c$. Next suppose that $cd < 1$. This gives us that

$$d(1+a-c)-b > 0 \Leftrightarrow d(a-c) > b-d,$$

and

$$c(1+b-d)-a > 0 \Leftrightarrow c(b-d) > a-c.$$

Again combining the two inequalities, $cd(a-c) > c(b-d) > a-c$, which in turn can be reduced to $cd > 1$. This again provides a contradiction, and therefore E_+ cannot exist.

(h) As $a = c$ then E_+ can be reduced to

$$\left(\frac{d-b}{1-cd}, \frac{c(b-d)}{1-cd}\right).$$

Note that one coordinate will be negative or if $d = b$, then $E_+ = E_0$. Therefore, E_+ cannot exist.

(i) Similar to case (h) we can reduce E_+ to

$$\left(\frac{d(a-c)}{1-cd}, \frac{c-a}{1-cd}\right).$$

This point will therefore not exist as one coordinate will be negative or if $c = a$, then $E_+ = E_0$.

Theorem 32 *Consider system* (43)*, and let* $a, b, c, d > 0$*,* $0 < \alpha, \beta < 1$*, and* $x_0, y_0 \geq 0$.

(a) If $c > a > 0$*,* $d > b > 0$*,* $b < d(1 + a - c)$*,* $a < c(1 + b - d)$ *and* E_+ *is a saddle point, it follows that* E_0 *is a repeller, and both* E_x *and* E_y *are locally asymptotically stable such that* $E_y \preceq_{se} E_+ \preceq_{se} E_x$*. Every solution which begins off of the stable manifold* $\mathcal{W}^s(E_+)$ *to the right of the manifold converges to* E_x*, and to the left of the manifold converges to* E_y*. Every solution which begins on the stable manifold* $\mathcal{W}^s(E_+)$ *converges to* E_+*. Every solution which begins on the* x*-axis without* E_0 *converges to* E_x*, and every solution which begins on the* y*-axis without* E_0 *converges to* E_y*.*

(b) If $c > a > 0$*,* $d > b > 0$*,* $b > d(1 + a - c)$*,* $a > c(1 + b - d)$*, and* E_+ *is locally asymptotically stable, then* E_0 *is a repeller, and both* E_x *and* E_y *are saddle points such that* $E_y \preceq_{se} E_+ \preceq_{se} E_x$*. Every solution which begins off the* x *and* y *axes converges to* E_+*. Every solution which begins on the* x*-axis without* E_0 *converges to* E_x*, and every solution which begins on the* y*-axis without* E_0 *converges to* E_y*.*

(c) *If* $c > a > 0$, $d > b > 0$, $b > d(1 + a - c)$, *and* $a < c(1 + b - d)$, *then* E_y *is a saddle point,* E_x *is locally asymptotically stable,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the y-axis converges to* E_x*, and every solution which begins on the y-axis without* E_0 *converges to* E_y*.*

(d) *If* $c > a > 0$, $d > b > 0$, $b < d(1 + a - c)$, *and* $a > c(1 + b - d)$, *then* E_y *is locally asymptotically stable,* E_x *is a saddle point,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the x-axis converges to* E_y*, and every solution which begins on the x-axis without* E_0*, converges to* E_x*.*

(e) *Suppose that* $c > a > 0$ *and* $b > d > 0$*. The equilibrium point* E_0 *will be a saddle point, while* E_x *will be locally asymptotically stable,and both* E_y *and* E_+ *will not exist. Every solution which begins off the y-axis converges to* E_x*, and every solution which begins on the y-axis converges to* E_0*.*

(f) *Suppose that* $a > c > 0$ *and* $d > b > 0$*. The equilibrium point* E_0 *will be a saddle point, while* E_y *will be locally asymptotically stable, and both* E_x *and* E_+ *will not exist. Every solution which begins off the x-axis converges to* E_y*, and every solution which begins on the x-axis converges to* E_0*.*

(g) *Suppose that* $0 < c < a$ *and* $0 < d < b$*. The equilibrium points of* E_x, E_y*, and* E_+ *do not exist and* E_0 *is globally asymptotically stable.*

Proof. For each case the existence of the equilibrium points is given by Lemma 7. The local stability results of E_0 are given in Lemma 8, of E_y are given in Lemma 9, and of E_x are given in Lemma 10.

For cases (a) and (b), Lemma 16 establishes that $cd < 1$ or $cd > 1$ respectively, conditions necessary to give the existence of E_+. Additionally, for case a the local behavior of E_+ was proved in Lemma 12 and for case b the local behavior of E_+ was proved in Lemma 14. In all other cases E_+ will not exist by Lemma 17. Using

Lemma 15 the $(\mathcal{O}^+)$ condition holds resulting in the fact that there are no minimal period two solutions. As the solutions are bounded by Lemma 11, then Theorem 30 will give us the global results of all the cases $(a) - (g)$.

Theorem 33 *Consider system (43), and let* $a, b, c, d > 0$, $0 < \alpha, \beta < 1$, *and* $x_0, y_0 \geq 0$.

(a) If $c > a > 0$, $d > b > 0$, $b = d(1 + a - c)$ *and* $a < c(1 + b - d)$, *then* E_y *is non-hyperbolic of the stable type,* E_x *is locally asymptotically stable,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the y-axis converges to* E_x, *and every solution which begins on the y-axis without* E_0 *converges to* E_y.

(b) If $c > a > 0$, $d > b > 0$, $b < d(1 + a - c)$ *and* $a = c(1 + b - d)$, *then* E_y *is a saddle point,* E_x *is non-hyperbolic point of the stable type,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the y-axis converges to* E_x, *and every solution which begins on the y-axis without* E_0 *converges to* E_y.

(c) If $c > a > 0$, $d > b > 0$, $b = d(1 + a - c)$ *and* $a > c(1 + b - d)$, *then* E_y *is non-hyperbolic of the stable type,* E_x *is a saddle point,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the x-axis converges to* E_y, *and every solution which begins on the x-axis without* E_0, *converges to* E_x.

(d) If $c > a > 0$, $d > b > 0$, $b > d(1 + a - c)$ *and* $a = c(1 + b - d)$, *then* E_y *is locally asymptotically stable,* E_x *is non-hyperbolic of the stable type,* E_0 *is a repeller, and* E_+ *does not exist. Every solution which begins off the x-axis converges to* E_y, *and every solution which begins on the x-axis without* E_0 *converges to* E_x.

(e) *If* $c > a > 0$, $d > b > 0$, $b = d(1 + a - c)$ *and* $a = c(1 + b - d)$, *then both* E_y *and* E_x *are non-hyperbolic of the stable type,* E_0 *is a repeller, and* E_+ *does not exist. Every solution on the* x*-axis without* E_0 *will converge to* E_x *and every solution on the* y*-axis without* E_0 *will converge to* E_y*. Every solution which begins off the* x *and* y *axis will converge to exactly one of* E_x *or* E_y*.*

(f) *Suppose that* $c = d = 1$ *and* $a = b$*. Then* $c > a > 0$ *and* $d > b > 0$, E_0 *will be a repeller,* E_x *and* E_y *will be non-hyperbolic of the stable type, and there will exist infinite number of solutions of the form* $E_K = \{(x, 1 - K - x) | 0 < x < 1 - K \text{ and } K = a = b\}$*. For each of the equilibrium points of the form* E_K*, there is a stable manifold* $W^s(E_K)$ *as its basins of attraction. All* $W^s(E_K)$ *have an end point at* E_0 *and they are graphs of continuous and non-decreasing functions. The equilibrium points* E_K *depends continuously on the initial point* (x_0, y_0)*.*

(g) *Suppose that* $c = a$ *and* $d > b > 0$*. The equilibrium point* E_0 *will be non-hyperbolic of the unstable type, while* E_y *will be locally asymptotically stable, and both* E_x *and* E_+ *will not exist. Then there will exists two curves,* C_1 *and* C_2, $C_2 \preceq_{se} C_1$ *that are continuous and non-decreasing with an endpoint at* E_0*. If the curves* C_1 *and* C_2 *coincide with each other or* C_2 *is not in the region, every solution which begins off the* x*-axis will converge to* E_y*. Every solution which begins on the* x*-axis will converge to* E_0*. If there exists both* C_1, C_2 *in the region then every solution to the left of* C_2 *will converge to* E_y *and every solution to the right of* C_2 *will converge to* E_0*.*

(h) *Suppose that* $c = a$ *and* $b > d > 0$ *or* $b = d$ *and* $a > c > 0$*. The equilibrium point* E_0 *will be non-hyperbolic of the stable type, while* E_y, E_x*, and* E_+ *will not exist. Every solution will converge to* E_0*.*

(i) *Suppose that $b = d$ and $c > a > 0$. The equilibrium point E_0 will be non-hyperbolic of the unstable type, while E_x is locally asymptotically stable, and both E_y and E_+ do not exist. Then there will exists two curves, C_1 and C_2, $C_2 \preceq_{se} C_1$ that are continuous and non-decreasing with an endpoint at E_0. If the curves C_1 and C_2 coincide with each other or C_2 is not in the region, every solution which begins off the y-axis will converge to E_x. Every solution which begins on the y-axis will converge to E_0. If there exists both C_1, C_2 in the region then every solution to the left of C_1 will converge to E_0 and every solution to the right of C_1 will converge to E_x.*

(j) *Suppose that $c = a$ and $b = d$. Then E_0 is non-hyperbolic of resonance type $(1, 1)$, and E_x, E_y, and E_+ will not exist. Every solution will converge to E_0.*

Proof. For each case the existence of the equilibrium points is given by Lemma 7. The local stability results of E_0 are given in Lemma 8, of E_y are given in Lemma 9, and of E_x are given in Lemma 10. Additionally, E_+ will not exist in all cases by Lemma 17.

For case (f), we need to check the eigenvalues of the Jacobian matrix for each of the infinite equilibrium points of the form E_K. As $c = d = 1$ and $a = b$ substituting in E_K into the Jacobian matrix will yield

$$J(x, 1-K-x) = \begin{bmatrix} \alpha + (1-\alpha)(1-x) & (\alpha-1)x \\ (\beta-1)(1-K-x) & \beta + (1-\beta)(k+x) \end{bmatrix},$$

The eigenvalues of this matrix will be $\lambda_1 = 1$ and $\lambda_2 = K + \alpha x + (1-K-x)\beta$. Note that as $|\lambda_2| < 1$, then $\lambda_1 > |\lambda_2| > 0$.

Using Lemma 15 the $(\mathcal{O}^+)$ condition holds resulting in the fact that there are no minimal period two solutions. As the solutions are bounded by Lemma 11, then Theorem 31 will give us the global results of all the cases $(a) - (j)$.

4.4 Example 2

4.4.1 Derivation of the System

We will begin by deriving the discrete system for example 2. The original continuous Lotka-Volterra system is

$$\frac{dN_1}{dt} = r_1 N_1(t)\left(1 - \frac{N_1(t) + a_1 N_2(t)}{k_1}\right),$$
$$\frac{dN_2}{dt} = r_2 N_2(t)\left(1 - \frac{N_2(t) + a_2 N_1(t)}{k_2}\right),$$

see [1]. Let $x(t) = N_1(t)$ and $y(t) = N_2(t)$

$$\frac{dx}{dt} = r_1 x(t)\left(1 - \frac{x(t) + a_1 y(t)}{k_1}\right),$$
$$\frac{dy}{dt} = r_2 y(t)\left(1 - \frac{y(t) + a_2 x(t)}{k_2}\right).$$

Using the method of semi implicit discretization this system can be converted to the discrete form

$$\frac{x_{n+1} - x_n}{h} = r_1\left(x_n - x_n \frac{x_{n+1} + a_1 y_n}{k_1}\right),$$
$$\frac{y_{n+1} - y_n}{h} = r_2\left(y_n - y_n \frac{y_{n+1} + a_2 x_n}{k_2}\right).$$

Solving for x_{n+1}, y_{n+1} we have

$$x_{n+1} = \frac{x_n(k_1(1 + hr_1) - hr_1 a_1 y_n)}{k_1 + hr_1 x_n},$$
$$y_{n+1} = \frac{y_n(k_2(1 + hr_2) - hr_2 a_2 x_n)}{k_2 + hr_2 y_n},$$

which yields

$$x_{n+1} = \frac{x_n(A - y_n)}{K_1 + x_n}, \quad y_{n+1} = \frac{y_n(A - x_n)}{K_2 + y_n}.$$

where $a_1 = a_2 = 1, K_1 = \frac{k_1}{hr_1}$, $K_2 = \frac{k_2}{hr_2}$, and $A = k_1(\frac{1}{hr_1} + 1) = k_2(\frac{1}{hr_2} + 1)$.

Assume that K_1, K_2, and A are all positive and $0 \leq x_0, y_0 < A$.

4.4.2 Local Stability Results

We will begin by proving local stability results for system (46). Additionally, we will prove that this system is bounded provided $0 \leq x_0, y_0 < A$.

Lemma 18 *The following holds true for system* (46) *where* $K_1, K_2, A > 0$.

(a) $E_0 = (0,0)$ *is always an equilibrium point.*

(b) If $A > K_2$*, then* $E_y = (0, A - K_2)$ *is an equilibrium point.*

(c) If $A > K_1$*, then* $E_x = (A - K_1, 0)$ *is an equilibrium point.*

(d) If $K_1 = K_2$ *and* $A > K_1, K_2$*, then there exists an infinite number of equilibrium points of the form* $E_K = \{(x, A - K - x) | 0 < x < A - K$ *and* $K = K_1 = K_2\}$.

Proof. The equilibrium points satisfy

$$\bar{x} = \frac{\bar{x}(A - \bar{y})}{K_1 + \bar{x}} \quad \text{and} \quad \bar{y} = \frac{\bar{y}(A - \bar{x})}{K_2 + \bar{y}}.$$

As this holds true when $\bar{x} = \bar{y} = 0$, then $E_0 = (0,0)$ is always an equilibrium point. Suppose that $\bar{x} = 0$ and $\bar{y} \neq 0$. Then the system can be reduced to

$$\bar{y} = \frac{\bar{y}A}{K_2 + \bar{y}} \Leftrightarrow K_2 + \bar{y} = A \Leftrightarrow \bar{y} = A - K_2.$$

Thus, when $A > K_2$, there will be the equilibrium point of $E_y = (0, A - K_2)$. Next suppose that $\bar{x} \neq 0$ and $\bar{y} = 0$. Using this information the system can be reduced to

$$\bar{x} = \frac{\bar{x}A}{K_1 + \bar{x}} \Leftrightarrow K_1 + \bar{x} = A \Leftrightarrow \bar{x} = A - K_1.$$

Thus, when $A > K_1$, there will be the equilibrium point of $E_x = (A - K_1, 0)$. Finally suppose that $\bar{x} \neq 0$ and $\bar{y} \neq 0$. Then the system can be reduced to

$$K_1 + \bar{x} = (A - \bar{y}),$$

$$K_2 + \bar{y} = (A - \bar{x}).$$

This can be rewritten as

$$A - K_1 = \bar{x} + \bar{y} = A - K_2.$$

When $A > K_1, K_2$ and furthermore, $K_1 = K_2$ then this will exist and result in infinite equilibrium points. Let $K = K_1 = K_2$, the infinite equilibrium points will be of the form $E_K = \{(x, A - K - x) | 0 < x < A - K\}$.

To find the local stability of E_0, E_x, and E_y we find the Jacobian matrix. For system (46) let f and g be defined as $f(u,v) = \frac{u(A-v)}{K_1+u}$ and $g(u,v) = \frac{v(A-u)}{K_2+v}$. Then the Jacobian matrix of

$$J(u,v) = \begin{bmatrix} \frac{K_1(A-v)}{(K_1+u)^2} & \frac{-u}{K_1+u} \\ \frac{-v}{K_2+v} & \frac{K_2(A-u)}{(K_2+v)^2} \end{bmatrix}.$$

When evaluated at an equilibrium point, the eigenvalues of λ_1 and λ_2 can be found from the matrix. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the equilibrium point is locally asymptotically stable. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$, then the equilibrium point is a saddle point. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then the equilibrium point is a repeller. Finally, if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then the equilibrium point is non-hyperbolic.

When the Jacobian matrix is evaluated at the equilibrium point of $E_0 = (0,0)$,

$$J(0,0) = \begin{bmatrix} \frac{A}{K_1} & 0 \\ 0 & \frac{A}{K_2} \end{bmatrix}.$$

Clearly we have the eigenvalues of $\lambda_1 = \frac{A}{K_1}$ and $\lambda_2 = \frac{A}{K_2}$.

Lemma 19 *The equilibrium point of $E_0 = (0,0)$ is*

(a) locally asymptotically stable if $A < K_1, K_2$.

(b) repeller if $A > K_1, K_2$.

(c) saddle point if $K_1 > A > K_2$ or $K_2 > A > K_1$.

(d) non-hyperbolic if $K_1 = A$ *or* $K_2 = A$.

Proof. The results follow from the eigenvalues.

Next we will investigate the Jacobian matrix at the equilibrium point of $E_y = (0, A - K_2)$. This will give the matrix

$$J(0, A - K_2) = \begin{bmatrix} \frac{K_2}{K_1} & 0 \\ \frac{-(A-K_2)}{A} & \frac{K_2}{A} \end{bmatrix},$$

where the eigenvalues will be $\lambda_1 = \frac{K_2}{K_1}$ and $\lambda_2 = \frac{K_2}{A}$.

Lemma 20 *When* $A > K_2$*, the equilibrium point of* $E_y = (0, A - K_2)$ *is*

(a) locally asymptotically stable if $K_1 > K_2$.

(b) saddle point if $K_1 < K_2$.

(c) non-hyperbolic if $K_1 = K_2$.

Proof. The results follow from the eigenvalues.

Finally, we evaluate Jacobian matrix at the equilibrium point of $E_x = (A - K_1, 0)$, which gives

$$J(A - K_1, 0) = \begin{bmatrix} \frac{K_1}{A} & \frac{-(A-K_1)}{A} \\ 0 & \frac{K_1}{K_2} \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = \frac{K_1}{A}$ and $\lambda_2 = \frac{K_1}{K_2}$.

Lemma 21 *When* $A > K_1$ *the equilibrium point of* $E_x = (A - K_1, 0)$ *is*

(a) locally asymptotically stable if $K_1 < K_2$.

(b) saddle point if $K_1 > K_2$.

(c) non-hyperbolic if $K_1 = K_2$.

Proof. The results follow from the eigenvalues.

Lemma 18 gives the necessary conditions for the equilibrium points to exist. Using Lemmas 19, 20, and 21 we can make the following conclusions regarding the local stability analysis of system (46).

Conditions	Stability of Equilibrium Points
$A < K_1, K_2$	E_0 is LAS
$A = K_1 = K_2$	E_0 is NH Resonance Type $(1, 1)$
$A = K_1 < K_2$, $A = K_2 < K_1$	E_0 is NHST
$K_1 < A < K_2$	E_0 is SP and E_x is LAS
$K_2 < A < K_1$	E_0 is SP and E_y is LAS
$K_1 < A = K_2$	E_0 is NHUT and E_x is LAS
$K_2 < A = K_1$	E_0 is NHUT and E_y is LAS
$K_2 < K_1 < A$	E_0 is R, E_x is SP, and E_y is LAS
$K_1 < K_2 < A$	E_0 is R, E_x is LAS, and E_y is SP
$K_1 = K_2 < A$	E_0 is R, E_x is NHST, E_y is NHST, and IEP

Table 1. The Local Stability of the Equilibrium Points of Example 2

As a reference, LAS stands for locally asymptotically stable, NH Resonance Type $(1, 1)$ stands for non-hyperbolic of resonance type $(1, 1)$, SP stands for saddle point, R stands for repeller, NHUT stands for non-hyperbolic of unstable type, NHST stands for non-hyperbolic of stable type, and IEP stands for infinite equilibrium points.

The following lemma will prove the criteria of boundedness needed for global analysis given that $0 \leq x_0, y_0 < A$.

Lemma 22 *Consider system* (46) *and assume that* $0 \leq x_0, y_0 < A$. *Then* $x_n, y_n \in [0, A]$ *for all* $n \geq 1$.

Proof. Note that if $x_0, y_0 \leq A$, then $x_1, y_1 < A$. Indeed as

$$x_1 = \frac{x_0}{K_1 + x_0}(A - y_0),$$

using inequalities we can conclude that

$$\frac{x_0}{K_1 + x_0}(A - y_0) < A - y_0 < A.$$

So it holds that $x_1 < A$. Additionally, as $A > x_0, y_0 \geq 0$, then $x_1 \geq 0$. Similarly as

$$y_1 = \frac{y_0}{K_2 + y_0}(A - x_0),$$

we can conclude using inequalities that

$$\frac{y_0}{K_2 + y_0}(A - x_0) < A - x_0 < A$$

resulting in the fact that $y_1 < A$. Additionally, as $A > x_0, y_0 \geq 0$, then $y_1 \geq 0$. Continuing with this technique for $n = 2, 3, \ldots$ we can conclude that $0 \leq x_n < A$ and $0 \leq y_n < A$.

4.4.3 Global Stability Results

In this section we will compile the local stability results and use Theorems 30 and 31 to give conclusions regarding the global dynamics of system (46). We will assume that $0 \leq x_0, y_0 < A$ so that the solutions are bounded. Otherwise, unbounded solutions could go to infinity or negative infinity. System (46) has no minimal period two solutions as was proved by *Mathematica*.

Theorem 34 *Consider System* (46) *and let* $A, K_1, K_2 > 0$ *as well as* $0 \leq x_0, y_0 < A$.

(a) If $A < K_1, K_2$, *then* E_x *and* E_y *will not exist, and every solution converges to* E_0.

(b) If $K_1 < A < K_2$, *then* E_0 *is a saddle point,* E_x *is locally asymptotically stable, and* E_y *does not exist. Every solution which begins off the* y*-axis converges to* E_x, *and every solution which begins on the* y*-axis converges to* E_0.

(c) If $K_2 < A < K_1$, then E_0 is a saddle point, E_y is locally asymptotically stable, and E_x does not exist. Every solution which begins off the x-axis converges to E_y, and every solution which begins on the x-axis converges to E_0.

(d) If $K_2 < K_1 < A$, then E_0 is a repeller, E_x is a saddle point, and E_y is locally asymptotically stable. Every solution which begins off the x-axis converges to E_y, and every solution which begins on the x-axis converges to E_x.

(e) If $K_1 < K_2 < A$, then E_0 is a repeller, E_x is locally asymptotically stable, and E_y is a saddle point. Every solution which begins off the y-axis converges to E_x, and every solution which begins on the y-axis converges to E_y.

Proof. The existence of the equilibrium points follow from Lemma 18. Lemma 19 gives the local dynamics of E_0, Lemma 20 gives the local dynamics of E_y, and Lemma 21 gives the local dynamics of E_x. As the solutions are bounded by Lemma 22 and there are no minimal period two solutions, Theorem 30 will give the global dynamics of cases $(a) - (e)$.

Theorem 35 *Consider System (46) and let $A, K_1, K_2 > 0$ as well as $0 \leq x_0, y_0 < A$.*

(a) If $A = K_1 = K_2$, then E_0 is non-hyperbolic of resonance type $(1, 1)$, and both E_x and E_y do not exist. Every solution converges to E_0.

(b) If $A = K_1 < K_2$, or $A = K_2 < K_1$, then E_0 is non-hyperbolic of stable type and both E_x and E_y do not exist. Every solution converges to E_0.

(c) If $K_1 < A = K_2$ then E_0 is non hyperbolic of the unstable type, E_x is locally asymptotically stable, and E_y does not exist. Then there will exists two curves, C_1 and C_2, $C_2 \preceq_{se} C_1$ that are continuous and non-decreasing with an endpoint at E_0. If the curves C_1 and C_2 coincide with each other

or C_2 is not in the region, every solution which begins off the y-axis will converge to E_x. Every solution which begins on the y-axis will converge to E_0. If there exists both C_1, C_2 in the region, then every solution to the left of C_1 will converge to E_0 and every solution to the right of C_1 will converge to E_x.

(d) *If $K_2 < A = K_1$ then E_0 is non hyperbolic of unstable type, E_y is locally asymptotically stable, and E_x does not exist. Then there will exists two curves, C_1 and C_2, $C_2 \preceq_{se} C_1$ that are continuous and non-decreasing with an endpoint at E_0. If the curves C_1 and C_2 coincide with each other or C_2 is not in the region, every solution which begins off the x-axis will converge to E_y. Every solution which begins on the x-axis will converge to E_0. If there exists both C_1, C_2 in the region, then every solution to the left of C_2 will converge to E_y and every solution to the right of C_2 will converge to E_0.*

(e) *If $K_1 = K_2 < A$, then E_0 is a repeller, E_x and E_y are non hyperbolic of stable type, and there will exist an infinite number of equilibrium solutions of the form $E_K = \{(x, A - K - x) | 0 < x < A - K\}$ where $K = K_1 = K_2$. For each of the equilibrium points E_K, there is the stable manifold $\mathcal{W}^s(E_K)$ as its basins of attraction. All $W^s(E_K)$ have an end point at E_0 and they are graphs of continuous and non-decreasing functions. The equilibrium points E_K, depends continuously on the initial point (x_0, y_0).*

Proof. The existence of the equilibrium points follow from Lemma 18. Lemma 19 gives the local dynamics of E_0, Lemma 20 gives the local dynamics of E_y, and Lemma 21 gives the local dynamics of E_x.

For case (e), we need to check the eigenvalues of the Jacobian matrix for the infinite equilibrium points of the form E_K where

$$J(x, A-K-x) = \begin{bmatrix} \frac{K}{x+K} & \frac{-x}{x+K} \\ -\frac{A-K-x}{A-x} & \frac{K}{A-x} \end{bmatrix}.$$

The eigenvalues will be $\lambda_1 = 1$ and $\lambda_2 = \frac{x^2+(K-A)x+K^2}{(A-x)(x+K)}$. Note that $|\lambda_2| < 1$ as $A > K$ and $A - K > x$. Therefore, $0 < |\lambda_2| < \lambda_1$.

As the solutions are bounded by Lemma 22 and there are no minimal period two solutions, Theorem 31 will give the global dynamics in cases $(a) - (e)$.

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