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Airway Wall Deformation Under Smooth Muscle Contraction

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AIRWAY WALL DEFORMATION UNDER SMOOTH MUSCLE

CONTRACTION

BY

CARLOS JAVIER

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE
IN MECHANICAL ENGINEERING AND APPLIED MECHANICS**

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OF

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ABSTRACT

The objective of this thesis is the development and evaluation of a mathematical model that captures the deformation behavior of the human airway under smooth muscle contraction. The problem consists of a multilayered circular cylindrical tube subjected to an active contractile stress generated by the smooth muscle layer. The problem formulation assumes axisymmetric deformation for an incompressible isotropic neo-Hookean material. The model is formulated as fully nonlinear and accommodates large deformation, since these are expected in the human airway and other soft biological materials. The equilibrium equations are obtained, as well as the strain-displacement relationship, boundary conditions, and constitutive equations that describe the contractile stress for neo-Hookean hyperelastic solids. The main focus of the project is to obtain the change in airway caliber after a prescribed contractile stress is introduced in the smooth muscle layer. A MATLAB code was written to numerically solve the resulting nonlinear algebraic equation, which obtains the change in airway caliber. The code allows the user to analyze and change the material parameters that govern the solution, as well as input different dimensions for the multilayer cylinder. In the deformation of the airway, the lung parenchyma plays a very important role in preventing full closure of the airway. Here, the lung parenchyma is treated as an infinitely large continuous solid, in which the airway is embedded. The airway generations selected for further investigation are generations 0, 4, 8, 12 and 16. For the lung parenchyma infinite domain assumption, the percent airway caliber change for the critical contractile stresses of the airway generations

were 23.86%, 23.68%, 23.95%, 24.56%, and 25.51% for generations 0,4,8,12, and 16, respectively.

The mathematical model does not account for the buckling that is observed in the deformation of the human airway, since the deformation is assumed to be axisymmetric. When the contractile stress in the smooth muscle layer exceeds a certain value, the airway buckles and folds. Therefore, finite element simulations were conducted using ABAQUS software to determine the critical contractile stress at which the airway will buckle. Buckling analyses were performed in order to obtain the buckling modes of the airway, as well as the critical load that will cause the it to buckle, for airway generations 0, 4, 8, 12, and 16. The critical load in which the airway buckles and folds for these generations of the airway are 1.755 kPa, 1.572 kPa, 1.493 kPa, 1.426 kPa, and 1.374 kPa respectively. Contractile stresses used in the analytical model that are larger than the critical buckling loads will provide invalid results since the assumptions made to develop the model will no longer be valid. After the buckling analysis, the first buckling mode is introduced in a post buckling analysis in order to obtain the average change in caliber and the active contractile stresses.

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NOMENCLATURE

<u>Notation</u>	<u>Description</u>
μ	... Shear modulus
E	... Young's modulus
ν	... Poisson's ratio
P_{tp}	... Transmural pressure
A	... Undeformed inner radius of stiff layer
B	... Undeformed outer radius of stiff layer
D	... Undeformed outer radius of smooth muscle layer
a	... Deformed inner radius of stiff layer
b	... Deformed outer radius of stiff layer
d	... Deformed outer radius of smooth muscle layer
$\mathbf{E}_r$	... Orthonormal basis vector for the R direction in reference configuration
$\mathbf{E}_\theta$	... Orthonormal basis vector for the θ direction in reference configuration
$\mathbf{E}_z$	... Orthonormal basis vector for the Z direction in reference configuration
$\mathbf{e}_r$	... Orthonormal basis vector for the R direction in current configuration
$\mathbf{e}_\theta$	... Orthonormal basis vector for the θ direction in current configuration
$\mathbf{e}_z$	... Orthonormal basis vector for the Z direction in current configuration
P	... particle on the reference configuration
$\mathbf{X}$	... position vector of P in the reference configuration
$\mathbf{x}$	... position vector of P in the current configuration

$\mathbf{u}$	... displacement vector of an arbitrary material point
u	... R direction component of displacement vector
v	... Θ direction component of displacement vector
w	... Z direction component of displacement vector
∇_0	... gradient of a tensor in the reference configuration
$\mathbf{F}$	... deformation tensor
$\mathbf{R}$	... finite rotation (polar decomposition)
$\mathbf{U}$	... right stretch tensor
$\mathbf{V}$	... left stretch tensor
$\mathbf{B}$	... left Cauchy–Green deformation tensor
$\mathbf{C}$	... right Cauchy–Green deformation tensor
u_r	... Partial derivative with respect to R of the R component of displacement vector
u_θ	... Partial derivative with respect to θ of the R component of displacement vector
u_z	... Partial derivative with respect to Z of the R component of displacement vector
v_r	... Partial derivative with respect to R of the θ component of displacement vector
v_θ	... Partial derivative with respect to θ of the θ component of displacement vector
v_z	... Partial derivative with respect to Z of the θ component of displacement vector
w_r	... Partial derivative with respect to R of the Z component of displacement vector
w_θ	... Partial derivative with respect to θ of the Z component of displacement vector
w_z	... Partial derivative with respect to Z of the Z component of displacement vector
R	... Radius in the reference configuration
r	... Radius in the current configuration
λ_1	... Principal stretch in the R direction

λ_2	...	Principal stretch in the θ direction
λ_3	...	Principal stretch in the Z direction
V_0	...	Volume of the body in the reference configuration
V	...	Volume of the body in the current configuration
J	...	Jacobian of the deformation gradient
$\mathbf{P}$	...	first Piola–Kirchhoff stress tensor
$\mathbf{T}$	...	Cauchy stress tensor
W	...	strain energy density function
I_1	...	First invariant of the right stretch tensor
I_2	...	Second invariant of the right stretch tensor
p	...	hydrostatic pressure
c_1	...	Material constant
div	...	divergence of a tensor
$\bar{k}$	...	Spring constant density
k	...	Hooke's law spring constant
n	...	Number of lung parenchyma springs per unit area
N	...	Total number of lung parenchyma springs
A_p	...	Cross-sectional area of lung parenchyma
L_0	...	Length of lung parenchyma at rest
$T_{\theta\theta_a}^{(2)}$	...	Active contractile stress in smooth muscle layer
$T_{\theta\theta_p}^{(2)}$	...	Passive stress in smooth muscle layer
C	...	Integration constant

$P1$	...	Pressure acting on airway caliber
A_{wall}	...	Cross-sectional area of airway wall
Π	...	Dimensionless $\bar{k}$ dimensionless parameter
ε_t	...	Thermal strain for ABAQUS simulation
ΔT	...	Change in temperature for ABAQUS simulation
α	...	Coefficient of thermal expansion

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SECTION 1 – INTRODUCTION

For organisms to survive, the intake and exchange of oxygen and carbon dioxide between the organism and the environment is necessary. The respiratory system is involved in the exchange of oxygen and carbon dioxide between an organism and its environment. In humans, the respiratory system includes the trachea, bronchioles, bronchi, diaphragm, and lungs. More importantly, the human airway consists of layers of anisotropic smooth muscle that wrap around the lumen, which contract and relax to decrease or increase the caliber of the airway. Due to its importance, the airway has been extensively studied, by experiments, mechanics, and by finite element simulations [5] [6] [8] [13] [22]. It is of special interest to understand the behavior under abnormal conditions, such as illnesses. An illness that is likely to affect the airway is asthma, since roughly 235 million people suffer from it worldwide [1]. Asthma is a disease of the airway, classified as the obstruction of air flow due to airway thickening from scarring and inflammation. It is also classified as bronchoconstriction, which is the narrowing of the airways in the lungs due to tightening of the surrounding smooth muscle.

Figure 1.1: (A) Cross-section of a normal airway. Figure (B) Cross-section of an airway during asthma symptoms. Figure reproduced from [2]

Figure 1.1 shows the difference between a normal airway and an airway under asthma symptoms. The airway wall is thicker, and the airway is narrower in figure 1 (B) due to swelling and smooth muscle constriction. It can be seen that the airway has folded and lobes are formed due to the reduction in airway caliber. Other causes for the narrowing of the airway are treatments such as endotracheal intubation, tracheostomy, radiotherapy, previous surgery, congenital conditions, external injury, and cancer.

To better understand the physiology and pathology of the human airway, it is important to understand the mechanics of the airway. Understanding the mechanics of the airway also aids in the engineering of airway tissue. Engineered airway tissue is used to test new drugs, which helps lower the screening time and cost of the developed drugs. Because of the complex nature of the airway's biological tissue, and from the

variations of material properties and dimensions from different organisms, understanding the airway behavior is a complicated task. Experts in the medical, scientific and engineering fields have sought to develop experimental procedures and theoretical models that describe certain aspects of the airway. Kamm [3] gives a brief summary of the literature on the mechanics of the airway. Experimental studies have utilized methods of freeze-drying the lung in order to measure the bronchiole diameter, and with the findings, develop mathematical equations relating the diameter of the bronchioles to the percentage of maximum volume of the lung. Other experimental studies include that of Codd [4], which aimed to obtain the elastic properties of the airway wall for the use in theoretical models. In the study by Codd, strips of trachea from sheep were tested in tension, in both the axial and circumferential directions. These studies did not aim to develop a theoretical model of their own. The studies aimed at obtaining elastic properties in order to be used in future theoretical models.

A theoretical model developed by Lai-Fook and his colleagues [5] tried to predict the diameter change of the airway wall. The model separates the airway into two portions, one which examines the elastic behavior of the parenchyma, and the other of the airway wall, separately. Once both have been studied, then the two models can be used to describe the entirety of the system. Initially, the derivation considers a homogeneous parenchymal structure, separated from the airway wall, and treated as an elastic continuum. As the volume within the lung varies, so does the imaginary cylindrical surface that runs through the lung parenchyma. According to Lai-Fook, the diameter of the imaginary hole varies as the cube root of the lung volume. Since lung

volume is a function of trans-pulmonary pressure, then it is possible to construct a pressure-diameter relationship for the imaginary hole. When the imaginary hole is replaced by the airway wall, then the change in diameter will also depend on the elastic characteristic of the airway wall.

Other theoretical models were developed from the need to have a tube law for all of the pulmonary airways. Lambert et al. [31] developed a theoretical model based on the pressure-area measurements given by a previous study conducted by Hyatt et al. [7]. Lambert extrapolated the data since Hyatt conducted the experiments on the first three generation of the human airway, only using a single transmural pressure. The Transmural pressure is the difference in pressure between two sides of a wall, or an equivalent separator. Elad et al. [8] constructed a similar model of the tube law. This model combines the effect of the lung parenchyma deformation with airway wall deformation. This tube law is both a function of transmural pressure, as well as lung volume.

Models aimed at solely studying the behavior of constricted airways have been developed as well. Gunst et al [9] developed a model which ignores the stiffness of the airway wall tissue, and instead focused on the role that the parenchymal attachments play in preventing the airway from full closure. This model however, does not provide a description for the entire airway. Wiggs and coworkers [10] developed a model addressing the importance of the geometrical parameters of the airway. The model is aimed at describing how the cross-sectional area of tissue inside of the smooth muscle layer affects luminal narrowing for smooth muscle constriction. This model includes the influences that the change in stiffness has on smooth muscle shortening. Lambert

et al [11] continued the earlier work of the tube law, and fully developed a model which includes a balance between the applied forces from smooth muscle constriction and the structural properties of the lung parenchymal and the airway wall. However, the model heavily relies on the tube law based on sparse extrapolated data.

The theoretical models previously developed mainly deal with pressure-volume relations, or with a tube law derived from extrapolated data. These models do not completely capture the entire deformation and contractile stress of the airway wall. These models also do not include the complete deformation, buckling, and folding of the airway wall.

Similar to the cylindrical geometry of the human airway, more rigorous theoretical models using continuum mechanics and hyperelastic constitutive relations have been developed to describe this type of problem. Zhu et al [12] developed a model that describes the nonlinear axisymmetric deformation of a hyperelastic tube under external pressure. This model does not include the multiple layers in the airway wall, nor does it include the lung parenchyma and active constriction present in the airway.

In this work, the problem of a multi-layer airway wall deformation under smooth muscle constriction is formulated. The approach taken utilizes continuum mechanics in order to model a hyperelastic material under large axisymmetric deformation, and implementing boundary conditions which captures the multiple layers of the airway. The formulation is fully nonlinear and accounts for large strains and large deformations. The analytical solution obtained describes the relationship between the caliber of the airway and the active smooth muscle contractile stress in

the human airway. The limitation of this model and the models reviewed by Kamm is that they do not consider the buckling and folding of the airway wall.

In order to understand the buckling and folding of the airway, analyses were completed using finite element analysis. Lambert [6] performed finite element analyses of the process of buckling with a tube law corresponding to the deformed buckled state. In the model, the inner layer corresponding to the subepithelial collagen layer is modeled as a stiff band. The smooth muscle layer is modeled as a compliant elastic thick layer. The smooth muscle layer is made to constrict uniformly, since the folding of the airway wall depends on the thickness of the stiff and smooth muscle layer, as well as the means that the airway wall constricts. Lambert discovered that for finite element simulations, simple application of external pressure only produced a “peanut shape” buckling, and not the multi-fold buckling that is seen in airways. The multi-lobe buckling is only seen when a band applies a circumferential displacement on the periphery of the smooth muscle layer.

In this study, it is necessary to perform finite element analysis to obtain the critical contractile stress that will cause the airway to buckle. The theoretical model developed will no longer be valid due to the axisymmetric deformation assumption. Once the airway buckles, the deformation will not be axisymmetric, and as the assumption that the airway caliber is perfectly circular will no longer hold as well.

SECTION 2 – MODEL DESCRIPTION

The physiology of the airway wall is extremely complex. The airway has different sections, as well as layers of tissue that play various roles in breathing. The cross section of a bronchiole of the airway is shown in figure 2.1, and an enlarged image is shown in figure 2.2.

Figure 2.1: Cross section of an airway bronchiole. Figure reproduced from [30]

Figure 2.2: Enlarged image of the bronchiole shown in figure 2.1. Figure reproduced from [30]

A detailed schematic of the human airway is seen in Figure 2.3.

Figure 2.3: Schematic for the cross-section along the Z-axis of the airway wall. White regions correspond to airspaces. Airway and alveolar liquid linings are not shown. Figure reproduced from [3]

As shown in Figure 2.3, the airway is divided into several layers of tissue. The multilayer nature of the airway makes the analytical and finite element analysis complex. Since the problem of interest is the behavior of the airway under smooth muscle constriction and the percent caliber change, the geometry can be simplified to a two layered cylindrical tube. The inner layer is the mucosal region, which includes the subepithelial collagen, the basement membrane, and the epithelium. The outer layer is the submucosal region, which includes all loose connective tissue on the region, as well as the smooth muscle. Finally, it must be noted that in figure 2.3, there are several attachments on the periphery of the airway, labeled as parenchymal attachments. The parenchymal attachments, or the lung parenchyma, are the portion of the lung that is involved in gas transfer, which include the alveoli, and alveolar ducts. However, other structures and tissues of the airway can be included in its definition, such as other connective tissues. As can be seen in figure 2.1, the entire bronchiole is surrounded by the alveoli, which is part of the lung parenchyma. The surrounding lung parenchyma is assumed to be a continuous elastic solid. For this assumption, the airway is modeled as three layers of continuous elastic solids. Figure 2.4 shows the simplified airway model.

Figure 2.4: Simplified three-layer model of the airway

In this assumption, the model is simplified into three layers. The inner-most layer is stiff in comparison to the softer middle layer. The outer layer is the connective tissue and the tethering lung parenchyma that surrounds the airway.

For the rest of this work, the passive load bearing inner-most layer will be referred to as the stiff layer; the contractile middle layer will be referred to as the smooth muscle layer, and the tethering outer layer will be the lung parenchyma layer. The stiff layer consists of material that is passive in the sense that it will deform and develop internal stress, strain, and displacement in response to internal or external loadings. The smooth muscle layer behaves in a different manner. It is the aim of this project to develop a model that captures the active contractile stress developed by the smooth muscle. It is necessary to separate the smooth muscle layer into two components. One component of the smooth muscle layer is passive and load bearing, similar to that of the stiff layer. The other component applies an active contractile stress that causes the airway to deform and contract. The lung parenchyma layer is also a passive layer, similar to the stiff layer. However, the lung parenchyma layer

plays a different role in the deformation of the airway. The constriction of the smooth muscle is due to the force that the muscle can develop in response to a stimulus. In intrapulmonary airway, the decrease in airway caliber due to smooth muscle constriction is opposed by the surrounding lung parenchyma and the stiff layer. This tethering effect of the lung parenchyma on the airway is an elastic constraint that opposes muscle contraction. In the airway, the elastic constraint from the lung parenchyma increases with the increase in lung volume. The shear modulus of the lung parenchyma increases in magnitude with increase in transpulmonary pressure, as well as increase in lung volume [13]. Maximal airway narrowing during constriction is greater at low lung volumes, when the elastic load from the lung parenchyma is lower, than at higher volumes. It can be seen that the material property of the lung parenchyma is a non-linear function of the pressure and lung volume.

It is assumed in this case, when the smooth muscle contract, the deformation of the airway is analyzed under a fixed volume of the lung in order to simplify the non-linear dependence for the mechanical properties of the lung parenchyma. To obtain the best solution for the effect of the lung parenchyma on airway deformation, two distinct descriptions of the lung parenchyma will be used. In the first formulation of the model, the lung parenchyma will be modeled as an infinite elastic continuum, as seen in figure 2.4. This assumption hinges on the fact that the numbers of parenchymal attachments that surround the airway are large in quantity, and packed close together, in which case it can be treated as a continuous solid. Treating the lung parenchyma as a homogeneous elastic solid has been previously done by Kamm [14], Lai-Fook [15]

and Ma [27]. It can be seen in figure 2.1 that this assumption is valid, since bronchiole is encased within the lung parenchyma tissue.

However, on a small scale, the lung parenchyma seems to be more appropriately described as a discrete spring network, as seen in figure 2.5.

Figure 2.5: Airway lung parenchyma described as a discrete network of springs. Figure reproduced from [28]

In the second solution of the model, the lung parenchyma will be modeled as a discrete network of springs that surrounds the airway. This network of springs, which represents the lung parenchyma, resists contractile deformation and prevents full closure of the airway. In this case, the lung parenchyma will be studied at a certain fixed volume, and previously obtained experimental results for the mechanical properties of the lung parenchyma will be used to determine the effects on airway contraction.

It is the aim of this project to obtain values for the percent change in airway caliber, as well as to understand the deformation throughout the airway. The percent change in the airway caliber is hypothesized to depend on the geometrical dimensions of the airway, as well as on the inner pressure of the airway, and finally the elastic properties of the layers. To obtain meaningful results from the models developed, general values of the physical geometry must be known.

Figure 2.6: human airway tree from the trachea extending out to the terminal bronchioles. Alveoli not shown. Figure reproduced from [1]

Figure 2.6 shows a cast of the airway tree. The diameter of the bronchioles differs depending on their location on the airway tree. Dimensions of the airway tree are given by Kamm [3] and can be seen in table 2.1. In this work, generations 0, 4, 8, 12, and 16 are chosen to be studied in detail.

Table 2.1: Dimensions of the human airway. Table reproduced from [1]

Generation	Diameter, 75% lung capacity (cm)	Wall area (mm ²)	t/R ^b
0	1.80	32.24	0.043
1	1.22	14.99	0.044
2	0.828	7.36	0.044
3	0.562	3.58	0.045
4	0.445	2.36	0.046
5	0.851	1.5	0.047
6	0.281	0.999	0.049
7	0.227	0.682	0.5
8	0.186	0.477	0.052
9	0.154	0.344	0.054
10	0.130	0.255	0.056
11	0.110	0.195	0.059
12	0.095	0.152	0.062
13	0.083	0.123	0.065
14	0.073	0.102	0.068
15	0.065	0.086	0.071
16	0.059	0.074	0.075

In physiology, airway generation means a division point in the airway in which one airway branches into two or smaller airways. Generation 0 is the trachea, or windpipe. Generation 16 is the terminal bronchioles. It can be seen in table 1 that there is a jump in diameter between generation 4 and 5, and then a steep drop in diameter once generation 6 is reached. This is to be expected, since the human airway generation will have different values of geometry, depending on the lung in question.

According to Codd [4], the values of Young's modulus for the smooth muscle layer averaged 3.3 kPa for circumferential strains. Codd also estimated the Young's modulus for the stiff layer to be 20 kPa in the circumferential direction. The shear modulus in terms of Young's modulus and Poisson's ratio is

$$\mu = \frac{E}{2(1+\nu)} \quad (2.1)$$

The Poisson's ratio in this case is 0.5 due to the incompressibility of the materials. The shear moduli for the outer and the stiff layer can be easily calculated to be 1.1 kPa and 6.67 kPa, respectively. The shear modulus for the lung parenchyma is given by

$$\mu^{(3)} \approx 0.7P_{tp} \quad (2.2)$$

Where P_{tp} is the transmural pressure. The transmural pressure will be taken at the functional residual capacity, and is usually given to be 0.49 kPa [21]. This gives the shear modulus for the lung parenchyma to be 0.343 kPa. Figure 2.7 shows the airway layers with their respective material properties labeled.

Table 2.2: Elastic values for the three layers of the airway

Airway Layers	Young's Modulus (kPa)	Shear Modulus (kPa)	Poisson's Ratio
stiff Layer	20	6.67	0.5
Smooth Muscle Layer	3.3	1.1	0.5
Lung Parenchyma	1.03	0.343	0.5

Figure 2.7: Schematic of airway with labeled material properties for the respective layers.

SECTION 3 – DEFORMATION

Since the geometry is cylindrical, it is convenient to develop the analytical model in cylindrical polar coordinates (R, θ, Z) . Consider the two layer cylinder to be stress free, as the initial or reference configuration. The geometry can be described as follows

$$A \leq R \leq D \quad 0 \leq \theta \leq 2\pi \quad 0 \leq Z \leq L$$

Where A is the inner radius of the stiff layer, B is the outer radius of the stiff layer, and D is the outer radius of the smooth muscle layer, as seen in figure 3.1. L is the length of the airway. Note that in figure 3.1, the schematic shown is that for the lung parenchyma modeled as an infinitely large domain.

Figure 3.1: Geometrical description of simplified airway problem.

Let $\mathbf{E}_R, \mathbf{E}_\theta, \mathbf{E}_Z$ denote the orthonormal unit bases vectors associated with the cylindrical polar coordinates in the reference configuration. Once the airway is deformed, the cylindrical polar coordinates are said to be in the current configuration (r, θ, z) . Similarly, let $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ be the orthonormal unit bases vectors in the current configuration.

Let P denote a particle on the reference configuration of the cylinder. From the origin of the cylinder, the position of P on the reference configuration can be denoted by $\mathbf{X} = \mathbf{X}(\mathbf{P})$. This is the position vector of P in the reference configuration. Once the airway deforms, the position of P in the current configuration can be described by $\mathbf{x} = \mathbf{x}(\mathbf{P})$. The relation between the position vector in the reference and the current configuration can be written as

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \quad (3.1)$$

Where $\mathbf{u}$ in equation (3.1) is the displacement vector, which simply denotes the displacement of the arbitrary material point. For each time step, the displacement of a material point is defined by the difference between its current position and its previous position. The displacement vector can be written in cylindrical polar coordinates as

$$\mathbf{u} = u(R, \theta, Z)\mathbf{E}_r + v(R, \theta, Z)\mathbf{E}_\theta + w(R, \theta, Z)\mathbf{E}_z \quad (3.2)$$

Since the mass in this system is conserved, it implies that every particle in the reference configuration has a one to one mapping to the current configuration. This deformation mapping function maps the reference position of every particle $\mathbf{X}$ to its

current position $\mathbf{x}$. The deformation mapping function is extended and can be expressed as the deformation tensor, denoted by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad (3.3)$$

$$\mathbf{F} = \nabla_0 \mathbf{x}$$

Where the operator ∇_0 is simply the gradient

$$\nabla_0 = \mathbf{E}_R \frac{\partial}{\partial R} + \mathbf{E}_\theta \frac{1}{R} \frac{\partial}{\partial \theta} + \mathbf{E}_Z \frac{\partial}{\partial Z} \quad (3.4)$$

Taking the gradient of the position vector in the current configuration gives

$$\begin{aligned} \mathbf{F} = & (1 + u_R) \mathbf{e}_r \otimes \mathbf{E}_R + \frac{1}{R} (u_\theta - v) \mathbf{e}_r \otimes \mathbf{E}_\theta + u_z \mathbf{e}_r \otimes \mathbf{E}_Z + v_r \mathbf{e}_\theta \otimes \mathbf{E}_R + \\ & \left[1 + \frac{1}{R} (v_\theta + u) \right] \mathbf{e}_\theta \otimes \mathbf{E}_\theta + v_z \mathbf{e}_\theta \otimes \mathbf{E}_Z + \\ & w_R \mathbf{e}_Z \otimes \mathbf{E}_R + \frac{1}{R} w_\theta \mathbf{e}_Z \otimes \mathbf{E}_\theta + (1 + w_Z) \mathbf{e}_Z \otimes \mathbf{E}_Z \end{aligned} \quad (3.5)$$

Where the subscripts $_R$, $_\theta$ and $_Z$ are the partial derivatives $\frac{\partial}{\partial R}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial Z}$, respectively.

For an axis-symmetric long tube, the deformation gradient reduces to

$$\begin{aligned} \mathbf{F} = & (1 + u_R) \mathbf{e}_r \otimes \mathbf{E}_R + u_z \mathbf{e}_r \otimes \mathbf{E}_Z + \left(1 + \frac{u}{R} \right) \mathbf{e}_\theta \otimes \mathbf{E}_\theta + w_R \mathbf{e}_Z \otimes \mathbf{E}_R + \\ & (1 + w_Z) \mathbf{e}_Z \otimes \mathbf{E}_Z \end{aligned} \quad (3.6)$$

The deformation gradient maps a material vector in the reference configuration, to a spatial vector in the current configuration, and vice versa, of the neighborhood of a material particle. The deformation gradient includes change in

lengths, or stretching, as well as changes in angles, or shearing. However, the deformation gradient also includes a rotation of the neighboring particles. Since rotation does not play a role in shape change, it is useful to decompose $\mathbf{F}$ into two tensors. One that accounts for shape change, and the other for rotation. Imagine two scenarios of the deformation gradient. One in which the deformation gradient only describes pure rigid body displacements, without any deformations. Then the deformation gradient $\mathbf{F}$ can be represented by a proper orthogonal tensor $\mathbf{R}$. The other deformation gradient describes pure stretching and in this case $\mathbf{F}$ is a symmetric positive definite tensor $\mathbf{U}$, or $\mathbf{V}$. Finally, the deformation gradient can be expressed by

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \quad (3.7)$$

The equation above describes the polar decomposition of the deformation gradient. In equation (3.7), $\mathbf{U}$ is the right stretch tensor, and $\mathbf{V}$ is the left stretch tensor. Taking equation (3.7) further, it can be said that

$$\mathbf{F}^T \mathbf{F} = (\mathbf{R}\mathbf{U})^T (\mathbf{R}\mathbf{U}) = \mathbf{U}^T \mathbf{U} = \mathbf{U}^2 \quad (3.8)$$

The deformation gradient gives a one-to-one mapping, for the position of a neighborhood of particles $\mathbf{X}$ in the reference configuration, to the current configuration, $\mathbf{x}$. This one-to-one mapping imposes a restriction on the deformation, that is, particles are neither created nor destroyed. It means that a single particle cannot be mapped to two positions, and that two particles cannot be mapped to one position. Mathematically this can be said that $\det \mathbf{F} \neq 0$. Since $\det \mathbf{F} > 0$, then the $\partial \mathbf{x}$ term in equation (3.3) cannot be equal to zero, unless the term $\partial \mathbf{X}$ is zero.

Therefore, $\partial \mathbf{x} \cdot \partial \mathbf{x}$ is a positive definite quadratic in the form of:

$$\partial x_k \partial x_k = (F_{kI} \partial X_I) (F_{kJ} \partial X_J) = C_{IJ} \partial X_I \partial X_J \quad (3.9)$$

Where $\mathbf{C}$ is a symmetric positive definite tensor. This means that its square roots exist. $\mathbf{C}$ is the right Cauchy-Green deformation tensor. From equation (3.9) it can be seen that the right Cauchy-Green deformation tensor can be mathematically described as

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \quad (3.10)$$

Also from equation (3.8), $\mathbf{U}^2 = \mathbf{C}$, $\mathbf{U} = \sqrt{\mathbf{C}}$. Equation (3.10) can be further expanded to show the relationship between the right Cauchy-Green deformation tensor, and the left Cauchy-Green deformation tensor, $\mathbf{B}$. The relationship is simply $\mathbf{B} = \mathbf{R} \mathbf{C} \mathbf{R}^T = \mathbf{F} \mathbf{F}^T$.

Using equation (3.6) the right Cauchy-Green deformation tensor can be expanded

$$\begin{aligned} \mathbf{C} = & [(1 + u_R)^2 + w_R^2] \mathbf{E}_R \otimes \mathbf{E}_R + \left(1 + \frac{u}{R}\right)^2 \mathbf{E}_\theta \otimes \mathbf{E}_\theta + [u_z^2 + (1 + w_Z)^2] \mathbf{E}_Z \otimes \\ & \mathbf{E}_Z + [u_z(1 + u_R) + (1 + w_Z)w_R] (\mathbf{E}_R \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_R) \end{aligned} \quad (3.11)$$

For the special case of axis symmetry and plain strain assumption, it is assumed that the displacement is given by $u = U(R)$. this means that the deformation and displacement is independent of Z , and $w=0$, therefore u_z, w_R and w_Z are zero.

The deformation gradient $\mathbf{F}$, and the right Cauchy-Green deformation tensor reduce to

$$\begin{aligned} \mathbf{F} = & (1 + u_R) \mathbf{e}_r \otimes \mathbf{E}_R + \left(1 + \frac{u}{R}\right) \mathbf{e}_\theta \otimes \mathbf{E}_\theta + \mathbf{e}_z \otimes \mathbf{E}_Z \\ \mathbf{C} = & [(1 + u_R)^2] \mathbf{E}_R \otimes \mathbf{E}_R + \left(1 + \frac{u}{R}\right)^2 \mathbf{E}_\theta \otimes \mathbf{E}_\theta + \mathbf{E}_Z \otimes \mathbf{E}_Z \end{aligned} \quad (3.12)$$

Since the right Cauchy-Green deformation tensor $\mathbf{C}$ is symmetric, its eigenvectors are automatically mutually perpendicular as long as no two eigenvalues are the same. If

two or all three eigenvalues are the same, the eigenvectors are not uniquely defined. In this case, any convenient mutually perpendicular set of eigenvectors can be used. It follows immediately from equation (3.12) that the principal axes of $\mathbf{C}$ are $\mathbf{E}_R, \mathbf{E}_\theta, \mathbf{E}_Z$. $\mathbf{C}$ can then be expressed in terms of eigenvectors and eigenvalues

$$\mathbf{C} = \lambda_1^2 \mathbf{E}_R \otimes \mathbf{E}_R + \lambda_2^2 \mathbf{E}_\theta \otimes \mathbf{E}_\theta + \lambda_3^2 \mathbf{E}_Z \otimes \mathbf{E}_Z \quad (3.13)$$

The square root of $\mathbf{C}$ gives the stretches in the principal directions, that is

$$\lambda_1 = 1 + u_R = 1 + \frac{du}{dR} \quad \lambda_2 = 1 + \frac{u}{R} \quad \lambda_3 = 1 \quad (3.14)$$

It is assumed that the materials are incompressible so that the volume is preserved.

This indicates that $\det \mathbf{F} = 1 = \lambda_1 \lambda_2 \lambda_3 = J$. Here J is the Jacobian. The definition of the Jacobian leads to a local condition for invertibility. As previously mentioned, the deformation gradient is a one-to-one mapping in the neighborhood of the particle $\mathbf{X}$.

Failure of this implies that $\frac{dV}{dV_0} \rightarrow 0$ which is physically unacceptable. This indicates that [24]

$$\frac{dV}{dV_0} = \sqrt{\det \mathbf{C}} = \sqrt{\mathbf{F}^T \mathbf{F}} = \det \mathbf{F} = J \quad (3.15)$$

Using equation (3.14) with the expression $J = \lambda_1 \lambda_2 \lambda_3 = 1$ gives

$$(1 + u_R) \left(1 + \frac{u}{R}\right) = 1 \quad (3.16)$$

Simplifying equation (3.16) into $u + (R + u)u_R = 0$ and integrating it with respect to R , with $r = R + u$ gives

$$r^2 = R^2 + a^2 - A^2 \quad (3.17)$$

The standard convention of continuum mechanics is applied, where everything related to the reference configuration is in upper case lettering, and everything related to the current configuration is in lower case lettering. Equation (3.17) gives the relationship for the radii in the reference and current configurations. Substituting the outer radius of the stiff layer, B and b for R and r gives $A^2 - B^2 = a^2 - b^2$. Similarly, the outer radius for layer two can be substituted, for the reference and current configuration. This yields the relationships between all of the radii in the reference and current configuration

$$\begin{aligned} A^2 - B^2 &= a^2 - b^2 \\ A^2 - D^2 &= a^2 - d^2 \\ B^2 - D^2 &= b^2 - d^2 \end{aligned} \quad (3.18)$$

As already discussed, u in the above equation is the displacement between the reference configuration to the current configuration. The displacement is $u = r - R$. This expression can be used in equation (3.16) to give further representation of u_R . It yields

$$u_R = \frac{R}{r} - 1 \quad (3.19)$$

Using equation (3.19) as well as $u = r - R$ and replacing these into equation (3.14) gives

$$\lambda_1 = \frac{R}{r} \quad \lambda_2 = \frac{r}{R} \quad \lambda_3 = 1 \quad (3.20)$$

The deformation gradient can be utilized to understand the stress within the continuous material through the means of displacements. However, since the deformation gradient only provides a linear mapping of deformation of a neighborhood of particles, and does not provide the responses of different materials, a constitutive equation and the stress equilibrium equations must be provided to obtain the full scope of the continuum.

SECTION 4 – HYPERELASTICITY

In the paper by Codd [4], mentioned earlier, the strips of membranous trachea were strained to roughly 20%. Biological materials can obtain strains much larger than that, it can be from 50-100% strain. Due to the large strains exhibited in biological materials, the use of a constitutive model that can accommodate large nonlinear displacements must be used. For biological materials, the constitutive model of linear elasticity does not accurately describe its behavior. The behavior of biological materials can be modeled as non-linear, elastic, anisotropic, incompressible, and independent of strain rate. For the sake of simplicity, isotropy of material properties is assumed here. Hyperelasticity provides a means of modeling the stress-strain behavior of biological materials. Hyperelasticity is a constitutive model for elastic materials in which the stress-strain relationship is derived from a strain energy density function. For the material to be modeled as isotropic, certain requirements must be met. The strain energy function must be $W(\mathbf{C}) = W(\mathbf{B})$. That is, it must be a function of both the right Cauchy-Green deformation tensor, as well as the left Cauchy-Green deformation tensor.

From traditional continuum mechanics, the First Piola-Kirchhoff Stress and the Cauchy stress tensors, in terms of a strain energy function, are

$$\begin{aligned}\mathbf{P} &= \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \\ \mathbf{T} &= \mathbf{T}^T = \frac{1}{J} \mathbf{P} \mathbf{F}^T\end{aligned}\tag{4.1}$$

Note that the first Piola-Kirchhoff stress can also be expressed as the Cauchy stress tensor by the relation $\mathbf{P} = J \mathbf{T} \mathbf{F}^{-T}$ (The expression $J \mathbf{T}$ is the Kirchhoff stress tensor, $\boldsymbol{\tau}$ such that $\boldsymbol{\tau} = J \mathbf{T}$).

Similarly to the first Piola-Kirchhoff stress, the Cauchy stress tensor can be written in terms of the strain energy function W .

$$\mathbf{T} = \mathbf{T}^T = \frac{1}{J} \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \mathbf{F}^T \quad (4.2)$$

If W is a function of the invariants of the right Cauchy-Green Deformation Tensor $\mathbf{C}$, it can be said that

$$W = W(I_1, I_2, J) \quad (4.3)$$

Where I_1 and I_2 are the invariants of the stretch tensor $\mathbf{U}$, defined as

$$I_1(\mathbf{U}^2) = \lambda_1^2 \lambda_2^2 \lambda_3^2$$

$$I_2(\mathbf{U}^2) = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_1^2 \lambda_3^2$$

The first Piola Kirchhoff stress tensor and the Cauchy stress tensor can be written as

$$\begin{aligned} \mathbf{P} &= \frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial W}{\partial J} \frac{\partial J}{\partial \mathbf{F}} \\ \mathbf{T} &= \frac{1}{J} \left(\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{F}} + \frac{\partial W}{\partial J} \frac{\partial J}{\partial \mathbf{F}} \right) \mathbf{F}^T \end{aligned} \quad (4.4)$$

The invariants of the right Cauchy-Green deformation Tensor are a function of the right stretch tensor, that is $I_1(\mathbf{U}^2), I_2(\mathbf{U}^2)$, where $\mathbf{U}$ is the right stretch tensor. $\mathbf{U}^2$ has the identity of

$$\mathbf{U}^2 = \mathbf{F}^T \mathbf{F} \quad (4.5)$$

With this, expressions to simplify equation (4.4) are given

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1\mathbf{F} - 2\mathbf{F}\mathbf{F}^T\mathbf{F}, \quad \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \quad (4.6)$$

Equations (4.4) simplifies to

$$\begin{aligned} \mathbf{P} &= \frac{\partial W}{\partial I_1} 2\mathbf{F} + \frac{\partial W}{\partial I_2} 2I_1\mathbf{F} - 2\mathbf{F}\mathbf{F}^T\mathbf{F} + \frac{\partial W}{\partial J} J\mathbf{F}^{-T} \\ \mathbf{T} &= \frac{2}{J} \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \mathbf{F}^T \mathbf{F} - \frac{2}{J} \frac{\partial W}{\partial I_2} (\mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{F}^T) + \frac{\partial W}{\partial J} \mathbf{I} \end{aligned} \quad (4.7)$$

In this paper, the Neo-Hookean hyperelastic model is employed. The strain-energy density function for a Neo-Hookean hyperelastic material is [29]

$$W = c_1(I_1 - 3) + p(J - 1) \quad (4.8)$$

Where p is a hydrostatic pressure acting on the body. Once the strain energy density function is given, expressions to further simplify the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor can be given

$$\frac{\partial W}{\partial I_1} = c_1, \quad \frac{\partial W}{\partial I_2} = 0, \quad \frac{\partial W}{\partial J} = p \quad (4.9)$$

Substituting equation (4.9) into equation (4.7) gives

$$\begin{aligned} P &= 2c_1 \mathbf{F} + Jp\mathbf{F}^{-T} \\ T &= 2c_1 \mathbf{F}\mathbf{F}^T + p\mathbf{I} \end{aligned} \quad (4.10)$$

For the case of incompressibility, the Jacobian is equal to 1. Where c_1 is a material constant. In terms of the bulk modulus, it can be expressed as

$$c_1 = \frac{\mu}{2} \quad (4.11)$$

Substituting equations (3.10) and (4.11) into equation (4.10) gives the expression for the Cauchy stress tensor for a Neo-Hookean hyperelastic material [25].

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{B} \quad (4.12)$$

The negative sign in equation (4.12) is introduced since the hydrostatic pressure is positive when in compression. This is the opposite of the sign convention for stress.

Due to the symmetric deformation assumption, the deformation gradient has the expression $\mathbf{F} = \mathbf{F}^T$ and $\mathbf{C} = \mathbf{B}$. Equation (3.12) can be substituted into equation (4.12) to give the Cauchy stress in component form. It can be written as

$$\begin{aligned}
\boldsymbol{T} &= [\mu(1 + u_R)^2 - p] \boldsymbol{E}_R \otimes \boldsymbol{E}_R + \\
&\left[\mu \left(1 + \frac{u}{R} \right)^2 - p \right] \boldsymbol{E}_\theta \otimes \boldsymbol{E}_\theta + (\mu - p) \boldsymbol{E}_Z \otimes \boldsymbol{E}_Z
\end{aligned} \tag{4.13}$$

SECTION 5 – EQUILIBRIUM AND BOUNDARY CONDITIONS

Through the use of the principle of conservation of mass, and the balance of linear momentum, the equilibrium equation has been obtained. If there are no body forces, such as gravity, or electromagnetic fields, the equilibrium equation for a static system reduces to

$$\begin{aligned} \text{div } \mathbf{T} &= 0 \\ \text{div } \mathbf{T} &= \left[\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} \right] \mathbf{e}_r + \left[\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{\theta r} + T_{r\theta}}{r} \right] \mathbf{e}_\theta + \\ &\quad \left[\frac{\partial T_{zr}}{\partial r} + \frac{T_{zr}}{r} + \frac{1}{r} \frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} \right] \mathbf{e}_z = 0 \end{aligned} \quad (5.1)$$

Equation (5.1) gives the equilibrium equations for the bases.

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} &= 0 \\ \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{\theta r} + T_{r\theta}}{r} &= 0 \\ \frac{\partial T_{zr}}{\partial r} + \frac{T_{zr}}{r} + \frac{1}{r} \frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} &= 0 \end{aligned} \quad (5.2)$$

These three equations are with regard to the base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$. The $\mathbf{e}_\theta$ component equation reduces to the requirement of p being independent of θ .

Simplifying the equilibrium equations gives the single equation for equilibrium

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) = 0 \quad (5.3)$$

The terms T_{rr} and $T_{\theta\theta}$ are given in equation (4.13). The terms in equation (5.3) can readily be expressed as

$$T_{rr} = \mu(1 + u_R)^2 - p, \quad T_{\theta\theta} = \mu\left(1 + \frac{u}{R}\right)^2 - p \quad (5.4)$$

$$T_{rr} = \mu\lambda_1^2 - p, \quad T_{\theta\theta} = \mu\lambda_2^2 - p$$

The field equations for an axisymmetric cylindrical airway under large deformation have been developed. The deformation mapping of the body under investigation, along with the stress-strain relationship, constitutive equation and the equilibrium of the body have been described. In order to complete the general formulation of the problem, boundary conditions must be developed that are properly coupled with the field equations described above. These conditions describe what occurs on the boundary of the body, the loading inputs that create stresses, strains, rotation and deformation within the body.

5.1 – INFINITE DOMAIN

It is possible for the airway to experience an internal on the periphery of the stiff layer. Let P_1 be the inside pressure, acting on the stiff layer. The boundary condition for the elastic tube under external pressure is given by The lung parenchyma layer is also modeled as an infinite solid continuum. In such a case, equation (5.5) is written as follows

$$\begin{aligned} T_{rr}^{(1)} &= -P_1 \text{ at } r = a \\ T_{rr}^{(1)} &= T_{rr}^{(2)} \text{ at } r = b \\ T_{rr}^{(2)} &= -T_p \text{ at } r = d \\ T_{rr}^{(3)} &= 0 \text{ at } r = \infty \end{aligned} \quad (5.9)$$

Where T_p is the stress exerted by the lung parenchyma on the airway. In this case, there will be three integration constants to solve, one for each individual layer.

5.2 – SPRING DOMAIN

As previously mentioned, the parenchymal attachments act as springs, with spring constant $\bar{k}$, which prevent deformation of the airway.

$$\begin{aligned} T_{rr}^{(1)} &= -P_1 \text{ at } r = a \\ T_{rr}^{(2)} &= -\bar{k}u \text{ at } r = d \\ T_{rr}^{(1)} &= T_{rr}^{(2)} \text{ at } r = b \end{aligned} \tag{5.5}$$

The superscripts in equation (5.5) denote the stress on the two layers of the airway. All things regarding the stiff layer is denoted with the superscript ⁽¹⁾, and all things regarding the smooth muscle layer is denoted with the superscript ⁽²⁾.

The second boundary condition incorporates a spring constant $\bar{k}$. This constant is similar to the spring constant found in Hooke's law; however it is a spring constant density. $\bar{k}$ is defined by

$$\bar{k} = nk \tag{5.6}$$

Where k is the traditional spring constant, relating the force needed to extend or compress a spring by some distance, n is the number of springs per unit area, defined by

$$n = \frac{N}{A} \tag{5.7}$$

Where N in equation is the total number of parenchymal attachments on the airway.

The relationship between the spring constant k and the material properties of the lung

parenchyma can be expressed in terms of its geometry and Young's modulus as follows

$$k = \frac{EA_p}{L_0} \quad (5.8)$$

Where A_p is the cross sectional area of the lung parenchyma, and L_0 is the length of the lung parenchyma at rest. The value for $\bar{k}$ is

$$\bar{k} = \frac{EA_p}{L_0} \frac{N}{A} \quad (5.9)$$

The second boundary condition aims to incorporate the effect that the lung parenchyma has on the caliber change. The stress on the boundary of the smooth muscle layer exhibits the stress that the lung parenchyma exerts, however in this case; it is modeled as a series of linear springs that undergo displacement. This displacement changes the magnitude of the stress exerted by the lung parenchyma, which is exactly what is seen in an actual lung, since the stress exerted by the lung parenchyma is a function of the lung volume.

SECTION 6 – MATHEMATICAL MODEL

6.1 – SOLUTION FOR THE STIFF LAYER

As shown in equation (5.5), the stress developed in the stiff layer is directly related to the stress developed in the smooth muscle layer. It is useful to solve the equilibrium equation for the stiff layer and for the smooth muscle layer, with their appropriate boundary conditions, and then use the direct relation of the boundary conditions to complete the formulation of the solution for the model.

The superscript ⁽¹⁾ will be used to describe all variables relating to the stiff layer, including the derivation for the solution of the equilibrium equation pertaining to the stiff layer. The values for T_{rr} and $T_{\theta\theta}$ found in equation (5.4) can be introduced into equation (5.3) to represent the equilibrium equation in terms of the principal stretches.

$$\frac{T_{rr}^{(1)}}{\partial r} + \frac{1}{r} [\mu^{(1)} \lambda_1^2 - p - (\mu^{(1)} \lambda_2^2 - p)] = 0 \quad (6.1)$$

It can be immediately seen from equation (6.1) that the hydrostatic pressure in this case does not play a role in the deformation of the body, and cancels itself. The values for the principal stretches found in equation (3.20) can be used in equation (6.1) to describe the equilibrium of the body based on the radii of the reference and current configurations.

$$\frac{T_{rr}^{(1)}}{\partial r} = \mu^{(1)} \left(\frac{r}{R^2} - \frac{R^2}{r^3} \right) \quad (6.2)$$

Using the value for R^2 found in equation (3.17) and substituting it into equation (6.2) gives the equilibrium equation for the stiff layer as a function of the radius in the current configuration, which is

$$\frac{T_{rr}^{(1)}}{\partial r} = \mu^{(1)} \left(\frac{r}{r^2 - a^2 + A^2} - \frac{r^2 - a^2 + A^2}{r^3} \right) \quad (6.3)$$

Finally, equation (6.3) can be integrated to give the solution for the stress developed in the stiff layer. Integrating with respect to r gives

$$T_{rr}^{(1)} = \frac{\mu^{(1)}}{2} \left[\ln \left(\frac{r^2 + A^2 - a^2}{r^2} \right) + \frac{A^2 - a^2}{r^2} \right] + C \quad (6.4)$$

Using the first boundary condition in equation (5.5) and substituting it into equation (6.4) gives the value for the integration constant C . Solving for the constant, substituting it back into equation (6.4) and rearranging gives the solution for the stress in the stiff layer that is

$$T_{rr}^{(1)} = \frac{\mu^{(1)}}{2} \left[\ln \left(\frac{r^2 + A^2 - a^2}{r^2} \right) + \left(\frac{A^2 - a^2 + r^2}{r^2} - \frac{A^2}{a^2} \right) + \ln \left(\frac{a^2}{A^2} \right) \right] - P_1 \quad (6.5)$$

This solution gives the Cauchy stress as a function of the radius in the current configuration. Of course, it should be noted that this solution is not completed because the deformed inner radius a is still unknown.

6.2 – SOLUTION FOR THE SMOOTH MUSCLE LAYER

Upon being stimulated, the smooth muscle layer of the airway contracts and develops stress. The Cauchy stress is composed of a passive component and an active component. The active stress is generated from the smooth muscle contraction. The

passive component is determined by the equilibrium equation previously discussed.

The active stress in this case is developed in the θ direction, therefore the Cauchy stress component in the θ direction will be broken down into an active and passive term. The total Cauchy stress component $T_{\theta\theta_{total}}^{(2)}$ is given by

$$T_{\theta\theta_{total}}^{(2)} = T_{\theta\theta_a}^{(2)} + T_{\theta\theta_p}^{(2)} \quad (6.6)$$

Where the subscripts $_a$ and $_p$ are the active and passive terms, respectively. As previously mentioned, all things with the superscript $^{(2)}$ represent all things relating to layer two. The passive term is already given by equation (5.4). Substituting this value, the total Cauchy stress in the θ direction, $T_{\theta\theta_{total}}^{(2)}$ is given by

$$T_{\theta\theta_{total}}^{(2)} = T_{\theta\theta_a}^{(2)} + \mu\lambda_2^2 - p \quad (6.7)$$

Equation (6.7) gives the total stress in the θ direction for both the passive and the active stress. Substituting the expression for the total stress in the θ direction, as well as the Substituting the known value for λ_1 and λ_2 in the equilibrium equation gives

$$T_{rr,r}^{(2)} = \mu^{(2)} \left(\frac{r}{R^2} - \frac{R^2}{r^3} \right) + \frac{T_{\theta\theta_a}^{(2)}}{r} \quad (6.8)$$

Solving this differential equation is done similarly as it was done for the stiff layer.

Substituting the values for R^2 gives

$$\frac{T_{rr}^{(2)}}{\partial r} = \mu^{(2)} \left(\frac{r}{r^2 - a^2 + A^2} - \frac{r^2 - a^2 + A^2}{r^3} \right) + \frac{T_{\theta\theta_a}^{(2)}}{r} \quad (6.9)$$

The differential equation is then integrated, and the second boundary condition is used to solve for the unknown constant of integration. Substituting the value for the unknown constant into the solution for the differential equation gives

$$T_{rr}^{(2)} = \frac{\mu^{(2)}}{2} \left[\ln \left(\frac{r^2 + A^2 - a^2}{r^2} \right) + \left(\frac{D^2 - d^2 + r^2}{r^2} - \frac{D^2}{d^2} \right) + \ln \left(\frac{d^2}{D^2} \right) \right] + T_{\theta\theta_a}^{(2)} \ln \left(\frac{r}{d} \right) - \bar{k}u \quad (6.10)$$

6.3 – SPRING DOMAIN SOLUTION

Since the equilibrium equation simplified to a first order differential equation, then the number of unknown integration constants was one. Both the solution for the stiff layer and the smooth muscle layer has been derived. However, this is not the full solution of the system, since small a is unknown. The third boundary condition explains how both of the layers interact with each other. The third boundary condition states that the radial stress is equal at the boundary of the stiff layer and the smooth muscle layer. Applying the third boundary condition and using the relationship between the reference and current radii, the above equation can be simplified further to give

$$-T_{rr}^{(2)} = \mu^{(1)} \left[\ln \left(\frac{Ba}{Ab} \right) + \frac{1}{2} \left(\frac{B^2}{b^2} - \frac{A^2}{a^2} \right) \right] - P_1 \quad (6.11)$$

Note that the solution for layer two was not included in equation (6.12). This is solely for simplicity. Finally, substituting the value for $T_{rr}^{(2)}$ and again using the relationship between reference and current radii gives

$$\mu^{(1)} \left[\ln \left(\frac{Ba}{Ab} \right) + \frac{1}{2} \left(\frac{B^2}{b^2} - \frac{A^2}{a^2} \right) \right] + \mu^{(2)} \left[\ln \left(\frac{Db}{Bd} \right) + \frac{1}{2} \left(\frac{D^2}{d^2} - \frac{B^2}{b^2} \right) \right] + T_{\theta\theta_a}^{(2)} \ln \left(\frac{b}{d} \right) - P_1 + \bar{k}u = 0 \quad (6.12)$$

The above equation fully relates the active stress to the displacements in the airway.

The desired results are for the deformed inner radius to be a function of the active contractile stress $T_{\theta\theta_a}^{(2)}$ that is $a = f(T_{\theta\theta_a}^{(2)})$.

$$T_{\theta\theta_a}^{(2)} = \frac{\mu^{(1)} \left[\ln \left(\frac{Ab}{Ba} \right) + \frac{1}{2} \left(\frac{A^2}{a^2} - \frac{B^2}{b^2} \right) \right] + \mu^{(2)} \left[\ln \left(\frac{Bd}{Db} \right) + \frac{1}{2} \left(\frac{B^2}{b^2} - \frac{D^2}{d^2} \right) \right] + P_1 - \bar{k}u}{\ln \left(\frac{b}{d} \right)} \quad (6.13)$$

The radii in the reference configuration A, B, D , are assumed to be known. The radii in the current configuration b and d are directly related to the inner radius a by equations (3.18). Therefore, equation (6.13) gives the contractile stress as a function of the inner radius in the current configuration. Equation is a nonlinear, logarithmic algebraic equation, making it difficult to obtain a solution for a in terms of $T_{\theta\theta_a}^{(2)}$. To solve the above equation in order to obtain the caliber change, a MATLAB program has been completed, employing the bisection method.

6.4 – INFINITE DOMAIN SOLUTION

In the solution for the smooth muscle layer, the term $\bar{k}u$ represents the effect of the lung parenchyma. The value for $\bar{k}$ depends on the total number of lung parenchyma, as well as the cross-sectional area and length of the lung parenchyma. This value heavily depends on the particular lung that is being examined, as well as the generation of the airway being examined. Obtaining an exact value or even an

estimate for the quantity of lung parenchyma that surrounds a single bronchiole, or the trachea is extremely difficult, therefore in order to obtain meaningful results, a range of reasonable values had to be selected. The second approach gives a different representation of the lung parenchyma, although it comes with essential assumptions of its own. The lung parenchyma that surround the bronchioles are assumed to be closely packed, as to leave little to no open space, and thus modeled as a continuous solid. In order to simplify the solution, the lung parenchyma continuous solid is modeled as infinitely large, since the surrounding bronchioles are widely apart in comparison to the dimensions of the bronchioles themselves. The solution for the lung parenchyma layer becomes similar to a well-known problem of linear elasticity, that is, a pressurized hole in an infinite medium. The solution for the lung parenchyma layer is taken from equation (6.4)

$$T_{rr}^{(3)} = \frac{\mu^{(3)}}{2} \left[\ln \left(\frac{r^2}{r^2} + \frac{A^2 - a^2}{r^2} \right) + \frac{A^2 - a^2}{r^2} \right] + C \quad (6.14)$$

Where the superscript ⁽³⁾ denotes everything related to the lung parenchyma infinite domain layer. Substituting the fourth boundary condition on equation (5.9) gives the value of the integration constant $C = 0$. Since the integration constant is equal to zero, this gives the solution to the lung parenchyma layer as

$$T_P = \mu^{(3)} \left[\ln \left(\frac{D}{d} \right) + \frac{A^2 - a^2}{2d^2} \right] \quad (6.15)$$

Note that this stress will not be prescribed, but will take a certain value depending on the displacement induced by the active contractile stress given in the smooth muscle

layer. Using the second boundary condition in equation (5.9) and introducing the obtained value of T_p in equation (6.15), and solving gives

$$\begin{aligned} & \mu^{(1)} \left[\ln \left(\frac{Ba}{Ab} \right) + \frac{1}{2} \left(\frac{B^2}{b^2} - \frac{A^2}{a^2} \right) \right] + \mu^{(2)} \left[\ln \left(\frac{Db}{Bd} \right) + \frac{1}{2} \left(\frac{D^2}{d^2} - \frac{B^2}{b^2} \right) \right] + \\ & T_{\theta\theta_a}^{(2)} \ln \left(\frac{d}{b} \right) + \mu^{(3)} \left[\ln \left(\frac{D}{d} \right) + \frac{A^2 - a^2}{2d^2} \right] - P_1 = 0 \end{aligned} \quad (6.16)$$

Finally, simplifying yields

$$T_{\theta\theta_a}^{(2)} = \frac{\mu^{(1)} \left[\ln \left(\frac{Ab}{Ba} \right) + \frac{1}{2} \left(\frac{A^2}{a^2} - \frac{B^2}{b^2} \right) \right] + \mu^{(2)} \left[\ln \left(\frac{Bd}{Db} \right) + \frac{1}{2} \left(\frac{B^2}{b^2} - \frac{D^2}{d^2} \right) \right] + P_1 + \mu^{(3)} \left[\ln \left(\frac{d}{D} \right) + \frac{a^2 - A^2}{2d^2} \right]}{\ln \left(\frac{d}{b} \right)} \quad (6.17)$$

This solution takes into consideration the stress developed by the lung parenchyma in response to the displacement in the airway. As previously mentioned, the mechanical properties of the lung parenchyma are dependent on the transmural pressure [15].

SECTION 7 – THE BISECTION METHOD

Certain solutions can be obtained with ease without the need of numerical analysis. For example, if f is a polynomial of degree 4 or less, formulas for its roots exist. However, if the degree of the formula is 5 or higher, with arbitrary coefficients, then it does not have a general algebraic solution, as stated by the Abel-Ruffini theorem [16]. Also, if f is a general nonlinear function, then no closed-form algebraic solution exists, as is the case with equation (6.13). This entails that a method that computes approximate roots must be utilized in order to obtain a solution for this equation. Considering a function $f(x) = 0$. An iterative method can be used to compute its solution numerically, by construction a sequence of $x_1, x_2, \dots, x_n, x_{n+1}, \dots$ that converge to a root of $f(x) = 0$. There are several mathematical methods used in order to converge this sequence to the root of $f(x) = 0$, such as the Secant method, Newton's method, or bisection method. The method used here is that the bisection method.

A bisection is the division of a given curve, or an interval, into two equal parts. The method of bisection is a mathematical tool employed for finding the roots of a given continuous function $f(x)$. Consider the function $f(x)$ is definite within the interval $[a, b]$, with $f(a)$ and $f(b)$ having opposite signs. According to the intermediate value theorem, there exists a number c_0 within (a, b) that is $f(c_0) = 0$. It is possible to halve the interval $[a, b]$, or bisect the interval, to subintervals that contain the value c_0 . This process can be performed several times, locating the half of the interval that contains c_0 . Note that the procedure will work even if there is more than one root in the interval $[a, b]$. However, for simplicity it is assumed that there is

one unique root. The bisection method is slow and time consuming. For the purpose of this project however, a faster method is not required since the solution necessary to produce satisfactory results does not require a large number of iterations.

In more detail, a bisection procedure is used in iterations, which converges on a solution that is known to lie inside of the given interval, $[a, b]$. The solution to the function is given by

$$c_0 = \frac{(a+b)}{2} \quad (7.1)$$

To start, set the first interval as $a_1 = a$ and $b_1 = b$ and let c_1 be the midpoint of $[a, b]$.

For the first iteration, the solution is

$$c_1 = \frac{(a_1+b_1)}{2} \quad (7.2)$$

Note that if $f(c_1) = 0$ then no further bisection is necessary, and the solution has been located. If $f(c_1) \neq 0$ then $f(c_1)$ has either a positive or negative value. If $f(c_1)$ and $f(a_1)$ have the same sign, then the solution is found in the interval $\left[\frac{(a_1+b_1)}{2}, b_1\right]$, then set $a_2 = c_1$ and $b_2 = b_1$. If $f(c_1)$ and $f(b_1)$ have the same sign, then the solution is found in the interval $\left[a_1, \frac{(a_1+b_1)}{2}\right]$. Then set $a_2 = a_1$ and $b_2 = c_1$. In either case, the interval is half as long as the initial interval. The process selects a subinterval that guarantees to have a root. The next interval will then be $[a_2, b_2]$. The advantage of the bisection method is that it is very straight forward and simple, as well as it always provides the roots of a given continuous function with the property $f(a)f(b) < 0$.

If c_1 is the midpoint of the initial interval, and c_n is the midpoint of the n_{th} interval, then the difference between c_n and a solution c is bounded by

$$|c_n - c| \leq \frac{|b-a|}{2^n} < \varepsilon, n \geq 1 \quad (7.3)$$

Where ε is a given error, or tolerance. Taking the natural logarithm of both sides gives

$$n > \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)} \quad (7.4)$$

Equation (7.4) can be used to determine in advance the number of iterations that the bisection method needs in order to converge to a root within this tolerance [17]. A pseudocode for the bisection method is given in Appendix II, as well as the MATLAB code that solves for the analytic solutions.

SECTION 8 – FINITE ELEMENT METHOD

8.1 – LINEAR BUCKLING

In a study by Klingele & Staub [18], the terminal bronchioles in cat lungs were studied, in which they freeze dried the lungs after inflating the lungs to 100%, 48%, 23%, and 8% of their total lung volume. The diameters were then measured. They observed that at the lowest lung volume, the airway walls had buckled into kinks and folds. Since the buckling of airway wall is a well-known and documented phenomenon, it is necessary to obtain reliable results when the buckling is included. Buckling is a mode of failure that is due to elastic instability of an equilibrium configuration, without fracture or separation of the continuum. Buckling, often referred to as loss of stability, refers to a large increase in displacement when there is a small increment in load. In most structures, the displacements increase gradually with increased applied load. If the applied load is too large, especially for compressive structures, a small increase in applied load can lead to a sudden large increase in the displacements. Buckling refers to this transition to large, often catastrophic displacements. Buckling mainly occurs due to thermal or mechanical loads. Buckling is a type of instability. Instability is the state in which small perturbations can cause large changes in the response of the structure. The instability in buckling is due to geometric effect and not a change in the material property, as is in material yielding and failure. In this course we will limit ourselves to instability of structures due to buckling. The differential equation for the loss of stability in columns is

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 W}{dx^2} \right) + N_0 \frac{d^2 W}{dx^2} = 0 \quad (8.1)$$

Where E is the modulus of elasticity, I is the moment of inertia, W is the non-trivial equilibrium mode of the buckled column, and N_0 is the axial compressive load [19].

This differential equation represents an eigenvalue problem.

An eigenvalue problem is defined as one in which the values of the parameter λ are obtained from the equation

$$A(\mathbf{u}) = \lambda B(\mathbf{u}) \quad (8.1)$$

The obvious solution for equation (8.1) is a trivial solution. Of course, this is of no importance, and it must be satisfied for nontrivial values of $\mathbf{u}$. Such a solution is called an eigenfunction. Any value for λ in which equation (8.1) is satisfied is an eigenvalue. For each value of λ there is a vector $\mathbf{u}$, called an eigenvector. The values A and B are matrix operators, or differential operators. The simple differential equation that governs the loss of stability in columns does not fully describe the buckling that occurs when the airway wall constricts, since it is of cylindrical geometry.

Consider a body in a base configuration, in equilibrium, with stress $\mathbf{T}$, traction $\mathbf{t}$ and body forces $\mathbf{q}$. If the stress, traction, and body force see a change, then the body is said to be in a current configuration. If there is additional loading, then it can be said that $\Delta\mathbf{T}, \Delta\mathbf{t}, \Delta\mathbf{q}$ are the change in stress, traction, and body forces, respectively. This deformation is a linear perturbation on the reference configuration. For a linear perturbation finite element buckling analysis, it is assumed that the deformation is small. In the present case however, due to the nature of the materials in question, the deformation is not small. This large deformation can be accounted for by performing a post buckling analysis in order to include large deformations. Due to the assumption

of linearity for the buckling analysis, since the change in stress $\Delta \mathbf{T}$ is a response to the change in displacement, traction, and body forces $\Delta \mathbf{u}, \Delta \mathbf{t}, \Delta \mathbf{q}$, then for the loading $\lambda \Delta \mathbf{u}, \lambda \Delta \mathbf{t}, \lambda \Delta \mathbf{q}$ the stress response will be $\lambda \Delta \mathbf{T}$. This change in displacement, traction, body forces, and stress are the perturbation loads.

As previously mentioned, the eigenvalues of interest are nontrivial solutions, which in this case are referred to as buckling modes. The buckling modes represent displacement fields with valid solutions, of arbitrary magnitude. The buckling modes are taken from the reference configuration with stress $\mathbf{T} + \lambda \Delta \mathbf{T}$, tractions $\mathbf{t} + \lambda \Delta \mathbf{t}$ and body forces $\mathbf{q} + \lambda \Delta \mathbf{q}$. The software package ABAQUS offers a very extensive and thorough discussion on the loss of instability in continuous media. The basic equilibrium equation for a continuum is transformed into the principle of virtual work, that is

$$\int_{V^B} \mathbf{P} : \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{X}} dV^B = \int_{S^B} \mathbf{a} \cdot \bar{\mathbf{v}} dS^B + \int_{V^B} \mathbf{b} \cdot \bar{\mathbf{v}} dV^B \quad (8.2)$$

Where $\mathbf{P}$ is the first Piola-Kirchhoff stress, $\bar{\mathbf{v}}$ is a virtual velocity field, $\mathbf{a}$ is a nominal traction on the boundary S^B of the reference configuration, with the expression $\mathbf{a} = \mathbf{t} \frac{dS}{dS^B}$, (where dS are the elements of surface area in the reference configuration), $\mathbf{b}$ is the body force per unit volume in the reference configuration, expressed as $\mathbf{b} = \mathbf{q} \frac{dV}{dV^B}$, (where dV are the elements of volume in the reference configuration), and V^B is the volume of the body in the reference configuration.

The equation is further rewritten in a rate form

$$\int_{V^B} \dot{\mathbf{P}} : \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{X}} dV^B = \int_{S^B} \dot{\mathbf{a}} \cdot \bar{\mathbf{v}} dS^B + \int_{V^B} \dot{\mathbf{b}} \cdot \bar{\mathbf{v}} dV^B \quad (8.3)$$

The ABAQUS theory guide makes use of several continuum mechanics identities and constitutive equations in order to rewrite the equation into a form that can fully describe elasticity, hypoelasticity, and hyperelasticity.

Using the finite element approach, the governing equation for buckling presented above takes the form of

$$(K_0^{NM} + \lambda_i K_{\Delta}^{NM})v_i^M = 0 \quad (8.4)$$

Where K_0^{NM} is the stiffness matrix in the reference configuration, which includes the effects of the preloads, K_{Δ}^{NM} is the differential initial stress and load stiffness matrix, λ_i are the eigenvalues, and v_i^M are the eigenvectors, the non-trivial displacement solutions, often referred as the buckling mode shapes. The magnitude of the eigenvectors is normalized in a linear buckling analysis. They do not represent the magnitude of deformation at the critical load. The buckling mode shapes do not have a magnitude since they are used to predict the likely failure mode of the structure. For this fully formulated finite element solution, the applied loads can consist of pressures, concentrated forces, nonzero prescribed displacements, and thermal loading. The loading in which the system will buckle, called the critical buckling load, is then

$$P^N + \lambda Q^N \quad (8.5)$$

Where P^N is simply the result from the applied forces t^B and q^B , as well as the prescribed displacements u^B , and Q^N is the perturbation loads, due to Δt^B , Δq^B and Δu^B .

8.2 – POST BUCKLING

The problem described is that of a nonlinear, hyperelastic multilayered cylindrical airway. The buckling analysis performed by ABAQUS is that for an elastic

buckling by eigenvalue extraction. The estimation is useful for stiff structures. However, the problem formulated is nonlinear, and cannot be fully described by the linear buckling analysis. In the case in which the structure responds in a nonlinear way prior to buckling, as is the case, a nonlinear analysis using the Riks method is required in order to obtain reliable results. The Riks method is used to predict the nonlinear collapse of a structure. The Riks method solves for loads and displacements, simultaneously. In order to do so, ABAQUS uses an “arc length” l , in a load-displacement space, in order to measure the progress of the solution. The Riks method in ABAQUS uses the solution for the buckling modes from the linear buckling analysis. Since this is a continuation of a previous history, the loads from the previous analysis are not redefined and are treated as “dead” loads P_0 , with constant magnitudes. The load defined during the Riks step is referred to as a “reference” load, P_{ref} . Since the load varies through the load-displacement space, the current load must always be proportional to the dead and reference load. The total load is given by

$$P_{total} = P_0 + \lambda(P_{ref} - P_0) \quad (8.6)$$

The value λ should not be confused with the eigenvalue given previously. Here λ is the load proportionality factor. This load proportionality factor is found as part of the solution given by the Riks method. The load proportionality factor is obtained at each increment in the load-displacement space. The initial load proportionality factor, $\Delta\lambda_{in}$ is computed as

$$\Delta\lambda_{in} = \frac{\Delta l_{in}}{l} \quad (8.7)$$

Where l is a user specified total arc length [20].

SECTION 9 – NUMERICAL RESULTS

9.1 – INFINITE DOMAIN

For generation 0, the radii to the inside and outside of the stiff layer are 8410.6 μm and 8578.3 μm , respectively.

Figure 9.1: Contractile stress versus deformed inner radius for the infinite domain

Figure 9.2: Contractile stress versus percent caliber change for the infinite domain

As can be seen in figure 9.1, the trend is similar to the general graph of a negative natural logarithm function. As expected, figure 9.2 follows a very similar trend, showing similarities to the exponential graph of e^x .

Table 9.1: Tabulated results for final airway caliber and percent caliber change for the infinite domain.

Generation	Initial inner radius (μm)	Final inner radius (μm)	Final percent caliber (%)	Buckling contractile stress (kPa)
0	8410.6	6407.3	23.70	1.755
4	2049.2	1562.7	23.74	1.572
8	844.43	642.22	23.95	1.493
12	421	317.62	24.56	1.426
16	251.93	187.66	25.51	1.374

The results in table 9.1 give the final airway caliber and the percent caliber change at their respective buckling stresses, which were obtained from the finite

element simulations. The percentage change in the airway caliber goes from 23.86 to 25.51 percent, justifying the large deformation solution. It can also be seen that the difference in percent caliber change between generations 0 and 16 is 1.65 percent.

9.2 – SPRING DOMAIN

In order to obtain results for the spring domain, a value for the spring constants $\bar{k}$ and k must be obtained. Values for material properties for the lung parenchyma have already been previously given in table 2.2. In order to finalize the value for $\bar{k}$, it is necessary to understand the lung parenchyma density, n . The value for n depends on the surface area of the airway, as well as the total number of lung parenchyma. In short, equation (5.9) has the unknowns of L_0, A_p, A and N . In order to obtain said parameters, it would be necessary to average a count of the total number of lung parenchyma in each generation of the airway, measure the length of the lung parenchyma, the cross-sectional area, and finally obtain the total surface area of the airway. This of course, is impractical. To obtain a satisfactory value of $\bar{k}$, a dimensionless parametric study is performed. The Buckingham Pi theorem is employed to accomplish this. The relevant parameters in the solution for the spring domain are

$$\bar{k} = f \left(T_{\theta\theta_a}^{(2)}, \mu^{(1)}, \mu^{(2)}, A_{wall}, a \right)$$

The Buckingham Pi theorem states that the total number of these relevant dimensional parameters (n) can be grouped into $n-m$ independent dimensionless groups. The number m is usually equal to the minimum of independent dimensions required to specify the dimensions of all relevant parameters. Since $\mu^{(1)}$ and $\mu^{(2)}$ have the same

dimensions as $T_{\theta\theta_a}^{(2)}$, these parameters will not be used in order to obtain the dimensionless parameter. The dimensions here are

$$\begin{aligned}\bar{k} &= \frac{M}{L^3 \cdot T^2} \\ T_{\theta\theta_a}^{(2)} &= \frac{M}{L \cdot T^2} \\ A_{wall} &= L^2 \\ a &= L\end{aligned}\tag{9.1}$$

The number of relevant dimensional parameters (n) in this case is four, and the number of individual dimensions is three, therefore this provides only one dimensionless group. The dimensionless parameter is

$$\Pi = \frac{\bar{k} \sqrt{A_{wall}}}{T_{\theta\theta_a}^{(2)}}\tag{9.2}$$

Figure 9.3: Dimensionless $\bar{k}$ parameter versus deformed inner radius.

Figure 9.4: Dimensionless k parameter versus percent caliber change

Figures 9.3 and 9.4 shows the trend of the dimensionless parameter as a function of the deformed inner radius and percent caliber change, respectively. To obtain the values in figures 9.3 and 9.4, the value for $\bar{k}$ is given to be 1 kPa/ μm for simplicity. The trend it exhibits is similar to that of the nonlinearity seen in the infinite domain. However, as can be seen in figure 9.4, the percent caliber change is extremely small. In order to allow large displacements, the value of $\bar{k}$ must be smaller by several orders of magnitude.

Figure 9.5: Contractile stress versus deformed inner radius for the spring domain

Figure 9.6: Contractile stress versus percent caliber change for the spring domain

Table 9.2: Tabulated results for final airway caliber and percent caliber change for the spring domain

Generation	Initial inner radius (μm)	Final inner radius (μm)	Final percent caliber (%)	Buckling contractile stress (kPa)
0	8410.6	8410.5	1.19e-03	1.755
4	2049.2	2049.1	4.88e-03	1.572
8	844.43	844.3071	0.0149	1.493
12	421	420.7747	0.0535	1.426
16	251.93	251.7119	0.088	1.374

As can be seen in table 9.2, the percent change in the airway caliber at their respective buckling contractile stress for the spring domain also exhibits the trend that the percent change varies with airway generation. The difference in percent caliber change between generations 0 and 16 is 0.08681 percent. The spring domain heavily depends on the spring constant values to allow for larger displacements.

Table 9.3: Parametric study of generation 0 for the lung parenchyma spring constant $\bar{k}$, at 1.755 kPa maximum contractile stress

$\bar{k}$ (kPa/ μm)	Final inner radius (μm)	Final percent caliber (%)
1	8410.5	1.19e-03
0.1	8409.6	1.19e-02
0.01	8400.5	0.120
0.001	8315.6	1.13
4e-04	8195.9	2.55
2e-04	8044.4	4.35
1e-04	7855.6	6.60
5e-05	7678.5	8.70
1e-06	7395.3	12.07
1e-07	7388.8	12.15

As can be seen in table 9.3, the values of $\bar{k}$ from 1 to 0.01 provide very small changes in the airway caliber, therefore those values will not be considered. The caliber change from $\bar{k}$ 5e-05 to 1e-06 have a dramatic jump, going from 8.70 percent

to 12.07 percent caliber change. This implies that somewhere in that range, the value of $\bar{k}$ stops governing the entire solution, and thus allows more displacement. Note also that after $\bar{k}$ of 1e-07, the value hardly changes.

SECTION 10 – FINITE ELEMENT SIMULATION

As was shown in the previous sections, the generation of the airway has a buckling critical load. Once this load is reached, the structure will buckle, and the axisymmetric deformation assumption will no longer be valid. These values will be considered when performing the numerical analysis.

In order to simplify the simulation for finite element analysis, only the infinite domain will be modeled due to the fact that it describes the problem more accurately than the spring domain. Generation 0 (trachea) is modeled using ABAQUS and presented here. To successfully complete the ABAQUS simulation, first a linear buckling analysis must be completed. The linear buckling analysis is necessary since it predicts the buckling mode of the structure, and gives the critical buckling load that will cause the structure to buckle. The mathematical model developed assumes plane strain, axisymmetric deformations. These assumptions carry to the finite element simulation as well.

Figure 10.1: ABAQUS geometry for generation 0.

Figure 10.1 shows the two dimensional ABAQUS geometry. Although the model is simple, it provides the fundamental features of an airway wall internal to the smooth muscle. The model is intended to help understand the mechanism of the buckling, and not to provide an exact numerical solution for the behavior of the airway. As can be seen in the figure, there are three layers. The two inner layers are relatively small compared to the outer most layer. The outermost layer in figure 10.1 models the lung parenchyma infinite domain. The inner radius A is $8410.6 \mu\text{m}$, the middle radius B is $8582.32 \mu\text{m}$, and the outer radius of the airway is $9000 \mu\text{m}$. The radius for the infinite domain is $12,000 \mu\text{m}$. The material properties for all three layers are found in table 2.2. ABAQUS software does not allow a prescribed Poisson's ratio of 0.5 as is

assumed in the mathematical model for hyperelastic solids. The highest Poisson's ratio that is accepted by ABAQUS is 0.48, and it is the value used for the simulations. The model is cut into four equal size parts. This is because the software has problems properly meshing the inner most layer because of its relatively thin dimension. The geometry is then cut into four equal parts to facilitate the meshing.

To perform a buckling analysis, under the step module, Linear Perturbation is selected, and then the Buckle option is chosen. The number of eigenvalues chosen will give the user the different buckling modes for the structure, and their corresponding critical buckling loads. For this case, only one eigenvalue is necessary. The loading will be simulated as a thermal contraction. Figure 10.2 gives a visual representation of the loading condition applied to the cylinder.

Figure 10.2: Loading conditions for the buckling and post buckling analysis of generation 0.

As can be seen, the soft muscle layer is the only layer which has a thermal load applied. The high temperature is given to be 0, built into the initial step of the simulation. The low temperature is given to be -1, built into the Buckle step of the simulation. It is only necessary to provide a temperature difference of magnitude 1, since the critical buckling load is given by the eigenvalue multiplied by the prescribed load, as seen in equation (8.5). The coefficient of thermal expansion for the material of layer two is 1 in order to simplify the results.

A boundary condition is applied on the periphery of the cylinder. A cylindrical coordinate system is created in order to fix the r and θ directions. Since the third layer is assumed to be infinite, then it will have zero stress, strains and displacements on the far boundary, and the infinite domain is simulated here approximately by a large domain and the outmost surface are held fixed for the simulation. Figure 10.3 shows the boundary condition for the simulation.

Figure 10.3: Boundary conditions for the buckling and post buckling analysis of generation 0.

The mesh is performed by choosing the seed part option under the Mesh module. The size of the mesh selected is 0.0001. Under the Assign Mesh Control option, the element shape is Quad. The technique is free, and the Algorithm is Advance front, selecting the option “Use mapped meshing where appropriate”.

Figure 10.4: Mesh utilized for the buckling and post buckling analysis of generation 0.

Different mesh sizes were attempted prior to concluding that the size of 0.0001 provided the desired results. As can be seen in figure 9.5, the mesh is evenly constant throughout the geometry.

Before submitting the job for completion, it is important to create an *.fil* file. The *.fil* file contains the geometry file of the buckling analysis. This file contains the imperfections in the geometry from the buckling. In order to create a *.fil* file, right clicking the “Part” tab on the left of the screen, the option for “Edit Keywords” is selected. At the end of the keyword file, the command

*NODE FILE

U

must be written, as seen in figure 10.5. Once this is completed, the job is submitted with the file name thermal_buckle.

Figure 10.5: Command for creating the *.fil* file for the linear buckling analysis

The resulting simulation is shown in figure 10.6.

Figure 10.6: Displacement contours for the linear buckling analysis of generation 0

Figure 10.6 shows the first buckling mode of the cylinder. The cylinder collapses into symmetric folds, as was described by Klingele & Staub. Kamm [22] performed similar simulations, and concluded that the number of folds exhibited is directly related to the ratios t_i/R , t_o/R , and E_i/E_o . In their model, R is the radius from the center to the middle layer, t_i is the thickness of the stiff, and t_o is the thickness of the smooth muscle layer. E_i and E_o are the Young's moduli of the stiff and smooth muscle layer, respectively. They noted that as the smooth muscle layer becomes thicker, the number of folds exhibited by buckling does not vary greatly. However, if the stiff layer becomes thicker, the number of folds decreases. Kamm uses the ratios that of $t_i/R = 0.02$, $t_o/R = 0.5$, and $E_i/E_o = 10$. In the simulation for generation 0

in this study, the ratios are $t_i/R = 0.02$, $t_o/R = 0.049$ and $E_i/E_o = 6.06$. Kamm concluded from his model that simple application of external pressure always leads to a two-lobe or “peanut-shaped” collapse mode, and not the desired multi-lobe collapse as seen in figure 10.6. In Kamm’s model, the multi-lobed collapse is seen only when a band that applies a uniform circumferential strain to the outer surface constricts the airway. The simulation presented here does not apply a circumferential strain as is done in Kamm’s simulations. The simulation presented here incorporates a third layer, aimed at modeling the lung parenchyma. The simulation in this paper applies a thermal stress on the smooth muscle layer, while restricting displacements on the boundary of the lung parenchyma layer. This seems to be a better means of simulating smooth muscle contraction. With the desired buckling mode, the post buckling simulation can be performed.

For the post buckling analysis, the same geometry is created, and the same material properties are used. The element size for the post buckling file must also be the same, since the imperfections from the linear buckling will be used in the post buckling analysis. Under the Step module, the option “Static, Riks” is chosen to perform the post buckling analysis, while also enabling the large deformation option. The loading condition is similar, where the initial hot temperature is chosen to be 0. The Cooler temperature is -0.53186, since in figure 9.7 it is given as the critical buckling load. To introduce the imperfections from the linear buckling analysis, right clicking the “Part” tab on the left of the screen, the option for “Edit Keywords” is selected once again. The command

```
*IMPERFECTION, FILE=thermal_buckle, STEP=1
```

1, 1e-4

This command obtains the imperfections from the *.fil* file created. Where 1 in the command given is the buckling mode 1, which will be used for the imperfection, and 1e-4 is the degree of the imperfection to be included.

Figure 10.7: Command for introducing imperfections found in the linear buckling analysis into the post buckling analysis

The final post buckling simulation is seen in figure 10.8.

Figure 10.8: Displacement contours for the post buckling analysis of generation 0

Note that in figure 10.8, the displacements are plotted. Figure 10.9 displays the Von Mises stresses.

Figure 10.9: Von Mises stress contours for the post buckling analysis for generation 0

As can be seen in figure 10.9, the highest level of stress is found on the stiffer stiff layer. The softer smooth muscle layer still exhibits stresses, mainly due to the active contraction. However, the lung parenchyma layer does not exhibit large stress. The stresses shown in the figure are small, having a 75 percent average Von Mises stress of 5.107 kPa. From the linear buckling analysis, the eigenvalue for mode 1 of buckling is -0.53286. This value can be used to obtain the critical contractile stress necessary to buckle the structure

$$T_{\theta\theta_a} = E\varepsilon_t \quad (10.1)$$

Where ε_t is the thermal strain, defined by

$$\varepsilon_t = E\alpha\Delta T \quad (10.2)$$

Where α is the coefficient of thermal expansion. Using equations 10.1 and 10.2 the critical contractile stress is calculated to be 1.755 kPa. Similarly, the critical buckling

contractile stress was calculated for generations 4, 8, 12, and 16. Table 10.1 lists the critical buckling contractile stresses.

Table 10.1: Critical buckling stress for each airway generation

Generation number	Critical buckling stress (kPa)
0	1.755
4	1.572
8	1.493
12	1.4265
16	1.374

Figure 10.10 gives a visual representation of the values presented in table 9.3.

Figure 10.10: Critical buckling stress for each considered generation number.

Figure 10.10 shows a relative linear trend. However, the sample number is not large enough, and the trend could be a logarithmic trend. This implies that the critical contractile buckling stress will increase with increase in the overall geometrical

dimensions of the cylinder. It is also worth noting that the structure will buckle according to the material properties given to each layer. Table 10.2 details the number of folds created upon buckling for each generation number. The figures for generations 4, 8, 12, and 16 can be seen in Appendix I for the linear buckling analysis. As Kamm [22] noted, the number of folds seen when the airway buckles depend on the ratios t_i/R , t_o/R , and E_i/E_o . The ratios used for the different airways are found on table 10.2.

Table 10.2: Parametric ratios for different airway generations

Generation	E_i/E_o	t_i/R	t_o/R	Number of folds
0	6.06	0.02	0.049	13
4	6.06	0.02	0.06	12
8	6.06	0.02	0.08	11
12	6.06	0.02	0.106	9
16	6.06	0.02	0.148	8

The ratios t_i/R and E_i/E_o does not vary for each generation. However, the ratio does increase for t_o/R , and as seen in table 9.2, the number of folds decreases with increase in the ratio t_o/R . According to Kamm, varying the ratio t_o/R does not produce a large change in the preferred buckling mode. However, the number of folds changed between 0 and 16 by 5 folds, where the difference in t_o/R between these two generations is 0.099. It appears then that increasing this ratio does play a role in the number of folds seen in the buckling of the airway. This conflicting conclusion is due to the fact that Kamm did not include an infinitely large domain to model the lung parenchyma as done here.

SECTION 11 – CONCLUSION

From the findings summarized in this study, it is clear that the objective of this study to formulate a theoretical model that describes the constriction of the airway was successfully accomplished. The model includes the important aspects of the stiff layer, as well as the active contractile stress produced by the smooth muscle layer. The model also includes the effect that the lung parenchyma has on the closure of the airway. It should be noted that the model can be used as simplified versions, by simply remodeling the boundary conditions, or removing the contractile stress term, if necessary for other problems. The lung parenchyma terms for the spring and infinite domains can be easily removed if it is desired to obtain the solution for a simple two layer cylinder. It is possible to replace the lung parenchyma term with radial pressure acting on the periphery, if the problem in question includes an outer pressure. Similarly, the inner pressure term can be removed and the solution will still hold true. Moreover, it is quite straightforward to only complete the solution for a single layer if it is desired. This formulation covers a several solutions for cylindrical problems. The formulation of the model can be applied for the lumen of the digestive track.

Figure 11.1: Schematic illustrating the form of ridges and folds in the lumen of the digestive track. Figure reproduced from [26]

As seen in figure 11.1, the digestive track also implements smooth muscle contraction. The solution presented in this paper can be used in studies for airway tissue engineering. An in vitro model based on the human airway musculature-on-chip technology has been developed [23]. The human airway musculature on chip technology consists of human bronchial smooth muscular thin films, comprised of a bottom layer of elastic polymer polydimethylsiloxane (PDMS), and a top layer of engineered bronchial smooth muscle. It is capable of simulating healthy and asthmatic bronchoconstriction and bronchodilation. When stimulated, the smooth muscle layer contracts and bends into a curvature, which is analogue for the constriction of the human airway.

Figure 11.1 shows a visual description of the human airway on a chip.

Figure 11.2: Basal tone, active contraction, and relaxation configurations of the human airway musculature on a chip. Figure reproduced from [23]

As can be seen in figure 11.2, the human airway on a chip is a flat cantilever, and does not include a real world representation of the caliber change in the airway.

In order to extrapolate the results of the musculature-on-chip flat cantilever to a real world representation of the caliber change, the theoretical model presented in this document can be used.

The finite element simulations give a better understanding of the behavior which the airway exhibits when it is under contractile stress. The simulations provided values of contractile stress for which the theoretical model will fail. The simulations presented here seem to be a better means of simulating airway contraction, since the contractile loads applied are more closely related to those seen in real world physiology. Also, the modeling of the lung parenchyma as an infinite domain is able to capture their effect on the airway deformation, and provide good results of the critical contractile stresses for the use in the theoretical model.

SECTION 12 – FURTHER WORK

To fully complete this work, if the assumption that the lung parenchyma is modeled as an infinite domain will be carried on to the finite element analysis, then the elements used to model the lung parenchyma will need to be modeled as infinite elements, and not finite elements as done in this work. It is also necessary to write a code for ABAQUS in order to incorporate a constitutive model that can more accurately describe the contractile stress, instead of modeling the contractile stress as thermal contraction. To obtain a more accurate mathematical model, it is possible to drop the assumption of isotropy, and model the materials as anisotropic, since this seems to be a more accurate way to model biological materials. Also, it should be mentioned that during abnormal conditions, the airway also swells and grows. This growth and swelling needs to be taken into account if the mathematical model is to have the most accurate results possible. Finally, to complete the spring domain solution for the airway problem, reliable and accurate values of $\bar{k}$ will need to be obtained. With these values, then the spring domain solution can be confidently used and compared with the assumption of the lung parenchyma infinite domain. However, the most important portion that is necessary in order to complete this work is to have experimental results to validate the mathematical model. Due to the nature of the problem, obtaining reliable results for the caliber change after a certain prescribed stress has been introduced is extremely difficult. In a physiological organism, knowing the exact contractile stress generated by the smooth muscle is difficult to obtain, as well as measuring the caliber once the stress has been induced. However, this is necessary in order to validate the model.

APPENDIX I

Figure A.1: Displacement contours for the linear buckling analysis of generation 4

Figure A.2: Displacement contours for the linear buckling analysis of generation 8

Figure A.3: Displacement contours for the linear buckling analysis of generation 12

Figure A.4: Displacement contours for the linear buckling analysis of generation 16

APPENDIX II

%Pseudo-code for the bisection method

```
INPUT: function f; endpoint a, b; tolerance TOL; maximum number of
iterations N0
CONDITIONS: Either  $f(a) < 0$  and  $f(b) > 0$ , or  $f(a) > 0$  and  $f(b) < 0$ 
OUTPUT approximate solution p or message of failure.
Step 1 i = 1;
FA = f (a).
Step 2 While i < N0 limit iterations to prevent infinite loop. Do
Steps 3-6.
Step 3 c = (b + a)/2; (Compute c_i.)
FC = f I.
Step 4 If FP = 0 or (b + a)/2 < TOL then
OUTPUT I; (Successfully completed)
STOP.
Step 5 Set i = i + 1.
Step 6 If FA*FP > 0 then set a = p; (Compute a_i, b_i.)
FA = FC
else set b = p. (FA is unchanged.)
Step 7 OUTPUT ('Method failed, Procedure unsuccessful, max number of
steps exceeded);
STOP.
```

```

%This MATLAB file provides the basic material properties, prescribed
%contractile stress, and the algebraic solution to the problem.

function main
%Carlos Javier

%University of Rhode Island, Mechanical Engineering Department

%03/14/2016

%MATLAB file for numerical solution using bisection method

close all

clc

elastic_modulus_stiff = 20; % kPa, Young's Modulus for the stiff
layer

elastic_modulus_smooth_muscle = 3.3; % kPa, Young's Modulus smooth
muscle layer

poisson_ratio_stiff = 0.5; % Poisson's Ratio for the stiff layer

poisson_ratio_smooth_muscle = 0.5; % Poisson's Ratio for the smooth
muscle layer

shear_modulus_stiff = (elastic_modulus_stiff
/(2*(1+poisson_ratio_stiff))); %
kPa, Shear modulus for the stiff layer

shear_modulus_smooth_muscle =
(elastic_modulus_smooth_muscle/(2*(1+poisson_ratio_smooth_muscle)));
% kPa, Shear modulus for the smooth muscle layer

shear_modulus_parenchyma = 0.343; % kPa, Shear modulus for the lung
parenchyma layer

inner_pressure = 0; %kPa

wall_area = 32.24e6; % um2

undeformed_outer_radius = (18e3)/2; % um

undeformed_inner_radius =
sqrt(undeformed_outer_radius*undeformed_outer_radius - wall_area/pi);
% um

undeformed_middle_radius = undeformed_inner_radius/0.98;

```

```

contractile_stress = linspace(0.00001,1.57,100);% kPa, prescribed
contractile_stress

deformed_inner_radius = zeros(size(contractile_stress)); %um, creates
an array of 0s the size of the contractile stress array

%This statement obtains the values for the deformed inner radius
solved by
%the (Bisec) file.
for i=1:length(contractile_stress)

    deformed_inner_radius(i) =
    Bisec(undeformed_middle_radius,shear_modulus_smooth_muscle,inner_pres
sure,wall_area,undeformed_outer_radius,shear_modulus_ridig,shear_modu
lus_parenchyma,contractile_stress (i));

end

% Obtaining the values for the middle radius and the outer radius

deformed_outer_radius = sqrt((-undeformed_inner_radius ^2)+(
undeformed_outer_radius ^2)+( deformed_inner_radius.^2)); % um

deformed_middle_radius =
sqrt(((deformed_inner_radius.^2)+(undeformed_middle_radius^2)-
(undeformed_inner_radius^2))); % um

%This expression is the algebraic equation (6.13)
contractile_stress_analytical_infinite =
(shear_modulus_smooth_muscle*(log((undeformed_middle_radius
.*deformed_outer_radius)./(
undeformed_outer_radius.*deformed_middle_radius)))+(1/2)*(((
undeformed_middle_radius ^2)./( deformed_middle_radius.^2))-((
undeformed_outer_radius
^2)./(deformed_outer_radius.^2))))+shear_modulus_parenchyma.*(log(def
ormed_middle_radius./(undeformed_middle_radius))+(deformed_inner_radi
us.^2-
undeformed_inner_radius^2)./(2*deformed_middle_radius.^2))+inner_pres
sure+ shear_modulus_ridig.*(log((undeformed_inner_radius
*deformed_middle_radius)./(undeformed_middle_radius.*
deformed_inner_radius)))+(1/2)*((( undeformed_inner_radius ^2)./(
deformed_inner_radius.^2))-(( undeformed_middle_radius
^2)./(deformed_middle_radius.^2)))))./(log(deformed_outer_radius./def
ormed_middle_radius));

percent_caliber_change = ((undeformed_inner_radius-
deformed_inner_radius)/undeformed_inner_radius)*100; % %

```

figure (1)

```

plot(deformed_inner_radius,contractile_stress_analytical_infinite)

xlabel('Deformed Inner Radius (um)','FontSize', 18)

ylabel('Contractile Stress (kPa)','FontSize', 18)

title('Contractile Stress vs Deformed Inner Radius, Infinite Domain
(Generation 0)','FontSize', 18)

grid on

figure (2)

plot(percent_caliber_change, contractile_stress_analytical_infinite)

xlabel('Percent Caliber Change','FontSize', 18)

ylabel('Contractile Stress (kPa)','FontSize', 18)

title('Contractile Stress vs Percent Caliber Change, Infinite Domain
(Generation 0)','FontSize', 18)

grid on

end

```



```
%This MATLAB code provides the equation that is being solved for
(contractile stress) where the right hand side is equal to 0.
```

```
% Carlos Javier
% University of Rhode Island, Mechanical Engineering Department
```

```
function bisection_function =
bisection_function(undeformed_middle_radius,shear_modulus_smooth_musc
le,inner_pressure,undeformed_inner_radius,
undeformed_outer_radius,shear_modulus_ridig,shear_modulus_parenchyma,
hoop_stress,deformed_inner_radius)
```

```
deformed_outer_radius = sqrt((-undeformed_inner_radius ^2)+(
undeformed_outer_radius ^2)+( deformed_inner_radius.^2)); %um.
```

```
deformed_middle_radius =
sqrt(((deformed_inner_radius.^2)+(undeformed_middle_radius^2)-
(undeformed_inner_radius^2))); %um.
```

```
%The (bisection_function) file gives equation (6.12).
```

```
bisection_function = contractile_stress-
((shear_modulus_smooth_muscle*(log((undeformed_middle_radius
.*deformed_outer_radius)./(
undeformed_outer_radius.*deformed_middle_radius)))+(1/2)*(((
undeformed_middle_radius ^2)./( deformed_middle_radius.^2))-((
undeformed_outer_radius
^2)./(deformed_outer_radius.^2))))+shear_modulus_parenchyma.*(log(def
ormed_middle_radius./(undeformed_middle_radius)))+(deformed_inner_radi
us.^2-
undeformed_inner_radius^2)./(2*deformed_middle_radius.^2))+inner_pres
sure+ shear_modulus_ridig.*(log((undeformed_inner_radius
*deformed_middle_radius)./(undeformed_middle_radius.*
deformed_inner_radius)))+(1/2)*((( undeformed_inner_radius ^2)./(
deformed_inner_radius.^2))-(( undeformed_middle_radius
^2)./(deformed_middle_radius.^2)))))./(log(deformed_outer_radius./def
ormed_middle_radius)));
```

```

%This MATLAB code solves the provided equation in the
(bisection_function) file.

% Carlos Javier
% University of Rhode Island, Mechanical Engineering Department

function deformed_inner_radius =
Bisec(undeformed_middle_radius,shear_modulus_smooth_muscle,inner_pres
sure,wall_area,undeformed_outer_radius,shear_modulus_ridig,shear_modu
lus_parenchyma,hoop_stress)

undeformed_inner_radius = sqrt(undeformed_outer_radius *
undeformed_outer_radius - wall_area/pi); % um

lower_bound = -0.1e-6; %This value is the lower value for the
interval in the bisection method given in equation (7.1)

upper_bound = undeformed_inner_radius; %This value is the upper value
for the interval in the bisection method given in equation (7.1)

error = 1e-6;

for i=1:1e6 %This value is the number of iterations to be performed

    %ensuring that the interval values have opposite sign as
mentioned in
    %section 7
    if
bisection_function(undeformed_middle_radius,shear_modulus_smooth_musc
le,inner_pressure,undeformed_inner_radius,undeformed_outer_radius,she
ar_modulus_ridig,shear_modulus_parenchyma, hoop_stress,
lower_bound)*bisection_function(undeformed_middle_radius,shear_modulu
s_smooth_muscle,inner_pressure,undeformed_inner_radius,undeformed_out
er_radius, shear_modulus_ridig,shear_modulus_parenchyma,hoop_stress,
upper_bound)>0
        disp('error')
        break
    end

solution = ( lower_bound + upper_bound)/2;

```

```

fx=bisection_function(undeformed_middle_radius,shear_modulus_smooth_m
uscle,inner_pressure,undeformed_inner_radius,undeformed_outer_radius,
shear_modulus_ridig,shear_modulus_parenchyma,hoop_stress,solution);

%Command to stop the iterations once a solution has been reached.
if fx==0
    break
end

%Command to stop the iterations once a satisfactory error has been
reached.
if abs(fx)<=error

    break
end

if fx < 0
    lower_bound = solution;
else
    upper_bound = solution;
end

end

%final solution to equation (6.13) for the deformed inner radius
deformed_inner_radius = solution;

```

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