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On Possible Limit Functions on a Fatou Component in Non-Autonomous Iteration

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ON POSSIBLE LIMIT FUNCTIONS ON A FATOU COMPONENT IN NON-AUTONOMOUS ITERATION

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ABSTRACT

The Schlicht class of functions, commonly denoted as $\mathcal{S}$, is the class of univalent functions defined on the unit disk such that $f(0) = 0$ and $f'(0) = 1$. This is a well-studied class for which many results are known. We prove that there exists a bounded sequence of polynomials, and a Fatou component for this sequence, such that for all $f \in \mathcal{S}$, there exists a subsequence of iterates of compositions of our polynomial sequence for which f is a limit function.

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CHAPTER 1

Introduction

We are concerned with non-autonomous iteration of bounded sequences of polynomials, a field in complex dynamics. In classical complex dynamics, one studies the iteration of a (fixed) rational function on the Riemann sphere. Often in applications of dynamical systems, noise is introduced, and thus it is natural to consider the iteration where the function at each stage is allowed to vary. Here, we study the situation where the functions being applied are polynomials with appropriate bounds on the coefficients and degrees.

Non-Autonomous Iteration, in our context, was first studied by Fornaess and Sibony [1]. Further work was done by Rainer Brück, Stefan Reitz, Matthias Büger [2, 3, 4, 5], and Michael Benedicks among others. Mark Comerford was one of the first to consider the scenario where the polynomials in the sequence are not in general monic [6].

One of the main topics of interest in non-autonomous iteration is discovering which results in classical complex dynamics generalize to the non-autonomous setting and which do not. For instance, Comerford proved there is a generalization of the Sullivan Straightening Theorem [7, 8, 9], while Sullivan's Non-Wandering Theorem [10, 7] no longer holds in this context [6]. One can construct polynomial sequences which provide counter examples or that have interesting properties in their own right.

1.1 Non-Autonomous Iteration

Following [9], let $d \geq 2$, $M \geq 0$, $K \geq 1$ and let $\{P_m\}_{m=1}^{\infty}$ be a sequence of polynomials where each $P_m(z) = a_{d_m,m}z^{d_m} + a_{d_m-1,m}z^{d_m-1} + \cdots\cdots + a_{1,m}z + a_{0,m}$ is a polynomial of degree $2 \leq d_m \leq d$ whose coefficients satisfy

$$1/K \leq |a_{d_m,m}| \leq K,\ m \geq 1, \quad |a_{k,m}| \leq M,\ m \geq 1,\ 0 \leq k \leq d_m - 1.$$

Such sequences are called *bounded sequences of polynomials* or simply *bounded sequences*. For a constant $C > 0$, we will say that a bounded sequence is *C-bounded* if all of the coefficients in the sequence are bounded above in modulus by C.

For each $1 \leq m$, let Q_m be the composition $P_m \circ \cdots\cdots \circ P_2 \circ P_1$ and for each $0 \leq m < n$, let $Q_{m,n}$ be the composition $P_n \circ \cdots\cdots \circ P_{m+2} \circ P_{m+1}$. Let the degrees of these compositions be D_m and $D_{m,n}$ respectively so that $D_m = \prod_{i=1}^{m} d_i$, $D_{m,n} = \prod_{i=m+1}^{n} d_i$.

For each $m \geq 0$ define the *mth iterated Fatou set* or simply the *Fatou set* at time m, $\mathcal{F}_m$, by

$$\mathcal{F}_m = \{z \in \overline{\mathbb{C}} : \{Q_{m,n}\}_{n=m}^{\infty} \text{ is a normal family on some neighborhood of } z\}$$

where we take our neighborhoods with respect to the spherical topology on $\overline{\mathbb{C}}$ and let the *mth iterated Julia set* or simply the *Julia set* at time m, $\mathcal{J}_m$, to be the complement $\overline{\mathbb{C}} \setminus \mathcal{F}_m$.

It is easy to show that these iterated Fatou and Julia sets are completely invariant in the following sense.

Theorem 1. *For any $m \leq n \in \mathbb{N}$, $Q_{m,n}(\mathcal{J}_m) = \mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m) = \mathcal{F}_n$, with Fatou components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$ by $Q_{m,n}$.*

If $\{P_m\}_{m=1}^{\infty}$ is a bounded sequence, we can find some radius R depending only on the bounds d, K, M above so that for any sequence $\{P_m\}_{m=1}^{\infty}$ as above and any $m \geq 0$, it is easy to see that

$$|Q_{m,n}(z)| \to \infty \qquad \text{as} \quad n \to \infty, \quad |z| > R$$

which shows in particular that as for classical polynomial Julia sets, there will be a *basin at infinity at time m*, denoted $\mathcal{A}_{\infty,m}$ on which all points escape to infinity under iteration. Such a radius will be called an *escape radius* for the bounds d, K, M. Note that the maximum principle shows that just as in the classical case (see [7]), there can be only one component on which ∞ is a normal limit function and so the sets $\mathcal{A}_{\infty,m}$ are completely invariant in the sense given in Theorem 1.

The complement of $\mathcal{A}_{\infty,m}$ is called the *filled Julia set* at time m for the sequence $\{P_m\}_{m=1}^{\infty}$ and is denoted by $\mathcal{K}_m$. As above, the same argument using Montel's theorem as in the classical case shows that $\partial\mathcal{K}_m = \mathcal{J}_m$ (see [7]).

1.2 The Schlicht Class

The *Schlicht* class of functions, commonly denoted as $\mathcal{S}$, is the set of univalent functions defined on the unit disk such that, for all $f \in \mathcal{S}$, we have $f(0) = 0$ and $f'(0) = 1$. This is a classical class of functions for which many useful results are known. By rescaling, one can often apply these results to an arbitrary univalent function, making the knowledge of this class quite useful in practice.

1.3 Statement of the Main Theorem

Our main goal is to prove the following result:

Theorem 2. *There exists a bounded sequence of quadratic polynomials* $\{P_m\}_{m=1}^{\infty}$ *and a Fatou component* V *for this sequence such that, for any* $f \in \mathcal{S}$*, there exists a subsequence* $\{P_{m_k}\}_{k=1}^{\infty}$ *of* $\{P_m\}_{m=1}^{\infty}$ *such that* $\{Q_{m_k}\}_{k=1}^{\infty}$ *converges locally uniformly to* f *on* V*.*

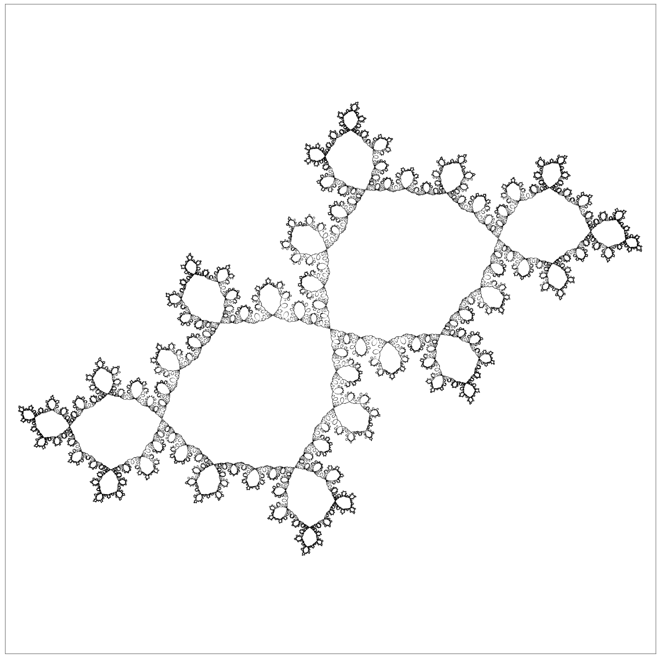
One of the strengths of this statement is that *every* member of $\mathcal{S}$ is a limit function on the same Fatou component for a *single* polynomial sequence.

The proof relies on a scaled version of the polynomial $P(z) = \lambda z(1 - z)$ where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$. As (the scaled) P is conjugate to an irrational rotation on its Siegel disk about 0, which we denote U, we may find a subsequence of iterates which converges uniformly to the identity on compact subsets of U. We scale so that $\mathcal{K}$, the filled Julia set for the scaled version of P, is contained in a small Euclidean disc about 0. This is done to control, using the distortion theorems, $|f'|$ if $f \in \mathcal{S}$ on a large hyperbolic disk inside U.

The proof of this result will follow from an inductive argument, and each step in the induction will be broken up into two phases:

- Phase I: Construct a bounded polynomial composition which approximates given functions from $\mathcal{S}$ on a subset of the unit disk.

Figure 1. Filled Julia Set for P



- Phase II: Construct a bounded polynomial composition which corrects the error of the previous sequence to arbitrary accuracy on a slightly smaller subset.

Great care is needed to control the error in the approximations and to ensure that the domain loss that occurs in each Phase II eventually stabilizes, and we are left with a region upon which the desired approximations hold.

To create our polynomial approximations, we use the Polynomial Implementation Lemma. Suppose we want to approximate a given univalent function f with a polynomial composition. Let γ and Γ be two Jordan curves outside $\mathcal{K}$ such that γ is inside Γ and $f(\gamma)$ is inside Γ. We define a homeomorphism of the sphere as follows: define it to be f inside γ, the identity outside Γ and extend by interpolation to the region between γ and Γ. The homeomophism can be made quasiconformal, with non-zero dilation (possibly) only on the region bewtween γ and Γ. If we then pull back with a high iterate of P, the support of the dilation becomes small, which will eventually allow us to conclude, that when we straighten, we get a polynomial composition that approximates f closely on a large compact subset of U. In Phase I, we then create a polynomial composition which approximates a finite set of functions from $\mathcal{S}$.

In Phase II, we wish to correct the error from the Phase I composition. This error is defined on a subset of the Siegel disk, but in order to apply the Polynomial Implementation Lemma to create a composition which corrects the error, we need the error to be defined on a region which contains $\mathcal{K}$.

To get around this, we conjugate so that the conjugated error is defined on a region which contains $\mathcal{K}$. This introduces a further problem, namely that we must now cancel the conjugacy with polynomial compositions. A key element of the proof is viewing the expanding map as a dilation in the correct conformal coordinates. An inevitable loss of domain occurs in using these conformal coordinates, but we are, in the end, able to create a Phase II composition which corrects the error of the Phase I approximation on a (slightly smaller) compact subset of U. What allows us to control the loss of domain, is that while the loss of domain is unavoidable, the accuracy of the Phase II correction is completely at our disposal.

This eventually allows us to control loss of domain. We then implement a fairly lengthly inductive argument to prove the theorem, getting better approximations to more functions in the Schlicht class with each stage in the induction, and ensuring that the region upon which the approximation holds does not shrink to nothing.

In [11], Gelfriech and Turaev show that an area-preserving two dimensional map with an elliptic periodic point can be renormalized so that the renormalized iterates are dense in the set of all real-analytic symplectic maps of a two dimensional disk.

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CHAPTER 2

Preliminaries

We will now discuss some background which will be instrumental in proving Theorem 2.

2.1 Results on $\mathcal{S}$

We now state some common results regarding the class $\mathcal{S}$. These can be found in many texts, in particular [1]. Before we state the first result, let us establish some notation. Throughout, let $\mathbb{D}$ be the unit disk and let $\mathrm{D}(z, R)$ be the Euclidean disk centered at z of radius R. The following is Theorem I.1.3 in [1].

Theorem 3. *(The Koebe one-quarter theorem) If $f \in \mathcal{S}$, then $f(\mathbb{D}) \supset \mathrm{D}(0, \frac{1}{4})$.*

Also of great importance are the well-known distortion theorems (Theorem I.1.6 in [1]).

Theorem 4. *(The distortion theorems): If $f \in \mathcal{S}$, then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$
$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

We also have that $\mathcal{S}$ is a normal family.

Theorem 5. *The family $\mathcal{S}$ is normal, and the limit of any sequence in $\mathcal{S}$ belongs to $\mathcal{S}$.*

2.2 The Hyperbolic Metric

We will be using the hyperbolic metric to measure both the accuracy of our approximations and the loss of domain that occurs in each Phase II. Let $\mathrm{d}\rho_D$ represent the hyperbolic length element for a hyperbolic domain D. We first establish some notation for hyperbolic disks. If D is a hyperbolic domain, let $\Delta_D(z, R)$ be the hyperbolic disk centered at z of radius R. If the domain is obvious in context, we may simply denote this disk $\Delta(z, R)$.

One of the key tools we will be using is the following relationship between the hyperbolic and Euclidean metrics (see [1] Theorem I.4.3). If D is a domain in $\mathbb{C}$ and $z \in D$, let $\delta_D(z)$ denote the Euclidean distance to ∂D.

Lemma 1. *Let $D \subset \mathbb{C}$ be a simply connected domain and let $z \in D$. Then*

$$\frac{1}{2}\frac{|\mathrm{d}z|}{\delta_D(z)} \leq \mathrm{d}\rho_D(z) \leq 2\frac{|\mathrm{d}z|}{\delta_D(z)}$$

We remark that there is a stronger version of this theorem for general hyperbolic domains in $\mathbb{C}$ (see [1] Theorem I.4.3). Next, we will need a notion of internal and external hyperbolic radii.

Definition 2.2.1. *Suppose $U \subset V \subset \overline{\mathbb{C}}$ are simply connected hyperbolic domains and $u \in U$. We define the internal hyperbolic radius of U in V about u, denoted $R^{int}_{(V,u)}U$, to be $\inf_{z \in V \setminus U} \rho_V(u, z)$. Further, define the external hyperbolic radius of U in V about u, denoted $R^{ext}_{(V,u)}U$, to be $\sup_{z \in U} \rho_V(u, z)$. If it happens that $R^{int}_{(V,u)}U =$*

$R^{ext}_{(V,u)}U$, *we will call the quantity the hyperbolic radius of* U *in* V *about* u, *denoted* $R_{(V,u)}U$.

We remark that if $U = V$, then for any $u \in U$, we have that $R^{int}_{(V,u)}U = R^{ext}_{(V,u)}U = \infty$. Also, if $\overline{U} \subset V$, then $R^{ext}_{(V,u)}U < \infty$. Further, we have the following formulation that will be more useful in practice.

Lemma 2. *If* V *is a simply connected hyperbolic domain, with* $U \subsetneq V$ *and* $u \in U \cap V$, *then*

1. $R^{int}_{(V,u)}U = \inf_{z \in (\partial U) \cap V} \rho_V(u,z)$.

   If, in addition, U *is simply connected, we have*

2. $R^{ext}_{(V,u)}U = \sup_{z \in (\partial U) \cap V} \rho_V(u,z)$.

We remark that if $U \subsetneq V$, then $R^{int}_{(V,u)}U < \infty$. Indeed, let $v \in V \setminus U$. Then

$$\begin{aligned} R^{int}_{(V,u)}U &= \inf_{z \in V \setminus U} \rho_V(u,z) \\ &\leq \rho_V(u,v) \\ &< \infty \end{aligned}$$

Proof. To prove 1., we first know $R^{int}_{(V,u)}U \leq \inf_{z \in (\partial U) \cap V} \rho_V(u,z)$ as $(\partial U) \cap V \subset V \setminus U$. Now we show $R^{int}_{(V,u)}U \geq \inf_{z \in (\partial U) \cap V} \rho_V(u,z)$. Take a point $w \in V \setminus U$ and connect u to w with a geodesic γ in V, which must intersect $\partial U \cap V$ at a point v. Clearly

$$\rho_V(u,v) \le \rho_V(u,w),$$

so

$$\inf_{z\in(\partial U)\cap V} \rho_V(u,z) \le \rho_V(u,w),$$

and thus

$$\inf_{z\in(\partial U)\cap V} \rho_V(u,z) \le R^{int}_{(V,u)}U.$$

This completes the proof of 1.

To prove 2., we first consider the case when $\sup_{z\in(\partial U)\cap V} \rho_V(u,z) = \infty$. Then there exists a sequence $\{w_n\} \in (\partial U) \cap V$ such that $\rho_V(u,w_n) \to \infty$. For each w_n, choose $u_n \in U$ such that $\rho_V(w_n,u_n) \le 1$. Then $\rho_V(u,u_n) \to \infty$ as well, which shows $R^{ext}_{(V,u)}U = \infty$.

Now consider the case when $\sup_{z\in(\partial U)\cap V} \rho_V(u,z) < \infty$. We first show $\sup_{z\in(\partial U)\cap V} \rho_V(u,z) \le R^{ext}_{(V,u)}U$. First take a sequence $\{w_n\} \in (\partial U) \cap V$ for which $\rho_V(u,w_n) \to \sup_{z\in(\partial U)\cap V} \rho_V(u,z)$. Then take a sequence $\{u_n\} \in U$ such that $\rho_V(u_n,w_n) < \frac{1}{n}$. As U is open, we must have

$$\rho_V(u,u_n) \le R^{ext}_{(V,u)}U,$$

and since $\rho_V(u_n, w_n) < \frac{1}{n}$, we must have that

$$\sup_{z \in (\partial U) \cap V} \rho_V(u, z) \leq R^{ext}_{(V,u)} U.$$

Now we show $\sup_{z \in (\partial U) \cap V} \rho_V(u, z) \geq R^{ext}_{(V,u)} U$. Let $\rho = \sup_{z \in (\partial U) \cap V} \rho_V(u, z)$.

Claim: $U \subset \overline{\Delta}_V(u, \rho)$.

Proof (of claim): Suppose not. Then there exists $\tilde{u} \in U$ such that $\rho_V(u, \tilde{u}) > \rho$. Set $\tilde{\rho} := \rho_V(u, \tilde{u})$ and define C to be the hyperbolic circle of radius $\tilde{\rho}$ with respect to the hyperbolic metric of V. Then we have $C \cap \partial U = \emptyset$. Now we have $\tilde{u} \in C$. We now show that each point of C must lie in U. Suppose z is another point on C such that $z \notin U$. Then z would be in $V \setminus U$. As $C \cap \partial U = \emptyset$, we have that $z \in V \setminus \overline{U} = \text{Int}(V \setminus U)$. But this is impossible as U and $\text{Int}(V \setminus U)$ would then form a separation of the connected set C. Thus $C \subset U$ and C induces a separation of $\overline{\mathbb{C}} \setminus U$. Indeed, ∂U is inside the Jordan curve C while $\overline{\mathbb{C}} \setminus V$ is outside C. This contradicts the fact that U is simply connected (cf. [2] Theorem 8.2.2). $\Diamond$

From the above, we see that $\sup_{z \in (\partial U) \cap V} \rho_V(u, z) \geq R^{ext}_{(V,u)} U$, and thus

$$\sup_{z \in (\partial U) \cap V} \rho_V(u, z) = R^{ext}_{(V,u)} U$$

as desired.

□

The utility of these hyperbolic radii is illustrated in the following proposition, of which the proof is easy.

Proposition 6. *Suppose $V \subset \overline{\mathbb{C}}$ is a simply connected hyperbolic domain and let $u \in V$. Further suppose U and $\tilde{U}$ are simply connected hyperbolic domains containing u such that $U \subset V$ and $\tilde{U} \subset V$. If $R^{ext}_{(V,u)}\tilde{U} \leq R^{int}_{(V,u)}U$, then $\tilde{U} \subset U$.*

Further, we will make use of the following, which comes from the theory of metric spaces:

Definition 2.2.2. *Suppose D is a hyperbolic domain and that U and V are compactly contained in D. For $u \in \overline{U}$, we define*

$$\rho_D(u, \partial V) = \inf_{v \in \partial V} \rho_D(u, v)$$

and

$$\rho_D(\partial U, \partial V) = \inf_{u \in \partial U} \rho_D(u, V)$$

We have the following lemma on hyperbolic convexity:

Lemma 3. *(The hyperbolic convexity lemma) For all $z, w \in \mathbb{D}$ are distinct and for all $R > 0$, if $\rho_{\mathbb{D}}(0, w) \leq R$ and $\rho_{\mathbb{D}}(0, z) \leq R$, then $\rho_{\mathbb{D}}(0, \zeta) \leq R$ for all $\zeta \in \gamma_{\mathbb{D}}[z, w]$.*

Proof. Let $z, w \in \mathbb{D}$ arbitrary. If z, w, and 0 are collinear, the result is obvious. Suppose not. Otherwise $\gamma_{\mathbb{D}}[z, w]$ is the arc of a circle C which is orthogonal to $\partial\mathbb{D}$. By applying a rotation, we may assume without loss of generality that the segment between 0 and the center of C, is a subset of $\mathbb{R}$ to the right of 0. Denote the center of C as c. Without loss of generality we may assume that z is at least as close to 0 as w, that w is below $\mathbb{R}$ and z is above $\mathbb{R}$, and that $\rho_{\mathbb{D}}(0, w) = R$. As $\Delta_{\mathbb{D}}(0, R)$ is circular in Euclidean coordinates as well, let r be such that $\Delta_{\mathbb{D}}(0, R) = \mathrm{D}(0, r)$, where $\mathrm{D}(0, r)$ is a Euclidean disk of radius r. Let S be the sector in $\mathbb{D}$ which contains $\gamma := C \cap \mathbb{D}$ with angle θ which satisfies $\theta < \pi$. Note that S is symmetric about $\mathbb{R}$.

If z_1 is the point above $\mathbb{R}$ at the top of the sector on $\partial\mathbb{D}$ and z_2 is the point below $\mathbb{R}$ at the bottom of the sector on $\partial\mathbb{D}$, let $\alpha = \sphericalangle z_1 0 c$ and let $\beta = z_1 c 0$. Then we have $\alpha + \beta + \frac{\pi}{2} = \pi$. If we denote as θ_z the angle, measured in C from the real axis to the right of c to z, and define θ_w in the same way, we have that $\theta_z, \theta_w \in (\frac{\pi}{2}, \frac{3\pi}{2})$, with that angle range measured as before. Note we are using the fact that $\beta < \frac{\pi}{2}$. Let $\delta_z := |\theta_z - \pi|$ and $\delta_w := |\theta_w - \pi|$, and notice that $\delta_z, \delta_w < \frac{\pi}{2}$. Using $|z| \leq |w|$ and the law of cosines, we see that $0 \leq \delta_z \leq \delta_w$. Thus $\theta_z \in [\pi - \delta_w, \pi + \delta_w]$. Now $\gamma_{\mathbb{D}}[z, w]$ is the shorter arc of C from z to w and in fact the range of angles for points in $\gamma_{\mathbb{D}}[z, w]$ is contained in $[\pi - \delta_w, \pi + \delta_w]$ (again measured in terms of C).

Observe that $\overline{w}$ is the point on C at angle $\pi - \delta_w$. Consider a point ζ on C

corresponding to an angle in $(\pi - \delta_w, \pi)$. The result will follow, using symmetry, if we can show $|\zeta| \le |\overline{w}| = |w| = r$. Let $\overline{w} = x_1 + iy_1$ and $\zeta = x_2 + iy_2$. Since the sine and cosine functions are decreasing on $[\frac{\pi}{2}, \pi] \supset [\pi - \delta_w, \pi]$ we see that $x_1 > x_2$ and $y_1 > y_2$ and thus $|\zeta| \le |\overline{w}|$ as desired. $\square$

Ordinary derivatives are useful for estimation when using the Euclidean metric. In our case, we will need a notion of a derivative taken with respect to the hyperbolic metric.

Let S, T be hyperbolic Riemann surfaces with metrics

$$\mathrm{d}\rho_S = \sigma_S(z)|\mathrm{d}z|$$

$$\mathrm{d}\rho_T = \sigma_T(z)|\mathrm{d}z|,$$

respectively. Let $f : W \subset S \to T$ be analytic. Define the hyperbolic derivative:

$$f^{\natural}(z) := f'(z)\frac{\sigma_T(f(z))}{\sigma_S(z)}$$

See the differential operations defined in [3]. Note that the hyperbolic derivative satisfies the chain rule, i.e. $(f \circ g)^{\natural} = (f^{\natural} \circ g) \cdot g^{\natural}$. Let $K \subset W$ be relatively compact. Define the hyperbolic Lipschitz bound as

$$\|f^\natural\|_K := \sup_{z\in K}|f^\natural(z)|$$

Lemma 4. *(Hyperbolic M-L estimates) Suppose S, T are hyperbolic Riemann Surfaces and $f : S \to T$ is holomorphic. Let $z, w \in S$ and let γ be a hyperbolic geodesic connecting z and w, with $|f^\natural| \le M$ on γ. Then*

$$\rho_T(f(z), f(w)) \le M\rho_S(z, w).$$

Proof. We calculate

$$\begin{aligned}
\rho_T(f(z), f(w)) &\le l(f(\gamma)) \\
&= \int_{f(\gamma)} \mathrm{d}\rho_T \\
&= \int_a^b \mathrm{d}\rho_T(f(\gamma(t))) \cdot |f'(\gamma(t))| \cdot |\gamma'(t)|\mathrm{d}t \\
&= \int_a^b |f^\natural(\gamma(t))| \cdot \mathrm{d}\rho_S(\gamma(t)) \cdot |\gamma'(t)|\mathrm{d}t \\
&= \int_\gamma |f^\natural|\mathrm{d}\rho_S \\
&\le M\int_\gamma \mathrm{d}\rho_S \\
&= M\rho_S(z, w)
\end{aligned}$$

□

Let D by a hyperbolic domain. As $\mathcal{S}$ is normal, we can, given a compact subset $\tilde{D}$ of D and $\varepsilon > 0$, find a finite set $\{f_i\}_{i=1}^N \in \mathcal{S}$ such that, given $f \in \mathcal{S}$, there exists $f_k \in \{f_i\}_{i=1}^N$ such that $\rho_D(f(z), f_k(z)) < \varepsilon$ on $\tilde{D}$, using Proposition VII.1.16 in [2] and 1. Such a set will be called an ε-*net* for $\mathcal{S}$, or simply a *net*.

2.3 The Carathéodory Topology

The Carathéodory topology is a topology on pointed domains, which are domains with a marked point referred to as the base point. In [4], Constantin Carathéodory defined a suitable topology for simply connected domains for which convergence in this topology is equivalent to the convergence of suitably normalized inverse Riemann maps. The work was then extended in an appropriate sense to hyperbolic domains by Adam Epstein in his Ph.D thesis [5]. This work was expanded upon further still by Comerford [6, 7]. This is a supremely useful tool in non-autonomous iteration; the domains on which one wishes to apply may be as variable as the polynomials themselves. We follow [6] for the following discussion.

A *pointed domain* is a pair (U, u) consisting of an open connected subset U of $\overline{\mathbb{C}}$, (possibly equal to $\overline{\mathbb{C}}$ itself) and a point u in U. We say that $(U_m, u_m) \to (U, u)$ in the Carathéodory topology if and only if:

1. $u_m \to u$ in the spherical topology,
2. for all compact sets $K \subset U$, $K \subset U_m$ for all but finitely many m,
3. for any *connected* open set N containing u, if $N \subset U_m$ for infinitely many m, then $N \subset U$.

We also wish to consider the degenerate case where $U = \{u\}$. In this case, condition 2. is omitted (U has no interior of which we can take compact subsets) while condition 3. becomes

3. for any *connected* open set N containing u, N is contained in at most finitely many of the sets U_m.

Convergence in the Carathéodory topology can also be described using the *Carathéodory kernel*. Originally defined by Carathéodory himself in [4], one first requires that $u_m \to u$ in the spherical topology. If there is no open set containing u which is contained in the intersection of all but finitely many of the sets U_m, then one defines the kernel of the sequence $\{(U_m, u_m)\}_{m=1}^{\infty}$ to be $\{u\}$. Otherwise one defines the Carathéodory kernel as the largest domain U containing u with the property 2. above. It is easy to check that a largest domain does indeed exist. Carathéodory convergence can also be described in terms of the Hausdorff topology. We have the following theorem in [6].

Theorem 7. *Let $\{(U_m, u_m)\}_{m=1}^{\infty}$ be a sequence of pointed domains and (U, u) be another pointed domain where we allow the possibility that $(U, u) = (\{u\}, u)$. Then the following are equivalent:*

1. *$(U_m, u_m) \to (U, u)$;*

2. *$u_m \to u$ in the spherical topology and $\{(U_m, u_m)\}_{m=1}^{\infty}$ has Carathéodory kernel U as does every subsequence;*

3. *$u_m \to u$ in the spherical topology and for any subsequence where the complements of the sets U_m converge in the Hausdorff topology (with respect to the spherical metric), U corresponds with the connected component of the com-*

plement of the Hausdorff limit which contains u (this component being empty in the degenerate case $U = \{u\}$).

Of particular use to us will be the following theorem in [6] regarding the equivalence of Carathéodory convergence and the local uniform convergence of suitably normalized covering maps:

Theorem 8. *Let $\{(U_m, u_m)\}_{m \geq 1}$ be a sequence of pointed hyperbolic domains and for each m let π_m be the unique normalized covering map from $\mathbb{D}$ to U_m satisfying $\pi_m(0) = 0$, $\pi'_m(0) > 0$.*

Then (U_m, u_m) converges in the Carathéodory topology to another pointed hyperbolic domain (U, u) if and only if the mappings π_m converge with respect to the spherical metric uniformly on compact subsets of $\mathbb{D}$ to the covering map π from $\mathbb{D}$ to U satisfying $\pi(0) = u$, $\pi'(0) > 0$.

In addition, in the case of convergence, if D us a simply connected subset of U and $v \in D$, then locally defined branches ω_m of $\pi_m^{\circ -1}$ on D for which $\omega_m(v)$ converges to a point in $\mathbb{D}$ will converge locally uniformly with respect to the spherical metric on D to a uniquely defined branch ω of $\pi^{\circ -1}$.

Finally, if π_m converges with respect to the spherical topology locally uniformly on $\mathbb{D}$ to the constant function u, the (U_m, u_m) converges to $(\{u\}, u)$.

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CHAPTER 3

The Polynomial Implementation Lemma

Let $\tilde{P} = \lambda z(1-z)$ where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$. Let $\tilde{\mathcal{K}}$ be the filled Julia set for $\tilde{P}$, and let $\tilde{U}$ be the Siegel disc containing 0 . Let $\kappa > 1$ and set $P = P_\kappa = \frac{1}{\kappa}\tilde{P}(\kappa z)$. Then if $\mathcal{K}$ is the filled Julia set for P, we have $\mathcal{K} \subset D(0, \frac{2}{\kappa})$. Let U be the Siegel disk for P and note that $U = \{z \in \mathbb{C} : z = \frac{w}{\kappa} \text{ for some } w \in \tilde{U}\}$.

Let $\Omega, \Omega' \subset \mathbb{C}$ be Jordan domains with analytic boundary curves γ and Γ, respectively, such that $\mathcal{K} \subset \Omega \subset \overline{\Omega} \subset \Omega' \subset \mathrm{D}(0, \frac{2}{\kappa})$, where we recall $\frac{2}{\kappa} < 2$ is an escape radius for P. Suppose f is analytic and injective on a neighborhood of $\overline{\Omega}$ such that $f(\gamma)$ is still inside Γ. Let $D = \Omega \setminus \overline{\Omega}'$ and D' be the conformal annulus bounded by $f(\gamma)$ and Γ. Define

$$F(z) = \begin{cases} f(z) & z \in \overline{\Omega} \\ z & z \in \overline{\mathbb{C}} \setminus \Omega' \end{cases}$$

We wish to extend F to a quasiconformal homeomorphism of $\overline{\mathbb{C}}$. To do this, we will use a lemma of Lehto [1]. In order to apply the lemma, we first need to show that F is an *admissible boundary function* for D in the sense that the positive orientations of γ and Γ with respect to the annulus D correspond to the positive orientations of $F(\gamma) = f(\gamma)$ and Γ with respect to D'.

Claim: F is an admissible boundary function for D.

To prove the claim, first let χ_1 be the orientation preserving Riemann map which maps D to the annulus $A(0,1,R)$, where $R>1$ is chosen so that D and $A(0,1,R)$ have the same modulus. Similarly, let $\tilde{\chi}_2$ be the Riemann map which maps the annulus D' to the annulus $A(0,1,\tilde{R})$, where $\tilde{R}>1$ is chosen so that the annuli D' and $A(0,1,\tilde{R})$ have the same modulus. Post-compose $\tilde{\chi}_2$ with a quasiconformal stretching to get a map χ_2 which maps D' to $A(0,1,R)$. As γ and $F(\gamma)=f(\gamma)$ are analytic arcs, we may use Schwarz reflection to analytically extend the maps χ_1 and χ_2 to neighborhoods of γ and $f(\gamma)$, respectively (see [2]). Let $\tilde{f}=\chi_2\circ f\circ\chi_1^{-1}$ be a lift of f which maps $C(0,1)$ to $C(0,1)$. Post-composing χ_2 with a rotation, if necessary, we may suppose that $\tilde{f}(1)=1$. Notice that $\tilde{f}$ is analytic on a neighborhood of $C(0,1)$, as f, χ_1, and a rotation function are.

Let $\tilde{f}(re^{i\theta})=\rho(r,\theta)e^{i\omega(r,\theta)}$, with $re^{i\theta}=x+iy$, be a polar representation of $\tilde{f}$. As $\tilde{f}$ is a homeomorphism of $C(0,1)$ with itself, we must have that $\omega_\theta(1,\theta)>0$ always or $\omega_\theta(1,\theta)<0$ always. We calculate

$$\tilde{f}'(1) = \lim_{y\to 0} \frac{\tilde{f}(1+iy) - \tilde{f}(1)}{iy}$$

$$= -i\tilde{f}_y(z)$$

$$= -i(\rho_r(1,0)r_y(1,0) + \rho_\theta(1,0)\theta_y(1,0) + i(\omega_r(1,0)r_y(1,0) + \omega_\theta(1,0)\theta_y(1,0))e^{i\omega(1,0)}$$

$$= -i(0+0+i(0+\omega_\theta(1,0)))e^{i\omega(1,0)}$$

$$= \omega_\theta(1,0).$$

Then, near 1, we have

$$\tilde{f} = \tilde{f}(1) + \tilde{f}'(1)(z-1) + \mathcal{O}(|z-1|^2)$$

$$= \tilde{f}(1) + \omega_\theta(1,0)(z-1) + \mathcal{O}(|z-1|^2)$$

If $\omega'(0) < 0$, $\tilde{f}$ locally maps the inside of $C(0,1)$ to the outside of $C(0,1)$. Then by the conjugacy f would map the inside of γ to the outside of $f(\gamma)$. But this is impossible, as f is a homeomorphism. Thus $\omega' > 0$, and the positive orientation of γ with respect to D corresponds to the positive orientation of $f(\gamma)$ with respect to D'. Since the identity function is orientation preserving, we have that F is an admissible boundary function for D. $\diamondsuit$

If f, F, γ, Γ, and D are all as above, with F an admissable boundary function on D, we will say that (f, Id) is *an admissible pair* on (γ, Γ).

Next let $N \in \mathbb{N}$ and set $\mu_n^N := (P^{N-n})^* \mu_F$ for $0 \leq n < N$. Let $\varphi_N^N := F$ and, for $0 \leq n \leq N-1$, let φ_n^N be the unique normalized solution of the Beltrami equation for μ_n^N which satisfies $\varphi_n^N(z) = z + \mathcal{O}(\frac{1}{|z|})$ near ∞ (see [3]). For $0 \leq n \leq N$, let

$$\tilde{P}_n^N(z) = \varphi_n^N \circ P \circ (\varphi_{n-1}^N)^{-1}(z).$$

Then for each n, $\tilde{P}_n^N$ is an analytic degree 2 branched cover of $\mathbb{C}$ which has a double pole at ∞ and no other poles. Thus $\tilde{P}_n^N$ is a quadratic polynomial. Let $\alpha_n^N := \varphi_n^N(0)$. Since the dilatation of φ_n^N is zero on $\overline{\mathbb{C}} \setminus \overline{D}(0, \frac{2}{\kappa})$, we know φ_n^N is univalent on $\overline{\mathbb{C}} \setminus \overline{D}(0, \frac{2}{\kappa})$. Thus $\frac{1}{\varphi_n^N(1/z)}$ is univalent on $D(0, \frac{\kappa}{2})$. It follows from the Koebe one-quarter theorem and the injectivity of φ_n^N that $|\alpha_n^N| \leq 4\frac{2}{\kappa} = \frac{8}{\kappa}$.

Define $\psi_n^N(z) := \varphi_n^N(z) - \alpha_n^N$. Then for each $0 \leq n \leq N$, if we define

$$P_n^N(z) = \psi_n^N \circ P \circ (\psi_{n-1}^N)^{-1}(z)$$

we have that $P_n^N(z)$ is a quadratic polynomial which fixes 0, as it is $\tilde{P}_n^N$ composed with (uniformly bounded) translations. We now turn to calculating bounds on the coefficients of each P_n^N.

Claim: Any sequence formed from the polynomials $P_n^N(z)$ for $0 \leq n \leq N$ is a bounded sequence of polynomials, with all coefficients bounded in modulus by $17 + \kappa$.

By construction, the constant term is zero. Now

$$P_n^N(z) = \lambda(z+\alpha_{n-1}^N+\mathcal{O}(\frac{1}{|z|}))(1-\kappa z-\kappa\alpha_{n-1}^N+\mathcal{O}(\frac{1}{|z|}))-\alpha_n^N+\mathcal{O}(\frac{1}{|P\circ(\psi_{n-1}^N)^{-1}(z))|}),$$

and for $|z|$ sufficiently large, we see that the $\mathcal{O}(\frac{1}{|P\circ(\psi_{n-1}^N)^{-1}(z))|})$ term is actually $\mathcal{O}(\frac{1}{|z|^2})$. Therefore the coefficient of the linear term is $\lambda - 2\lambda\kappa\alpha_{n-1}^N$, and thus is bounded in modulus by $1 + 2 \cdot 1 \cdot \kappa \cdot \frac{8}{\kappa} = 17$. The leading term is $-\lambda\kappa$, and thus we have constructed a bounded sequence of polynomials, proving the claim. $\diamondsuit$

We have a subsequence $\{P^{\circ n_k}\}_{k=1}^{\infty}$ of iterates of P for which $P^{\circ n_k}$ converges uniformly to the identity of compact subsets of its Siegel disc containing 0. In fact, we can choose $\{n_k\}_{k=1}^{\infty}$ to be the Fibonacci sequence (see [4]).

Lemma 5. *ψ_0^N converges locally uniformly to the identity on $\mathbb{C}$ and $(\psi_0^N)^{-1}$ converges locally uniformly to the identity on $\mathbb{C}$, both with respect to the Euclidean metric.*

Proof. Recall that Γ is the boundary of Ω'. Let $G(z)$ be the Green's function for P and let $h := \sup_{z\in\Gamma} G(z)$. Then $\operatorname{supp}\mu_n^N \subset \{0 < G(z) \leq h \cdot 2^{n-N}\}$ and in particular $\operatorname{supp}\mu_0^N \subset \{0 < G(z) \leq h\cdot 2^{-N}\}$. Thus $\operatorname{supp}\mu_0^N \to 0$ almost everywhere as $N \to \infty$. By Theorem 7.5 on page 24 of [3] (see also Lemma 1 on page 93 of

[5]), we have that φ_0^N converges uniformly to the identity on $\mathbb{C}$ (recall that the unique solution for $\mu \equiv 0$ is the identity).

For the inverses, let $\varepsilon > 0$ and, if $z \in \mathbb{C}$ and $z = \varphi_0^N(w)$ (recall that φ_0^N is a homeomorphism of $\mathbb{C}$), then

$$|(\varphi_0^N)^{-1}(z) - z| = |w - \varphi_0^N(w)|$$
$$< \varepsilon$$

for all N large enough. Since φ_0^N is a homeomorphism of $\mathbb{C}$, we have that φ_0^N and $(\varphi_0^N)^{-1}$ both converge uniformly to the identity on $\mathbb{C}$. Then $\alpha_0^N = \varphi_0^N(0) \to 0$ as $N \to \infty$, and since $\psi_0^N = \varphi_0^N(z) - \alpha_0^N$, the result follows.

□

The support of μ_0^N is contained the basin of infinity for P, A_∞. Since $2^{-N} \inf_{z\in\gamma} G(z) > 0$, ψ_0^N is analytic on a neighborhood of $\overline{U}$. Then if we define $U^N = \psi_0^N(U)$, we have that $(\psi_0^N)^{-1}$ is analytic on a neighborhood of $\overline{U^N}$. On the other hand, we have the following lemma:

Lemma 6. $(U^N, 0) \to (U, 0)$ *in the Carathéodory topology.*

Proof. Define $\psi^{-1} : \mathbb{D} \to U$ to be the unique inverse Riemann map from $\mathbb{D}$ to U satisfying $\psi^{-1}(0) = 0$, $(\psi^{-1})'(0) > 0$. Proposition 3.2 in [6] gives that $\psi_0^N \circ \psi^{-1}$ converges locally uniformly to $Id \circ \psi^{-1}$ on $\mathbb{D}$. The result follows from Theorem 8.

□

In the following, let $A \subset U$ open and relatively compact be arbitrary. By Lemma 6, we have that $A \subset U^N$ for all N large. Further, let $\delta > 0$ and let $\hat{A}$ and $\hat{\hat{A}}$ be a δ-neighborhood and a 2δ-neighborhood, respectively, of A with respect to ρ_U. Further, let $\check{A}$ be $\Delta_U(0, \hat{\hat{R}})$, where $\hat{\hat{R}} = R^{ext}_{(U,0)}\hat{\hat{A}}$. The domain $\check{A}$ will be useful to us as it is hyperbolically convex by Lemma 3, so that we can apply the hyperbolic M-L estimates (Lemma 4) to a function whose hyperbolic derivative is bounded on $\check{A}$. We now turn to another lemma.

Lemma 7. *For any $\varepsilon > 0$ and any open and relatively compact subset A of U, there exists an N_0 such that*

$$|(\psi_0^N)^\natural - 1| < \varepsilon$$

$$|((\psi_0^N)^{-1})^\natural - 1| < \varepsilon$$

for all z in A, $N \geq N_0$.

Proof. Let $d\rho_U = \sigma(z)|dz|$, where the hyperbolic density σ is continuous on U and bounded away from 0 on any relatively compact subset of U. Since ψ_0^N and $(\psi_0^N)^{-1}$ are analytic on any relatively compact subset of U for N sufficiently large, we have that by Lemma 5 both $(\psi_0^N)'$ and $((\psi_0^N)^{-1})'$ converge uniformly to 1 on A. Using the local equivalence of the Euclidean and hyperbolic metrics there exists a $\delta_0 > 0$ such that $\hat{A}$ contains a Euclidean δ_0-neighborhood of A, which we denote

$\tilde{A}$. By Lemma 5 we can choose N_0 large enough such that $\psi_0^N(A) \subset \tilde{A} \subset \hat{A}$ for all $N \geq N_0$. Then, since σ is uniformly continuous on the relatively compact subset $\hat{A}$ of U, there exists $\eta > 0$ such that $|\sigma| > \eta$ on $\hat{A}$. Then for $z \in A$ we have that

$$|(\psi_0^N)^\natural(z)| = \left| \frac{(\psi_0^N)'(z)\sigma(\psi_0^N(z))}{\sigma(z)} \right|$$

converges uniformly to 1 on A, using the uniform continuity of σ, as desired. The proof for $((\psi_0^N)^{-1})^\natural$ is similar.

□

Statement and Proof of the Polynomial Implementation Lemma:

Recall we had defined $P_n^N(z) = \psi_n^N \circ P \circ (\psi_{n-1}^N)^{-1}(z)$. For convenience, we change the subindex to m, so that we have defined P_m^N for $0 \leq m \leq N$. Recall that we have a subsequence n_k for which $P^{\circ n_k}$ coverges uniformly to the identity on compact subsets of U. Define $Q_{n_k}^{n_k}(z) = P_{n_k}^{n_k} \circ P_{n_{k-1}}^{n_k} \circ \cdots \circ P_{n_1}^{n_k} \circ P_{n_0}^{n_k}(z)$ and note that this simplifies so that $Q_{n_k}^{n_k}(z) = \psi_{n_k}^{n_k} \circ P^{\circ n_k} \circ (\psi_0^{n_k})^{-1}(z)$. We now state the Polynomial Implementation Lemma. It is by means of this lemma that we create all polynomials in the proofs of Phases I and II.

Lemma 8. *(The Polynomial Implementation Lemma) Let $\tilde{P}$, $\tilde{U}$, κ, P, U, Ω, Ω', γ, Γ, and f be as above. Suppose $A \subset U$ is open and relatively compact. Then for all $\varepsilon > 0$ and $\delta > 0$, if $\hat{A}$ and $\hat{\hat{A}}$ are δ- and 2δ-neighborhoods of A with respect to*

ρ_U as above, $\check{A}$ is as above, and M is such that $\|f^\natural\|_{\check{A}} \leq M$, there exists $k_0 > 0$ and a (17+κ)-bounded sequence of quadratic polynomials $\{P_m^{n_k}\}_{m=1}^{n_k}$ such that $Q_{n_k}^{n_k}$ is univalent on A and

1. *$\rho_U(Q_{n_k}^{n_k}(z), f(z)) < \varepsilon$ for all $z \in A$, $k \geq k_0$,*

2. *$\|(Q_{n_k}^{n_k})^\natural\|_A \leq M(1+\varepsilon)$,*

3. *$Q_{n_k}^{n_k}(0) = 0$.*

Proof. By construction, $Q_{n_k}^{n_k}(0) = 0$. Let $\varepsilon, \delta > 0$ and without loss of generality take $\varepsilon < \delta$ and $\varepsilon < 3$. As the Euclidean and hyperbolic metrics are equivalent on compact subsets of U (c.f. Lemma 1), we can use Lemma 5 to make k_0 larger if needed so that

$$\rho_U((\psi_0^{n_k})^{-1}(z), z) < \frac{\varepsilon}{3M+1}, \qquad z \in A \tag{1}$$

for all $k \geq k_0$. This also implies

$$(\psi_0^{n_k})^{-1}(A) \subset \hat{A}. \tag{2}$$

Next, by Lemma 7, we can choose k_0 large enough such that

$$|((\psi_0^{n_k})^{-1})^\natural(z) - 1| < \frac{\varepsilon}{3}, \qquad z \in \Delta_U(0, R_{(U,0)}^{ext} A) \supset A \tag{3}$$

for all $k \geq k_0$. We will need this to hold on $\Delta_U(0, R^{ext}_{U,0}A)$ as we wish to apply the hyperbolic M-L estimates and we need these to hold on a set that is hyperbolically convex ($\Delta_U(0, R^{ext}_{U,0}A)$ is hyperbolically convex by 3). Since the hyperbolic metric of U is locally equivalent to the Euclidean metric, we have that $P^{\circ n_k}$ converges locally uniformly to the identity with respect to ρ_U. Then we can make k_0 larger if necessary to ensure

$$\rho_U(P^{\circ n_k}(z), z) < \frac{\varepsilon}{3M+1}, \qquad z \in \hat{A} \tag{4}$$

for all $k \geq k_0$. This also implies

$$P^{\circ n_k}(\hat{A}) \subset \hat{\hat{A}}. \tag{5}$$

Using a similar argument to Lemma 7, we can make k_0 large enough such that

$$|(P^{\circ n_k})^{\natural}(z) - 1| < \frac{\varepsilon}{3}, \qquad z \in \Delta_U(0, R^{ext}_{(U,0)}\hat{A}) \supset \hat{A} \tag{6}$$

for all $k \geq k_0$, where we again insist this holds on the hyperbolically convex set $\Delta_U(0, R^{ext}_{U,0}\hat{A})$ so that we may apply the hyperbolic M-L estimates. Now (2) and (5) imply that $Q^{n_k}_{n_k}$ is univalent on A. Lastly by hypothesis we have $|f^{\natural}(z)| \leq M$ for $z \in \check{A} \supset \hat{\hat{A}}$. Then if $z \in A$ we have, using the above in conjuction with the hyperbolic convexity lemma (Lemma 3) and the hyperbolic M-L estimates (Lemma 4), that

$$\begin{aligned}\rho_U(Q^{n_k}_{n_k}(z), f(z)) &= \rho_U(\psi^{n_k}_{n_k} \circ P^{\circ n_k} \circ (\psi^{n_k}_0)^{-1}(z), f(z)) \\ &\leq \rho_U(f \circ P^{\circ n_k} \circ (\psi^{n_k}_0)^{-1}(z), f \circ P^{\circ n_k}(z)) + \rho_U(f \circ P^{\circ n_k}(z), f(z)) \\ &< M(1+\frac{\varepsilon}{3})(\frac{\varepsilon}{3M+1}) + M(\frac{\varepsilon}{3M+1}) \\ &< \varepsilon.\end{aligned}$$

Also,

$$\begin{aligned}|((Q^{n_k}_{n_k})^\natural)(z)| &= |f^\natural(P^{\circ n_k} \circ (\psi^{n_k}_0)^{-1}(z)) \cdot (P^{\circ n_k})^\natural((\psi^{n_k}_0)^{-1}(z)) \cdot ((\psi^{n_k}_0)^{-1})^\natural(z)| \\ &\leq M(1+\frac{\varepsilon}{3})(1+\frac{\varepsilon}{3}) \\ &< M(1+\varepsilon)\end{aligned}$$

as desired.

□

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CHAPTER 4

Phase I

We begin by finding a suitable disk on which $f \circ g^{-1}$ is defined for $f, g \in \mathcal{S}$.

Lemma 9. *If $f, g \in \mathcal{S}$, then $f \circ g^{-1}$ is defined on* $\mathrm{D}(0, \frac{1}{12})$ *and* $(f \circ g^{-1})(\mathrm{D}(0, \frac{1}{12})) \subset \mathrm{D}(0, \frac{1}{3})$.

Proof. Let $f, g \in \mathcal{S}$. By the Koebe one-quarter theorem we have $\mathrm{D}(0, \frac{1}{4}) \subset g(\mathbb{D})$ so g^{-1} is defined on $\mathrm{D}(0, \frac{1}{12})$. Then if $h(z) := 4g^{-1}(\frac{z}{4})$ for $z \in \mathbb{D}$ we have that $h \in \mathcal{S}$ and $g^{-1}(w) = \frac{1}{4}h(4w)$ for $w \in \mathrm{D}(0, \frac{1}{4})$, where $z = 4w$. Then if $|w| \leq \frac{1}{12}$, then $|z| \leq \frac{1}{3}$. By the distortion theorems we have that $|h(z)| \leq \frac{3}{4}$ and $|g^{-1}(w)| \leq \frac{3}{16}$. Then by distortion the distortion theorems again, if $z \in \mathrm{D}(0, \frac{1}{12})$ we have that $f \circ g^{-1}(z) \leq \frac{48}{169} < \frac{1}{3}$. Thus $f \circ g^{-1}$ is defined on $\mathrm{D}(0, \frac{1}{12})$ for all $f, g \in \mathcal{S}$ and maps $\mathrm{D}(0, \frac{1}{12})$ into $\mathrm{D}(0, \frac{1}{3})$.

□

In the proof of Phase I, we scale the filled Julia set for the polynomial $\tilde{P}(z) = \lambda z(1-z)$, where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$, so that it is a subset of $\mathrm{D}(0, \frac{1}{12})$. We are then able to apply $f \circ g^{-1}$ for $f, g \in \mathcal{S}$ on this filled Julia set. We wish to find a suitable subdomain of this scaled filled-Julia set so that we may control $|f^{\natural}|$ on that subdomain. There are two ways of doing this. One can either consider a small hyperbolic disk in the Siegel disk, or scale $\tilde{P}$ so that the scaled filled Julia set lies inside a small Euclidean disk about 0. We found the second option more convenient, as it allows us to consider an arbitrarily large hyperbolic disk inside the scaled Siegel disk on which $|f^\natural|$ is tame. Lemmas 10 through 15 regard finding a suitable scaling and estimating $|f^\natural|$.

Lemma 10. *There exists $K_1 > 0$ such that for all $f, g \in \mathcal{S}$, if $|z| \leq \frac{1}{24}$, then*

$$|f \circ g^{-1}(z) - z| \leq K_1|z|^2$$

Proof. Let $f, g \in \mathcal{S}$. By Lemma 9 the function $f \circ g^{-1}$ is defined on $\mathrm{D}(0, \frac{1}{12})$. Let $w \in \mathbb{D}$, $z = \frac{1}{12}w$ (note $z \in \mathrm{D}(0, \frac{1}{12})$) and define $h(w) = 12(f \circ g^{-1})(\frac{w}{12})$. We have $h \in \mathcal{S}$. Then setting $K_0 = \sum_{n=2}^{\infty} n^3 (\frac{1}{2})^{n-2}$, if $|w| \leq \frac{1}{2}$ we have

$$\begin{aligned}
|h'(w) - 1| &= \left| w \sum_{n=2}^{\infty} n a_n w^{n-2} \right| \\
&\leq |w| \sum_{n=2}^{\infty} n|a_n||w|^{n-2} \\
&\leq |w| \sum_{n=2}^{\infty} n^3 (\frac{1}{2})^{n-2} \\
&= K_0|w|
\end{aligned}$$

where we used that $|a_n| \leq n^2$ as $h \in \mathcal{S}$. Let $\gamma = [0, z]$ be the radial line segment from 0 to w. Then, if $|w| \leq \frac{1}{2}$.

$$\begin{aligned}
|h(w) - w| &= \left| \int_\gamma h'(\zeta) - 1 \mathrm{d}\zeta \right| \\
&\leq K_0|w| \int_\gamma |\mathrm{d}\zeta| \\
&= K_0|w|^2.
\end{aligned}$$

Then if $|z| \leq \frac{1}{24}$ (note $|w| \leq \frac{1}{2}$) we have

$$|h(w) - w| \leq K_0|w|^2$$
$$|12f \circ g^{-1}(\frac{w}{12}) - w| \leq K_0|w|^2$$
$$|12f \circ g^{-1}(z) - 12z| \leq K_0|12z|^2$$
$$|f \circ g^{-1}(z) - z| \leq 12K_0|z|^2,$$

from which the lemma follows by setting $K_1 = 12K_0$. □

Let $\tilde{P}$ be as above and let $\tilde{U}$ be the Siegel disc corresponding to $\tilde{P}$. Now fix $R > 0$ and let $\tilde{U}_R = \Delta_{\tilde{U}}(0, R)$. Let $\tilde{\psi} : \tilde{U} \to \mathbb{D}$ be the unique Riemann map satisfying $\tilde{\psi}(0) = 0$, $\tilde{\psi}'(0) > 0$. Let $\tilde{r}_0 = \tilde{r}_0(R) = d(\tilde{U}_R, \tilde{U})$. If $\kappa > 1$, then set $P = P_\kappa = \frac{1}{\kappa}\tilde{P}(\kappa z)$ and note that P obviously depends on κ. Then if $\mathcal{K} = \mathcal{K}(\kappa)$ is the filled Julia set for P, we have $\mathcal{K} \subset \mathrm{D}(0, \frac{2}{\kappa})$. Let $U = U(\kappa)$ be the Siegel disk for P and note that $U = \{z \in \mathbb{C} : z = \frac{w}{\kappa} \text{ for some } w \in \tilde{U}\}$. Let $U_R = \Delta_U(0, R)$. Define $\psi = \psi_\kappa = \tilde{\psi}(\kappa z)$ and observe that ψ is the unique Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$ and $\psi'(0) > 0$. Also define $r_0 = r_0(\kappa, R) = d(\partial U_R, \partial U)$ and note $r_0 = \frac{\tilde{r}_0}{\kappa}$. Observe that $\tilde{r}_0$ and r_0 are decreasing in R.

Lemma 11. *(Local Distortion) Let $\tilde{P}$, $\tilde{U}$, $\tilde{\psi}$, κ, P. U and ψ be as above. Then for all $R > 0$, there exists $C_0 = C_0(R)$ depending on R (in particular, C_0 is independent of κ) which is increasing, real valued, and (thus) bounded on any bounded subset of $(0, \infty)$ such that if $\tilde{U}_R$, $\tilde{r}_0$, U_R and r_0 are all as above, and $z_0 \in \overline{U_R}$, $z \in U$ with $|z - z_0| \leq s < r_0$, we have*

1. $|\psi(z)-\psi(z_0)| \leq \frac{C_0 \frac{s}{r_0}}{(1-\frac{s}{r_0})^2}$

2. $\frac{1-\frac{s}{r_0}}{(1+\frac{s}{r_0})^3} \leq |\frac{\psi'(z)}{\psi'(z_0)}| \leq \frac{1+\frac{s}{r_0}}{(1-\frac{s}{r_0})^3}$

Proof. Set $C_0 = C_0(R) = 2\max_{z\in\overline{\tilde{U}}_R} |\tilde{\psi}'(z)|$. Then $C_0(R)$ is clearly increasing, real valued, and therefore bounded on any bounded subinterval of $(0,\infty)$. Let $\zeta = \frac{z-z_0}{r_0}$ and note that if we define $\varphi(\zeta) = \frac{\psi(r_0\zeta+z_0)-\psi(z_0)}{r_0\psi'(z_0)}$ we have that $\varphi \in \mathcal{S}$. Applying the distortion theorems to φ we see

$$\begin{aligned}
|\varphi(\zeta)| &\leq \frac{|\zeta|}{(1-|\zeta|^2)} \\
&\leq \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2}
\end{aligned}$$

and thus

$$\begin{aligned}
|\psi(z)-\psi(z_0)| &\leq \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot r_0 \cdot |\psi'(z_0)| \\
&\leq \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot r_0 \cdot \max_{w\in\overline{U_R}} |\psi'(w)| \\
&= \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot r_0 \cdot \kappa \max_{w\in\overline{\tilde{U}_R}} |\tilde{\psi}'(w)| \\
&= \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot \tilde{r}_0 \cdot \max_{w\in\overline{\tilde{U}_R}} |\tilde{\psi}'(w)| \\
&\leq \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot 2\max_{w\in\overline{\tilde{U}_R}} |\tilde{\psi}'(w)| \\
&= \frac{\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \cdot C_0
\end{aligned}$$

which proves 1. For 2. we again apply the distortion theorems to φ and observe

$$\frac{1-\frac{s}{r_0}}{(1+\frac{s}{r_0})^3} \leq \frac{1-|\zeta|}{(1+|\zeta|)^3} \leq |\varphi'(\zeta)| \leq \frac{1+|\zeta|}{(1-|\zeta|)^3} \leq \frac{1+\frac{s}{r_0}}{(1-\frac{s}{r_0})^3},$$

from which 2. follows as $\varphi'(\zeta) = \frac{\psi'(z)}{\psi'(z_0)}$. □

Lemma 12. *Let $\tilde{P}$, $\tilde{U}$, $\tilde{U}_R$, $\tilde{r}_0$, κ, P, U, U_R, and r_0 be as above. Suppose $f, g \in \mathcal{S}$ and $z \in U$. Then for all $\eta > 0$ and $R > 0$, there exists $\kappa_0 = \kappa_0(\eta, R)$ such that for all $\kappa \geq \kappa_0$*

$$|f \circ g^{-1}(z) - z| \leq \eta r_0.$$

In particular, this holds for $z \in \overline{U_R}$.

Proof. Fix $\kappa_0 \geq 48$. By Lemma 10 we have, on $U \subset \mathrm{D}(0, \frac{2}{\kappa}) \subset \mathrm{D}(0, \frac{1}{24})$, that $|f \circ g^{-1}(z) - z| < K_1|z|^2$ for some $K_1 > 0$ (note that $f \circ g^{-1}$ is defined by Lemma 9). So $|f \circ g^{-1}(z) - z| < \frac{4K_1}{\kappa^2}$ since $|z| < \frac{2}{\kappa}$. Then make κ_0 larger if necessary to ensure that $\frac{4K_1}{\kappa^2} < \eta r_0$ for all $\kappa \geq \kappa_0$. In fact, $\kappa_0 = \max\{48, \sqrt{\frac{4K_1}{\eta r_0}}\}$ will suffice.

□

Lemmas 10 through 12 are technical lemmas that assist in proving the following result:

Lemma 13. *Let $\tilde{P}$, $\tilde{U}$, $\tilde{U}_R$, $\tilde{\psi}$, κ, P, U, U_R, and ψ be as above. Suppose $f, g \in \mathcal{S}$ and $z \in \overline{U_R}$. Then, for all $R > 0$ there exists $\kappa_0 = \kappa_0(R) > 0$ such that for all $\kappa \geq \kappa_0$*

1. $\frac{1-|\psi(z)|^2}{1-|\psi((f\circ g^{-1})(z))|^2} \le \frac{10}{9}$

2. $\frac{|\psi'((f\circ g^{-1})(z))|}{|\psi'(z)|} \le \frac{9}{8}$

Proof. If $z \in \overline{U_R}$ we have that $|\psi(z)| \le \frac{e^R-1}{e^R+1} := c_R$ (recall that $\rho_{\mathbb{D}} = \log(\frac{1+|z|}{1-|z|})$). Thus $c_R < 1$ and $1 - |\psi(z)|^2 > 1 - c_R^2 > 0$. As in the proof of Lemma 11, set $C_0 = C_0(R) = 2\max_{z\in\overline{\tilde{U}}_R} |\tilde{\psi}'(z)|$ Let $0 < \eta_0 = \eta_0(R) < 1$ be such that

$$\frac{C_0\eta_0}{(1-\eta_0)^2} \le \frac{\log 10 - \log 9}{2(1-c_R^2)}. \tag{7}$$

and note that η_0 depends only on R. Using Lemma 12, we can pick $\kappa_1 = \kappa_1(\eta(R), R) = \kappa_1(R) > 0$ such that if $\kappa \ge \kappa_1$, then $|f \circ g^{-1}(z) - z| < \eta_0 r_0$ (recall the definitions of $\tilde{r_0}$ and r_0 before Lemma 11). Fix $z_0 \in \overline{U_R}$. Then for some $s > 0$ we have $|f \circ g^{-1}(z_0) - z_0| = s < \eta_0 r_0 < r_0$ as $\eta_0 < 1$, so $f \circ g^{-1}(z_0) \in U$ using the definition of r_0 (note that $\frac{s}{r_0} < \eta_0$). We may then apply 1. of Lemma 11 to see

$$\begin{aligned} |\psi(z_0) - \psi((f\circ g^{-1})(z_0))| &\le \frac{C_0\frac{s}{r_0}}{(1-\frac{s}{r_0})^2} \\ &\le \frac{C_0\eta_0}{(1-\eta_0)^2} \\ &\le \frac{\log 10 - \log 9}{2(1-c_R^2)}. \end{aligned}$$

Further, using the triangle inequality we see that

$$|(1-|\psi(z_0)|^2) - (1-|\psi((f\circ g^{-1})(z_0))|^2)| < \frac{\log 10 - \log 9}{1-c_R^2}.$$

Applying the Mean Value Theorem to the logarithm function on the interval $[1 - c_R^2, \infty)$ we have

$$|\log(1 - |\psi(z_0)|^2) - \log(1 - |\psi((f \circ g^{-1})(z_0))|^2)| < \log 10 - \log 9.$$

from which 1. follows easily. For 2., let $0 < \eta_1 < 1$ be such that

$$\frac{1 + \eta_1}{(1 - \eta_1)^3} < \frac{9}{8}.$$

By Lemma 12 we can pick $\kappa_2 = \kappa_2(R) > 0$ such that for all $\kappa \geq \kappa_2$, if $z \in \overline{U_R}$

$$|(f \circ g^{-1})(z) - z| < \eta_1 r_0.$$

Let $z_0 \in \overline{U_R}$. Then, similarly to the proof of 1., we can use 2. of Lemma 11 to see

$$\frac{|\psi'((f \circ g^{-1})(z))|}{|\psi'(z)|} \leq \frac{9}{8}$$

as desired. The result follows if we set $\kappa_0 = \kappa_0(R) = \max\{\kappa_1(R), \kappa_2(R)\}$.

□

Lemma 14. *Let $\tilde{P}$, $\tilde{U}$, κ, P and U be as above. Suppose $f, g \in \mathcal{S}$ and $z \in \overline{U}$. There exists $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$*

$$|(f \circ g^{-1})'(z)| \leq \frac{6}{5}.$$

Proof. Set $\kappa_0 = 576$ and let $w \in \mathbb{D}$. As in the proof of Lemma 10, define $h(w) = 12(f \circ g^{-1})(\frac{w}{12})$. Note that $h \in \mathcal{S}$ and h is defined by Lemma 9. Let $z = \frac{w}{12}$. Using the distortion theorems we have that

$$|h'(w)| \leq \frac{1 + |w|}{(1 - |w|)^3},$$

so

$$|(f \circ g^{-1})'(z)| \leq \frac{1 + |12z|}{(1 - |12z|)^3}. \tag{8}$$

If $\kappa \geq \kappa_0$ we have that $\mathrm{D}(0, \frac{2}{\kappa}) \subset \mathrm{D}(0, \frac{2}{\kappa_0}) = \mathrm{D}(0, \frac{1}{288})$. Let $z \in \overline{U}$ and since $\overline{U} \subset \mathcal{K} \subset \mathrm{D}(0, \frac{2}{\kappa}) \subset \mathrm{D}(0, \frac{1}{288})$, we have $|z| < \frac{1}{288}$ for $\kappa \geq \kappa_0$. Thus (8) is less than $\frac{25 \cdot 24^2}{23^3}$, which in turn is less than $\frac{6}{5}$ for all $\kappa \geq \kappa_0$ as desired. ∎

As all the previous lemmas hold for all κ sufficiently large, applying them in tandem in the next result is valid. In general each lemma may require a different choice of κ_0, but we may choose the maximum one so that all results hold simultaneously. Lemmas 13 and 14 are designed to prove:

Lemma 15. *Let $\tilde{P}$, $\tilde{U}$, $\tilde{U}_R$, $\tilde{\psi}$, κ, P, U, U_R, and ψ be as above. Suppose $f, g \in \mathcal{S}$ and $z \in \overline{U_R}$. Then*

$$|(f \circ g^{-1})^{\natural}(z)| \leq \frac{3}{2}.$$

Proof. Applying Lemmas 13 and 14, we have that there exists a κ_0 such that for all $\kappa \geq \kappa_0$

$$\begin{aligned}|(f \circ g^{-1})^{\natural}(z)| &= \frac{1 - |\psi(z)|^2}{1 - |\psi((f \circ g^{-1})(z))|^2} \cdot \frac{|\psi'((f \circ g^{-1})(z))|}{|\psi'(z)|} \cdot |(f \circ g^{-1})'(z)| \\ &\leq \frac{10}{9} \cdot \frac{9}{8} \cdot \frac{6}{5} \\ &= \frac{3}{2}\end{aligned}$$

as desired.

□

Statement and Proof of Phase I

Lemma 16. *(Phase I) Let $\tilde{P}$, $\tilde{U}$, $\tilde{U}_R$, κ, P, U, and U_R be as above. For all $\varepsilon > 0$, $R > 0$, and $N \in \mathbb{N}$, if $\{f_i\}_{i=0}^{N+1}$ is a collection of mappings with $f_i \in \mathcal{S}$ for $i = 0, 1, 2, \cdots, N+1$ with $f_0 = f_{N+1} = \mathrm{Id}$, there exists $\kappa_0 = \kappa_0(R) > 0$, $M_N = M_N(\varepsilon, N) \in \mathbb{N}$, and a $(17 + \kappa)$-bounded finite sequence of quadratic polynomials $\{P_m\}_{m=1}^{(N+1)M_N}$ such that, for all $\kappa \geq \kappa_0$ and all $1 \leq i \leq N+1$,*

1. *Q_{iM_N} is univalent on U_{5R}.*
2. *$Q_{iM_N}(U_{2R}) \subset U_{4R}$.*
3. *$\rho_U(f_i(z), Q_{iM_N}(z)) < \varepsilon$ on U_{2R}.*
4. *$\|Q^{\natural}_{iM_N}\|_{U_R} \leq C$ for some $0 < C < \infty$.*

5. $Q_{(N+1)M_N}(0) = 0$.

In fact, we can take $C = 7$ above.

Proof. **Step 1:** Setup

Let $\varepsilon > 0$ and $R > 0$. Let $\kappa_0 = 576$ (we will make κ_0 larger if necessary later). Then for all $\kappa \geq \kappa_0$ we have $U \subset \mathcal{K} \subset \mathrm{D}(0, \frac{2}{\kappa}) \subset \mathrm{D}(0, \frac{1}{288}) \subset \mathrm{D}(0, \frac{1}{12})$. Note that the last inclusion implies that if $f, g \in \mathcal{S}$ that $f \circ g^{-1}$ is defined on U by Lemma 9.

Step 2: Polynomial Implementation Lemma.

Let $\delta > 0$ and, making κ_0 larger if necessary, apply Lemma 15 so that, if $f, g \in \mathcal{S}$, we have for all $\kappa \geq \kappa_0$

$$\|(f \circ g^{-1})^\natural\|_{U_{5R+2\delta}} \leq \frac{3}{2}. \tag{9}$$

Note that, by Lemma 4, this implies

$$f \circ g^{-1}(U_{2R}) \subset U_{3R}. \tag{10}$$

Remark: Since $Id \in \mathcal{S}$ we have $f(U_{2R}) \subset U_{3R}$ for all $f \in \mathcal{S}$.

For each $0 \leq i \leq N$, apply the Polynomial Implementation Lemma (Lemma 8), with $\Omega = \mathrm{D}(0, \frac{1}{24})$, $\Omega' = \mathrm{D}(0, \frac{1}{2})$, $\gamma = \mathrm{C}(0, \frac{1}{24})$, $\Gamma = \mathrm{C}(0, \frac{1}{2})$, $A = U_{5R}$, $\hat{\hat{A}} = \check{A} = U_{5R+2\delta}$ and $f = f_{i+1} \circ f_i^{-1}$. From this lemma we obtain M_N (taking a suitable maximum, if necessary) and $\{P_m \; : \; iM_N + 1 \leq m \leq (i+1)M_N, \; 0 \leq i \leq N\}$ such that for $z \in U_{5R}$ we have that $Q_{iM_N,(i+1)M_N}$ is univalent on U_{5R} and

$$\rho_U(Q_{iM_N,(i+1)M_N}(z), f_{i+1} \circ f_i^{-1}(z)) < \frac{\varepsilon}{3^{N+1}} \tag{11}$$

It also follows from the Polynomial Implementation Lemma that $Q_{(N+1)M_N}(0) = 0$, proving 5.

Step 3: Estimates on the compositions $\{Q_{iM_N}\}_{i=1}^{N+1}$

We use the following claim to prove 1., 2., and 3. in the statement of Phase I:

Claim: For each $1 \leq i \leq N+1$, we have that Q_{iM_N} is univalent on U_{2R} and

1. $\rho_U(Q_{iM_N}(z), f_i(z)) < \dfrac{\varepsilon}{3^{N+1-i}}$
2. $\rho_U(Q_{iM_N}(z), 0) < 4R$

for $z \in U_{2R}$.

Note that the error in the polynomial approximation for $i = 1$ is the smallest as this error needs to pass through the greatest number of mappings.

We prove the claim by induction on i. Let $z \in U_{2R}$. For the base case $i = 1$, we have that univalence and 1. in the claim follow immediately from the Polynomial Implementation Lemma. For 2., recall that $f_0 = Id$ and compute

$$\begin{aligned}\rho_U(Q_{M_N}(z), 0) &\leq \rho_U(Q_{M_N}(z), f_1(z)) + \rho_U(f_1(z), 0) \\ &< \frac{\varepsilon}{3^{N+1}} + 3R \\ &< 4R,\end{aligned}$$

which completes the proof of the base case. Now suppose the claim holds for some $0 < k < N + 1$. Then

$$\begin{aligned}\rho_U(Q_{(k+1)M_N}(z), f_{k+1}(z)) \leq \rho_U(Q_{kM_N,(k+1)M_N} \circ Q_{kM_N}(z), (f_{k+1} \circ f_k^{-1}) \circ Q_{kM_N}(z)) + \\ \rho_U((f_{k+1} \circ f_k^{-1}) \circ Q_{kM_N}(z), (f_{k+1} \circ f_k^{-1}) \circ f_k(z))\end{aligned}$$

Now since $Q_{kM_N}(z) \in U_{4R}$ by the induction hypothesis, so (11) implies that the first term in the inequality above is less than $\frac{\varepsilon}{3^{N+1}}$. By induction hypothesis $Q_{kM_N}(z) \in U_{4R} \subset U_{5R}$ while we also have $f_k(z) \in U_{3R} \subset U_{5R}$ by (10). Thus (9) implies that the second term in the inequality is less than $\frac{1}{2} \cdot \frac{\varepsilon}{3^{N+1-k}}$. Thus we have $\rho_U(Q_{(k+1)M_N}(z), f_{k+1}(z)) < \frac{\varepsilon}{3^{N+1-(k+1)}}$, proving the first part of the claim.

Also,

$$
\begin{aligned}
\rho_U(Q_{(k+1)M_N}(z),0) &\leq \rho_U(Q_{(k+1)M_N}(z), f_{k+1}(z)) + \rho_U(f_{k+1}(z),0) \\
&< \frac{\varepsilon}{3^{N+1-(k+1)}} + 3R \\
&< 4R
\end{aligned}
$$

using what we just proved and (10), which proves 2. in the claim. Univalence of each Q_{iM_N} follows as $Q_{kM_N}(U_{2R}) \subset U_{4R}$, and $Q_{kM_n,(k+1)M_N}$ is univalent on $A = U_{5R} \supset U_{4R}$ by the Polynomial Implementation Lemma. This completes the proof of the claim, from which 1., 2., and 3. in the statement of Phase I follow. $\Diamond$

Step 4: Proving 4. in the statement.

Let $\mathrm{d}\rho_U(z)$ be the hyperbolic length element in U. Write $\mathrm{d}\rho_U(z) = \sigma_U(z)|\mathrm{d}z|$, where σ_U is continuous on U, and therefore uniformly continuous on U_{3R}, as U_{3R} is relatively compact in U. Let $\sigma > 0$ be the infimum of σ_U on U_{3R} Let $z \in U_{2R}$.

Now 3. in the statement together with the Schwarz lemma for the hyperbolic metric give for $i \leq i \leq N+1$ $\rho_{\mathbb{D}}(Q_{iM_N}(z), f_i(z)) < \rho_U(Q_{iM_N}(z), f_i(z)) < \varepsilon$. Then if γ is a geodesic in $\mathbb{D}$ from $Q_{iM_N}(z)$ to $f_i(z)$ we see that

$$\begin{aligned}
\varepsilon &> \rho_U(Q_{iM_N}(z), f_i(z)) \\
&\geq \rho_{\mathbb{D}}(Q_{iM_N}(z), f_i(z)) \\
&= \int_\gamma \mathrm{d}\rho_{\mathbb{D}} \\
&= \int_\gamma \frac{2|\mathrm{d}w|}{1-|w|^2} \\
&\geq \int_\gamma 2|\mathrm{d}w| \\
&= 2l(\gamma) \\
&\geq 2|Q_{iM_N}(z) - f_i(z)|
\end{aligned}$$

so in particular, for all $z \in U_{2R}$,

$$|Q_{iM_N}(z) - f_i(z)| < \varepsilon$$

Now suppose further that $z \in U_R$. If we set $\delta = \min_{w \in \partial U_R} \operatorname{dist}(w, \partial U_{\frac{3}{2}R})$ then using Corollary IV.5.9 in [1] and the Jordan curve theorem we see

$$\begin{aligned}
|Q'_{iM_N}(z)| &\leq |f'_i(z)| + |Q'_{iM_N}(z) - f'_i(z)| \\
&= |f'_i(z)| + \left| \int_{\partial U_{\frac{3}{2}R}} \frac{Q_{iM_N}(w) - f_i(w)}{(w-z)^2} \right| \\
&\leq \frac{1 + \frac{1}{576}}{(1 - \frac{1}{576})^3} + \frac{\varepsilon}{\delta^2} l(\partial U_{\frac{3}{2}R})
\end{aligned}$$

Where $l(\partial U_{\frac{3}{2}R})$ is the Euclidean length of $\partial U_{\frac{3}{2}R}$. By making ε smaller if needed, we can ensure, for $z \in U_R$, that

$$|Q'_{iM_N}(z)| \leq \frac{3}{2} \tag{12}$$

We can make ε smaller still to guarantee that if $z, w \in U_{3R}$, and $|z - w| < \varepsilon$, then, by uniform continuity of σ_U,

$$|\sigma_U(z) - \sigma_U(w)| < \sigma$$

Then if $z \in U_R$ we have

$$\begin{aligned}
|Q^\natural_{iM_N}(z)| &\leq |f^\natural_i(z)| + |Q^\natural_{iM_N}(z) - f^\natural_i(z)| \\
&= |f^\natural_i(z)| + \left| \frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} f'_i(z) \right| \\
&\leq |f^\natural_i(z)| + \left| \frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z) \right| + \\
&\left| \frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} f'_i(z) \right|
\end{aligned}$$

We need to bound each of the three pieces on the right hand side of the above inequality. Recall that, as $g = iD \in \mathcal{S}$, we have that $|f^\natural_i(z)| \leq \frac{3}{2}$. For the second term, we have

$$\left|\frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)}Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)}Q'_{iM_N}(z)\right| = \frac{1}{|\sigma_U(z)|}\cdot|Q'_{iM_N}(z)|\cdot|\sigma_U(Q_{iM_N}(z)) - \sigma_U(f_i(z))|$$
$$\leq \frac{1}{\sigma}\cdot\frac{3}{2}\cdot\sigma$$
$$= \frac{3}{2}$$

where we note $|\sigma_U(Q_{iM_N}(z)) - \sigma_U(f_i(z))| \leq \sigma$ using 3. in the statement of Phase I, the local equivalence of the Euclidean and hyperbolic metrics, and the uniform continuity of σ. For the third term, we can apply Lemma 13 to make κ_0 larger if necessary to ensure

$$\left|\frac{\sigma_U(f_i(z))}{\sigma_U(z)}Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)}f'_i(z)\right| \leq \left|\frac{\sigma_U(f_i(z))}{\sigma_U(z)}\right|\cdot(|Q'_{iM_N}(z)| + |f'_i(z)|)$$
$$\leq \frac{10}{9}\cdot\frac{9}{8}(\frac{3}{2} + \frac{6}{5})$$
$$\leq 4$$

Thus

$$|Q^{\natural}_{iM_N}(z)| \leq \frac{3}{2} + \frac{3}{2} + 4$$
$$= 7$$

as desired.

□

List of References

[1] J. Conway, *Functions of One Complex Variable I.* Berlin, Germany: Springer Verlag, 1978.

CHAPTER 5

Phase II

Our first objective here is to prove The Fitting Lemma, one of the key tools in controlling loss of domain in the proof of Phase II and indeed in controlling loss of domain in the main result. We will be interpolating functions between Green's Lines of a scaled version of the polynomial $\tilde{P} = \lambda z(1-z)$ where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$. If we denote the Green's function by G, we will want to be able to choose h small enough so that the region between the sets $\{G = h\}$ and $\{G = 2h\}$ has small area. This will eventually allow us to conclude that we get good approximations to the inverse of a given error function on the domain, which will in turn allow us to control the loss of domain. On the other hand, we will want h to be large enough so that, if we distort the Green's lines slightly (with that same error function), the distorted region between them will still be a conformal annulus. We must first prove several technical lemmas.

Several Technical Lemmas

Definition 5.0.1. *Let* $\mathcal{U} = \{(U_h, u_h)\}_{h \in I}$ *be a sequence of pointed domains indexed by a set* $I \subset \mathbb{R}$*. We say that* $\mathcal{U}$ *varies continuously in the Carathéodory topology at* h_0 *or is continuous at* h_0 *if* $(U_h, u_h) \to (U_{h_0}, u_{h_0})$ *as* $h \to h_0$*. If this property holds for all* $h \in I$*, we say* $\mathcal{U}$ *varies continuously in the Carathéodory topology over* I*.*

Definition 5.0.2. *Let $I \subset \mathbb{R}$ and let $\{\gamma_t\}_{t \in I}$ be a family of Jordan curves indexed by I. We say that $\{\gamma_t\}$ is a continuously varying family of Jordan curves if we can find a parameterization $F(z,t)$ of γ_t, where $F : \mathbb{T} \times I \to \mathbb{C}$ is continuous on $\mathbb{T} \times I$ and injective in the first coordinate.*

Lemma 17. *Let $\{\gamma_t\}$ be a continuously varying family of Jordan curves. Then if U_t is the Jordan domain which is the bounded component of $\overline{\mathbb{C}} \setminus [\gamma_t]$ and there exists $u : I \to \mathbb{C}$ continuous with $u(t) \in U_t$ for all t, then the family $\{(U_t, u(t))\}$ varies continuously in the Carathéodory topology.*

Proof. The continuity of u implies (i) of Carathéodory convergence. For (ii), fix $t_0 \in I$. Let $K \subset U_{t_0}$ be compact and let $z \in K$. Let $\delta = d(K, \partial U_{t_0})$, and for each $z \in K$, form the open ball $\mathrm{D}(z, \frac{\delta}{2})$. The union of these balls forms an open cover of K, so we may select a finite subcover $\{\mathrm{D}(z_i, \frac{\delta}{2})\}_{i=1}^{n}$. If $w \in K$, then $w \in \mathrm{D}(z_i, \frac{\delta}{2})$ for some $1 \leq i \leq n$. Now $n(\gamma_{t_0}, w) = 1$ by the Jordan curve theorem. By the uniform continuity of F on compact subsets of $\mathbb{T} \times I$, we can find δ_i such that $n(\gamma_t, w) = 1$ for all t satisfying $|t - t_0| < \delta_i$ (recall that the winding number is integer valued). Setting $\delta_0 = \min_{1 \leq i \leq n} \delta_i$, we have that $n(\gamma_t, w) = 1$ for all $w \in K$ if t satisfies $|t - t_0| < \delta_0$. Thus (ii) of Carathéodory convergence is satisfied. Now let $\{t_n\}$ be any sequence in I which converges to t_0 and suppose N is an open connected set containing $u(t_0)$ such that $N \subset U_{t_n}$ for infinitely many n. Without loss of generality we may pass to a subsequence to assume that $N \subset U_{t_n}$ for all n. Let $z \in N$ and connect z to $u(t_0)$ by a curve η in N. As $[\eta]$ is compact, there exists $\delta > 0$ such that a Euclidean δ-neighborhood of $[\eta]$ avoids γ_{t_n} for all n.

By the continuity of F, this neighborhood also avoids γ_{t_0}. Since $u(t_0)$ and z are connected by η which avoids γ_{t_0}, they are in the same region determined by γ_{t_0}. Thus $n(\gamma_{t_0}, z) = n(\gamma_{t_0}, u(t_0)) = 1$. Hence $z \in U_{t_0}$ by the Jordan curve theorem. As z is arbitrary, we have $N \subset U_{t_0}$ and (iii) of Carathéodory convergence and the result then follow. □

Lemma 18. *Let I be an interval, $\mathcal{U} = \{(U_h, v_h)\}_{h \in I}$ be a sequence of pointed Jordan domains, and $\mathcal{V} = \{(V_h, v_h)\}_{h \in I}$ be a sequence of pointed simply connected domains with the same base points, indexed over I. If $\mathrm{pt} \sqsubset \mathcal{U} \sqsubset \mathcal{V} \sqsubset \overline{\mathbb{C}}$, $\mathcal{U}$ and $\mathcal{V}$ vary continuously in the Carathéodory topology over I, and ∂U_h is a continuously varying family of Jordan curves on I, then $R^{ext}_{(V_h, v_h)} U_h$ is continuous on I.*

Proof. Let $h_0 \in I$ be arbitrary. As ∂U_h is a continuously varying family of Jordan curves, let $F : \mathbb{T} \times I \to \mathbb{C}$ be a continuous mapping (with $I \subset \mathbb{R}$ an interval), injective in the first coordinate, where for each h fixed, $F(t, h)$ maps $\mathbb{T}$ to ∂U_h. Let φ_h be the unique normalized Riemann map from V_h to $\mathbb{D}$ satisfying $\varphi_h(v_h) = 0$, $\varphi_h'(v_h) > 0$. Let $\Phi = \{\varphi_h\}_{h \in I}$ be the corresponding family. As $\mathrm{pt} \sqsubset \mathcal{V} \sqsubset \overline{\mathbb{C}}$, we have $\Phi \bowtie \mathcal{V}$ by [1] Theorem 4.15. Since $\mathrm{pt} \sqsubset \mathcal{U} \sqsubset \mathcal{V}$ there exists $0 < \rho < 1$ such that $\varphi_h(U_h) \subset \mathrm{D}(0, \rho)$ and thus $R^{ext}_{(V_h, v_h)} U_h \le \rho$. Also, as $h \to h_0$, we know that $\varphi_h \rightrightarrows \varphi_{h_0}$ locally uniformly on V_{h_0} as $(V_h, v_h) \to (V_{h_0}, v_{h_0})$ by Theorem 8. If we then let π_h be the (unique) inverse Riemann map from $\mathbb{D}$ to U_h, normalized so that $\pi_h(0) = 0$ and $\pi_h'(0) > 0$, then $\varphi_h \circ \pi_h$ converges to $\varphi_{h_0} \circ \pi_{h_0}$ locally uniformly on $\mathbb{D}$ by [2] Proposition 3.2. Using Theorem 8 again, we see $(\varphi_h(U_h), 0) \to (\varphi_{h_0}(U_{h_0}), 0)$. Now, let $\tilde{\varphi}(z, h) = \varphi_h(z)$.

Claim: For all $h_0 \in I$ and $z_0 \in V_{h_0}$, $\tilde{\varphi}(z, h)$ is jointly continuous in z, h on a suitable neighborhood of (z_0, h_0).

Proof (of claim): Let $\varepsilon > 0$. Let $\{h_n\}$ be a sequence in I which converges to h_0 and $\{z_n\}$ be a sequence in V_{h_0} which converges to z_0. Using (ii) of Carathéodory convergence and the fact that V_{h_0} is open, we have that $z_n \in V_{h_n}$ for all sufficiently large n. Then for z_n and h_n sufficiently close to z_0 and h_0, respectively, we have

$$\begin{aligned}
|\tilde{\varphi}(z_n, h_n) - \tilde{\varphi}(z_0, h_0)| &= |\varphi_{h_n}(z_n) - \varphi_{h_0}(z_0)| \\
&\leq |\varphi_{h_n}(z_n) - \varphi_{h_0}(z_n)| + |\varphi_{h_0}(z_n) - \varphi_{h_0}(z_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

which proves the claim. Note that we are using the fact that $\varphi_h \rightrightarrows \varphi_{h_0}$ locally uniformly on V_{h_0} and that φ_{h_0} is continuous. $\Diamond$

Thus if we define $\psi(t, h) := \tilde{\varphi}(F(t, h), h)$, we have that $\psi(t, h)$ is jointly continuous in t and h. If we write $R_n = R^{ext}_{(V_{h_n}, v_{h_n})} U_{h_n}$ and $R_0 = R^{ext}_{(V_{h_0}, v_{h_0})} U_{h_0}$, we wish to show that as $h \to h_0$ that $R_n \to R_0$. As pt $\sqsubset \mathcal{U} \sqsubset \mathcal{V}$, we may choose a subsequence $\{R_{n_k}\}$ which converges. If we can show that the limit is R_0, we will have completed the proof. For each k, we have that R_{n_k} is attained at some

$z_{n_k} \in \partial U_{n_k}$, so we may write $R_{n_k} = \rho_{V_{n_k}}(0, \tilde{\varphi}(z_{n_k}, h_{n_k}))$ for some $z_{n_k} \in \partial U_{h_{n_k}}$. Now $z_{n_k} = F(t_{n_k}, h_{n_k})$ for some $t_{n_k} \in \mathbb{T}$, so $R_{n_k} = \rho_{V_{n_k}}(0, \psi(t_{n_k}, h_{n_k}))$. As $h_{n_k} \to h_0$, in passing to a further subsequence if necessary, we have that $(t_{n_k}, h_{n_k}) \to (t_0, h_0)$ for some $t_0 \in \mathbb{T}$. Observe that there is no loss of generality in passing to a further subsequence.

Claim: $R_0 = \rho_{\mathbb{D}}(0, \psi(t_0, h_0))$

Proof (of claim): Suppose not. Then there exists $(\tilde{t}_0, h_0) \in \mathbb{T} \times I$ such that $|\psi(\tilde{t}_0, h_0)| > |\psi(t_0, h_0)|$. Choose a sequence $\{(\tilde{t}_{n_k}, h_{n_k})\}$ in $\mathbb{T} \times I$ which converges to $(\tilde{t}_0, h_0)$. Then by joint continuity of ψ there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have that $|\psi(\tilde{t}_{n_k}, h_{n_k})| > |\psi(t_{n_k}, h_{n_k})|$, which contradicts the fact that $R_{n_k} = \rho_{\mathbb{D}}(0, \psi(t_{n_k}, h_{n_k}))$. This completes the proof of both the claim and the lemma.

□

Let $\tilde{P} = \lambda z(1-z)$ where $\lambda = e^{\frac{2\pi i(\sqrt{5}-1)}{2}}$. For $\kappa > 1$, define $P = \frac{1}{\kappa}\tilde{P}(\kappa z)$ and let G be the Green's function for this polynomial. Set $V_h := \{z \in \overline{\mathbb{C}} \; : \; G(z) < h\}$.

Lemma 19. *The family $\{(\partial V_h, 0)\}_{h>0}$ gives a continuously varying family of Jordan curves.*

Proof. Let P be as above, let $\mathcal{K}$ be the filled Julia set for P, and let $\varphi : \overline{\mathbb{C}} \backslash \mathcal{K} \to \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ be the Böttcher map. Then the map $F : \mathbb{T} \times (0, \infty) \to \mathbb{C}$, $F(\theta, h) \mapsto \varphi^{-1}(e^{h+i\theta})$ is the desired mapping for a continuously varying family of Jordan curves. □

Before the statement of the next lemma, we note the following fact: suppose $\varphi : (0,\infty) \to (0,\infty)$ is a continuous function. Then, for all $y > \inf\{\varphi(x) : x \in (0,\infty)\}$, the set $\{x : \varphi(x) \le y\}$ is nonempty.

Lemma 20. *Let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous function such that $\varphi(x) \to \infty$ as $x \to 0_+$. Then, for all $y > \inf\{\varphi(x) : x \in (0,\infty)\}$, if we let*

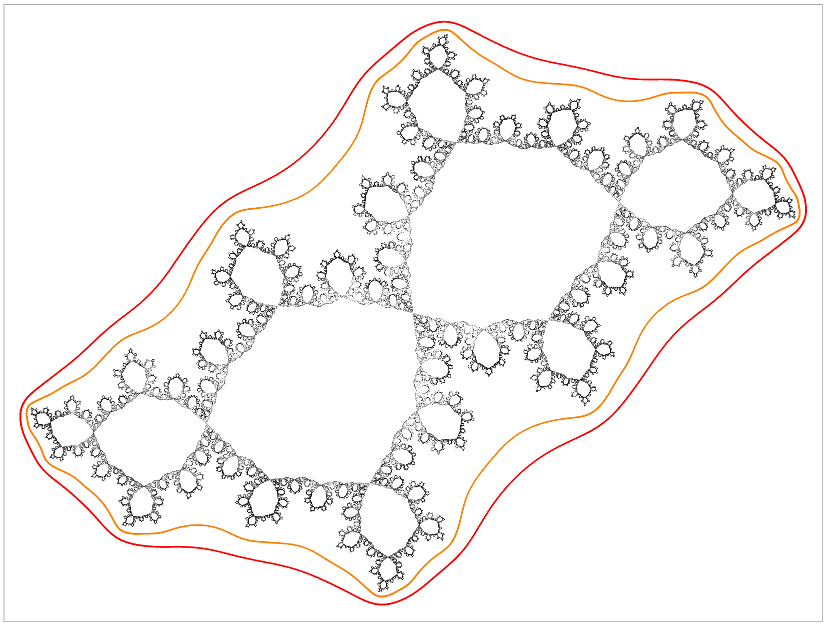
$$x(y) = \inf\{x : \varphi(x) \le y\},$$

we have $x(y) \to 0_+$ as $y \to \infty$.

Proof. Suppose not. Suppose there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \to \infty$ but $x(y_n) \not\to 0_+$ as $n \to \infty$. Set $x_n = x(y_n)$. By continuity of φ we actually have $\varphi(x_n) = y_n$. Otherwise we could make x_n smaller, contradicting the fact that it is an infimum. Since $x_n \not\to 0_+$, taking a subsequence if necessary, there exists $\delta > 0$ such that $x_n \ge \delta$ for all n. Since $\varphi(x) \to \infty$ as $x \to 0_+$, we may assume that this sequence is contained in the bounded interval $[\delta, 1]$ for n sufficiently large. We can then take a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to a limit x_0, with $x_0 \ge \delta$. As φ is continuous, we have

$$\lim_{k\to\infty} \varphi(x_{n_k}) = \varphi(x_0) < \infty,$$

Figure 2. Filled Julia Set $\mathcal{K}$ with Green's Lines $\{G = h\}$ and $\{G = 2h\}$



On the other hand, we must have that

$$\lim_{k\to\infty} \varphi(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = \infty,$$

a contradiction. □

Now, let $\mathcal{K}$ be the filled Julia set for P, and let U be the Siegel disc about 0 for P. Fix $R > 0$ and define $U_R := \Delta_U(0, R)$. Define $\tilde{R} = R^{int}_{(V_{2h},0)} U_R$ and let $\tilde{V}_{2h} := \Delta_{V_{2h}}(0, \tilde{R})$.

Lemma 21. $(V_{2h}, 0) \to (U, 0)$ *as* $h \to 0_+$.

Proof. We will use the Carathéodory Kernel version of Carathéodory convergence to prove this. Let W be the Carathéodory Kernel of $\{(V_{2h}, 0)\}$. Clearly $U \subset W$. To show containment in the other direction, let h_n be a sequence of positive numbers such that $h_n \to 0_+$ as $n \to \infty$. Let $\{(V_{2h_{n_k}}, 0)\}_{k=1}^{\infty}$ be a subsequence of $\{(V_{2h_{n_k}}, 0)\}_{k=1}^{\infty}$ and let $z \in W$ be arbitrary. Construct a path from 0 to z in W and denote this path by γ. Since W is the Carathéodory kernel, we have that the track $\{\gamma\}$ is contained in $V_{h_{n_k}}$ for all k sufficiently large. Thus the iterates of P are bounded on W. This implies $W \subset \mathcal{K}$. Since W is open, $W \subset \text{int}\mathcal{K}$. Then W is contained in a Fatou component for P, and since $0 \in W$, $W \subset U$. Since we have already shown $U \subset W$, we have $W = U$ as desired. $\square$

Further, let $\varphi_{2h} : \tilde{V}_{2h} \to V_{2h}$ be the unique Riemann mapping from $\tilde{V}_{2h}$ to V_{2h} normalized so that $\varphi_{2h}(0) = 0$ and $\varphi'_{2h}(0) > 0$. We now prove a small lemma:

Lemma 22. *Given any finite upper bound $h_0 \in (0, \infty)$, there exists ρ_0, depending on P and h_0, such that for all $R \in [\frac{1}{20}, 1]$ and all $h \in (0, h_0]$, we have that the hyperbolic radius of $\tilde{V}_{2h}$ in V_{2h} about 0 satisfies*

$$R_{(V_{2h},0)}\tilde{V}_{2h} \geq \rho_0.$$

Proof. Since $R \in [\frac{1}{20}, 1]$, we may use compactness or Lemma 1 to find d_0 such that $d(0, \partial U_R) \geq d_0$, where $d(0, \partial U_R)$ denotes the Euclidean distance from 0 to ∂U_R. Further, from the definition of $\partial\tilde{V}_{2h}$ as a hyperbolic incircle of ∂U_R, we have that for all $h \in (0, h_0]$, there exists $z_h \in \partial\tilde{V}_{2h} \cap \partial U_R$ with $|z_h| \geq d_0$. On the other hand, as the domains $\{V_{2h}\}_{h\in(0,h_0]}$ are nested, there exists D_0 depending only on h_0 such that for all $z \in U$, and for all $h \in (0, h_0]$, we have $\delta_{V_{2h}}(z) \leq D_0$. Letting ρ_h be the hyperbolic radius about 0 of $\tilde{V}_{2h}$ in V_{2h}, we have

$$\rho_h = \int_\gamma \mathrm{d}\rho_{V_{2h}(z)},$$

where γ is a geodesic in V_{2h} from 0 to z_h. Then, using Lemma 1, we have

$$\begin{aligned}
\rho_h &= \int_\gamma \mathrm{d}\rho_{V_{2h}(z)} \\
&\geq \frac{1}{2} \int_\gamma \frac{1}{\delta_{V_{2h}}(z)} |\mathrm{d}z| \\
&\geq \frac{1}{2D_0} l(\gamma) \\
&\geq \frac{1}{2D_0} |z_h| \\
&\geq \frac{d_0}{2D_0},
\end{aligned}$$

from which the result follows by setting $\rho_0 = \frac{d_0}{2D_0}$ (note that l denotes the Euclidean length above). □

Now define $\tilde{V}_h := \varphi_{2h}^{-1}(V_h)$ and note that $\tilde{V}_{2h} = \varphi_{2h}^{-1}(V_{2h})$. Further, define $\check{R}(h) := R^{ext}_{(V_{2h},0)} V_h$. Then it follows from Lemmas 17, 18, and 19 that $\check{R}$ is continuous in h on closed subsets of $(0, \infty)$, and

Lemma 23. *$\check{R}(h) \to \infty$ as $h \to 0_+$.*

Note that the function $\check{R}$ depends on P.

Proof. Let $z \in \partial V_h$ and, using the hyperbolic metric of V_{2h}, connect 0 to z using a hyperbolic geodesic γ in V_{2h}. Then γ must meet ∂U. Choose $w \in \gamma \cap \partial U$. Then $\rho_{V_{2h}}(0, z) \geq \rho_{V_{2h}}(0, w)$. As $h \to 0_+$ we can choose h sufficiently small such that for all $\varepsilon > 0$, $d(w, \partial V_{2h}) < \varepsilon$ for all $w \in \partial U$. Choose $\tilde{z} \in \partial V_{2h}$ with $|w - \tilde{z}| < \varepsilon$ and let $\zeta \in \gamma_{V_{2h}}[0, w]$ be arbitrary. We have that

$$
\begin{aligned}
\delta_{V_{2h}}(\zeta) &\leq |\zeta - \tilde{z}| \\
&\leq |\zeta - w| + |w - \tilde{z}| \\
&< |\zeta - w| + \varepsilon
\end{aligned}
$$

Let $\tilde{d}_0 = \inf_{w \in \partial U} |w|$. Now write $\zeta = \gamma_{V_{2h}}[0, w](t) = w + r(t)e^{i\theta(t)}$, for $t \in [0, 1]$ and note that, as $\gamma_{V_{2h}}[0, w]$ is a geodesic in V_{2h} from w to 0, $r(1)e^{i\theta(1)} = -w$. Since $\gamma_{V_{2h}}[0, w]$ is not self-intersecting, we have $r(t) > 0$ for all $t \in (0, 1]$. Then, using Lemmas 1 and 2 we have that

$$
\begin{aligned}
\check{R}(h) &= R^{ext}_{(V_{2h},0)} V_h \\
&\geq \rho_{V_{2h}}(0,z) \\
&\geq \rho_{V_{2h}}(0,\zeta) \\
&= \int_\gamma \mathrm{d}\rho_{V_{2h}}(\zeta) \\
&\geq \frac{1}{2}\int_\gamma \frac{|\mathrm{d}\zeta|}{\delta_{V_{2h}}(\zeta)} \\
&\geq \frac{1}{2}\int_\gamma \frac{|\mathrm{d}\zeta|}{|\zeta - w| + \varepsilon} \\
&\geq \frac{1}{2}\int_0^1 \frac{|r'(t)e^{i\theta(t)} + i\theta'(t)r(t)e^{i\theta(t)}|}{r(t)+\varepsilon}\mathrm{d}t \\
&\geq \frac{1}{2}\int_0^1 \frac{|r'(t)|}{r(t)+\varepsilon}\mathrm{d}t, \\
&\geq \frac{1}{2}\left|\int_0^1 \frac{r'(t)}{r(t)+\varepsilon}\mathrm{d}t\right| \\
&= \frac{1}{2}\int_0^{|w|} \frac{\mathrm{d}x}{x+\varepsilon} \\
&\geq \frac{1}{2}\int_0^{\tilde{d}_0} \frac{\mathrm{d}x}{x+\varepsilon} \\
&\geq \frac{1}{2}\int_\varepsilon^{\tilde{d}_0+\varepsilon} \frac{\mathrm{d}u}{u} \\
&= \ln(\tilde{d}_0+\varepsilon) - \ln(\varepsilon),
\end{aligned}
$$

from which the result follows. □

Let $\varepsilon_1 > 0$ and construct a $3\varepsilon_1$ open neighborhood of $\partial\tilde{V}_{2h}$ using the hyperbolic metric of U, which we will denote by $\hat{N}$. Recall h_0 from the statement of Lemma 22 and the scaling κ. We now state and prove

Lemma 24. *(The Target Lemma) Let* ρ_0 *be as in the statement of Lemma 22. There exists an upper bound* $\tilde{\varepsilon}_1 > 0$*, depending on* P*,* κ*, and the interval for* R*, namely* $[\frac{1}{20}, 1]$ *such that for all* $\varepsilon_1 \in (0, \tilde{\varepsilon}_1)$ *there exists* $T(\varepsilon_1) > 0$*, depending on* κ*,* P*,* h_0*, and the interval of* R*, namely* $[\frac{1}{20}, 1]$*, such that for all* $h > 0$ *and* $R \in [\frac{1}{20}, 1]$*, we have*

1. $R^{int}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N}) \geq T(\varepsilon_1)$

2. $T(\varepsilon_1) \to \infty$ *as* $\varepsilon_1 \to 0_+$*.*

We remark that part 1. of Lemma 24 allows us to interpolate for the "during" portion of Phase II. Conclusion 2. will be vital for the Fitting Lemma; it allows us to conclude that $h \to 0_+$ as $\varepsilon_1 \to 0_+$ (see the statement of the Fitting Lemma).

Proof. As is the case in Phase I, we will assume that $U \subset \mathrm{D}(0, \frac{1}{288})$. Regarding the upper bound $\frac{\rho_0}{3}$: we note that if ε_1 is too large, then we would actually have $\tilde{V}_{2h} \subset \hat{N}$ so that $\tilde{V}_{2h} \setminus \hat{N} = \emptyset$. Recall that by Lemma 22, we have that ρ_0 is such that for all $h > 0$ and $R \in [\frac{1}{20}, 1]$, we have $R_{(V_{2h},0)}\tilde{V}_{2h} \geq \rho_0$. Using the Schwarz Lemma for the hyperbolic metric, we see that $R^{int}_{(U,0)}\tilde{V}_{2h} \geq R_{(V_{2h},0)}\tilde{V}_{2h} \geq \rho_0$, so ensuring that $\varepsilon_1 < \frac{\rho_0}{3}$ implies that $\tilde{V}_{2h} \setminus \hat{N} \neq \emptyset$.

For all $R \in [\frac{1}{20}, 1]$, it follows from compactness that if $\xi \in \partial U_R$, then $|\xi| \geq d_0$ for some $d_0 > 0$ (this is the same d_0 from the proof of Lemma 22). With the distortion theorems in mind, we define

$$r_1 := \frac{e-1}{e+1}$$
$$D_1 := \frac{(1+r_1)^2}{(1-r_1)^2} = e^2.$$

Note that r_1 is chosen so that $\mathrm{D}(0, r_1)$ has hyperbolic radius 1 in $\mathbb{D}$, that is, $D(0, r_1) = \Delta_{\mathbb{D}}(0, 1)$. By the Schwarz Lemma, since $U \subset V_{2h}$ and $R \leq 1$, we have $R_{(V_{2h},0)}\tilde{V}_{2h} \leq 1$ (recall that $\tilde{V}_{2h}$ is round in the conformal coordinates of V_{2h} so that the internal and external hyperbolic radii are equal). If follows from the distortion theorems that if $\xi \in \partial\tilde{V}_{2h}$, then $|\xi| \geq \frac{d_0}{D_1}$.

Now suppose $\zeta_0 \in \partial\tilde{V}_{2h}$. If $\zeta \in \partial\Delta_U(\zeta_0, 3\varepsilon_1)$, we wish to find a Euclidean disk about ζ_0 in which ζ is contained. Let γ_0 be a geodesic in U from ζ_0 to ζ. Then, using Lemma 1, we calculate

$$\begin{aligned} 3\varepsilon_1 &= \int_{\gamma_0} \mathrm{d}\rho_U \\ &\geq \frac{1}{2}\int_{\gamma_0} \frac{|\mathrm{d}w|}{\delta_U(w)} \\ &\geq 144 \int_{\gamma_0} |\mathrm{d}w| \\ &= 144 l(\gamma_0) \\ &\geq 144|\zeta - \zeta_0| \end{aligned}$$

where $l(\gamma_0)$ is the Euclidean length of γ_0. Note that we use the fact that

$U \subset \mathrm{D}(0, \frac{1}{288})$ in the above calculation. Thus $|\zeta - \zeta_0| \leq \frac{\varepsilon_1}{48}$. As ζ_0 was arbitrary, this implies that for any $\xi \in \partial\tilde{V}_{2h}$, we have $\Delta_U(\xi, 3\varepsilon_1) \subset \mathrm{D}(\xi, \frac{\varepsilon_1}{48})$.

Now we aim to define the quantity $T(\varepsilon_1)$. Let $z \in (\partial\hat{N}) \cap \tilde{V}_{2h}$. Pick $z_0 \in \partial\tilde{V}_{2h}$ which is closest to z. Then $\rho_U(z, z_0) \leq 3\varepsilon_1$, which by what we have just shown implies $|z - z_0| \leq \frac{\varepsilon_1}{48}$. Note that as $|z_0| \geq \frac{d_0}{D_1}$ we have that $|z| \geq \frac{d_0}{D_1} - \frac{\varepsilon_1}{48}$. Note that we need ε_1 to satisfy $\varepsilon_1 < \frac{48d_0}{D_1}$, and since the constants d_0 and D_1 depend on κ, P, the interal of R, ε_1 depends on these same constants. Let γ be a geodesic in $\tilde{V}_{2h}$ from z to 0. If $w \in \{\gamma\}$, we have

$$\begin{aligned}\delta_{\tilde{V}_{2h}}(w) &\leq |w - z_0| \\ &\leq |w - z| + |z - z_0| \\ &\leq |w - z| + \frac{\varepsilon_1}{48}.\end{aligned}$$

So, once more using Lemma 1,

$$\begin{aligned}\rho_{\tilde{V}_{2h}}(0, z) &\geq \frac{1}{2}\int_\gamma \frac{|\mathrm{d}w|}{\delta_{\tilde{V}_{2h}}(w)} \\ &\geq \frac{1}{2}\int_\gamma \frac{|\mathrm{d}w|}{|w - z| + \frac{\varepsilon_1}{48}}\end{aligned}$$

Now write $w = \gamma(t) = z + r(t)e^{i\theta(t)}$, for $t \in [0, 1]$ and note that, as γ is a geodesic in $\tilde{V}_{2h}$ from z to 0, $r(1)e^{i\theta(1)} = -z$. Since γ is not self-intersecting, we have $r(t) > 0$ for all $t \in (0, 1]$. Then

$$
\begin{aligned}
\frac{1}{2}\int_\gamma \frac{|\mathrm{d}w|}{|w-z|+\frac{\varepsilon_1}{48}} &\geq \frac{1}{2}\int_0^1 \frac{|r'(t)e^{i\theta(t)}+i\theta'(t)r(t)e^{i\theta(t)}|}{r(t)+\frac{\varepsilon_1}{48}}\mathrm{d}t \\
&\geq \frac{1}{2}\int_0^1 \frac{|r'(t)|}{r(t)+\frac{\varepsilon_1}{48}}\mathrm{d}t, \\
&\geq \frac{1}{2}\left|\int_0^1 \frac{r'(t)}{r(t)+\frac{\varepsilon_1}{48}}\mathrm{d}t\right| \\
&= \frac{1}{2}\int_0^{|z|} \frac{1}{u+\frac{\varepsilon_1}{48}}\mathrm{d}u \\
&\geq \frac{1}{2}\int_0^{\frac{d_0}{D_1}-\frac{\varepsilon_1}{48}} \frac{1}{u+\frac{\varepsilon_1}{48}}\mathrm{d}u \\
&= \frac{1}{2}\int_{\frac{\varepsilon_1}{48}}^{\frac{d_0}{D_1}} \frac{1}{x}\mathrm{d}x \\
&= \frac{1}{2}(\ln(\frac{d_0}{D_1})-\ln(\frac{\varepsilon_1}{48})).
\end{aligned}
$$

The result follows from Lemma 2 and setting $T(\varepsilon_1)=\frac{1}{2}(\ln(\frac{d_0}{D_1})-\ln(\frac{\varepsilon_1}{48}))$.

□

As $\check{R}(h)$ is continuous, for $T(\varepsilon_1)$ sufficiently large, we may choose the smallest $h(T(\varepsilon_1))$ such that $\check{R}(h(T(\varepsilon_1)))=T(\varepsilon_1)$. Lemmas 20, 23 and 24 imply:

Lemma 25. *(The Fitting Lemma) $h(T(\varepsilon_1))\to 0_+$ as $\varepsilon_1\to 0_+$.*

Proof. By Lemma 24, we have that $T(\varepsilon_1)\to\infty$ as $\varepsilon_1\to 0_+$. Lemma 23, together with the fact that $\check{R}$ is continuous as discussed above, ensure that the hypotheses of Lemma 20 are met. Lemma 20 then implies that $h\to 0_+$. □

As we remarked earlier, the Fitting Lemma will be essential in controlling loss of domain in Phase II. The idea is that we can find an h such that the domain $\tilde{V}_h$ fits inside $\tilde{V}_{2h}\setminus\hat{N}$ with the desired properties, one of which being $h\to 0_+$ as $\varepsilon_1\to 0_+$.

The fact that $\tilde{V}_h \subset \tilde{V}_{2h} \setminus \hat{N}$ will allow us to apply the Polynomial Implementation Lemma which we will need to correct the error from the Phase I immediately prior to this. As will use the Target Lemma to choose an appropriote h, we can be sure that we can distort ∂V_h slightly, and this distorted curve will still lie well inside V_{2h}. This is vital for the hypothesis of the Polynomial Implementation Lemma; it ensures we have a conformal annulus on which to interpolate functions. The fact that $h \to 0_+$ as $\varepsilon_1 \to 0_+$ will allow us to control the unavoidable loss of domain when we correct the error from the previous Phase I. Before we move on to the statement and proof of Phase II, we state a further technical lemma that will be useful in the proof of Phase II:

Lemma 26. *Let $D \subset \mathbb{C}$ be a bounded simply connected domain containing 0 and let $\varepsilon > 0$. Then there exists $R > 0$ such that if $\tilde{D}$ is any simply connected domain compactly contained in D such that $R^{int}_{(D,0)}\tilde{D} > R$, then $\mathrm{d}(\partial\tilde{D}, \partial D) \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$ and define $D_\varepsilon = \{z \in D \,:\, \mathrm{d}(z, \partial D) \geq \varepsilon\}$. As D_ε is compactly contained in D, we can find an R_ε such that $D_\varepsilon \subset D_{R_\varepsilon}$, where $D_{R_\varepsilon} := \Delta_D(0, R_\varepsilon)$. Then if $R > R_\varepsilon$, and $D_R \subset \tilde{D} \subset D$, we must have $\partial\tilde{D} \subset \{z \in D \,:\, \mathrm{d}(z, \partial D) < \varepsilon\}$. Indeed, we know $\partial\tilde{D} \cap \tilde{D} = \emptyset$. Then $\partial\tilde{D} \cap D_R = \emptyset$ as $D_R \subset \tilde{D}$. Further, $\partial\tilde{D} \cap D_{R_\varepsilon} = \emptyset$, as $D_{R_\varepsilon} \subset D_R$, and finally $\partial\tilde{D} \cap D_\varepsilon = \emptyset$ as $D_\varepsilon \subset D_{R_\varepsilon}$. Then from $\partial\tilde{D} \subset \{z \in D \,:\, \mathrm{d}(z, \partial D) < \varepsilon\}$ and the compactness of $\partial\tilde{D}$, we get $\mathrm{d}(\partial\tilde{D}, \partial D) < \varepsilon$, as desired. □

Statement and Proof of Phase II

The idea behind Phase II is this; in Phase I we construct a polynomial composition which is close to the identity and approximates given functions at prescribed stages in the composition. In Phase II we wish to correct the error in the Phase I composition with another polynomial composition. In doing this, we will lose domain on which we can correct the error, but the correction can be chosen arbitrarily good on this smaller domain. Further, this loss of domain will become arbitrarily small as the initial Phase I error becomes small.

Recall the scaling factor $\kappa > 1$ and h_0 from the statement of Lemma 22.

Lemma 27. *(Phase II) There exists an upper bound $\tilde{\varepsilon}_1 > 0$, depending on P, κ, h_0, and the interval of values for R, namely $[\frac{1}{20}, 1]$, and a function $\delta : (0, \tilde{\varepsilon}_1) \to (0, \frac{1}{80})$, with $\delta(x) \to 0_+$ as $x \to 0_+$, such that for all $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$, there exists an upper bound $\tilde{\varepsilon}_2 > 0$, depending on ε_1, P, κ, h_0, and the interval of values of R, namely $[\frac{1}{20}, 1]$, such that, for all $\varepsilon_2 \in (0, \tilde{\varepsilon}_2]$, $R \in [\frac{1}{20}, 1]$, and all functions $\mathcal{E}$ univalent on U_R with $\rho_U(\mathcal{E}(z), z) < \varepsilon_1$ for $z \in U_R$, there exists a $17 + \kappa$-bounded quadratic composition $\mathbf{Q}$ such that*

i) $\mathbf{Q}$ is univalent on a neighborhood of $\overline{U_{R-\delta(\varepsilon_1)}}$, and

ii) For all $z \in U_{R-\delta(\varepsilon_1)}$, we have

$$\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) < \varepsilon_2.$$

iii) $\mathbf{Q}(0) = 0$

Because we will be using the Polynomial Implementation Lemma repeatedly to construct our polynomial composition, we need to interpolate functions outside of $\mathcal{K}$, the filled Julia set for P. Indeed, as we saw in the Polynomial Implementation Lemma, the solutions to the Beltrami equation converge to the identity because the supports of the Beltrami data become small in measure. However, $\mathcal{E}$ is defined on a subset of U, where we can use high iterates of P which converge to the identity to assist us in approximating $\mathcal{E}$. Hence, we will need to map a suitable subset of U to a domain which contains $\mathcal{K}$, and correct the conjugated error using the Polynomial Implementation Lemma. The trick to doing this is that we choose our subset of U such that the mapping to blow this subset up to U can be expressed as a high iterate of a map which is defined on the whole of the Green's domain V_h, not just on this subset. This will allow us to interpolate outside $\mathcal{K}$. Further, we will then use the Polynomial Implementation Lemma to "undo" the conjugated map and its inverse.

The two key considerations in the proof are controlling loss of domain, and showing that the error in our polynomial approximation to the function $\mathcal{E}$ is mild. In controlling loss of domain, the main difficulty will arise in converting between the hyperbolic metrics of different domains, U and V_{2h}. The techniques for doing this will be the Fitting Lemma, and the fact that $(V_{2h}, 0) \to (U, 0)$ in the Carathéodory topology as $h \to 0_+$. As stated above, we will apply the Polynomial

Implementation Lemma to $\varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$ in what we call the "During" portion of the error calculations. We then wish to "cancel" the conjugacy, so "During" is bookended by "Up" and "Down" portions, in which we apply the Polynomial Implementation Lemma to get polynomial compositions which are arbitrarily close to φ_{2h} and φ_{2h}^{-1}, respectively, on suitable domains.

We begin the proof of Phase II by considering "Ideal Loss of Domain." In creating polynomial approximations, error will be created that will have an impact on the loss of domain that occurs. We first describe the loss of domain that is forced on us before this error is taken into account.

Proof. **Ideal Loss of Domain:**

We first turn our attention to controlling loss of domain. Let $\psi : U \to \mathbb{D}$ be the unique normalized Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$, $\psi'(0) > 0$. Similarly, let $\psi_{2h} : V_{2h} \to \mathbb{D}$ be the unique normalized Riemann map from V_{2h} to $\mathbb{D}$ satisfying $\psi_{2h}(0) = 0$, $\psi'_{2h}(0) > 0$. Recall that $\tilde{R} = R^{int}_{(V_{2h},0)} U_R$ and $\tilde{V}_{2h} = \Delta_{V_{2h}}(0, \tilde{R})$. Define $R' = R^{int}_{(U,0)} \tilde{V}_{2h}$. We prove the following claim:

Claim: $R - R' \to 0_+$ as $h \to 0_+$.

As $(V_{2h}, 0) \to (U, 0)$ in the Carathéodory topology as $h \to 0_+$ (Lemma 21), we have that ψ_{2h} converges locally uniformly to ψ on U by Theorem 8. Let h_n be an arbitrary sequence such that $h_n \to 0_+$ as $n \to \infty$. By the definitions of $\tilde{V}_{2h}$ and R' and Lemma 2, there exists $w_{h_n,1} \in \partial\tilde{V}_{2h_n} \cap \partial U_R$ and $w_{h_n,2} \in \partial\tilde{V}_{2h_n} \cap \partial U_{R'}$. Let $0 < s, s', s_{2h_n} < 1$ be such that $\psi(\partial U_R) = C(0,s)$, $\psi(\partial U_{R'}) = C(0,s')$, and $\psi_{2h_n}(\partial\tilde{V}_{2h_n}) = C(0, s_{2h_n})$.

Let $\varepsilon_0 > 0$. By compactness of ∂U_R we have that there exists a point $w_{h_*,1} \in \partial U_R$ and a subsequence $\{w_{h_{n_k},1}\}_{k=1}^{\infty}$ of $\{w_{h_n,1}\}$ such that $w_{h_{n_k},1} \to w_{h_*,1}$ as $k \to \infty$, whence by uniform convergence we have $|\psi_{2h_{n_k}}(w_{h_{n_k},1}) - \psi(w_{h_0,1})| < \frac{\varepsilon_0}{2}$, for all k sufficiently large. Thus

$$|s_{2h_{n_k}} - s| < \frac{\varepsilon_0}{2}.$$

for all k sufficiently large. We may apply a similar argument to the sequence $\{w_{h_n,2}\}$ to find a subsequence $\{w_{h_{n_k},2}\}$ and see, for h sufficiently small, that

$$|s_{2h_{n_k}} - s'| < \frac{\varepsilon_0}{2}.$$

Thus $|s - s'| < \varepsilon_0$ for any subsequence h_{n_k} for which $h_{n_k} \to 0_+$. Thus $|s - s'| < \varepsilon_0$ for all h sufficiently small, and using the continuity of ψ^{-1}, the claim follows. $\diamondsuit$

Now define the *Internal Siegel disc*, $\tilde{U} := \varphi_{2h}^{-1}(U)$, and let $R'' = R^{int}_{(U,0)}\tilde{U}$.

Next, we show

Claim: $R - R'' \to 0_+$ as $h \to 0_+$

First we show $R^{int}_{(V_{2h},0)}U \to \infty$ as $h \to 0_+$. Fix $R_0 > 0$ and set $X = U_{R_0}$ and $Y = U_{R_0+1}$. Now $\psi(X) = \Delta_{\mathbb{D}}(0, R_0)$ and $\psi(Y) = \Delta_{\mathbb{D}}(0, R_0 + 1)$. As $\overline{\Delta}_{\mathbb{D}}(0, R_0) \subset \Delta_{\mathbb{D}}(0, R_0 + 1)$, let $\eta = d(\partial\Delta_{\mathbb{D}}(0, R_0), \partial\Delta_{\mathbb{D}}(0, R_0 + 1)) > 0$. Now let $z \in \partial Y$ and $w \in \Delta_{\mathbb{D}}(0, R_0)$. We have that $(V_{2h}, 0) \to (U, 0)$ as $h \to 0_+$ (Lemma 21), so by Theorem 8 we have that ψ_{2h} converges to ψ uniformly on compact subsets of U. So for all h sufficiently small, we have

$$
\begin{aligned}
|(\psi_{2h}(z) - w) - (\psi(z) - w)| &= |\psi_{2h}(z) - \psi(z)| \\
&< \frac{\eta}{2} \\
&< \eta \\
&\leq |\psi(z) - w|
\end{aligned}
$$

So by Rouché's theorem, since the convergence was uniform and $w \in \Delta_{\mathbb{D}}(, R_0)$ was arbitrary, $\Delta_{\mathbb{D}}(0, R_0) \subset \psi_{2h}(Y)$. Then $\psi_{2h}^{-1}(\Delta_{\mathbb{D}}(0, R_0)) \subset Y$, so $R^{int}_{(V_{2h},0)}Y \geq R_0$. We also have that $Y \subset U$ so $R^{int}_{(V_{2h},0)}U \geq R^{int}_{(V_{2h},0)}Y$, and thus $R^{int}_{(V_{2h},0)}U \geq R_0$. Since R_0 was arbitrary, we have that $R^{int}_{(V_{2h},0)}U \to \infty$ as $h \to 0_+$.

Now, by conformal invariance, $R^{int}_{(V_{2h},0)}U = R^{int}_{(\tilde{V}_{2h},0)}\tilde{U}$. For a constant c and a domain D, define the scaled domain $cD := \{z \in \mathbb{C} : z = cw \text{ for some } w \in D\}$. As $\psi_{2h}(\tilde{V}_{2h}) = \Delta_{\mathbb{D}}(0, r_h)$, for some $r_h > 0$, depending on h, and then $\frac{1}{r_h}\psi_{2h}(\tilde{V}_{2h}) = \mathbb{D}$. As $R^{int}_{(\tilde{V}_{2h},0)}\tilde{U} \to \infty$ as $h \to 0_+$, it follows that $R^{int}_{(\frac{1}{r_h}\psi_{2h}(\tilde{V}_{2h}),0)}(\frac{1}{r_h}\psi_{2h}(\tilde{U})) \to \infty$ as $h \to 0_+$. We can the apply Lemma 26 to conclude that

$$\mathrm{d}(\partial(\frac{1}{r_h}\psi_{2h}(\tilde{U})), \partial(\frac{1}{r_h}\psi_{2h}(\tilde{V}_{2h}))) \to 0_+ \text{ as } h \to 0_+$$

and thus, scaling by r_h, we have

$$\mathrm{d}(\partial(\psi_{2h}(\tilde{U})), \partial(\psi_{2h}(\tilde{V}_{2h}))) \to 0_+ \text{ as } h \to 0_+.$$

Further, we have that

$$\psi_{2h}(\tilde{U}) \subset \psi_{2h}(\tilde{V}_{2h}) \subset \psi_{2h}(U_R) \subset \psi_{2h}(U_1) \subset \psi_{2h}(\Delta_{V_{2h}}(0,1)),$$

where we use the Schwarz Lemma for the hyperbolic metric for the last inclusion. Since ψ_{2h}^{-1} converges to ψ^{-1} uniformly on compact subsets of $\mathbb{D}$ by Theorem 8, we have that $\mathrm{d}(\partial\tilde{U}, \partial\tilde{V}_{2h}) \to 0_+$ as $h \to 0_+$. Using Lemma 1, we see that $\rho_U(\partial\tilde{U}, \partial\tilde{V}_{2h}) \to 0_+$ as $h \to 0_+$.

Fix $\varepsilon_0 > 0$. Pick $z \in \partial\tilde{U}$ such that $\rho_U(0, z) = R''$ using Lemma 2. Since $\rho_U(\partial\tilde{U}, \partial\tilde{V}_{2h}) \to 0_+$ as $h \to 0_+$, we can pick $w_h \in \partial\tilde{V}_{2h}$ such that $\rho_U(z, w_h) < \frac{\varepsilon_0}{2}$. Now let γ be the unique geodesic in U passing through $0, w_h$. As γ must eventually leave U_R, let w be the closest point on $\gamma \cap \partial U_R$ to w_h. Then $0, w_h$, and w are on the same geodesic. We now have $\rho_U(0, w) = R$ and $\rho_U(0, w_h) \geq R'$ using lemma 2. Then, since $0, w_h$, and w are on the same geodesic, we have

$$\begin{aligned} \rho_U(w, w_h) &= \rho_U(0, w) - \rho_U(0, w_h) \\ &\leq R - R' \\ &< \frac{\varepsilon_0}{2} \end{aligned}$$

for h sufficiently small. Further, we have

$$\begin{aligned} R - R'' &= \rho_U(0, w) - \rho_U(0, z) \\ &\leq \rho_U(z, w) \end{aligned}$$

Finally, we have

$$\begin{aligned} R - R'' &\leq \rho_U(z, w) \\ &\leq \rho_U(z, w_h) - \rho_U(w_h, w) \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\ &= \varepsilon_0 \end{aligned}$$

for h sufficiently small, and thus $R - R'' \to 0_+$ as $h \to 0_+$. $\Diamond$

Since $R - R'' \to 0_+$ as $h \to 0_+$, we have that

$$R - R'' \to 0_+ \text{ as } \varepsilon_1 \to 0_+ \tag{13}$$

by the Fitting Lemma (Lemma 25).

We now fix h. Let $T(\varepsilon_1)$ be as in the statement of the Target Lemma and set $h = \min\{h : \check{R}(h) = T(\varepsilon_1)\}$. We know $h \to 0_+$ as $\varepsilon_1 \to 0_+$ by the Fitting Lemma. The Target Lemma will guarantee that our fundamental annuli in the "During" portion of the proof are indeed conformal annuli.

Controlling Error: "Up"

Recall that φ_{2h} is the unique Riemann map which sends $\tilde{V}_{2h}$ to V_{2h} and ψ_{2h} is the unique Riemann map which sends V_{2h} to $\mathbb{D}$. Notice that, in the conformal coordinates of V_{2h}, φ_{2h} is a dilation of $\tilde{V}_{2h}$. To estimate the error, we wish to break this dilation into many smaller dilations, and apply the Polynomial Implementation Lemma so as to approximate each of these small dilations with a polynomial composition.

Let $r \in (0, 1)$ be such that $\psi_{2h}(\tilde{V}_{2h}) = \mathrm{D}(0, r)$. Also, we have that $\psi_{2h}(V_h) \subset \mathrm{D}(0, s)$ for some $s \in (0, 1)$ with $s > r$. Fix N such that $(\sqrt[N]{\frac{1}{r}})s < \sqrt{s}$. This will ensure that g in the composition does not distort ∂V_h so much so that we lack a conformal annulus for interpolation in the Polynomial Implementation

Lemma. Put more specifically, we have $\overline{g(V_h)} \subset V_{2h}$.

Define on $\psi_{2h}^{-1}(\mathrm{D}(0, s))$ the map

$$g(z) = \psi_{2h}^{-1}\left(\sqrt[N]{\frac{1}{r}}\psi_{2h}(z)\right)$$

and note in particular that g is defined on V_h as $\psi_{2h}(V_h) \subset \mathrm{D}(0, s)$.

Remarks (for $1 \leq k \leq N$):

i) $\psi_{2h}^{-1}(\mathrm{D}(0, \mathrm{r}^{\mathrm{k/N}})) \supset \psi_{2h}^{-1}(\mathrm{D}(0, \mathrm{r})) = \tilde{V}_{2h}$

ii) $g^{\circ k}(z) = \psi_{2h}^{-1}(\frac{1}{r^{k/N}}\psi_{2h}(z))$ on $\psi_{2h}^{-1}(\mathrm{D}(0, \mathrm{r}^{\mathrm{k/N}}))$

iii) $g^{\circ N}(z) = \varphi_{2h}(z)$ on $\tilde{V}_{2h}$ as φ_{2h} is unique.

Thus, as $\psi_{2h}^{-1}(\mathrm{D}(0, sr^{k/N}))$ is a relatively compact subset of $\psi_{2h}^{-1}(\mathrm{D}(0, r^{k/N}))$, we can use the chain rule to find $M_1' > 0$ independent of both N and k such that on $\psi_{2h}^{-1}(\mathrm{D}(0, sr^{k/N}))$ we have that $|(g^{\circ N-k})'| \leq M_1'$. Recall that the hyperbolic density in V_{2h}, $\sigma_{V_{2h}}$, is uniformly continuous and positive on compact sets of V_{2h}. Let $K_1' \geq 1$ be such that $\frac{\sigma_{V_{2h}}(z)}{\sigma_{V_{2h}}(w)} \leq K_1'$ for all $z, w \in \psi_{2h}^{-1}(\mathrm{D}(0, s))$, and note that $\psi_{2h}^{-1}(\mathrm{D}(0, s)) \supset \psi_{2h}^{-1}(\mathrm{D}(0, sr^{k/N}))$. Set $M_1 := K_1'M_1'$ and then, on $\psi_{2h}^{-1}(\mathrm{D}(0, sr^{k/N}))$, we have

$$
\begin{aligned}
|(g^{\circ N-k})^{\natural}(z)| &= \frac{\sigma_{V_{2h}}(g^{\circ N-k}(z))}{\sigma_{V_{2h}}(z)}|(g^{\circ N-k})'(z)| \\
&\leq K_1' M_1' \\
&= M_1
\end{aligned}
$$

We also have that ψ_{2h} maps U compactly inside $\mathrm{D}(0, s)$, the function $z \mapsto \sqrt[N]{\frac{1}{r}}z$ maps $\mathrm{D}(0, s)$ inside $\mathrm{D}(0, \sqrt{s})$, and ψ_{2h}^{-1} maps $\mathrm{D}(0, \sqrt{s})$ compactly back inside V_{2h}. Combining these three observations with the chain rule, we can make M_1 larger if necessary to ensure that, on U, we have

$$
\|g^{\natural}\|_U \leq M_1 \tag{14}
$$

Set $B = \cup_{k=0}^{N} g^{\circ k}(U_{R''-5\varepsilon_1})$ for $0 \leq k \leq N$ and let A be a 1-neighborhood of B in the hyperbolic metric of U. Then if $\hat{\hat{A}}$ is a 2-neighborhood of A, again using the hyperbolic metric of U, and $\check{A} = \Delta_U(0, R^{ext}_{(U,0)}\hat{\hat{A}})$, we have $\|g^{\natural}\|_{\check{A}} \leq M_1$, as $\check{A}$ is a (relatively compact) subset of U. Define $\Gamma_h = \partial V_h$ and $\Gamma_{2h} = \partial V_{2h}$, and set ε in the statement of the Polynomial Implementation Lemma to be $\frac{\varepsilon_2}{3(2M_1)^{N-1}M_2M_3}$, where M_2 and M_3 are equicontinuity bounds which will be chosen later. For now we just assume that $M_i > 1$ for $i = 1, 2, 3$ (these are bounds, and we can always choose a larger upper bound). Without loss of generality, assume that $\frac{\varepsilon_2}{3(2M_1)^{N-1}} < 1$ which implies that both $\varepsilon < 1$ and $\varepsilon_2 < 1$. Further, note that $\varepsilon < \varepsilon_2$. Now apply the Polynomial Implementation Lemma, with $\gamma = \Gamma_h$, $\Gamma = \Gamma_{2h}$, and $\varepsilon = \frac{\varepsilon_2}{3(2M_1)^{N-1}M_2M_3}$ to g to get $M_N > 0$, and a $(17+\kappa)$-bounded finite sequence of quadratics polynomials $\{Q_m\}_{m=1}^{M_N}$ such that the composition of these polynomials, Q_{M_N}, is univalent on A and satisfies

$$\rho_U(Q_{M_N}(z), g(z)) < \frac{\varepsilon_2}{3(2M_1)^{N-1}M_2M_3}, \qquad z \in A \tag{15}$$

$$\|Q^\natural_{M_N}\|_A \leq M_1(1 + \frac{\varepsilon_2}{3(2M_1)^{N-1}M_2M_3}) \tag{16}$$

Define $Q_{kM_N} := Q_{M_N}^{\circ k}$. We prove the following claim, which will allow us to control the error in the "Up" portion of Phase II:

Claim: For each $1 \leq k \leq N$, we have Q_{kM_N} is univalent on A and, for $z \in U_{R''-5\varepsilon_1}$,

$$1.\ \rho_U(Q_{kM_N}(z), g^{\circ k}(z)) < \frac{\varepsilon_2}{3(2M_1)^{N-k}M_2M_3} \tag{17}$$

$$2.\ Q_{kM_N}(z) \in A \tag{18}$$

For the base case $k = 1$, we have that 1. is obvious while 2. follows from the fact that $\varepsilon < 1$ and $g(z) \in B$. Now assume this is true for some $1 \leq k \leq N$. For $z \in U_{R''-5\varepsilon_1}$ we have

$$\rho_U(Q_{(k+1)M_N}(z), g^{\circ k+1}(z)) \leq \rho_U(Q_{(k+1)M_N}(z), g \circ Q_{kM_N}(z)) + \rho_U(g \circ Q_{kM_N}(z), g^{\circ k+1}(z))$$

Now $Q_{kM_N}(z) \in A$ by hypothesis so the first term in the inequality above is less than ε by (15). We have that $\rho_U(Q_{kM_N}(z), g^{\circ k}(z)) < \varepsilon$ by hypothesis so the second term is less than $M_1\varepsilon$ using the hyperbolic M-L estimates and (14). Thus we have

$$\begin{aligned}\rho_U(Q_{(k+1)M_N}(z), g^{\circ k+1}(z)) &\leq \varepsilon + M_1\varepsilon \\ &= \frac{1}{(2M_1)^k}\cdot\frac{\varepsilon_2}{3(2M_1)^{N-(k+1)}M_2M_3} + \frac{1}{2}\cdot\frac{\varepsilon_2}{3(2M_1)^{N-(k+1)}M_2M_3} \\ &\leq \frac{\varepsilon_2}{3(2M_1)^{N-(k+1)}M_2M_3}\end{aligned}$$

which proves 1. in the claim. Now $Q_{(k+1)M_N}(U_{R''-5\varepsilon_1})$ lies in a 1-neighborhood of $g^{\circ k+1}(U_{R''-5\varepsilon_1})$ by 1. But $g^{\circ k+1}(U_{R''-5\varepsilon_1}) \in B$ so $Q_{(k+1)M_N}(z) \in A$ if $z \in U_{R''-5\varepsilon_1}$ (note that $k+1 \leq N$), which finishes the proof of 2. To see that $Q_{(k+1)M_N}(z)$ is univalent, we know $Q_{(k+1)M_N}(z) = Q_{M_N} \circ Q_{kM_N}(z)$. Since $Q_{kM_N}(z) \in A$, and Q_{M_N} is univalent on A by the Polynomial Implementation Lemma (Lemma 8), we have that $Q_{(k+1)M_N}$ is univalent on A. This completes the proof of the claim. $\Diamond$

For convenience, set $\mathbf{Q_1} = Q_{NM_N}$. Recall that on $\tilde{V}_{2h}$, we have $g^{\circ N} = \varphi_{2h}$. Thus on $U_{R''-5\varepsilon_1}$ we have

$$\rho_U(\mathbf{Q_1}(z), \varphi_{2h}(z)) < \frac{\varepsilon_2}{3M_2M_3}, \tag{19}$$

and

$$\mathbf{Q_1}(z) \in A \tag{20}$$

Controlling Error: "During"

Let $z \in \partial U_{R''-6\varepsilon_1}$, $w \in \partial U_{R''-5\varepsilon_1}$ and note $\rho_U(z,w) \geq \varepsilon_1$. As φ_{2h} is a homeomorphism, we have that $\varphi_{2h}(w) \in \partial\varphi_{2h}(U_{R''-5\varepsilon_1})$ and $\varphi_{2h}(z) \in \text{int}\varphi_{2h}(U_{R''-5\varepsilon_1})$. If we set $R_0 := R^{ext}_{(U,0)}\varphi_{2h}(U_{R''-5\varepsilon_1})$ and consider $\Delta_U(0,R_0)$, we have that $\Delta_U(0,R_0)$ is compactly contained in U as $U_{R''-5\varepsilon_1}$ is compactly contained in U and φ_{2h} is a homeomorphism. Note that $\varphi_{2h}^{-1}(\Delta_U(0,R_0)) \supset U_{R''-5\varepsilon_1} \supset U_{R''-6\varepsilon_1}$. Thus we can find $\eta > 0$ such that $\|(\varphi_{2h}^{-1})^\natural\|_{\Delta_U(0,R_0)} \leq \eta$. Lemma 3 ensures that $\Delta_U(0,R_0)$ is hyperbolically convex and we may use the hyperbolic M-L estimates. Thus we have $\rho_U(\varphi_{2h}(z),\varphi_{2h}(w)) \geq \frac{\varepsilon_1}{\eta}$, which implies the hyperbolic distance from $\varphi_{2h}(z)$ to $\partial(\varphi_{2h}(U_{R''-5\varepsilon_1}))$ is at least $\frac{\varepsilon_1}{\eta}$. Making ε_2 smaller if necessary to ensure that $\varepsilon_2 < \frac{\varepsilon_1}{\eta}$, we have that

$$\begin{aligned}\rho_U(\mathbf{Q_1}(z),\varphi_{2h}(z)) &< \frac{\varepsilon_2}{2M_2M_3}\\ &< \varepsilon_2\\ &< \frac{\varepsilon_1}{\eta},\end{aligned}$$

and thus $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-5\varepsilon_1})$. As z was arbitrary, we have that

$$\mathbf{Q_1}(U_{R''-6\varepsilon_1}) \subset \varphi_{2h}(U_{R''-5\varepsilon_1}) \tag{21}$$

On the other hand, we observe the following hold:

1. We have that $\tilde{U} \subset \tilde{V}_h$. Then by the Target Lemma, h was chosen such that an ε_1-neighborhood (in the hyperbolic metric of U) of $\tilde{V}_h$ avoids an ε_1-neighborhood of $\partial\tilde{V}_{2h}$.

2. It follows from 1. above that an ε_1 neighborhood (in the hyperbolic metric of U) of $\tilde{U}$ has finite external hyperbolic radius in $\tilde{V}_{2h}$

Thus if we define $\hat{\mathcal{E}} = \varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$, we have that $|\hat{\mathcal{E}}^\natural|$ is uniformly bounded on U. Let $\hat{\hat{N}}$ be a 2-neighborhood (using the hyperbolic metric of U) of $\varphi_{2h}(U_{R''-5\varepsilon_1})$ and let $\check{N} = \Delta_U(0, R^{ext}_{(U,0)}\hat{\hat{N}})$. As $\check{N}$ is compactly contained in U, we have that $|\hat{\mathcal{E}}^\natural|$ is uniformly bounded on this neighborhood. As we used the Target Lemma to choose h, $\overline{\hat{\mathcal{E}}(\partial V_h)} \subset V_{2h}$ and so we have $(\hat{\mathcal{E}}, Id)$ are an admissible pair on (Γ_h, Γ_{2h}). Now fix $M_2 > 1$ such that

$$|\hat{\mathcal{E}}^\natural(z)| \leq M_2, \qquad z \in U \tag{22}$$

Note that this doesn't affect our earlier assertion that $\varepsilon < 1$. We can then apply the Polynomial Implementation Lemma (Lemma 8) to construct a $(17+\kappa)$-bounded composition of quadratic polynomials, $\mathbf{Q_2}$, univalent on $\varphi_{2h}(U_{R''-5\varepsilon_1})$ such that, for $z \in \varphi_{2h}(U_{R''-5\varepsilon_1})$, we have

$$\rho_U(\mathbf{Q_2}(z), \hat{\mathcal{E}}(z)) < \frac{\varepsilon_2}{3M_3} \tag{23}$$

$$\|\mathbf{Q_2}^\natural\|_{\varphi_{2h}(U_{R''-5\varepsilon_1})} \leq M_2(1 + \frac{\varepsilon_2}{3M_3}) \tag{24}$$

Controlling Error: "Down"

It is easy to see that

$$\hat{\mathcal{E}}(\varphi_{2h}(U_{R''-5\varepsilon_1})) \subset \varphi_{2h}(U_{R''-4\varepsilon_1}). \tag{25}$$

Now $\overline{U_{R''-4\varepsilon_1}} \subset \tilde{U} \subset \tilde{V}_{2h}$. Thus there exists a finite constant $M_0 > 0$, depending on ε_1, such that $R^{ext}_{(\tilde{U},0)} U_{R''-4\varepsilon_1} = M_0$, whence by conformal invariance $R^{ext}_{(U,0)} \varphi_{2h}(U_{R''-4\varepsilon_1}) = M_0$. Also by (23) we have that $\mathbf{Q_2}(\varphi_{2h}(U_{R''-4\varepsilon_1}))$ is contained in an $\frac{\varepsilon_2}{3M_3}$-neighborhood of $\varphi_{2h}(U_{R''-4\varepsilon_1})$ (using the hyperbolic metric of U). Thus

$$\begin{aligned} R^{ext}_{(U,0)} \mathbf{Q_2}(\varphi_{2h}(U_{R''-4\varepsilon_1})) &\leq M_0 + \frac{\varepsilon_2}{3M_3} \\ &\leq M_0 + \varepsilon_2, \end{aligned}$$

and so

$$R^{ext}_{(U,0)} \mathbf{Q_2}(\varphi_{2h}(U_{R''-4\varepsilon_1})) \leq M_0 + 1 \tag{26}$$

as $\varepsilon_2 < 1$. Now $\mathbf{Q_2}(\varphi_{2h}(U_{R''-4\varepsilon_1})) \subset U_{M_0+1} \subset U_{M_0+3} \subset \overline{U} \subset V_{2h}$. Now fix $M_3 > 1$ such that

$$|(\varphi_{2h}^{-1})^\natural| \leq M_3, \qquad z \in U_{M_0+3}. \tag{27}$$

Further, we have that (φ_{2h}^{-1}, Id) is easily seen to be an admissible pair on (Γ_h, Γ_{2h}). We can then apply the Polynomial Implementation Lemma (Lemma 8) to construct a (17+κ)-bounded quadratic polynomial composition $\mathbf{Q_3}$ that is univalent on U_{M_0+1}, and for $z \in U_{M_0+1}$, we have

$$\rho_U(\mathbf{Q_3}(z), \varphi_{2h}^{-1}(z)) < \frac{\varepsilon_2}{3} \tag{28}$$

$$\|\mathbf{Q_3}\|_{U_{M_0+1}} \leq M_3(1 + \frac{\varepsilon_2}{3}) \tag{29}$$

Concluding the Proof of Phase II

Now, as $\mathbf{Q_1}$, $\mathbf{Q_2}$, and $\mathbf{Q_3}$ were all constructed using the Polynomial Implementation Lemma, they are all (17+κ)-bounded compositions of quadratic polynomials. Then, if we define $\mathbf{Q} := \mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}$, we have that $\mathbf{Q}$ is a (17+κ)-bounded composition of quadratic polynomials. We have that $\mathbf{Q_1}$ is univalent on $A \supset U_{R''-5\varepsilon_1} \supset U_{R''-6\varepsilon_1}$, $\mathbf{Q_2}$ is univalent on $\varphi_{2h}(U_{R''-5\varepsilon_1}) \supset \mathbf{Q_1}(U_{R''-6\varepsilon_1})$, and $\mathbf{Q_3}$ is univalent on $U_{M_0+1} \supset \varphi_{2h}(U_{R''-4\varepsilon_1}) \supset \varphi_{2h}(U_{R''-5\varepsilon_1})$.

Thus $\mathbf{Q}$ is univalent on $U_{R''-6\varepsilon_1}$, and analytic on a neighborhood of $\overline{U_{R''-7\varepsilon_1}}$, namely $U_{R''-6\varepsilon_1}$. As all compositions were created with the Polynomial Implemetation Lemma, we have that $\mathbf{Q}(0) = 0$. Set $\delta = R - R'' + 7\varepsilon_1$. It follows from (13) that $\delta \to 0_+$ as $\varepsilon_1 \to 0_+$. Choose $\tilde{\varepsilon}_1$ sufficiently small such that $\delta(\varepsilon_1) < \frac{1}{80}$, which ensures that $U_{R-\delta(\varepsilon_1)} \neq \emptyset$.

Then for $z \in U_{R-\delta} = U_{R''-7\varepsilon_1} \subset U_{R''-6\varepsilon_1}$, we have

$$\begin{aligned}\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) \leq & \rho_U(\mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z)) + \\ & \rho_U(\varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}(z)) + \\ & \rho_U(\varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}, \mathcal{E}(z)) \end{aligned} \tag{30}$$

We now estimate the three terms on the right hand side of the inequality above. We have that $z \in U_{R''-7\varepsilon_1} \subset U_{R''-6\varepsilon_1}$, so $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-5\varepsilon_1})$ by (21). Then $\mathbf{Q_2} \circ \mathbf{Q_1}(z) \in U_{M_0+1}$ by (26). Thus

$$\rho_U(\mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z)) < \frac{\varepsilon_2}{3} \tag{31}$$

by (28). For the second term, we still have $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-5\varepsilon_1})$ and $\mathbf{Q_2} \circ \mathbf{Q_1}(z) \in U_{M_0+1} \subset U_{M_0+3}$ as above. Also, we have $\hat{\mathcal{E}} \circ \mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-4\varepsilon_1}) \subset U_{M_0+1} \subset U_{M_0+3}$ by (25) and (26). Thus, using the hyperbolic convexity lemma (Lemma 3) and the hyperbolic M-L estimates, we have

$$\rho_U(\varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}(z)) < M_3 \cdot \frac{\varepsilon_2}{3M_3}$$
$$< \frac{\varepsilon_2}{3} \quad (32)$$

by (23) and (27). For the third term we note that $\mathcal{E}(z) = \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \varphi_{2h}$. Recall that $\hat{\hat{N}}$ is a 2-neighborhood of $\varphi_{2h}(U_{R''-5\varepsilon_1})$ in the hyperbolic metric of U. We still have $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-5\varepsilon_1}) \subset \hat{\hat{N}}$, and clearly $\varphi_{2h}(z) \in \varphi_{2h}(U_{R''-6\varepsilon_1}) \subset \varphi_{2h}(U_{R''-5\varepsilon_1}) \subset \hat{\hat{N}}$. We know $|\hat{\mathcal{E}}^\natural|$ is bounded on $U \supset \hat{\hat{N}}$ using (22). Also, $\hat{\mathcal{E}} \circ \varphi_{2h}(z) \in \varphi_{2h}(U_{R''-5\varepsilon_1}) \subset \varphi_{2h}(U_{R''-4\varepsilon_1}) \subset U_{M_0} \subset U_{M_0+3}$ while $\hat{\mathcal{E}} \circ \mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-4\varepsilon_1}) \subset U_{M_0} \subset U_{M_0+3}$ using (21) and the definition of $\hat{\mathcal{E}}$, where we know $|(\varphi_{2h}^{-1})^\natural|$ is bounded using (27). Then, using (19), the hyperbolic convexity lemma (Lemma 3), and the hyperbolic M-L estimates, we have

$$\rho_U(\varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}, \mathcal{E}(z)) < M_3 \cdot M_2 \cdot \frac{\varepsilon_2}{3M_2M_3}$$
$$< \frac{\varepsilon_2}{3} \quad (33)$$

Finally, using (30), (31),(32), and (33), we have

$$\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) < \varepsilon_2$$

which completes the proof of Phase II. □

List of References

[1] M. Comerford, "The carathéodory topology for multiply connected domains ii," *Central European Journal of Mathematics*, vol. 12(5), pp. 721–741, 2014.

[2] M. Comerford, "The carathéodory topology for multiply connected domains i," *Central European Journal of Mathematics*, vol. 11(2), pp. 322–340, 2013.

CHAPTER 6

Proving The Main Theorem

In this chapter we prove Theorem 2. The proof of the theorem will follow from a large inductive argument. We first have some additional lemmas.

Lemma 28. *(The Jordan Curve Argument) Let* U *and* U_R *as above. Given* $\varepsilon > 0$*, suppose* g *is a univalent function defined on* U *such that* $g(0) = 0$ *and* $\rho_U(g(z), z) < \varepsilon$*. Then* $g(U_R) \supset U_{R-\varepsilon}$*.*

Proof. We have that $g(\partial U_R)$ avoids $U_{R-\varepsilon}$ and is a Jordan curve. Also, $g(0) = 0$, so 0 lies inside $g(U_R)$. The function g is a homeomorphism, so $\partial(g(U_R)) = g(\partial U_R)$. Then 0 lies inside the Jordan curve $\partial(g(U_R))$. Since this curve avoids $U_{R-\varepsilon}$, all of the connected set $U_{R-\varepsilon}$ lies inside $\partial(g(U_R))$. Hence $U_{R-\varepsilon} \subset g(U_R)$. □

Lemma 29. *There exist*

a) a sequence of positive real numbers $\{\varepsilon_i\}_{i=1}^{\infty}$*,*

b) a sequence $\{J_i\}_{i=1}^{\infty}$ *of natural numbers and a sequence of quadratic polynomial compositions* $\{\mathbf{Q^i}\}_{i=1}^{\infty}$ *where each* $\mathbf{Q^i}$ *is a composition of* J_i *(17+*κ*)-bounded quadratics,*

c) a sequence of strictly decreasing hyperbolic radii $\{R_i\}_{i=1}^{\infty}$*, and*

d) a sequence of strictly increasing hyperbolic radii $\{S_i\}_{i=1}^{\infty}$*,* *such that*

i) $S_i < \frac{1}{10} < \frac{1}{5} < R_i$ *for all* $i \geq 1$,

ii) $\mathbf{Q^i}$ *is univalent on a neighborhood of* $U_{R_{i-1}}$ *and* $\mathbf{Q^i}(U_{R_{i-1}}) \supset U_{R_i}$,

iii) $\mathbf{Q^i} \circ \cdots \circ \mathbf{Q^1}(U_{\frac{1}{20}}) \subset U_{S_i} \subset U_{\frac{1}{10}}$ *for each* $1 \leq i \leq 2n-1$, *and*

iv) if $\mathbf{Q^i_m}$ *denotes the partial composition of the first* m *quadratics of* $\mathbf{Q^i}$, *then for all* $f \in \mathcal{S}$, *there exists* $1 \leq m \leq J_i$ *such that, if* $z \in U_{\frac{1}{20}}$, *we have*

$$\rho_U(\mathbf{Q^i_m} \circ \mathbf{Q^{i-1}} \circ \cdots \circ \mathbf{Q^1}, f(z)) < \varepsilon_{i+1}.$$

The next lemma follows as an immediate corollary:

Lemma 30. *There exists a sequence of quadratic polynomials* $\{P_m\}_{m=1}^{\infty}$ *such that the following hold:*

i) $\{P_m\}_{m=1}^{\infty}$ *is (17+*κ*)-bounded.*

ii) $Q_m(U_{\frac{1}{20}}) \subset U_{\frac{1}{10}}$ *for all* $m \geq 1$.

iii) For all $f \in \mathcal{S}$, *there exists a subsequence* $\{Q_{m_k}\}_{k=1}^{\infty}$ *such that* $Q_{m_k} \rightrightarrows f$ *on* $U_{\frac{1}{20}}$ *as* $k \to \infty$.

Proof of Lemma 29:

Let C be the equicontinuity constant from the statement of Phase I and let $\delta(x)$ be the function measuring loss of hyperbolic radius from the statement of Phase II. The proof of Lemma 29 will follow immediately from the following claim, which we prove by induction.

Claim: There exist sequences of positive real numbers $\{\varepsilon_i\}_{i=1}^n$, $\{\eta_i\}_{i=1}^n$, and $\{\sigma_i\}_{i=1}^n$, sequences of hyperbolic radii $\{R_i\}_{i=0}^{2n-1}$ and $\{S_i\}_{i=0}^{2n-1}$, integers $\{J_i\}_{i=1}^{2n-1}$, and polynomial compositions $\{\mathbf{Q^i}\}_{i=1}^{2n-1}$ such that the following hold:

i) The sequences $\{\eta_i\}_{i=1}^n$ and $\{\sigma_i\}_{i=1}^n$ are given by

$$\eta_i = \begin{cases} \frac{4\varepsilon_1}{3} + \delta(\varepsilon_1), & i = 1, \\ (\frac{4}{3} + \frac{1}{3C})\varepsilon_i + \delta(\varepsilon_i), & 2 \le i \le n, \end{cases}$$

$$\sigma_i = \begin{cases} \frac{4\varepsilon_1}{3}, & i = 1, \\ (\frac{4}{3} + \frac{1}{3C})\varepsilon_i, & 2 \le i \le n, \end{cases}$$

and $0 < \sigma_i < \eta_i < \frac{1}{40 \cdot 2^i}$, $1 \le i \le n$.

ii) The sequence $\{R_i\}_{i=0}^{2n-1}$ is strictly decreasing and is given by $R_0 = \frac{1}{4}$, $R_1 = \frac{1}{4} - \frac{\varepsilon_1}{3} - \delta(\varepsilon_1)$, and

$$R_i = \begin{cases} \frac{1}{4} - (\sum_{j=1}^k \eta_j) - \frac{\varepsilon_{k+1}}{3C} & i = 2k \text{ for some } 1 \le k \le n-1 \\ \frac{1}{4} - (\sum_{j=1}^k \eta_j) - (\frac{1}{3} + \frac{1}{3C})\varepsilon_{k+1} - \delta(\varepsilon_{k+1}) & i = 2k+1 \text{ for some } 1 \le k \le n-1 \end{cases}$$

The sequence $\{S_i\}_{i=0}^{2n-1}$ is strictly increasing and is given by $S_0 = \frac{1}{20}$, $S_1 = \frac{1}{20} + \frac{\varepsilon_1}{3}$, and

$$S_i = \begin{cases} \frac{1}{20} + (\sum_{j=1}^k \sigma_j) + \frac{\varepsilon_{k+1}}{3C} & i = 2k \text{ for some } 1 \le k \le n-1 \\ \frac{1}{20} + (\sum_{j=1}^k \sigma_j) + (\frac{1}{3} + \frac{1}{3C})\varepsilon_{k+1} & i = 2k+1 \text{ for some } 1 \le k \le n-1 \end{cases}$$

iii) $S_i < \frac{1}{10} < \frac{1}{5} < R_i$ for each $1 \le i \le 2n-1$.

iv) Each $\mathbf{Q^i}$ is a $(17{+}\kappa)$-bounded composition of J_i quadratics. For each $1 \le i \le 2n-1$, set $I_i = \sum_{j=1}^{i} J_j$.

v) For each $1 \le i \le 2n-1$, $\mathbf{Q^i}$ is univalent on a neighborhood of $\overline{U_{R_{i-1}}}$ and

$$\mathbf{Q^i}(U_{R_{i-1}}) \supset U_{R_i}$$

Thus the branch of $(\mathbf{Q^i} \circ \cdots \circ \mathbf{Q^1})^{-1}$ which maps 0 to 0 exists and is univalent on U_{R_i}.

vi) For each $1 \le i \le 2n-1$,

$$\mathbf{Q^i} \circ \cdots \circ \mathbf{Q^1}(U_{\frac{1}{20}}) \subset U_{S_i} \subset U_{\frac{1}{10}}$$

vii) If $i = 2k$ with $1 \le k \le n-1$ is even, and $z \in U_{R_{i-1}}$,

$$\rho_U(\mathbf{Q^i}(z), (\mathbf{Q^{i-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z)) < \frac{\varepsilon_{k+1}}{3C}$$

For the final three hypotheses, let $i = 2k+1$ with $0 \le k \le n-1$ be odd.

viii) If $z \in U_{R_i} \subset U_{R_{i-1} - \frac{\varepsilon_k}{3}}$, using the same inverse branch mentioned in v) we have

$$\rho_U((\mathbf{Q^i} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) < \varepsilon_{k+1}$$

ix) If $z \in U_{\frac{1}{4}}$

$$\rho_U(\mathbf{Q^i}(z), z) < \frac{\varepsilon_{k+1}}{3}$$

x) If, for each $1 \leq m \leq J_i$, $\mathbf{Q^i_m}$ denotes the partial composition of the first m quadratics of $\mathbf{Q^i}$, then for all $f \in \mathcal{S}$ there exists an m, $1 \leq m \leq J_i$, such that, if $z \in U_{\frac{1}{20}}$, we have

$$\rho_U(\mathbf{Q^i_m} \circ \mathbf{Q^{i-1}} \circ \cdots \circ \mathbf{Q^1}(z), f(z)) < \varepsilon_{k+1}$$

Remarks

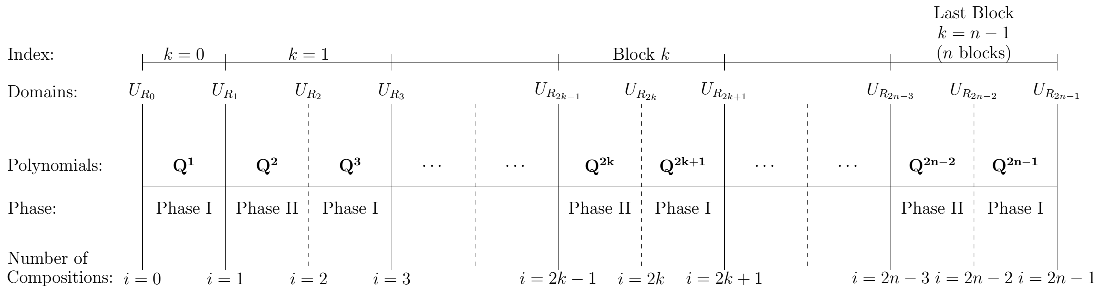
1. Statements i)-iii) are designed for keeping track of the domains on which estimates are holding.

2. Statements v) and vi) in regards to keeping track of domains under iteration of the polynomial compositions.

3. Statement vii) is a “Phase II” statement regarding error correction of a polynomial composition constructed with Phase II.

4. Statements viii) -x) are “Phase I” statements. Statement viii) is a bound on the error to be corrected by the next Phase II approximation. Statement x) is the key piece for proving Theorem 2.

Figure 3. A block diagram illustrating the induction.



Base Case: $n = 1$.

Let $\delta(x)$ be the function whose existence is guaranteed by Phase II. Now pick $\varepsilon_1 > 0$ such that if we set

$$\eta_1 = \frac{4}{3}\varepsilon_1 + \delta(\varepsilon_1)$$
$$\sigma_1 = \frac{4}{3}\varepsilon_1,$$

then $0 < \sigma_1 < \eta_1 < \frac{1}{40 \cdot 2}$. This verifies i). Now set $R_0 = \frac{1}{4}$, $S_0 = \frac{1}{20}$, and

$$R_1 = \frac{1}{4} - \frac{\varepsilon_1}{3} - \delta(\varepsilon_1)$$
$$S_1 = \frac{1}{20} + \frac{\varepsilon_1}{3},$$

which verifies ii) and iii). Let $\{f_0, f_1, \cdots f_{N_1+1}\}$ be an $\frac{\varepsilon_1}{3}$-net for $\mathcal{S}$, where $N_1 \in \mathbb{N}$, with $f_0 = f_{N_1+1} = Id$. Apply Phase I for these functions with $R = \frac{1}{4}$, $\varepsilon = \frac{\varepsilon_1}{3}$, to obtain $M_1 = M_1(\varepsilon_1, N_1) \in \mathbb{N}$, and the polynomial composition $\mathbf{Q^1} = Q_{(N_1+1)M_1}$, which satisfies, for $1 \leq i \leq N_1 + 1$,

1. $Q_{iM_1}(0) = 0$

2. Q_{iM_1} is univalent on U_{5R}.

3. $Q_{iM_1}(U_{2R}) \subset U_{4R}$.

4. $\rho_U(f_i(z), Q_{iM_1}(z)) < \frac{\varepsilon_1}{3}$ on U_{2R}.

5. $\|Q^{\natural}_{iM_1}\|_{U_R} \leq C$.

As $f_{N_1+1} = Id$, ix) is verified in view of 4. above. If $z \in U_{R_1}$, let $z = \mathbf{Q^1}(w)$ for some $w \in U_{R_0}$. Then

$$\begin{aligned}\rho_U((\mathbf{Q^1})^{-1}(z), z) &= \rho_U(w, \mathbf{Q^1}(w)) \\ &< \frac{\varepsilon_1}{3} \\ &< \varepsilon_1\end{aligned}$$

which verifies viii). Next, for $f \in \mathcal{S}$, if f_m is the member of the net for which $\rho_U(f(z), f_m(z)) < \frac{\varepsilon_1}{3}$, and $\mathbf{Q^1_m}$ is the partial composition which satisfies $\rho_U(\mathbf{Q^1_m}(z), f_m(z)) < \frac{\varepsilon_1}{3}$. Then

$$\begin{aligned}\rho_U(\mathbf{Q^1_m}(z), f(z)) &\leq \rho_U(\mathbf{Q^1_m}(z), f_m(z)) + \rho_U(f_m(z), f(z)) \\ &\leq \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} \\ &< \varepsilon_1,\end{aligned}$$

which verifies x). Now $\mathbf{Q^1}$ fixes 0 and is univalent on $U_{\frac{5}{4}} \supset U_{\frac{1}{4}} = U_{R_0}$. Further, if $\rho_U(z,0) = \frac{1}{4}$, then $\rho_U(\mathbf{Q^1}(z),0) > \frac{1}{4} - \frac{\varepsilon_1}{3}$, so by the Jordan curve argument $\mathbf{Q^1}(U_{R_0}) \supset U_{\frac{1}{4}-\frac{\varepsilon_1}{3}} \supset U_{R_1}$, which verifies v). Similarly, if $\rho_U(z,0) = \frac{1}{20}$, then $\rho_U(\mathbf{Q^1},0) < \frac{1}{20} + \frac{\varepsilon_1}{3}$. This implies $\mathbf{Q^1}(S_0) \subset S_1$ and verifies vi).

Set $J_1 = N_1 + 2$, $I_1 = J_1$, which verifies iv). Finally, we note that vii) is vacuously true, which completes the base case.

Induction Hypothesis: Assume i)-x) hold for some arbitrary n.

Induction Step: We show this is true for $n+1$.

Since the above hypotheses hold for n, we've already defined $R_{2n-1} = R_{2n-2} - \frac{\varepsilon_n}{3} - \delta(\varepsilon_n)$, where we recall that $\delta(x)$ is the function whose existence is guaranteed by Phase II. Using viii) for $i = 2n-1$ we have

$$\rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) < \varepsilon_n, \qquad z \in U_{R_{2n-1}}. \tag{34}$$

We can pick ε_{n+1} sufficiently small such that if we set

$$\eta_{n+1} = (\frac{4}{3} + \frac{1}{3C})\varepsilon_{n+1} + \delta(\varepsilon_{n+1})$$
$$\sigma_{n+1} = (\frac{4}{3} + \frac{1}{3C})\varepsilon_{n+1},$$

then we can ensure

$$0 < \sigma_{n+1} < \eta_{n+1} < \frac{1}{40 \cdot 2^{n+1}}$$

which verifies i) for n+1. If we now apply Phase II, with $R = R_{2n-2} - \frac{\varepsilon_n}{3}, \varepsilon_1 = \varepsilon_n$, and $\varepsilon_2 = \frac{\varepsilon_{n+1}}{3C}$, we can find a quadratic polynomial composition $\mathbf{Q^{2n}}$ which is univalent on a neighborhood of $\overline{U_{R_{2n-1}}}$ (as $U_{R_{2n-1}} \subsetneq R_{2n-2} - \frac{\varepsilon_n}{3}$) and satisfies

$$\rho_U(\mathbf{Q^{2n}}(z), (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z)) < \frac{\varepsilon_{n+1}}{3C}, \qquad z \in U_{R_{2n-1}} \tag{35}$$

which verifies vii) for $n+1$. Note that in light of $\tilde{\varepsilon}_2$ in the statement of Phase II, we may need to make ε_2 smaller, if necessary. However, this does not affect η_{n+1} or σ_{n+1} above. If we let J_{2n} be the number of quadratics in $\mathbf{Q^{2n}}$ and set $I_{2n} = I_{2n-1} + J_{2n}$, we see that the first half of iv) for $n+1$ is also verified. Then, by hypotheses i) and ii) for n, if we set

$$R_{2n} = R_{2n-1} - \varepsilon_n - \frac{\varepsilon_{n+1}}{3C}$$
$$S_{2n} = S_{2n-1} + \varepsilon_n + \frac{\varepsilon_{n+1}}{3C},$$

then

$$
\begin{aligned}
R_{2n} &= (\frac{1}{4} - \sum_{j=1}^{n-1} \eta_j - (\frac{1}{3} + \frac{1}{3C})\varepsilon_n - \delta(\varepsilon_n)) - \varepsilon_n - \frac{\varepsilon_{n+1}}{3C} \\
&= \frac{1}{4} - \sum_{j=1}^{n} \eta_j - \frac{\varepsilon_{n+1}}{3C} \\
S_{2n} &= (\frac{1}{20} + \sum_{j=1}^{n-1} \sigma_j + (\frac{1}{3} + \frac{1}{3C})\varepsilon_n) + \varepsilon_n + \frac{\varepsilon_{n+1}}{3C} \\
&= \frac{1}{20} + \sum_{j=1}^{n} \sigma_j + \frac{\varepsilon_{n+1}}{3C},
\end{aligned}
$$

which verifies half of ii) for $n+1$. Further,

$$
\begin{aligned}
R_{2n} &= \frac{1}{4} - \sum_{j=1}^{n} \eta_j - \frac{\varepsilon_{n+1}}{3C} \\
&> \frac{1}{4} - \sum_{j=1}^{n} \frac{1}{40 \cdot 2^j} - \frac{1}{40 \cdot 2^{n+1}} \\
&= \frac{1}{4} - \frac{1}{40}(1 - \frac{1}{2^n} - \frac{1}{2^{n+1}}) \\
&> \frac{1}{4} - \frac{1}{40} \\
&= \frac{9}{40} \\
&> \frac{1}{5},
\end{aligned}
$$

and

$$
\begin{aligned}
S_{2n} &= \frac{1}{20} + \sum_{j=1}^{n} \eta_j + \frac{\varepsilon_{n+1}}{3C} \\
&< \frac{1}{20} + \sum_{j=1}^{n} \frac{1}{40 \cdot 2^j} + \frac{1}{40 \cdot 2^{n+1}} \\
&= \frac{1}{20} + \frac{1}{40}(1 - \frac{1}{2^n} + \frac{1}{2^{n+1}}) \\
&< \frac{1}{20} + \frac{1}{40} \\
&= \frac{3}{40} \\
&< \frac{1}{10},
\end{aligned}
$$

which verifies the first half of iii) for $n+1$. Combining (34) and (35) we have, on $U_{R_{2n-1}}$,

$$
\begin{aligned}
\rho_U(\mathbf{Q^{2n}}(z), z) &\leq \rho_U(\mathbf{Q^{2n}}(z), (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z)) + \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) \\
&< \frac{\varepsilon_{n+1}}{3C} + \varepsilon_n \qquad (36)
\end{aligned}
$$

This, combined with the Jordan curve argument, implies that

$$
\mathbf{Q^{2n}}(U_{R_{2n-1}}) \supset U_{R_{2n}},
$$

which verifies the first half of v) for $n+1$. Further, (36) implies that

$$
\mathbf{Q^{2n}}(U_{S_{2n-1}}) \subset U_{S_{2n-1}+\varepsilon_n+\frac{\varepsilon_{n+1}}{3C}} = U_{S_{2n}},
$$

which verifies half of vi) for $n+1$ and finishes the Phase II portion of the induction step.

Construct an $\frac{\varepsilon_{n+1}}{3}$-net $\{f_0, f_1, \cdots f_{N_{n+1}+1}\}$ for $\mathcal{S}$, where we obtain $N_{n+1} = N_{n+1}(\varepsilon_{n+1}) \in \mathbb{N}$ and require $f_0 = f_{N_{n+1}+1} = Id$. We apply Phase I with $R = \frac{1}{4}$ and $\varepsilon = \frac{\varepsilon_{n+1}}{3}$ to obtain $M_{n+1} = M_{n+1}(\varepsilon_{n+1}, N_{n+1}) \in \mathbb{N}$, and a $(17+\kappa)$-bounded sequence of quadratic polynomials $\{P_m\}_{m=I_{2n}+1}^{I_{2n}+M_{n+1}(N_{n+1}+1)}$. Set $\mathbf{Q^{2n+1}} = Q_{I_{2n}+1, I_{2n}+M_{n+1}(N_{n+1}+1)}$ and let $J_{2n+1} = M_{n+1}(N_{n+1}+1)$ be the number of quadratics. Setting $I_{2n+1} = I_{2n} + J_{2n+1}$, we verifies iv) for $n+1$. Let $\mathbf{Q^{2n+1}_m}$ denote the composition of the first m quadratics of $\mathbf{Q^{2n+1}}$, with $1 \le m \le J_{2n+1}$. By Phase I this composition is univalent on a neighborhood of $\overline{U}_{R_{2n}}$, fixes 0, and satisfies

a) $\mathbf{Q^{2n+1}}(U_{\frac{1}{2}}) \subset U_1$

b) $\rho_U(f_i(z), \mathbf{Q^{2n+1}_{iM_{n+1}}}(z)) < \frac{\varepsilon_{n+1}}{3}$, $z \in U_{\frac{1}{2}}$, $1 \le i \le N_{n+1}+1$

c) $\|(\mathbf{Q^{2n+1}_{iM_{n+1}}})^{\natural}\|_{U_{\frac{1}{2}}} \le C$, $1 \le i \le N_{n+1}+1$

Using the fact that $f_{N_{n+1}+1}$ is the identity function, ix) is verified in view of b). Next define

$$R_{2n+1} = R_{2n} - \frac{\varepsilon_{n+1}}{3} - \delta(\varepsilon_{n+1})$$
$$S_{2n+1} = S_{2n} + \frac{\varepsilon_{n+1}}{3}$$

Then, using i) and ii) for R_{2n} and S_{2n}

$$\begin{aligned}
R_{2n+1} &= (\frac{1}{4} - \sum_{j=1}^{n} \eta_j - \frac{\varepsilon_{n+1}}{3C}) - \frac{\varepsilon_{n+1}}{3} - \delta(\varepsilon_{n+1}) \\
&= \frac{1}{4} - \sum_{j=1}^{n} \eta_j - \frac{\varepsilon_{n+1}}{3C} - \frac{\varepsilon_{n+1}}{3} - \delta(\varepsilon_{n+1}) \\
S_{2n+1} &= (\frac{1}{20} + \sum_{j=1}^{n} \sigma_j + \frac{\varepsilon_{n+1}}{3C}) + \frac{\varepsilon_{n+1}}{3} \\
&= \frac{1}{20} + \sum_{j=1}^{n} \sigma_j + (\frac{1}{3} + \frac{1}{3C})\varepsilon_{n+1}
\end{aligned}$$

Thus we have verified ii) and iii) for $n+1$ (note that completing the verification of iii) is similar to a calculation above verifying the first half of iii) for $n+1$).

By b) applied to the function $f_{N_{n+1}+1} = Id$, together with the Jordan curve argument we have

$$\mathbf{Q^{2n+1}}(U_{R_{2n}}) \supset U_{R_{2n} - \frac{\varepsilon_{n+1}}{3}} \supset R_{2n+1},$$

which verifies v) for $n+1$. Again by b) applied to the function $f_{N_{n+1}+1} = Id$, we see

$$\mathbf{Q^{2n+1}}(U_{S_{2n}}) \subset U_{S_{2n} + \frac{\varepsilon_{n+1}}{3}} = U_{S_{2n+1}}.$$

This, together with vi) for n and iii) for $n+1$, verifies vi) for $n+1$.

Now let $z \in U_{R_{2n}}$. By v) for $n+1$ $z = \mathbf{Q^{2n}}(w)$ for some $w \in U_{R_{2n-1}}$. Also, the branch of $(\mathbf{Q^{2n}})^{-1}$ which fixes 0 is defined and univalent on $U_{R_{2n}}$, so

$$\begin{aligned}\rho_U((\mathbf{Q^{2n}} \circ \mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) &= \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n}})^{-1}(z), z) \\ &= \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w), \mathbf{Q^{2n}}(w)) \\ &< \frac{\varepsilon_{n+1}}{3C} \qquad (37)\end{aligned}$$

by (35). By b) for $f_{N_{n+1}+1} = Id$, using this same inverse branch which fixes 0,

$$\rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) < \frac{\varepsilon_{n+1}}{3}, \qquad z \in \mathbf{Q^{2n+1}}(U_{R_{2n}}) \supset U_{R_{2n+1}} \qquad (38)$$

Then, if $z \in U_{R_{2n+1}}$ and we let $w = (\mathbf{Q^{2n+1}})^{-1}(z) \in U_{R_{2n}}$, we have

$$\begin{aligned}\rho_U((\mathbf{Q^{2n+1}} \circ \mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) &= \rho_U((\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n+1}})^{-1}(z), z) \\ \leq \rho_U((\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n+1}})^{-1}(z), (\mathbf{Q^{2n+1}})^{-1}(z)) &+ \rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) \\ = \rho_U((\mathbf{Q^{2n}} \circ \mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w), w) &+ \rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) \\ &< \frac{\varepsilon_{n+1}}{3C} + \frac{\varepsilon_{n+1}}{3} < \varepsilon_{n+1}\end{aligned}$$

(39)

using (37) and (38). This verifies xiii).

Let $f \in \mathcal{S}$ be arbitrary and let $z \in U_{\frac{1}{20}}$. Let f_i be the element of the $\frac{\varepsilon_1}{3}$-net which approximates f to within $\frac{\varepsilon_{n+1}}{3}$ on $D(0, \frac{1}{24}) \supset U \supset U_{\frac{1}{20}}$. Let $\mathbf{Q^{2n+1}_m} = \mathbf{Q^{2n+1}_{iM_{n+1}}}$

be the corresponding partial composition of $\mathbf{Q^{2n+1}}$ which approximates f_i to within $\frac{\varepsilon_{n+1}}{3}$ on $U_{\frac{1}{2}} \supset U_{\frac{1}{20}}$.

Also, (recall $z \in U_{\frac{1}{20}}$) we have $\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(z) \in U_{R_{2n}}$ using vi) for $n+1$. Then using the univalence from v) for $n+1$ $z = (\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w)$ for some $w \in U_{R_{2n}}$. Then, if $w = \mathbf{Q^{2n}}(\zeta)$ for $\zeta \in U_{R_{2n-1}}$, we have

$$\begin{aligned}\rho_U(\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(\mathbf{z}), z) &= \rho_U(w, (\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w)) \\ &= \rho_U(\mathbf{Q^{2n}}(\zeta), (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(\zeta)) < \frac{\varepsilon_{n+1}}{3C} \qquad (40)\end{aligned}$$

by (35). Using (40), b), c) and the fact that f_i approximates f, we have

$$\begin{aligned}&\rho_U(\mathbf{Q^{2n+1}_m} \circ \mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(\mathbf{z}), f(z)) \\ &\leq \rho_U(\mathbf{Q^{2n+1}_m} \circ \mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(z), \mathbf{Q^{2n+1}_m}(z)) + \rho_U(\mathbf{Q^{2n+1}_m}(z), f_i(z)) + \rho_U(f^i(z), f(z)) \\ &\leq C \cdot \frac{\varepsilon_{n+1}}{3C} + \frac{\varepsilon_{n+1}}{3} + \frac{\varepsilon_{n+1}}{3} \\ &= \varepsilon_{n+1}\end{aligned}$$

which verifies x). Note that the first term uses (40) and c), the second uses b), and the third uses the net approximation. This completes the proof of the claim from which Lemma 29 follows. □

We now prove theorem 2:

Proof of Theorem 2: Let $f \in \mathcal{S}$ be arbitrary. Lemma 30 implies that there exists a bounded sequence of quadratic polynomials $\{P_m\}_{m=1}^{\infty}$, and a subsequence $\{P_{m_k}\}_{k=1}^{\infty}$ of $\{P_m\}_{m=1}^{\infty}$ such that $\{Q_{m_k}\}_{k=1}^{\infty}$ converges locally uniformly to f on $U_{\frac{1}{20}}$. Further, Lemma 30 also implies that $U_{\frac{1}{20}}$ is contained in a bounded Fatou component V for this sequence. Since $\{Q_{m_k}\}_{k=1}^{\infty}$ is normal on V, we may pass to a further subsequence, if necessary, to ensure this subsequence of iterates will converge locally uniformly on all of V. By the identity principle, the limit must be f. $\square$

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