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Global Dynamics of Some Competitive and Cooperative Discrete Dynamical Systems

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GLOBAL DYNAMICS OF SOME COMPETITIVE AND COOPERATIVE DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT

This dissertation investigates the dynamics of some second-order difference equations and systems of difference equations whose defining functions satisfy certain monotonicity properties. In each study we utilize the theory for specific classes of monotone difference equations to establish local and global dynamics.

Manuscript 1 is an introduction that provides fundamental definitions and important results for difference equations that are used throughout the rest of the thesis.

Manuscript 2 presents some potential global dynamic scenarios for competitive systems of difference equations in the plane. These results are extended to apply to the class of second-order difference equations whose transition functions are decreasing in the first variable and increasing in the second. In particular, these results are applied to investigate the following equation as a case study:

$$x_{n+1} = \frac{Cx_{n-1}^2 + Ex_{n-1}}{ax_n^2 + dx_n + f}, \quad n = 0, 1, \ldots, \tag{1}$$

where the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that the solution is defined and the parameters satisfy $C, E, a, d, f \geq 0$, $C + E > 0$, $a + C > 0$, and $a + d > 0$. A rich collection of additional dynamical behaviors for Equation (1) are established to provide a nearly complete characterization of its global dynamics with the basins of attraction of equilibria and periodic solutions.

Manuscript 3 considers the following second-order generalization of the classical Beverton-Holt equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \ldots. \tag{2}$$

Here f is a continuous function nondecreasing in both arguments, the parameter a is a positive real number, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that the solution is defined. Local and global dynamics

of Equation (2) are presented in the event f is chosen to be a certain type of linear or quadratic polynomial. Particular consideration is given to the existence problem of period-two solutions.

Manuscript 4 presents an order-k generalization of Equation (2),

$$x_{n+1} = \frac{af(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + f(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1, \tag{3}$$

where f remains a function nondecreasing in all of its arguments, $a > 0$, and $x_0, x_{-1}, \ldots, x_{1-k} \geq 0$. We examine several interesting examples in which f is a transcendental function. This manuscript establishes conditions under which Equation (3) possesses a unique positive equilibrium that is a global attractor of all solutions with positive initial conditions. In particular, results are presented for the special case in which $f(x, \ldots, x)$ is chosen to be a concave function.

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I have enjoyed the warm hospitality of the URI Department of Mathematics and have been privileged to have there completed my entire college education and begun my own teaching career. Each member of the faculty has improved my education and teaching ability in some way. In particular, I would like to thank Dr. Tom Bella, Dr. Li Wu, Dr. James Baglama, and Laura Barnes for offering their mentorship and for involving me in several departmental service projects. From each of them I learned unique skills and was engaged in work dedicated to student success that motivated me to continue in the pursuit of my degree. I would also like to thank Deborah Beagan, our department administrator, who has provided considerable assistance and support every step of the way.

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DEDICATION

To the man who inspired me to become a mathematician in the first place:

Professor Lewis Pakula, URI Department of Mathematics

(*November 8, 1946 – October 1, 2012*)

PREFACE

This thesis has been prepared in manuscript form. The main content of the thesis is made up of three research papers: Manuscripts 2, 3, and 4. Manuscript 2 was submitted for publication on March 25, 2018 to *Advances in Difference Equations*, and Manuscripts 3 and 4 will be submitted for publication in the near future.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

MANUSCRIPT 1

Introduction

1.1 Second-Order Difference Equations

Discrete dynamical systems describe the evolution of a quantity or population whose changes are measured over discrete time intervals. Difference equations may be thought of more specifically as recurrence relations that describe a discrete dynamical system by relating the size of the next state (or generation), often denoted x_{n+1}, to some function of the sizes of several past states $x_n, x_{n-1}, \ldots$. For example, a **second-order autonomous difference equation** may take the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{1}$$

where $f : I \times I \to I$ with $I \subseteq \mathbb{R}$, and the initial conditions x_0 and x_{-1} are arbitrary elements from I. For each choice of initial conditions, Equation (1) has a unique solution $\{x_n\}_{n=-1}^{\infty}$. Much initial investigation in this field of research focuses on describing the *local dynamics* of such difference equations by examining the short-term trajectory of solutions for different choices of initial conditions. The paramount goal is to determine the *global dynamics* of a difference equation by analytically characterizing the end behavior of all solutions as $n \to \infty$.

1.2 Local Stability Analysis

To develop the necessary vocabulary we will utilize to study second-order difference equations, we will first reference some fundamental definitions provided in [5]. All definitions will accommodate the second-order Equation (1), but analogous statements will hold for equations of higher order or systems of difference equations. In particular, related preliminary material may be found in [6].

Definition 1 A number $\overline{x} \in I$ satisfying $\overline{x} = f(\overline{x}, \overline{x})$ is called an equilibrium, or fixed point, of Equation (1).

Definition 2 Let $\overline{x}$ be an equilibrium of Equation (1).

(i) $\overline{x}$ is called **locally stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0,\, x_{-1} \in I$ with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \delta$, we have

$$|x_n - \overline{x}| < \varepsilon \text{ for all } n \geq -1.$$

(ii) $\overline{x}$ is called **locally asymptotically stable** if it is locally stable and if there exists $\gamma > 0$ such that for all $x_0,\, x_{-1} \in I$ with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \gamma$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) $\overline{x}$ is called a **global attractor** if for every $x_0,\, x_{-1} \in I$ we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) $\overline{x}$ is called **globally asymptotically stable** if it is locally stable and a global attractor.

(v) $\overline{x}$ is called **unstable** if it is not stable.

(vi) $\overline{x}$ is called a **repeller** if there exists $r > 0$ such that for all $x_0,\, x_{-1} \in I$ with $0 < |x_0 - \overline{x}| + |x_{-1} - \overline{x}| < r$, there exists $N \geq 1$ such that

$$|x_N - \overline{x}| \geq r.$$

A repeller is an unstable equilibrium.

Let

$$P = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}) \quad \text{and} \quad Q = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$$

denote the partial derivatives of the function $f(u, v)$ used in Equation (1) evaluated at an equilibrium $\overline{x}$. The equation

$$y_{n+1} = Py_n + Qy_{n-1}, \quad n = 0, 1, \ldots \tag{2}$$

is called the linearized equation associated with Equation (1) about $\overline{x}$. The quadratic equation

$$\lambda^2 - P\lambda - Q = 0 \tag{3}$$

is called the characteristic equation of the linearized equation (2) associated with Equation (1). The nature of the solutions of Equation (3) provide a classification of the local character of an equilibrium $\overline{x}$. The following result (Theorem 2.13 of [6] or Theorem 1.1.1 of [5]) summarizes the potential cases that will be used to classify the local stability of equilibria.

Theorem 1 *Consider an equilibrium $\overline{x}$ of Equation* (1)*.*

(i) $\overline{x}$ is ***locally asymptotically stable*** *if and only if every solution of Equation* (3) *lies inside the unit circle, which is true if and only if*

$$|P| < 1 - Q < 2.$$

(ii) $\overline{x}$ is a ***repeller*** *if and only if every solution of Equation* (3) *lies outside the unit circle, which is true if and only if*

$$|P| < |1 - Q| \quad \textit{and} \quad |Q| > 1.$$

(iii) $\overline{x}$ is a ***saddle point*** *if and only if Equation* (3) *has one root that lies inside the unit circle and one root that lies outside the unit circle, which is true if and only if*

$$|P| > |1 - Q|.$$

(iv) $\overline{x}$ is ***nonhyperbolic*** *if and only if Equation* (3) *has at least one root that lies on the unit circle, which is true if and only if*

$$|P| = |1 - Q| \quad \textit{or} \quad (Q = -1 \quad \textit{and} \quad |P| \leq 2)\,.$$

Much of our work will investigate the existence of periodic solutions of prime period two. The general definition of a periodic solution is given below.

Definition 3 A solution $\{x_n\}$ of Equation (1) is said to be **periodic** with period p if

$$x_{n+p} = x_n \text{ for all } n \geq -1. \tag{4}$$

A solution $\{x_n\}$ is said to be **periodic with prime period** p, or a **minimal period-p solution**, if it is periodic with period p and p is the least positive integer for which Equation (4) holds.

1.3 Monotone Systems of Difference Equations

One can also consider systems of difference equations of the form

$$\left\{ \begin{array}{lcl} x_{n+1} & = & g(x_n, y_n) \\ y_{n+1} & = & h(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots, \tag{5}$$

where g and h are given functions and the initial condition (x_0, y_0) comes from some considered set in the intersection of the domains of g and h. A great deal of theory has been established for such systems in the event the defining functions obey certain monotonicity restrictions.

Definition 4 Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T : R \to R$ be a continuous map. Set $T(x, y) = (g(x, y), h(x, y))$. The map T is **competitive** if $g(x, y)$ is nondecreasing in x and nonincreasing in y while $h(x, y)$ is nonincreasing in x and nondecreasing in y. If both g and h are nondecreasing in x and y, we say that T is **cooperative**. If T is competitive (resp. cooperative), the associated system of difference equations (5) is said to be competitive (resp. cooperative). The map T and the associated system of difference equations are said to be **strongly competitive** (resp. **strongly cooperative**) if the adjectives nondecreasing and nonincreasing are replaced by increasing and decreasing, respectively.

Competitive and cooperative systems have been widely studied, largely due to their applicability to biological modeling. These monotone systems rank among the most important classes of systems that model interspecies relationships. Relevant research in evolutionary biology may be found in [2, 4]. The theory developed for such systems provides useful insight into the global dynamics of difference equations such as Equation (1) above.

Difference equations defined by Equation (1) are of particular interest when the function f is monotone in each of its variables. Such difference equations have direct applications to the study of two-generation population dynamics. In particular, this dissertation examines two main classes of difference equations that satisfy prescribed monotonicity characteristics, and we may now elucidate their connection to competitive and cooperative systems. In general, Equation (1) may always be transformed via a suitable change of coordinates to a corresponding system of difference equations. Set $x_{n-1} = u_n$ and $x_n = v_n$ to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \qquad n = 0, 1, \ldots .$$

Let $T(u,v) = (v, f(v,u))$. The second iterate T^2 is given by

$$T^2(u,v) = (f(v,u),\ f(f(v,u),v)) .$$

If f is a function nonincreasing in the first argument and nondecreasing in the second, the second iterate of its corresponding map, T^2, is competitive; see [8]. General dynamic scenarios for Equation (1) when f exhibits this monotonic character (and, more generally, for competitive systems) will be presented in Manuscript 2. If f is a function nondecreasing in both arguments, then T^2 is cooperative. Indeed, Manuscripts 3 and 4 will investigate Equation (1) for a class of functions that are always nondecreasing in both arguments.

We now present some general results discussed in [3, 8] for order-preserving maps that provide an essential foundation for many consequential results established for competitive and cooperative systems. More specific background material is presented in Manuscript 2 that is tailored to competitive systems.

Let $\preceq$ be a partial order on $\mathbb{R}^n$ with nonnegative cone P. For $\vec{x}, \vec{y} \in \mathbb{R}^n$ the **order interval** $[\![\vec{x}, \vec{y}]\!]$ is the set of all $\vec{z}$ such that $\vec{x} \preceq \vec{z} \preceq \vec{y}$. We say $\vec{x} \prec \vec{y}$ if $\vec{x} \preceq \vec{y}$ and $\vec{x} \neq \vec{y}$, and $\vec{x} \ll \vec{y}$ if $\vec{y} - \vec{x} \in int\, P$. A map T on a subset of $\mathbb{R}^n$ is **order-preserving** if $T(\vec{x}) \preceq T(\vec{y})$ whenever $\vec{x} \prec \vec{y}$, **strictly order-preserving** if $T(\vec{x}) \prec T(\vec{y})$ whenever $\vec{x} \prec \vec{y}$, and **strongly order-preserving** if $T(\vec{x}) \ll T(\vec{y})$ whenever $\vec{x} \prec \vec{y}$. The next result is stated for order-preserving maps on $\mathbb{R}^n$.

Theorem 2 *For a nonempty set $R \subseteq \mathbb{R}^n$ and a partial order $\preceq$ on $\mathbb{R}^n$, let $T : R \to R$ be an order-preserving map, and let $\vec{a}, \vec{b} \in R$ be such that $\vec{a} \prec \vec{b}$ and $[\![\vec{a}, \vec{b}]\!] \subseteq R$. If $\vec{a} \preceq T(\vec{a})$ and $T(\vec{b}) \preceq \vec{b}$, then $[\![\vec{a}, \vec{b}]\!]$ is invariant and:*
(i) There exists a fixed point of T in $[\![\vec{a}, \vec{b}]\!]$.
(ii) If T is strongly order-preserving, then there exists a fixed point in $[\![\vec{a}, \vec{b}]\!]$ which is stable relative to $[\![\vec{a}, \vec{b}]\!]$.
(iii) If there is only one fixed point in $[\![\vec{a}, \vec{b}]\!]$, then it is a global attractor in $[\![\vec{a}, \vec{b}]\!]$ and therefore asymptotically stable relative to $[\![\vec{a}, \vec{b}]\!]$.

We say that $\{\vec{x}_n\}_{n\in\mathbb{Z}}$ is an entire orbit of a map $T : A \to A$, $A \subseteq \mathbb{R}^n$ if $\vec{x}_{n+1} = T(\vec{x}_n)$ for all $n \in \mathbb{Z}$. This orbit is said to join $\vec{u}_1$ to $\vec{u}_2$ if $\vec{x}_n \to \vec{u}_1$ as $n \to -\infty$ and $\vec{x}_n \to \vec{u}_2$ as $n \to \infty$. The following result is for strictly order-preserving maps.

Theorem 3 (Order Interval Trichotomy of Dancer and Hess) *Let $\vec{u}_1 \preceq \vec{u}_2$ be distinct fixed points of a strictly order-preserving map $T : A \to A$, where $A \subseteq \mathbb{R}^n$, and let $I = [\![\vec{u}_1, \vec{u}_2]\!] \subseteq A$. Then at least one of the following holds.*
(a) T has a fixed point in I distinct from $\vec{u}_1$ and $\vec{u}_2$.
(b) There exists an entire orbit $\{\vec{x}_n\}_{n\in\mathbb{Z}}$ of T in I joining $\vec{u}_1$ to $\vec{u}_2$ and satisfying $\vec{x}_n \preceq \vec{x}_{n+1}$.

(c) There exists an entire orbit $\{\vec{x}_n\}_{n\in\mathbb{Z}}$ *of* T *in* I *joining* $\vec{u}_2$ *to* $\vec{u}_1$ *and satisfying* $\vec{x}_{n+1} \preceq \vec{x}_n$.

We also have the following powerful corollaries.

Corollary 1 *If* $\vec{a}$ *and* $\vec{b}$ *are stable fixed points, then there exists a third fixed point in* $[\![\vec{a}, \vec{b}]\!]$.

Corollary 2 *If the nonnegative cone of* $\preceq$ *is a generalized quadrant in* $\mathbb{R}^n$*, and if* T *has no fixed points in* $[\![\vec{u}_1, \vec{u}_2]\!]$ *other than* $\vec{u}_1$ *and* $\vec{u}_2$*, then the interior of* $[\![\vec{u}_1, \vec{u}_2]\!]$ *is either a subset of the basin of attraction of* $\vec{u}_1$ *or a subset of the basin of attraction of* $\vec{u}_2$.

These results have been utilized in papers such as [1] to determine the basins of attraction of certain fixed points; moreover, they provide a theoretical foundation for the investigation of the dynamics of competitive and cooperative systems. In particular, Kulenović and Merino have proven general results for monotone systems in [7, 8, 9, 10] that establish the existence of certain invariant curves that may separate regions of different dynamical behaviors for special cases of System (5). In many cases such curves will be classified as the stable or unstable manifolds for saddle-point equilibria. We will utilize such results to establish the global dynamics of several monotone difference equations in Manuscripts 2 and 3.

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MANUSCRIPT 2

Global Dynamic Scenarios for Competitive Maps in the Plane

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Abstract

In this paper we present some global dynamic scenarios for general competitive maps in the plane. We apply these results to the class of second-order autonomous difference equations whose transition functions are decreasing in the variable x_n and increasing in the variable x_{n-1}. We illustrate our results with the application to the difference equation

$$x_{n+1} = \frac{Cx_{n-1}^2 + Ex_{n-1}}{ax_n^2 + dx_n + f}, \quad n = 0, 1, \ldots,$$

where the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that the solution is defined and the parameters satisfy $C, E, a, d, f \geq 0$, $C+E > 0$, $a + C > 0$, and $a + d > 0$. We characterize the global dynamics of this equation with the basins of attraction of its equilibria and periodic solutions.

2.1 Introduction

Consider the second-order quadratic-fractional difference equation

$$x_{n+1} = \frac{Cx_{n-1}^2 + Ex_{n-1}}{ax_n^2 + dx_n + f}, \quad n = 0, 1, \ldots, \tag{1}$$

where the parameters satisfy $C, E, a, d, f \geq 0$, $C + E > 0$, and $a + C > 0$, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1}x_0 > 0$ when $f = 0$. We also stipulate that $a + d > 0$ to avoid overlap with the study of quadratic difference equations in [1]. Notice that Equation (1) is a special case of the equation

$$x_{n+1} = \frac{Cx_{n-1}^2 + Ex_{n-1} + F}{ax_n^2 + dx_n + f}, \quad n = 0, 1, \ldots, \tag{2}$$

where $F = 0$. For Equation (1) we will precisely define the basins of attraction of all attractors, which consist of the equilibrium points, period-two solutions, and points at infinity. Our investigation of the global character of Equation (1) will be based on the theory of competitive systems.

The special case of Equation (1) where $C = a = 0$ is one of the semi-implicit discretizations of the logistic differential equation

$$\frac{dy}{dt} = ry(t)\left(1 - \frac{y(t)}{K}\right),$$

where r and K are positive constants that represent the growth rate and sustainable population level, respectively. The more general logistic differential equation

$$\frac{dy}{dt} = ry(t)\left(1 - \frac{y(t)}{K} - \frac{y(t)^2}{M}\right),$$

where r, K, M are positive constants, will have Equation (1) as one of its discretizations. Thus Equation (1) has potential applications in population dynamics. In particular, the special case of Equation (2) with $C = a = 0$ and $d = 1$, or

$$x_{n+1} = \frac{Ex_{n-1} + F}{x_n + f}, \quad n = 0, 1, \ldots,$$

was thoroughly studied in [12] and led to the formulation of the global period-doubling bifurcation result in [18]. We thus exclude the case when both C and a are zero to avoid overlap with previously studied results.

Both Equations (1) and (2) are special cases of the general second-order quadratic-fractional difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \ldots, \tag{3}$$

where all parameters are nonnegative numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that the solution is defined. A great deal of special cases of Equation (3) have been studied in [2, 11, 13, 14, 21, 22] that may engender various different dynamical phenomena. For example, the equation

$$x_{n+1} = \frac{x_{n-1}^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots,$$

was studied in [11] and also uses the theory of monotone maps given in [18, 19]. However, the global dynamics of this equation is vastly dissimilar to that of Equation (1). Indeed, the authors in [11] reveal the coexistence of a sole locally asymptotically stable equilibrium point and a locally asymptotically stable minimal period-two solution. Equation (1), on the other hand, can have as many as three isolated fixed points with a saddle-point period-two solution. The possible dynamic scenarios for Equation (1) will provide motivation for obtaining corresponding results for general second-order difference equations in Section 2.3.

Many other interesting special cases of Equation (3) have been studied in [13, 21, 22, 23] and exhibit rich dynamical behaviors that include the Allee effect, period-doubling bifurcation, Neimark-Sacker bifurcation, and chaos. More special cases in which the numerator of Equation (3) is quadratic and the denominator is linear are treated in [7, 8, 14].

The following theorem from [5] applies to Equation (1):

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is nonincreasing in the first variable and nondecreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f\left(x_n, x_{n-1}\right), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots, \tag{4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (4) converges to either an equilibrium, a period-two solution, or a singular point on the boundary, as in the case of the difference equation

$$x_{n+1} = \frac{ax_{n-1}^2}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots, \quad a \in (0, 1),$$

where $x_{-1}, x_0 > 0$ and all solutions converge to 0. Thus we aim to determine the basins of attraction for both bounded and unbounded solutions. Herein lies the utility of the theory of monotone systems, of which several important results are introduced in the Preliminaries.

This paper is organized as follows. Section 2.2 gives some preliminary results about monotone maps in the plane which will be used in Section 2.3 to give some global dynamic scenarios for such maps and for Equation (4), where the transition function f is nonincreasing in the first variable and nondecreasing in the second variable. Section 2.4 will apply the results of Section 2.3 to the study of the global dynamics of Equation (1). The global dynamics of Equation (1) is interesting and includes five major dynamic scenarios described in Theorem 9 as well as several additional scenarios that include the existence of an infinite number of equilibrium solutions in Theorem 10, an infinite number of period-two solutions in Theorem 11, and a case when the solution is explicitly exhibited in Theorem 10.

2.2 Preliminaries

In this section we provide some basic facts about competitive maps and systems of difference equations in the plane from [18, 19, 20].

Definition 1 Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T : R \to R$ be a continuous map. Set $T(x, y) = (f(x, y), g(x, y))$. The map T is *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y while $g(x, y)$ is nonincreasing in x and nondecreasing in y. If both f and g are nondecreasing in x and y, we say that T is *cooperative*. If T is competitive (resp. cooperative), the associated system of difference equations

$$\begin{cases} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{cases}, \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in R \tag{5}$$

is said to be competitive (resp. cooperative). The map T and the associated system of difference equations are said to be *strongly competitive* (resp. *strongly cooperative*) if the adjectives nondecreasing and nonincreasing are replaced by increasing and decreasing, respectively.

Definition 2 A fixed point $\bar{\mathrm{x}}$ of the map T is *hyperbolic* if no root of the characteristic equation evaluated at $\bar{\mathrm{x}}$ is on the unit circle. A fixed point $\bar{\mathrm{x}}$ of T is *nonhyperbolic of stable* (resp. *unstable*) *type* if one root of the characteristic equation evaluated at $\bar{\mathrm{x}}$ is on the unit circle and the other one is inside (resp. outside) the unit circle. Finally the fixed point $\bar{\mathrm{x}}$ of the map T is *nonhyperbolic of resonant type* if both roots of the characteristic equation evaluated at $\bar{\mathrm{x}}$ are on the unit circle.

Definition 3 The *southeast partial order* on $\mathbb{R}^2$ is defined such that $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \le x_2$ and $y_1 \ge y_2$. A strict inequality between points may be defined such that $(x_1, y_1) \prec_{se} (x_2, y_2)$ if $(x_1, y_1) \preceq_{se} (x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$. An even stronger inequality may be defined such that $(x_1, y_1) \ll_{se} (x_2, y_2)$ if $x_1 < x_2$ and $y_1 > y_2$. (Similar orderings may be defined for the *northeast partial order* defined such that $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$.)

Remark 1 A competitive map $T : R \to R$ is monotone with respect to the southeast order; that is, $\vec{x} \preceq_{se} \vec{y}$ implies that $T(\vec{x}) \preceq_{se} T(\vec{y})$ for all $\vec{x}$ and $\vec{y}$ in R. A strongly competitive map T satisfies the property that, for all $\vec{x}$ and $\vec{y}$ in R, if $\vec{x} \prec_{se} \vec{y}$, then $T(\vec{x}) \ll_{se} T(\vec{y})$.

The following definition comes from [25].

Definition 4 A competitive map $T : R \to R$ is said to satisfy condition $(O+)$ if for every $\vec{x}, \vec{y} \in R$, $T(\vec{x}) \preceq_{ne} T(\vec{y})$ implies $\vec{x} \preceq_{ne} \vec{y}$.

A result of deMottoni-Schiaffino [9] generalized by Smith [25] yields that all bounded solutions of a competitive map satisfying condition $(O+)$ must converge.

Now we provide some theorems from [18, 19, 20] that will be of particular importance in our investigation of the global dynamics of Equation (1). The first two results hold for any kind of unstable fixed points of competitive maps; see [20].

Theorem 2 *Let $\mathcal{R} = (a_1, a_2) \times (b_1, b_2)$, and let $T : \mathcal{R} \to \mathcal{R}$ be a strongly competitive map with a unique fixed point $\bar{\mathrm{x}} \in \mathcal{R}$, and such that T is twice continuously differentiable in a neighborhood of $\bar{\mathrm{x}}$. Assume further that at the point $\bar{\mathrm{x}}$ the map T has associated characteristic values μ and ν satisfying $1 < \mu$ and $-\mu < \nu < \mu$, with $\nu \neq 0$, and that no standard basis vector is an eigenvector associated to one of the characteristic values.*

Then there exist curves $\mathcal{C}_1$, $\mathcal{C}_2$ in $\mathcal{R}$ and there exist $\mathrm{p}_1, \mathrm{p}_2 \in \partial\mathcal{R}$ with $\mathrm{p}_1 \ll_{se} \bar{\mathrm{x}} <$ $<_{se} \mathrm{p}_2$ such that

(i) *For $\ell = 1, 2$, $\mathcal{C}_\ell$ is invariant, north-east strongly linearly ordered, such that $\bar{\mathrm{x}} \in \mathcal{C}_\ell$ and $\mathcal{C}_\ell \subset \mathcal{Q}_3(\bar{\mathrm{x}}) \cup \mathcal{Q}_1(\bar{\mathrm{x}})$; the endpoints q_ℓ, r_ℓ of $\mathcal{C}_\ell$, where $\mathrm{q}_\ell \preceq_{ne} \mathrm{r}_\ell$, belong to the boundary of $\mathcal{R}$. For $\ell, j \in \{1, 2\}$ with $\ell \neq j$, $\mathcal{C}_\ell$ is a subset of the closure of one of the components of $\mathcal{R} \setminus \mathcal{C}_j$. Both $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangential at $\bar{\mathrm{x}}$ to the eigenspace associated with ν.*

(ii) *For $\ell = 1, 2$, let $\mathcal{B}_\ell$ be the component of $\mathcal{R} \setminus \mathcal{C}_\ell$ whose closure contains p_ℓ. Then $\mathcal{B}_\ell$ is invariant. Also, for $\mathrm{x} \in \mathcal{B}_1$, $T^n(\mathrm{x})$ accumulates on $\mathcal{Q}_2(\mathrm{p}_1) \cap \partial\mathcal{R}$, and for $x \in \mathcal{B}_2$, $T^n(\mathrm{x})$ accumulates on $\mathcal{Q}_4(\mathrm{p}_2) \cap \partial\mathcal{R}$.*

(iii) *Let* $\mathcal{D}_1 := \mathcal{Q}_1(\bar{\mathrm{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ *and* $\mathcal{D}_2 := \mathcal{Q}_3(\bar{\mathrm{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$*.*

Then $\mathcal{D}_1 \cup \mathcal{D}_2$ *is invariant.*

Corollary 1 *Let a map* T *with fixed point* $\bar{\mathrm{x}}$ *be as in Theorem 2. Let* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *be the sets as in Theorem 2. If* T *satisfies (O+), then for* $\ell = 1, 2$*,* $\mathcal{D}_\ell$ *is invariant, and for every* $\mathrm{x} \in \mathcal{D}_\ell$*, the iterates* $T^n(\mathrm{x})$ *converge to* $\bar{\mathrm{x}}$ *or to a point of* $\partial\mathcal{R}$*. If* T *satisfies (O−), then* $T(\mathcal{D}_1) \subset \mathcal{D}_2$ *and* $T(\mathcal{D}_2) \subset \mathcal{D}_1$*. For every* $\mathrm{x} \in \mathcal{D}_1 \cup \mathcal{D}_2$*, the iterates* $T^n(\mathrm{x})$ *either converge to* $\bar{\mathrm{x}}$*, or converge to a period-two point, or to a point of* $\partial\mathcal{R}$*.*

In the case of a saddle point or nonhyperbolic fixed point of stable type we have more precise results given in [18, 19].

Theorem 3 *Let* T *be a competitive map on a rectangular region* $\mathcal{R} \subset \mathbb{R}^2$*. Let* $\bar{\mathrm{x}} \in \mathcal{R}$ *be a fixed point of* T *such that* $\Delta := \mathcal{R} \cap int\,(Q_1(\bar{\mathrm{x}}) \cup Q_3(\bar{\mathrm{x}}))$ *is nonempty (i.e.,* $\bar{\mathrm{x}}$ *is not the NW or SE vertex of* $\mathcal{R}$*), and* T *is strongly competitive on* Δ*. Suppose that the following statements are true.*

a. The map T *has a* C^1 *extension to a neighborhood of* $\bar{\mathrm{x}}$*.*

b. The Jacobian $J_T(\bar{\mathrm{x}})$ *of* T *at* $\bar{\mathrm{x}}$ *has real eigenvalues* λ*,* μ *such that* $0 < |\lambda| < \mu$*, where* $|\lambda| < 1$*, and the eigenspace* E^λ *associated with* λ *is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ *through* $\bar{\mathrm{x}}$ *that is invariant and a subset of the basin of attraction of* $\bar{\mathrm{x}}$*, such that* $\mathcal{C}$ *is tangential to the eigenspace* E^λ *at* $\bar{\mathrm{x}}$*, and* $\mathcal{C}$ *is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of* $\mathcal{C}$ *in the interior of* $\mathcal{R}$ *are either fixed points or minimal period-two points. In the latter case, the set of endpoints of* $\mathcal{C}$ *is a minimal period-two orbit of* T*.*

We shall see in Theorem 5 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 4 *For the curve $\mathcal{C}$ of Theorem 3 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \bar{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\bar{\mathrm{x}}) < 0$, and $T(x) = \bar{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 3 reduces just to $|\lambda| < 1$. This follows from a change of variables that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 5 *(A) Assume the hypotheses of Theorem 3, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 3. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\} \text{ and}$$
$$\mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\},$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\bar{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\bar{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\bar{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\bar{\mathrm{x}}$, then T has no periodic

points in the boundary of $Q_1(\bar{x}) \cup Q_3(\bar{x})$ except for $\bar{x}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_2(\bar{x})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_4(\bar{x})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\bar{x}$ is a fixed point of T, the stable set $\mathcal{W}^s(\bar{x})$ of $\bar{x}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \bar{x}\}$ and the unstable set $\mathcal{W}^u(\bar{x})$ of $\bar{x}$ is the set

$$\left\{ x \in \mathcal{R} : \exists \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \bar{x} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\bar{x})$ may not be connected and be made up of infinitely many curves, or $\mathcal{W}^u(\bar{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\bar{x})$ and $\mathcal{W}^u(\bar{x})$ are the stable and unstable manifolds of $\bar{x}$.

Theorem 6 *In addition to the hypotheses of part (B) of Theorem 5, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 3 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\bar{x})$ of $\bar{x}$, and the unstable set $\mathcal{W}^u(\bar{x})$ of $\bar{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\bar{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\bar{x})$ in $\mathcal{R}$ are fixed points of T.*

Remark 2 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and second partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (4) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (4) is a strongly competitive map on $I \times I$.

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (4) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \qquad n = 0, 1, \ldots .$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v)),$$

which is strongly competitive on $I \times I$; see [18, 19].

Remark 3 The characteristic equation of Equation (4) at an equilibrium point $(\overline{x}, \overline{x})$,

$$\lambda^2 - D_1 f(\overline{x}, \overline{x})\lambda - D_2 f(\overline{x}, \overline{x}) = 0,$$

has two real roots λ, μ which satisfy $\mu < 0 < \lambda$ and $|\lambda| < \mu$ whenever f is strongly decreasing in the first variable and strongly increasing in the second variable. Thus the applicability of Theorems 3-6 depends on the existence and nonexistence of a minimal period-two solution.

2.3 Main Results

In this section we present some global dynamic scenarios for competitive maps which are motivated by some dynamic scenarios for Equation (1). Thus different global dynamic scenarios for Equation (1) will be examples of general global results for competitive maps.

Theorem 7 *Consider the competitive map T generated by system* (5) *on a rectangular region $\mathcal{R}$. Suppose T has no minimal period-two solutions in $\mathcal{R}$, is strongly competitive on int $\mathcal{R}$, and is C^2 in a neighborhood of any fixed point.*

(a) Assume T has a saddle fixed point E_2 and either a singular point or another fixed point E_1, $E_1 \ll_{ne} E_2$, where E_1 is the southwest corner of the region $\mathcal{R}$. If E_1 is a fixed point, assume it is a repeller or nonhyperbolic. Then every nonconstant solution which starts off the stable manifold $\mathcal{W}^s(E_2)$ will approach the boundary of the region $\mathcal{R}$. See Figure 1 for visual illustration.

In Cases (b)–(e), assume T has at least three fixed points E_1, E_2, E_3, where $E_1 \prec_{se} E_2 \prec_{se} E_3$, E_1, E_3 are saddle points, and E_2 is locally asymptotically stable and is the southwest corner of the region $\mathcal{R}$. Assume that the Jacobian $J_T(\bar{\mathrm{x}})$ of T evaluated at both E_1 and E_3 has real eigenvalues λ, μ such that $0 < |\lambda| < 1 < \mu$ and the eigenspace E^{λ} associated with λ is not a coordinate axis. Finally, suppose that the left vertical (resp. bottom horizontal) boundary of $\mathcal{R}$ without E_2 is $\mathcal{W}^u(E_1)$ (resp. $\mathcal{W}^u(E_3)$).

(b) In addition to the hypotheses listed above, suppose T has two additional fixed points E_4 and E_5 such that $E_i \ll_{ne} E_4 \ll_{ne} E_5$ for $i = 1, 2, 3$, E_4 is a repeller, and E_5 is a saddle point. Then every solution which starts below (resp. above) the union of the stable manifolds $\mathcal{W}^s(E_3) \cup \mathcal{W}^s(E_5)$ (resp. $\mathcal{W}^s(E_1) \cup \mathcal{W}^s(E_5)$) will approach the boundary of the region $\mathcal{R}$. Every solution which starts between the stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2. See Figure 2 for visual illustration.

(c) Assume exactly the hypotheses listed above. Then every solution which starts below (resp. above) the manifold $\mathcal{W}^s(E_3)$ (resp. $\mathcal{W}^s(E_1)$) will approach the boundary of the region $\mathcal{R}$. Every solution which starts between the stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2. See Figure 3 for visual illustration.

(d) In addition to the hypotheses listed above, suppose T has an additional fixed

point E_4 such that $E_i \ll_{ne} E_4$ for $i = 1, 2, 3$ and E_4 is nonhyperbolic of unstable type. Assume that no standard basis vector is an eigenvector associated to either of the characteristic values of E_4. Then there exist continuous, nondecreasing, and invariant curves $\mathcal{C}_1, \mathcal{C}_2$ (with $\mathcal{C}_1$ above $\mathcal{C}_2$) which emanate from E_4 such that the region between the curves is invariant. The region below (resp. above) the union of invariant curves $\mathcal{W}^s(E_3) \cup \mathcal{C}_2$ (resp. $\mathcal{W}^s(E_1) \cup \mathcal{C}_1$) is invariant, and every solution which starts in either region will approach the boundary of $\mathcal{R}$. If T satisfies condition $(O+)$, for every initial point (x_0, y_0) between $\mathcal{C}_1$ and $\mathcal{C}_2$ the corresponding solution either converges to E_4 or approaches the boundary of $\mathcal{R}$. See Figure 4 for visual illustration.

(e) *In addition to the hypotheses listed above, suppose T has an additional fixed point E_4 such that $E_i \ll_{ne} E_4$ for $i = 1, 2, 3$ and E_4 is a repeller. Assume that no standard basis vector is an eigenvector associated to either of the characteristic values of E_4. Then there exist continuous, nondecreasing, and invariant curves $\mathcal{C}_1, \mathcal{C}_2$ (with $\mathcal{C}_1$ above $\mathcal{C}_2$) which emanate from E_4 such that the region between the curves is invariant. The region below (resp. above) the union of invariant curves $\mathcal{W}^s(E_3) \cup \mathcal{C}_2$ (resp. $\mathcal{W}^s(E_1) \cup \mathcal{C}_1$) is invariant, and every solution which starts in either region will approach the boundary of $\mathcal{R}$. If T satisfies condition $(O+)$, for every initial point (x_0, y_0) between $\mathcal{C}_1$ and $\mathcal{C}_2$ the corresponding solution approaches the boundary of $\mathcal{R}$. See Figure 5 for visual illustration.*

Proof.

(a) The existence of the global stable and unstable manifolds of the saddle point equilibrium is guaranteed by Theorems 3-6. In any case $\mathcal{W}^s(E_2)$ has end-points on the boundary of $\mathcal{R}$. In view of Theorem 5 every solution which starts in $\mathcal{W}_-$ eventually enters $int\, Q_2(E_2)$ and every solution which starts in

$\mathcal{W}_+$ eventually enters $int\, Q_4(E_2)$. If $\vec{x}_0 = (x_0, y_0) \in \mathcal{W}_+$, then there exists $m \in \mathbb{N}$ such that $\vec{z} = T^m(\vec{x}_0) \in int\, Q_4(E_2)$. Regardless of whether $\vec{z}$ is above or below $\mathcal{W}^u(E_2)$, one can find $\vec{u} \in \mathcal{W}^u(E_2)$ such that $\vec{u} \preceq_{se} \vec{z}$. By monotonicity of the map T, this implies that $T^n(\vec{u}) \preceq_{se} T^n(\vec{z})$ for all $n \in \mathbb{N}$, and so

$$\lim_{n\to\infty} T^n(\vec{u}) \preceq_{se} \lim_{n\to\infty} T^n(\vec{z}).$$

In a similar way the case when the initial point $\vec{x}_0 \in \mathcal{W}_-$ can be handled.

(b) The existence of the global stable manifolds of $E_1,\ E_3,\ E_5$ and the global unstable manifold of E_5 is guaranteed by Theorems 3-6; see also [24]. Indeed, by Theorems 3 and 4, both $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ have endpoints at E_4, and $\mathcal{W}^s(E_5)$ has endpoints at E_4 and some point on the boundary of $\mathcal{R}$. Since no other equilibria exist in $Q_2(E_5) \cup Q_4(E_5)$, $\mathcal{W}^u(E_5)$ has endpoints on the boundary of $\mathcal{R}$. Furthermore, the left vertical boundary of the region $\mathcal{R}$ with the exception of E_2 is the unstable manifold of E_1 and the bottom horizontal boundary of the region $\mathcal{R}$ with the exception of E_2 is the unstable manifold of E_3.

Let $[\![\vec{a}, \vec{b}]\!]$ be the *order interval* consisting of all $\vec{c} \in \mathbb{R}^2$ such that $\vec{a} \preceq_{ne} \vec{c} \preceq_{ne} \vec{b}$. Consider an arbitrary initial point $\vec{x}_0 = (x_0, y_0) \in int\, [\![E_1, E_3]\!]$. Then there exist some projections onto the unstable manifolds $\mathcal{W}^u(E_1)$ and $\mathcal{W}^u(E_3)$, P_y and P_x, respectively, such that $P_y \preceq_{se} \vec{x}_0 \preceq_{se} P_x$, which implies that

$$T^n(P_y) \preceq_{se} T^n(\vec{x}_0) \preceq_{se} T^n(P_x)$$

for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} T^n(P_y) = \lim_{n\to\infty} T^n(P_x) = E_2$ we obtain that $\lim_{n\to\infty} T^n(\vec{x}_0) = E_2$. If $\vec{x}_0 \in \partial\left([\![E_1, E_3]\!]\right) \setminus \left(\mathcal{W}^u(E_1) \cup \mathcal{W}^u(E_3) \cup E_2\right)$ then $T(\vec{x}_0) \in int\, [\![E_1, E_3]\!]$ and the result follows.

Now suppose $\vec{x_0} \in \mathscr{B} \backslash [\![E_1, E_3]\!]$, where $\mathscr{B}$ denotes the region between the stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$. Then there must exist $\vec{s_l} \in \mathcal{W}^s(E_1)$ and $\vec{s_u} \in \mathcal{W}^s(E_3)$ such that $\vec{s_l} \preceq_{se} \vec{x_0} \preceq_{se} \vec{s_u}$. But then

$$T^n(\vec{s_l}) \preceq_{se} T^n(\vec{x_0}) \preceq_{se} T^n(\vec{s_u}),$$

and thus $T^n(\vec{x_0}) \in [\![E_1, E_3]\!]$ for n sufficiently large, which implies that $\lim_{n\to\infty} T^n(\vec{x_0}) = E_2$.

Now suppose $\vec{x_0} \in int\, Q_4(E_5)$. Then there exists $\vec{u} \in \mathcal{W}^u(E_5)$ so that $\vec{u} \preceq_{se} \vec{x_0}$, which implies

$$T^n(\vec{u}) \preceq_{se} T^n(\vec{x_0}),$$

and thus the solution approaches the boundary of the region $\mathcal{R}$. The treatment is similar for $\vec{x}_0 \in int\, Q_2(E_5)$.

Suppose $\vec{x}_0 \in Q_1(E_5)$. Without loss of generality suppose $\vec{x}_0$ is to the right of $\mathcal{W}^s(E_5)$ (otherwise the treatment is analogous) so that there exists some $\vec{p} \in \mathcal{W}^s(E_5)$ such that $\vec{p} \preceq_{se} \vec{x}_0$. We claim that there exists some n such that $T^n(\vec{x}_0) \in int\, Q_4(E_5)$. Certainly for any n it is the case that $T^n(\vec{p}) \preceq_{se} T^n(\vec{x}_0)$. For a contradiction suppose $T^n(\vec{x}_0) \to E_5$ as $n \to \infty$. But then for some n, $T^n(\vec{x}_0) \in \mathcal{W}^s_{\text{loc}}(E_5)$, the local stable manifold tangential to the eigenspace E^λ. Since in a small neighborhood of E_5 we have that $\mathcal{W}^s_{\text{loc}}(E_5) \subseteq \mathcal{W}^s(E_5)$, we now have the relation $T^n(\vec{p}) \preceq_{se} T^n(\vec{x}_0)$, but any points on this invariant curve are not comparable with respect to the southeast ordering. By continuity of T the only finite points to which any solution may converge are fixed points, and therefore it must be the case that eventually the solution enters $int\, Q_4(E_5)$.

Suppose $\vec{x}_0 \in [\![E_2, E_5]\!] \backslash [\![E_2, E_4]\!]$. In any case we can compare $\vec{x}_0$ to a point on $\mathcal{W}^s(E_5)$ and show using a similar argument as that used above that the corresponding solution must enter either $int\, Q_4(E_5)$ or $int\, Q_2(E_5)$ (in which case we can apply the previous results to establish the long-term behavior of the solution).

Finally suppose $\vec{x}_0 \in [\![E_2, E_4]\!] \backslash \overline{\mathscr{B}}$. By comparing $\vec{x}_0$ to some point on either $\mathcal{W}^s(E_3)$ or $\mathcal{W}^s(E_1)$ as appropriate, we may utilize a similar argument as before to deduce that the corresponding solution cannot converge to E_3 or E_1. Thus there exists some $n \in \mathbb{N}$ such that $T^n(\vec{x}_0) \in int\, Q_4(E_5)$ (or $T^n(\vec{x}_0) \in int\, Q_2(E_5)$), and we can apply the results of the previous case to complete the proof.

(c) The proofs used to show that the region between the stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ is the basin of attraction of E_2 and that solutions with initial conditions starting outside this region will approach the boundary of the region are similar to those provided in case (b) and will be omitted.

(d) The proof used to show that the region between the stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ is the basin of attraction of E_2 is the same as in case (b) and will be omitted. In view of the main result in [24] there exists a most unstable manifold $\mathcal{W}^u_{\max}(E_4)$, which is the graph of a decreasing function passing through E_4, which at E_4 is tangent to the eigenspace that corresponds to the largest (in absolute value) eigenvalue. The existence of the invariant curves $\mathcal{C}_1, \mathcal{C}_2$ is guaranteed by Theorem 2 applied to the open rectangular region $\mathcal{R}' = int\, \mathcal{R}$, in which T has only the interior fixed point E_4. The endpoints q_1 and q_2 of the full curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, should coincide with the fixed points E_1 and E_3 on the boundary. The proofs that nth iterates of points which start in the invariant region below $\mathcal{W}^s(E_3) \cup \mathcal{C}_2$ (resp. above $\mathcal{W}^s(E_1) \cup \mathcal{C}_1$) are approaching the boundary of the region $\mathcal{R}$

are similar to those provided in case (b); also, see Theorem 2 (ii). If an initial point $\vec{x}_0 = (x_0, y_0) \in Q_1(E_4)$ is between $\mathcal{C}_1$ and $\mathcal{C}_2$, then there exist points $\vec{c}_i \in \mathcal{C}_i, i = 1, 2$, such that $\vec{c}_1 \preceq_{se} \vec{x}_0 \preceq_{se} \vec{c}_2$. In view of Corollary 1, if T additionally satisfies condition $(O+)$ then the solution approaches the boundary of the region or $T^n(\vec{x}_0) \to E_4$ as $n \to \infty$.

(e) The proof for this case is analogous to that provided in case (d) and will be omitted. Note that if T satisfies condition $(O+)$ then every solution with initial point $\vec{x}_0 = (x_0, y_0) \in Q_1(E_4)$ between the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ must approach the boundary of the region since in this case E_4 is a repeller and has trivial basin of attraction.

□

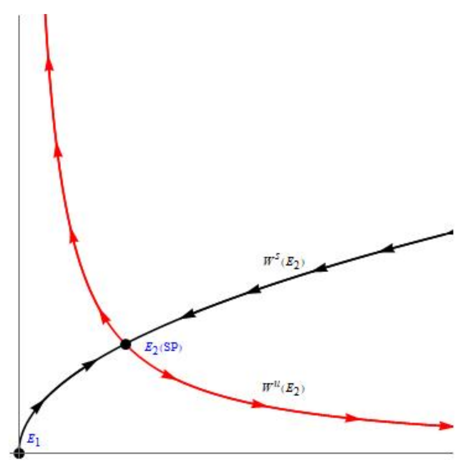

Figure 1. Visual illustration of part (a) of Theorem 7.

In the case of Equation (4) we have the following results which are direct applications of Theorem 7. See [10] for similar results.

Theorem 8 *Consider Equation* (4) *on a rectangular region* $\mathcal{R} = [a, b) \times [a, b)$, *where* $b \leq \infty$. *Assume that* f *is decreasing in the first variable and increasing in the second variable on* $(a, b)^2$ *such that* f *is* C^2 *in a neighborhood of any fixed point.*

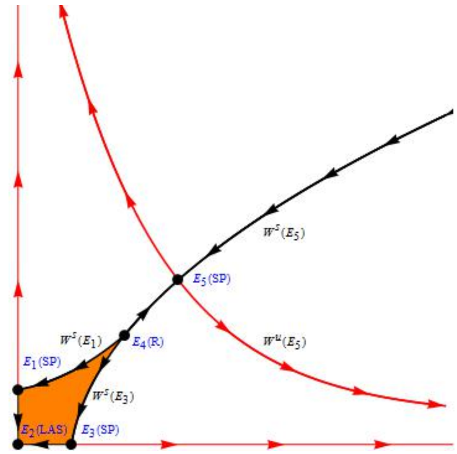

Figure 2. Visual illustration of part (b) of Theorem 7.

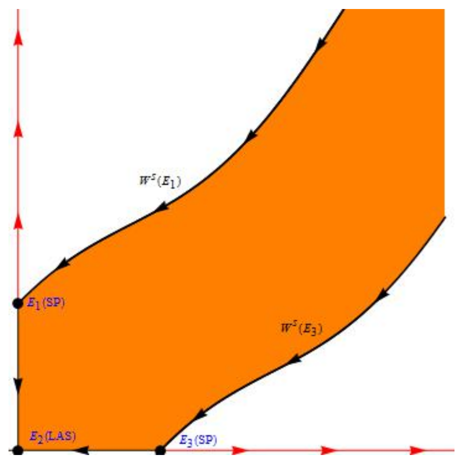

Figure 3. Visual illustration of part (c) of Theorem 7.

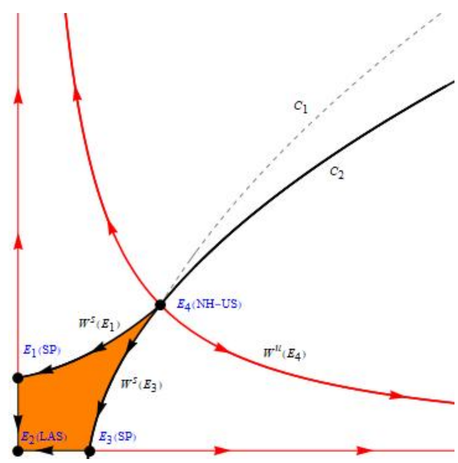

Figure 4. Visual illustration of part (d) of Theorem 7.

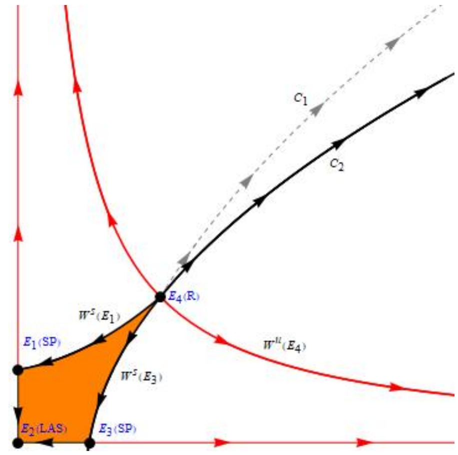

Figure 5. Visual illustration of part (e) of Theorem 7.

(a) *Assume that Equation* (4) *has one saddle equilibrium point* $\overline{x} > a$ *and that* a *is either a repelling (or nonhyperbolic) equilibrium point or a singular point of* $\mathcal{R}$. *If Equation* (4) *has no minimal period-two solutions, then every nonconstant solution which starts off the stable manifold* $\mathcal{W}^s((\overline{x}, \overline{x}))$ *will approach the boundary of the region* $\mathcal{R}$.

In Cases (b)–(e), assume that Equation (4) *has a locally asymptotically stable equilibrium point* a *and the unique minimal period-two solution* $\{a, p, a, p, \ldots\}$, *with* $p > a$, *such that* $P_1 = (a, p)$ *and* $P_2 = (p, a)$ *are saddle points. Assume further that the Jacobian* $J_{T^2}(\bar{\mathrm{x}})$ *of* T^2, *where* T *is the map corresponding to Equation* (4), *evaluated at both* P_1 *and* P_2 *has real eigenvalues* λ, μ *such that* $0 < |\lambda| < 1 < \mu$ *and the eigenspace* E^λ *associated with* λ *is not a coordinate axis. Finally, suppose that* $\mathcal{W}^u(P_1) = \{(x, y) : x = a,\ y \neq a\}$ *and* $\mathcal{W}^u(P_2) = \{(x, y) : y = a,\ x \neq a\}$.

(b) *In addition to the hypotheses listed above, assume that Equation* (4) *has two additional equilibrium points* $\overline{x}_2, \overline{x}_1$ *such that* $\overline{x}_2 > \overline{x}_1 > a$, $\overline{x}_1$ *is a repeller, and* $\overline{x}_2$ *is a saddle point. Then every solution which starts between the stable manifolds* $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(P_2)$ *converges to* (a, a) *while every solution which* *starts below* $\mathcal{W}^s((\overline{x}_2, \overline{x}_2)) \cup \mathcal{W}^s(P_2)$ *(resp. above* $\mathcal{W}^s((\overline{x}_2, \overline{x}_2)) \cup \mathcal{W}^s(P_1)$*) is approaching the boundary of the region* $\mathcal{R}$*.*

(c) Assume exactly the hypotheses listed above. Then every solution which starts between the stable manifolds $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(P_2)$ *converges to* (a, a) *while every solution which starts below* $\mathcal{W}^s(P_2)$ *(resp. above* $\mathcal{W}^s(P_1)$*) is approaching the boundary of the region* $\mathcal{R}$*.*

(d) In addition to the hypotheses listed above, assume that Equation (4) *has an additional equilibrium point* $\overline{x}$ *such that* $\overline{x} > a$ *and* $\overline{x}$ *is nonhyperbolic of unstable type. Assume that no standard basis vector is an eigenvector associated to either of the eigenvalues of the Jacobian* $J_{T^2}(\bar{\mathrm{x}})$ *evaluated at* $(\overline{x}, \overline{x})$*. Then there exist two continuous and nondecreasing curves* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *(with* $\mathcal{C}_1$ *above* $\mathcal{C}_2$*) which start at* $(\overline{x}, \overline{x})$ *and serve as the boundary of the region containing the basin of attraction of* $(\overline{x}, \overline{x})$*. Every solution which starts between the stable manifolds* $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(P_2)$ *converges to* (a, a)*, while every solution which starts below* $\mathcal{W}^s(P_2) \cup \mathcal{C}_2$ *(resp. above* $\mathcal{W}^s(P_1) \cup \mathcal{C}_1$*) is approaching the boundary of the region* $\mathcal{R}$*. Every solution which starts between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *converges to* $(\overline{x}, \overline{x})$ *or approaches the boundary of the region.*

(e) In addition to the hypotheses listed above, assume that Equation (4) *has an additional equilibrium point* $\overline{x}$ *such that* $\overline{x} > a$ *and* $\overline{x}$ *is a repeller. Assume that no standard basis vector is an eigenvector associated to either of the eigenvalues of the Jacobian* $J_{T^2}(\bar{\mathrm{x}})$ *evaluated at* $(\overline{x}, \overline{x})$*. Then there exist two continuous and nondecreasing curves* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *(with* $\mathcal{C}_1$ *above* $\mathcal{C}_2$*) which start at* $(\overline{x}, \overline{x})$*. Every solution which starts between the stable manifolds* $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(P_2)$ *converges to* (a, a)*, while every solution which*

starts below $\mathcal{W}^s(P_2) \cup \mathcal{C}_2$ (resp. above $\mathcal{W}^s(P_1) \cup \mathcal{C}_1$) is approaching the boundary of the region $\mathcal{R}$. Every solution which starts between $\mathcal{C}_1$ and $\mathcal{C}_2$ approaches the boundary of the region.

Proof.

In all cases recall that the applicability of Theorem 7 to a map T requires the nonexistence of minimal period-two solutions. Since we seek to apply the results of this theorem to T^2, where T is the map corresponding to Equation (4), we must rule out the possibility of minimal period-four solutions for Equation (4). However, realize that Theorem 1 specifically precludes the existence of periodic solutions of prime period greater than two.

(a) In view of Remark 2 the second iterate T^2 of the map T associated with Equation (4) is strongly competitive on $(a, b)^2$. Applying Theorem 7 part (a) to T^2 we complete the proof.

(b) In view of Remark 2 the second iterate T^2 of the map T associated with Equation (4) is strongly competitive and has five equilibrium points $E_1 = P_1$, $E_2 = (a, a)$, $E_3 = P_2$, $E_4 = (\overline{x}_1, \overline{x}_1)$, and $E_5 = (\overline{x}_2, \overline{x}_2)$. Applying Theorem 7 part (b) to T^2 we conclude that

$$\lim_{n\to\infty} T^{2n}((x_0, y_0)) = E_2$$

for every (x_0, y_0) between the stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(P_2)$. Furthermore, we also have that

$$\begin{aligned} \lim_{n\to\infty} T^{2n+1}((x_0, y_0)) &= \lim_{n\to\infty} T\left(T^{2n}\left((x_0, y_0)\right)\right) \\ &= T\left(\lim_{n\to\infty} T^{2n}\left((x_0, y_0)\right)\right) \\ &= T(E_2) = E_2, \end{aligned}$$

where we utilize continuity of the map T. Consequently $\lim_{n\to\infty} T^n((x_0, y_0)) = E_2$. The remaining conclusions follow from Theorem 7 part (b).

(c)–(e) The proofs of parts (c), (d), and (e) follow in a similar way by using the same reasoning as in parts (a) and (b). For parts (d) and (e), make the observation that condition (O^+) is automatically satisfied for the second iterate of the map T corresponding to Equation (4); see [18, 19].

$\square$

Remark 4 As shown in [20], the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ may coincide on one or both sides of the fixed point. Different global dynamic scenarios for competitive or cooperative maps and corresponding difference equations were established in the cases when these maps have a finite or infinite number of period-two solutions in [2, 4, 20].

Remark 5 Some special cases of Theorems 7 and 8 have appeared in a number of papers. For example, the global dynamics of the system

$$x_{n+1} = \frac{x_n}{a + y_n}, \quad y_{n+1} = \frac{y_n}{b + x_n}, \quad n = 0, 1, \ldots,$$

where $a, b \in (0, 1)$ and $x_0, y_0 \in [0, \infty)$, as studied in [6], follows from Theorem 7 case (a). Furthermore, several cases of the global dynamics of the system

$$x_{n+1} = \frac{a x_n^2}{1 + x_n^2 + c y_n}, \quad y_{n+1} = \frac{b y_n^2}{1 + d x_n + y_n^2}, \quad n = 0, 1, \ldots,$$

where $a, b, c, d \in (0, \infty)$ and $x_0, y_0 \in [0, \infty)$, as studied in [3], follow from Theorem 7 cases (a)–(d).

The global dynamics of the difference equations

$$x_{n+1} = \frac{x_{n-1}(x_n + \gamma)}{x_n(x_n + B x_{n-1})}, \quad n = 0, 1, \ldots,$$

where $B, \gamma > 0$ and $B < 4\gamma + 1$, and

$$x_{n+1} = \frac{x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{x_n^2}, \quad n = 0, 1, \dots,$$

where $\beta, \gamma > 0, \beta + \gamma \geq 1, 4\gamma + 2\beta + \beta^2 > 3$, is described by Theorem 8 case (a).

The global dynamics of the difference equation

$$x_{n+1} = \frac{x_{n-1}^2}{b x_n x_n + c x_{n-1}^2 + f}, \quad n = 0, 1, \dots,$$

where $b, c, f \geq 0$ and $b + c + f > 0$, is described by Theorem 8 cases (a)–(d) for several regions of parameters.

Finally, the global dynamics of the well-known difference equations

$$x_{n+1} = a + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots,$$

where $a \in (0, 1)$, and

$$x_{n+1} = \frac{p + q x_{n-1}}{1 + x_n}, \quad n = 0, 1, \dots,$$

where $p > 0, q > 1$, is described by Theorem 8 case (a). See [12] as well as [15], pp. 60-64 and pp. 89-91, and references therein.

It is worth noticing that case (e) in both Theorems 7 and 8 has been identified for the first time in the case of Equation (1).

2.4 Case Study: Equation (1)

In this section we apply the results of Theorem 8 to the study of the global dynamics of Equation (1). We begin by investigating the existence and local stability of equilibria and periodic solutions.

2.4.1 Equilibrium Solutions of Equation (1)

An equilibrium point $\overline{x}$ of Equation (1) satisfies

$$a\overline{x}^3 + (d - C)\overline{x}^2 + (f - E)\overline{x} = 0. \tag{6}$$

In the case when $f > 0$, it is clear that Equation (1) always has the zero equilibrium. The following cases will investigate the existence of any remaining positive equilibrium points.

Case 1 ($af > 0$):

When $f > 0$ and $a > 0$, denote by $\overline{x}_+$ and $\overline{x}_-$ the two possibly remaining positive equilibria:

$$\overline{x}_\pm = \frac{(C-d) \pm \sqrt{(C-d)^2 - 4a(f-E)}}{2a}. \tag{7}$$

Let $R = (C-d)^2 - 4a(f-E)$. A routine checking will find the parametric conditions under which the above solutions $\overline{x}_+$ and $\overline{x}_-$ are both real and nonnegative. Tables 1 and 2 summarize the values of parameters for which Equation (1) has one, two, or three equilibrium points and possibly period-two solutions (the existence of which we will investigate in Section 2.4.3).

	$C,\ d$	$f,\ E$	Equilibria		Period-two solutions
$C > 0$	$C \leq d$	$f = E$	$\overline{x}_0 = 0$		none
		$f > E$	$\overline{x}_0 = 0$		one
	$C > d$	$f = E$	$\overline{x}_0 = 0,\ \overline{x}_+ > 0$		none
	arbitrary	$f < E$	$\overline{x}_0 = 0,\ \overline{x}_+ > 0$		none
	$C > d$	$f > E$	$R < 0$	$\overline{x}_0 = 0$	one
			$R = 0$	$\overline{x}_0 = 0,\ \overline{x}_\pm > 0$	one
			$R > 0$	$\overline{x}_0 = 0,\ \overline{x}_-,\ \overline{x}_+ > 0$	one

Table 1. Existence of equilibria and period-two solutions for $a > 0,\ f > 0,\ C > 0$.

	d	$f,\ E$	Equilibria	Period-two solutions
$C = 0$	$d > 0$	$f = E$	$\overline{x}_0 = 0$	infinitely many
	$d = 0$	$f = E$	$\overline{x}_0 = 0$	infinitely many
	$d \geq 0$	$f > E$	$\overline{x}_0 = 0$	none
		$f < E$	$\overline{x}_0 = 0,\ \overline{x}_+ > 0$	none

Table 2. Existence of equilibria and period-two solutions for $a > 0,\ f > 0,\ C = 0$.

Case 2 $(af = 0)$**:**

When $f > 0$ but $a = 0$ notice that Equation (6) reduces to

$$\overline{x}\left((d - C)\overline{x} + (f - E)\right) = 0,$$

which has the isolated solutions $\overline{x}_0 = 0$ and possibly $\overline{x}_+ = \dfrac{f - E}{C - d}$. Existence of equilibria is summarized in Table 3.

$C,\ d$	$f,\ E$	Equilibria	Period-two solutions
$C \leq d$	$f > E$	$\overline{x}_0 = 0$	one
$C \geq d$	$f < E$	$\overline{x}_0 = 0$	none
$C \neq d$	$f = E$	$\overline{x}_0 = 0$	none
$C < d$	$f < E$	$\overline{x}_0 = 0,\ \overline{x}_+ > 0$	none
$C > d$	$f > E$	$\overline{x}_0 = 0,\ \overline{x}_+ > 0$	one
$C = d$	$f = E$	Any $\overline{x} \geq 0$ is a fixed point.	none

Table 3. Existence of equilibria and period-two solutions for $a = 0,\ f > 0$.

When $a > 0$ and $f = 0$, Equation (6) becomes

$$\overline{x}\left(a\overline{x}^2 + (d - C)\overline{x} - E\right) = 0,$$

and since necessarily $\overline{x} \neq 0$ in this case, Descartes' Rule of Signs yields that there may exist at most one positive fixed point $\overline{x}_+ > 0$. See Table 4 for a summary of the parametric conditions under which an equilibrium point exists.

$C,\ d$	E	Equilibria
$C \leq d$	$E = 0$	No equilibria
$C > d$	$E = 0$	$\overline{x}_+ > 0$
arbitrary	$E > 0$	$\overline{x}_+ > 0$

Table 4. Existence of equilibria for $a > 0,\ f = 0$.

In the case $a = f = 0$ the solutions of Equation (6) must satisfy

$$\overline{x}\left((d - C)\overline{x} - E\right) = 0.$$

Since we must have $\overline{x} > 0$, this equation has the isolated solution $\overline{x}_+ = \dfrac{E}{d - C}$ only when $d > C$ and $E > 0$. All remaining subcases may be summarized in Table 5.

$C,\ d$	E	Equilibria
$C \geq d$	$E > 0$	No equilibria
$C \neq d$	$E = 0$	No equilibria
$C < d$	$E > 0$	$\overline{x}_{+} > 0$
$C = d$	$E = 0$	Any $\overline{x} > 0$ is a fixed point.

Table 5. Existence of equilibria for $a = 0,\ f = 0$.

2.4.2 Local Stability Analysis of the Equilibrium Solutions

Define the function g such that

$$g(u, v) = \frac{Cv^2 + Ev}{au^2 + du + f}$$

so that Equation (1) becomes $x_{n+1} = g(x_n, x_{n-1})$. The partial derivatives of g are given by

$$g_u(u, v) = \frac{-(Cv^2 + Ev)(2au + d)}{(au^2 + du + f)^2} \qquad \text{and} \qquad g_v(u, v) = \frac{2Cv + E}{au^2 + du + f}.$$

The characteristic equation of the linearization of Equation (1) about $\overline{x}$ is $\lambda^2 = P\lambda + Q$, where $P = g_u(\overline{x}, \overline{x})$ and $Q = g_v(\overline{x}, \overline{x})$. Using Equation (6), this becomes

$$\lambda^2 = \frac{-\overline{x}(2a\overline{x} + d)}{a\overline{x}^2 + d\overline{x} + f}\lambda + \frac{2C\overline{x} + E}{a\overline{x}^2 + d\overline{x} + f}. \tag{8}$$

Lemma 1 *The zero equilibrium $\overline{x}_0 = 0$, which exists whenever $f > 0$, has the following stability:*

$$\overline{x}_0 = 0 \text{ is } \begin{cases} \textit{locally asymptotically stable} & \textit{if } E < f \\ \textit{a repeller} & \textit{if } E > f \\ \textit{nonhyperbolic (resonant } (1, -1) \textit{ type)} & \textit{if } E = f \end{cases}.$$

Proof. Notice that, evaluated at $(\overline{x}_0, \overline{x}_0)$, $P = 0$ and $Q = \frac{E}{f}$. Using Theorem 2.13 of [16], the first two results of the claim are immediate by checking the necessary inequalities. If $E = f$, then the characteristic equation of the linearized equation of Equation (1) (given in Equation (8)) is $\lambda^2 = 1$ and hence $\lambda_1 = 1,\ \lambda_2 = -1$ so that $\overline{x}_0$ is nonhyperbolic of resonant type $(1, -1)$. $\square$

Case 1 ($af > 0$):

Lemma 2 *Assume that $af > 0$.*

(a) If $C > d$, $f > E$, and $(C-d)^2 = 4a(f-E)$, then $\overline{x}_{\pm} = \frac{C-d}{2a}$ is a nonhyperbolic equilibrium point of unstable type.

(b) Suppose one of the following conditions holds:

1. $f < E$

2. $C > d,\ f = E$

3. $C > d,\ f > E,\ (C-d)^2 > 4a(f-E)$.

Then the positive equilibrium $\overline{x}_+$ is a saddle point.

(c) If $C > d$, $f > E$, and $(C-d)^2 > 4a(f-E)$, then $\overline{x}_-$ is a repeller.

Proof.

(a) Notice that, for $\overline{x} \neq 0$, we have the following:

$$\begin{aligned} |P| - 1 + Q &= \frac{\overline{x}(2a\overline{x} + d) - (a\overline{x}^2 + d\overline{x} + f) + 2C\overline{x} + E}{a\overline{x}^2 + d\overline{x} + f} \\ &= \frac{2a\overline{x}^2 + d\overline{x} - (a\overline{x}^2 + (d-C)\overline{x} + (f-E)) + C\overline{x}}{a\overline{x}^2 + d\overline{x} + f} \\ &= \frac{2a\overline{x}^2 + d\overline{x} + C\overline{x}}{a\overline{x}^2 + d\overline{x} + f} > 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} |P| + 1 - Q &= \frac{\overline{x}(2a\overline{x} + d) + (a\overline{x}^2 + d\overline{x} + f) - (2C\overline{x} + E)}{a\overline{x}^2 + d\overline{x} + f} \\ &= \frac{2a\overline{x}^2 + d\overline{x} + (a\overline{x}^2 + (d-C)\overline{x} + (f-E)) - C\overline{x}}{a\overline{x}^2 + d\overline{x} + f} \\ &= \frac{\overline{x}(2a\overline{x} + (d-C))}{a\overline{x}^2 + d\overline{x} + f}. \end{aligned} \tag{10}$$

From Equation (10) it is clear that, for $\overline{x}_{\pm} = \frac{C-d}{2a}$, $|P| + 1 - Q = 0$, and hence this equilibrium is indeed nonhyperbolic. Using the equilibrium equation and the fact that $2a\overline{x}_{\pm} + d = C$, notice that the characteristic equation (8) becomes

$$\lambda^2 = \frac{-\overline{x}(2a\overline{x}+d)}{a\overline{x}^2 + d\overline{x} + f}\lambda + \frac{2C\overline{x}+E}{a\overline{x}^2 + d\overline{x} + f} \iff \lambda^2 + \frac{C\overline{x}}{C\overline{x}+E}\lambda - \frac{2C\overline{x}+E}{C\overline{x}+E} = 0$$
$$\iff (\lambda - 1)\left(\lambda + \frac{2C\overline{x}+E}{C\overline{x}+E}\right) = 0.$$

Since $\lambda_1 = 1$ and $\lambda_2 = -\dfrac{2C\overline{x}+E}{C\overline{x}+E} < -1$, this nonhyperbolic equilibrium point is of the unstable type.

(b) Note that in all but the last case $\overline{x}_+$ is the unique positive equilibrium. It is clear from Equation (10) that $|P| + 1 - Q > 0$ if and only if $2a\overline{x} > C - d$. However, by definition in Equation (7) we have that

$$2a\overline{x}_+ = (C-d) + \sqrt{(C-d)^2 - 4a(f-E)} > (C-d).$$

Therefore $|P| > 1 - Q > -|P| \iff |1-Q| < |P|$, and thus by Theorem 2.13 of [16], $\overline{x}_+$ is a saddle point for all values of parameters for which it exists.

(c) Since

$$2a\overline{x}_- = (C-d) - \sqrt{(C-d)^2 - 4a(f-E)} < C - d,$$

by Equation (10) we have that $|P| + 1 - Q < 0$ and hence $|P| < |1-Q|$. Now

$$\begin{aligned} |Q| > 1 &\iff 2C\overline{x}_- + E > a\overline{x}_-^2 + d\overline{x}_- + f \\ &\iff C\overline{x}_- > a\overline{x}_-^2 + (d-C)\overline{x}_- + (f-E) \\ &\iff C\overline{x}_-^2 > 0, \end{aligned}$$

after we use Equation (6). Thus by Theorem 2.13 of [16], $\overline{x}_-$ is indeed a repeller.

$\square$

Case 2 ($af = 0$):

Lemma 3 *(a) Suppose $a = 0$ and $f > 0$.*

1. If $C < d$ and $f < E$, then $\overline{x}_+$ is a saddle point.

2. If $C > d$ and $f > E$, then $\overline{x}_+$ is a repeller.

3. If $C = d$ and $f = E$, then every point $\overline{x} > 0$ is a nonhyperbolic equilibrium of unstable type.

(b) If $a > 0$, $f = 0$, and either $E > 0$ or ($C > d$ and $E = 0$), then $\overline{x}_+$ is a saddle point.

(c) Suppose $a = f = 0$.

1. If $C < d$ and $E > 0$, then $\overline{x}_+$ is a saddle point.

2. If $C = d$ and $E = 0$, then any $\overline{x} > 0$ is a nonhyperbolic equilibrium of unstable type.

Proof.

(a) By Equation (10), we have that

$$|P| + 1 - Q = \frac{\overline{x}(d - C)}{d\overline{x} + f} = \frac{E - f}{d\overline{x} + f} \begin{cases} > 0, & E > f \\ < 0, & E < f \\ = 0, & E = f. \end{cases}$$

By Equation (9) we have that $|P| - 1 + Q > 0$. In the case when $C > d$, we can also check immediately that $|Q| > 1$. Thus when $E > f$ and $C < d$, $\overline{x}_+$ is a saddle point, and when $E < f$ and $C > d$, $\overline{x}_+$ is a repeller, which establishes Cases 1 and 2. In the nonhyperbolic case when $E = f$ and $C = d$, each point $\overline{x} > 0$ is an equilibrium point and the characteristic equation (8) reduces to

$$\lambda^2 + \frac{d\overline{x}}{d\overline{x} + f}\lambda - \frac{2d\overline{x} + f}{d\overline{x} + f} = 0 \iff (\lambda - 1)\left(\lambda + \frac{2d\overline{x} + f}{d\overline{x} + f}\right) = 0.$$

But then $\lambda_1 = 1$ and $\lambda_2 < -1$ so that each nonhyperbolic equilibrium is of the unstable type.

Thus we have verified Case 3, and the proof is complete.

(b) The result immediately follows from Equations (9) and (10).

(c) 1. When $d > C$ and $E > 0$, Equation (10) reduces to

$$|P| + 1 - Q = \frac{E}{d\overline{x}} > 0,$$

and coupling this result with Equation (9) shows that $\overline{x}_+$ is a saddle point.

2. If $d = C$ and $E = 0$, Equation (10) implies that any $\overline{x} > 0$ is nonhyperbolic. Equation (8) reduces to

$$\lambda^2 + \lambda - 2 = 0,$$

whence we deduce that $\lambda_1 = 1$ and $\lambda_2 = -2$, so a nonhyperbolic equilibrium $\overline{x} > 0$ is of the unstable type in this case.

□

2.4.3 Periodic Solutions

Lemma 4 *Consider Equation* (1).

(a) There exists no strictly positive minimal period-two solution to Equation (1).

(b) If $f > E$ and $C > 0$, Equation (1) *possesses the minimal period-two solution* $\{0, \frac{f-E}{C}, 0, \frac{f-E}{C}, \ldots\}$. *If $C = 0$ and $f = E$, then every point on the positive x- or y-axis is a period-two point.*

Proof.

(a) Suppose on the contrary that there exists a strictly positive periodic solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi \neq \psi$. Now ϕ and ψ satisfy:

$$\begin{aligned}\phi = g(\psi, \phi) &= \frac{C\phi^2 + E\phi}{a\psi^2 + d\psi + f} \\ \psi = g(\phi, \psi) &= \frac{C\psi^2 + E\psi}{a\phi^2 + d\phi + f},\end{aligned}$$

which together imply

$$\begin{aligned} & (C\psi + E)(a\psi^2 + d\psi + f) - (C\phi + E)(a\phi^2 + d\phi + f) = 0, \\ \iff \; & (\psi - \phi)[aC(\psi^2 + \psi\phi + \phi^2) + (Cd + aE)(\psi + \phi) + (Cf + dE)] = 0. \end{aligned} \tag{11}$$

Since $a + d > 0$, $a + C > 0$, and $C + E > 0$, it is clear that the latter factor of Equation (11) is strictly positive in *any* case, so we deduce that $\psi = \phi$, a contradiction. Thus no positive minimal period-two solution exists to Equation (1).

(b) In light of (a) there exists no interior period-two solution of Equation (1). Therefore, suppose there exists a periodic solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi \neq \psi$ and $\phi + \psi > 0$. Without loss of generality, we may set $\phi = 0$. Now

$$\psi = g(0, \psi) = \frac{C\psi^2 + E\psi}{f} \iff f - E = C\psi,$$

whence the result follows. Notice that if $C = 0$ and $f = E$, then any $\psi > 0$ will satisfy the above equation, establishing the second claim.

□

The following result gives the relation between the equilibria and period-two solutions.

Lemma 5 *(a) If $af > 0$, $C > d$, $f > E$, and $(C - d)^2 \geq 4a(f - E)$, $\overline{x}_-$ (or $\overline{x}_\pm$) is defined as in Equation (7), and $\psi = \frac{f-E}{C}$, then $\psi < \overline{x}_-$ (or $\psi < \overline{x}_\pm$).*

(b) If $a = 0$, $f > 0$, $C > d$, and $f > E$, then $\psi < \overline{x}_+$.

Proof.

(a) We need the check the following inequality:

$$\begin{aligned} \frac{f-E}{C} &< \frac{(C-d) - \sqrt{(C-d)^2 - 4a(f-E)}}{2a} \\ \iff \sqrt{(C-d)^2 - 4a(f-E)} &< (C-d) - \frac{2a(f-E)}{C}. \end{aligned} \tag{12}$$

Notice that the right-hand side of Inequality (12) is positive since $a > 0$:

$$\frac{C(C-d) - 2a(f-E)}{C} > \frac{(C-d)^2 - 4a(f-E)}{C} \geq 0.$$

If $(C-d)^2 = 4a(f-E)$ the result immediately follows. If $(C-d)^2 > 4a(f-E)$, we may square both sides of Inequality (12) to obtain

$$\begin{aligned} (C-d)^2 - 4a(f-E) &< \left((C-d) - \frac{2a(f-E)}{C}\right)^2 \\ \iff 0 &< C^2 - C(C-d) + a(f-E) = dC + a(f-E), \end{aligned}$$

which is always true by assumption. Thus indeed $\psi < \overline{x}_-$.

(b) In this case $\overline{x}_+ = \frac{f-E}{C-d}$, so $\psi < \overline{x}_+$ by definition since necessarily $d > 0$.

□

2.4.4 Local Stability Analysis of the Period-Two Solution

Lemma 6 *Consider Equation* (1)*.*

(a) If $f > E$ and $C > 0$, the period-two points $(\frac{f-E}{C}, 0)$ and $(0, \frac{f-E}{C})$ are saddle points.

(b) If $C = 0$ and $f = E$, then each point on the positive x- or y-axis is a nonhyperbolic period-two point of stable type.

Proof. Using the substitution $x_{n-1} = u_n$, $x_n = v_n$, Equation (1) becomes

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{Cu_n^2 + Eu_n}{av_n^2 + dv_n + f}. \end{aligned}$$

The corresponding map T is thus given by

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ g(v,u) \end{pmatrix}.$$

The second iteration T^2 of the map is given by

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} v \\ g(v,u) \end{pmatrix} = \begin{pmatrix} g(v,u) \\ g(g(v,u),v) \end{pmatrix} \overset{\text{set}}{=} \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix},$$

where

$$F(u,v) = g(v,u) = \frac{Cu^2 + Eu}{av^2 + dv + f}, \quad G(u,v) = \frac{Cv^2 + Ev}{aF^2(u,v) + dF(u,v) + f}.$$

Notice that the map T^2 is strongly competitive. The Jacobian of T^2 is given by

$$\begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix},$$

where

$$\begin{aligned} \frac{\partial F}{\partial u} &= \frac{2Cu + E}{av^2 + dv + f}, \\ \frac{\partial F}{\partial v} &= \frac{-(Cu^2 + Eu)(2av + d)}{(av^2 + dv + f)^2}, \\ \frac{\partial G}{\partial u} &= \frac{-(Cv^2 + Ev)(2aF(u,v) + d) \cdot \frac{\partial F}{\partial u}}{(aF^2(u,v) + dF(u,v) + f)^2}, \\ \frac{\partial G}{\partial v} &= \frac{(2Cv + E)(aF^2(u,v) + dF(u,v) + f) - (2aF(u,v) + d) \cdot \frac{\partial F}{\partial v} \cdot (Cv^2 + Ev)}{(aF^2(u,v) + dF(u,v) + f)^2}. \end{aligned}$$

Notice that if Equation (1) has the period-two solution $\{0, \psi, 0, \psi, \ldots\}$ for $\psi > 0$, then $(0, \psi)$ and $(\psi, 0)$ are both fixed points of T^2. The Jacobian of T^2 at the point $(0, \psi)$ has the following form:

$$\mathrm{Jac}_{T^2}\begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{E}{a\psi^2 + d\psi + f} & 0 \\ -\frac{Ed\psi}{f(a\psi^2 + d\psi + f)} & \frac{2C\psi + E}{f} \end{pmatrix},$$

which has eigenvalues $\lambda_1 = \frac{E}{a\psi^2 + d\psi + f}$ and $\lambda_2 = \frac{2C\psi + E}{f}$.

(a) If $f > E$ and $C > 0$, then $\psi = \frac{f-E}{C}$. Therefore

$$|\lambda_1| < 1 \iff E < a\psi^2 + d\psi + f, \text{ and}$$

$$|\lambda_2| > 1 \iff 2C\psi + E > f \iff 2f - 2E + E > f \iff f > E.$$

Moreover, one can check that no eigenvector corresponding to λ_1 is aligned with a coordinate axis if $E > 0$ and $d > 0$. A similar calculation will hold for $(\frac{f-E}{C}, 0)$. Thus the minimal period-two points are indeed saddle points.

(b) The eigenvalues of the Jacobian of T^2 evaluated at the point $(0, \psi)$ are given above for an arbitrary $\psi > 0$. But since $\lambda_2 = \frac{2C\psi+E}{f}$, by our hypothesis $\lambda_2 = 1$ and $\lambda_1 = \frac{E}{a\psi^2+d\psi+f} < 1$. Thus each minimal period-two solution is nonhyperbolic of stable type.

□

2.4.5 Global Dynamics of Equation (1)

The following result will establish the axes as the unstable manifolds for the isolated period-two points on the axes and will establish the axes as a repelling set when the period-two solution does not exist.

Lemma 7 *Consider Equation* (1).

(a) Suppose $fC > 0$.

If $f > E$, *then every solution with initial conditions* $x_{-1}x_0 = 0$ *and* $x_{-1} + x_0 > 0$ *will break into two subsequences of odd- and even-indexed terms. One subsequence will be identically zero, and the other will converge to* 0 *if* $x_i < \dfrac{f-E}{C} = \psi$ *and will be monotonically increasing (and hence unbounded) if* $x_i > \psi$ *for* $i = -1$ *or* $i = 0$.

If $f \leq E$, *as above, one subsequence will be identically zero and the other will be unbounded.*

(b) *Suppose $f > 0$ and $C = 0$.*

Then every solution with initial conditions $x_{-1}x_0 = 0$ and $x_{-1} + x_0 > 0$ will break into two subsequences of odd- and even-indexed terms. One subsequence will be identically zero, and the other will converge to 0 if $E < f$ and will be monotonically increasing (and hence unbounded) if $E > f$. Every point on the axes will be a period-two point if $E = f$.

Proof.

(a) Suppose $fC > 0$. Without loss of generality suppose $x_{-1} = 0$ and $x_0 > 0$. Then

$$x_1 = 0, \text{ and } x_2 = \frac{Cx_0^2 + Ex_0}{f} \begin{cases} < x_0 & \text{if } x_0 < \dfrac{f-E}{C} \\ = x_0 & \text{if } x_0 = \dfrac{f-E}{C} \\ > x_0 & \text{if } x_0 > \dfrac{f-E}{C}. \end{cases}$$

Since $x_3 = 0$, we may show a similar inequality as above for x_4 and x_2. By induction we may establish the claim.

(b) Now suppose $f > 0$ and $C = 0$. Again without loss of generality we may assume $x_{-1} = 0$ and $x_0 > 0$. Now

$$x_1 = 0 \text{ and } x_2 = \frac{Ex_0}{f} \begin{cases} < x_0 & \text{if } E < f \\ = x_0 & \text{if } E = f \\ > x_0 & \text{if } E > f, \end{cases}$$

and we again use induction to establish the claim.

□

If T is the map corresponding to Equation (1), then the strongly competitive map T^2 inherits as equilibria all corresponding fixed points and period-two points of Equation (1). With this in mind, the map T^2 may have as many as five isolated fixed points, listed below:

$$E_0 = (0,0),\ E_1 = (\overline{x}_-, \overline{x}_-),\ E_2 = (\overline{x}_+, \overline{x}_+),\ P_1 = \left(\frac{f-E}{C}, 0\right),\ P_2 = \left(0, \frac{f-E}{C}\right).$$

One can verify that no eigenvector associated with either characteristic value of $(\overline{x}_+, \overline{x}_+)$ (or $(\overline{x}_\pm, \overline{x}_\pm)$) is aligned with a coordinate axis. Using Lemmas 1-7 and Theorem 8, we may now deduce the global dynamics of Equation (1). Again, assume $C + E > 0$, $a + C > 0$, and $a + d > 0$.

Theorem 9 *Consider Equation* (1).

(a) Suppose one of the following conditions holds:

1. $f > 0,\ a > 0,\ C > d,\ f = E$
2. $f > 0,\ a > 0,\ f < E$
3. $f > 0,\ a = 0,\ C < d,\ f < E$
4. $f = 0,\ a > 0,\ C > d,\ E = 0$
5. $f = 0,\ a > 0,\ E > 0$
6. $f = 0,\ a = 0,\ C < d,\ E > 0.$

In Cases 1-3, Equation (1) *possesses the equilibrium point* 0*, which is non-hyperbolic of resonant type in Case 1 and a repeller in Cases 2 and 3. In Cases 4-6,* 0 *is an isolated point. In all cases, Equation* (1) *also possesses the saddle-point equilibrium* $\overline{x}_+$*. The global dynamics of Equation* (1) *is described by Theorem 8 part (a).*

In the following cases, assume $E > 0$ *and* $d > 0$.

(b) Suppose $f > 0,\ a > 0,\ C > d,\ f > E,$ *and* $(C - d)^2 > 4a(f - E)$.

Then Equation (1) *has three equilibrium points:* 0*, which is locally asymptotically stable,* $\overline{x}_-$*, which is a repeller, and* $\overline{x}_+$*, which is a saddle point. Equation* (1) *also has the minimal period-two solution* $\{0, \frac{f-E}{C}, 0, \frac{f-E}{C}, \ldots\}$*, which is a saddle point. The global dynamics of Equation* (1) *is described by Theorem 8 part (b).*

(c) Suppose either $f > 0,\ a > 0,\ C > 0,$ *and one of the following conditions*

holds:

1. $C \leq d, \ f > E$

2. $C > d, \ f > E, \ (C-d)^2 < 4a(f-E),$

or suppose $f > 0$, $a = 0$, $C \leq d$, *and* $f > E$. *Equation* (1) *possesses the equilibrium point* 0, *which is locally asymptotically stable, and a saddle-point minimal period-two solution. The global dynamics of Equation* (1) *is described by Theorem 8 part (c).*

(d) Suppose $f > 0$, $a > 0$, $C > d$, $f > E$, *and* $(C-d)^2 = 4a(f-E)$.

Equation (1) *possesses the equilibrium point* 0, *which is locally asymptotically stable,* $\overline{x}_{\pm}$, *which is nonhyperbolic of unstable type, and a saddle-point minimal period-two solution. The global dynamics of Equation* (1) *is described by Theorem 8 part (d).*

(e) Suppose $f > 0$, $a = 0$, $C > d$, *and* $f > E$.

Equation (1) *possesses the equilibrium point* 0, *which is locally asymptotically stable, and* $\overline{x}_{+}$, *which is a repeller. There also exists a saddle-point period-two solution. The global dynamics of Equation* (1) *is described by Theorem 8 part (e).*

The following results are not covered by the more general dynamic scenarios from Theorem 8 and require separate consideration.

Theorem 10 *Consider Equation* (1).

(a) Suppose $f > 0$, $a = 0$, $C \geq d$, *and* $f < E$.

Then Equation (1) *possesses only the zero equilibrium, and it is a repeller. All nonzero solutions are unbounded.*

(b) Suppose $a = 0$ *and one of the following conditions holds:*

1. $f > 0, \ C = d, \ f = E$

2. $f = 0,\ C = d,\ f = E$.

In either case Equation (1) *possesses every positive number as an equilibrium. (In the first case,* 0 *is also an equilibrium.) All non-equilibrium solutions are unbounded and will oscillate between approaching* 0 *and* ∞.

(c) *Suppose* $f = a = 0$, $C \geq d$, *and* $E > 0$. *Then Equation* (1) *has no equilibrium points, and all solutions are unbounded.*

(d) *Suppose* $f = a = E = 0$ *and* $C \neq d$. *Equation* (1) *is solvable in closed form. All solutions are unbounded and oscillate between approaching* 0 *and* ∞.

Proof.

(a) By Theorem 1 any bounded solution must converge to an equilibrium, a period-two solution, or a singular point on the boundary. Since the only member of the aforementioned set is a repelling fixed point, all solutions in this case must be unbounded.

(b) The strongly competitive map T^2 possesses an infinity of equilibria along the bisector in the first quadrant, where each equilibrium with positive coordinates is nonhyperbolic of unstable type. Through each fixed point E there exists a strictly decreasing curve $\mathcal{W}^u(E)$ that serves as its unstable manifold, and the union of these manifolds foliate the first quadrant. (In the first case the union of the axes serve as the unstable manifold for the origin.) See [18, 24] for the necessary results.

(c) By Theorem 1 any bounded solution must converge to an equilibrium, a period-two solution, or a singular point on the boundary. Since in this case no equilibria or period-two solutions exist, either the sequence is unbounded or it converges to a point on the boundary.

First suppose $\lim_{n\to\infty} x_n = 0$. Then subsequences of even- and odd-indexed terms are monotonically decreasing, so there must exist some $k \in \mathbb{N}$ such that for all $n > k$, both $x_{n+1} < x_{n-1}$ and $x_{n+2} < x_n$. Since $C \geq d$, we may use the first inequality to show that

$$\frac{x_{n-1}(dx_{n-1}+E)}{dx_n} \leq \frac{Cx_{n-1}^2 + Ex_{n-1}}{dx_n} < x_{n-1} \implies dx_{n-1} + E < dx_n.$$

In a similar way, our second assumed inequality implies that $dx_n + E < dx_{n+1}$. But then

$$x_{n+1} < x_{n-1} < x_n - \frac{E}{d} < x_{n+1} - \frac{2E}{d},$$

and this is a contradiction. Thus no sequence may converge to the isolated point at the origin.

Now suppose there exists a sequence $\{x_n\}$ such that, without loss of generality, the subsequence $\{x_{2n}\}$ converges to some positive limit. If $\lim_{n\to\infty} x_{2n} = L > 0$, then

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}\left(\frac{Cx_{2n}^2 + Ex_{2n}}{dx_{2n+2}}\right) = \frac{CL^2 + EL}{d\lim_{n\to\infty} x_{2n+2}} = \frac{CL+E}{d} > 0.$$

However, this contradicts the fact that Equation (1) has no minimal period-two solution. Consequently, every solution has an unbounded subsequence.

(d) Notice that Equation (1) reduces to

$$x_{n+1} = \frac{Cx_{n-1}^2}{dx_n}. \tag{13}$$

After taking the logarithm of both sides and seting $u_n = \ln(x_n)$ and $K = \ln\left(\frac{C}{d}\right)$ we obtain the linear, second-order, nonhomogeneous equation

$$u_{n+1} + u_n - 2u_{n-1} = K \iff (u_{n+1} - u_n) + 2(u_n - u_{n-1}) = K$$

which, after the substitution $v_n = u_n - u_{n-1}$, reduces to

$$v_{n+1} + 2v_n = K. \tag{14}$$

Equation (14) is of first order and has the general solution

$$v_n = (-2)^n \left(v_0 - \frac{K}{3} \right) + \frac{K}{3},$$

and hence

$$u_n - u_{n-1} = (-2)^n \left((u_0 - u_{-1}) - \frac{K}{3} \right) + \frac{K}{3}.$$

This first-order nonautonomous equation now has solution

$$\begin{aligned} u_n &= u_{-1} + \sum_{i=0}^{n} \left((-2)^i \left(u_0 - u_{-1} - \frac{K}{3} \right) + \frac{K}{3} \right) \\ &= u_{-1} + \frac{1 - (-2)^{n+1}}{3} \left(u_0 - u_{-1} - \frac{K}{3} \right) + \frac{(n+1)K}{3}. \end{aligned}$$

Finally, Equation (13) has solution

$$x_n = x_{-1} \left(\frac{C}{d} \right)^{\frac{n+1}{3}} \left(\frac{x_0}{x_{-1}} \left(\frac{d}{C} \right)^{\frac{1}{3}} \right)^{(1-(-2)^{n+1})/3}.$$

Thus we see that, as $n \to \infty$, every solution $\{x_n\}$ will oscillate between approaching 0 and ∞. We should remark that the above solution is valid for Equation (13) for all $C, d > 0$, even when $C = d$, the condition treated in part (b). If $C = d$ the solution reduces to

$$x_n = x_{-1} \left(\frac{x_0}{x_{-1}} \right)^{(1-(-2)^{n+1})/3}.$$

□

Theorem 11 *Assume* $C = 0$.

(a) *Suppose* $f > 0$, $a > 0$, $d > C = 0$, *and* $f = E$.
Then Equation (1) *possesses the zero equilibrium, which is nonhyperbolic of resonant type, and an infinity of minimal period-two solutions of the form* $\{0, s, 0, s, \ldots\}$ *for* $s > 0$, *which are nonhyperbolic of stable type. All solutions converge to a (not necessarily prime) period-two solution on the axes.*

(b) *Suppose $f > 0$, $a > 0$, $d \geq C = 0$, and $f > E$.*

Then Equation (1) possesses only the zero equilibrium and it is globally asymptotically stable.

Proof.

(a) In view of Lemma 6 the strongly competitive map T^2 possesses the nonhyperbolic zero equilibrium as well as infinitely many equilibria along the continuum of the x- and y-axes (where each equilibrium is nonhyperbolic of stable type). Through each fixed point E there exists a strictly increasing curve $\mathcal{W}^s(E)$ that serves as its stable manifold and is the basin of attraction of E. The result follows from an application of Theorems 1-4 or Theorems 3.2 and 3.6 in [4].

(b) Suppose $C = 0$. In view of $E < f$, Equation (1) implies:

$$x_{n+1} = \frac{Cx_{n-1}^2 + Ex_{n-1}}{ax_n^2 + dx_n + f} < \frac{E}{f}x_{n-1} < x_{n-1}. \tag{15}$$

By Inequality (15) it is clear that the susbsequences of even- and odd-indexed terms of Equation (1) are monotonically decreasing, which is consistent with Theorem 1. Since Equation (1) is bounded below, all solutions must converge to $\overline{x}_0$.

□

We leave the following conjectures for a few parametric situations not covered by the theorems above. First, we leave conjectures for the values of parameters for which zero is the sole equilibrium of Equation (1) and is nonhyperbolic of resonant type or for which no equilibria exist. We conjecture in these cases that all solutions remain unbounded, but it remains to be seen if there exist any bounded solutions converging to either the sole fixed point or to a point on the boundary.

Conjecture 1 *Suppose $f > 0$, $a > 0$, $0 < C \leq d$, and $f = E$, or suppose $f > 0$, $a = 0$, $C \neq d$, and $f = E$.*

Equation (1) *possesses only the zero equilibrium, which is nonhyperbolic of resonant type. All solutions are unbounded.*

Conjecture 2 *Suppose $f = E = 0$, $a > 0$, and $C \leq d$.*

Equation (1) *has no equilibrium points, and all solutions are unbounded.*

Further, we have added the stipulation $Ed > 0$ in parts (b) through (e) of Theorem 9 to ensure the applicability of Theorem 3, which requires that the eigenspace associated with the eigenvalue λ_1 does not align with a coordinate axis. We believe the established results for $Ed > 0$ in which the period-two solution exists will still hold for $Ed = 0$, and thus we leave the following conjecture.

Conjecture 3 *(a) Suppose $Ed = 0$. Then the results of Theorem 9 still hold in parts (b)–(e).*

(b) Suppose $f > 0$, $a > 0$, $d = C = 0$, and $f = E$. The global dynamics of Equation (1) *is described by the conclusions of Theorem 11 (a).*

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MANUSCRIPT 3

Generalized Second-Order Beverton-Holt Equations of Linear and Quadratic Type

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Abstract

We investigate second-order generalized Beverton-Holt difference equations of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \ldots,$$

where f is a function nondecreasing in both arguments, the parameter a is a positive constant, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. We will discuss several interesting examples of such equations and present some general theory. In particular, we will investigate the local and global dynamics in the event f is a certain type of linear or quadratic polynomial, and we explore the existence problem of period-two solutions.

3.1 Introduction and Preliminaries

Consider the following second-order difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \ldots. \tag{1}$$

Here f is a continuous function nondecreasing in both arguments, the parameter a is a positive real number, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. Equation (1) is a generalization of the first-order Beverton-Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{2}$$

where $a > 0$ and $x_0 \geq 0$. The global dynamics of Equation (2) may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a \leq 1 \\ a - 1 & \text{if } a > 1 \text{ and } x_0 > 0. \end{cases} \tag{3}$$

Many variations of Equation (2) have been studied. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913; see [18]. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942; see [18]. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics. The so-called Monod differential equation is given by

$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{rS}{a+S}, \tag{4}$$

where $N(t)$ is the concentration of bacteria at time t, $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, and a is a half-saturation constant (when $S = a$, the right-hand side of Equation (4) equals $r/2$). Based on experimental data, the following system of two differential equations for the nutrient S and bacteria N, as presented in [18], is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma} N \frac{rS}{a+S}, \qquad \frac{dN}{dt} = N \frac{rS}{a+S}, \tag{5}$$

where the constant γ is called the growth yield. Both Equation (4) and System (5) contain the function $f(x) = rx/(a+x)$ known as the Monod function, Michaelis-Menten function, Beverton-Holt function, or Holling function of the first kind; see [4, 9].

One possible two-generation population model based on Equation (2),

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}}{1+x_{n-1}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [16]. The global dynamics of Equation (6) may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a_1 + a_2 \leq 1 \\ a_1 + a_2 - 1 & \text{if } a_1 + a_2 > 1 \text{ and } x_0 + x_{-1} > 0. \end{cases}$$

This result was extended in [4] to the case of a k-generation population model based on Equation (2) of the form

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{a_i x_{n-i}}{1+x_{n-i}}, \quad n = 0, 1, \ldots, \tag{7}$$

where $a_i \geq 0$ for $i = 0, 1, \ldots, k-1$, $\sum_{i=0}^{k-1} a_i > 0$, and $x_{1-k}, \ldots, x_0 \geq 0$. It was shown that the global dynamics of Equation (7) is given precisely by (3), where $a = \sum_{i=0}^{k-1} a_i$ and we consider all initial conditions positive.

The simplest model of Beverton-Holt type which exhibits two coexisting attractors and the Allee effect is the sigmoid Beverton-Holt or (second-type Holling) difference equation

$$x_{n+1} = \frac{ax_n^2}{1 + x_n^2}, \quad n = 0, 1, \ldots, \tag{8}$$

where $a > 0$ and $x_0 \geq 0$. The dynamics of Equation (8) may be concisely summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a < 2 \text{ or } (a \geq 2 \text{ and } x_0 < \overline{x}_-) \\ \overline{x}_+ & \text{if } a \geq 2 \text{ and } x_0 > \overline{x}_-, \end{cases} \tag{9}$$

where $\overline{x}_-$ and $\overline{x}_+$ are the two positive equilibria when $a \geq 2$; see [1, 4]. One possible two-generation population model based on Equation (8),

$$x_{n+1} = \frac{a_1 x_n^2}{1 + x_n^2} + \frac{a_2 x_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{10}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [3]. However, the summary of the global dynamics of Equation (10) is not an immediate extension of the global dynamics of Equation (8) as given in (9); see [3]. Equation (10) can have up to three equilibrium solutions and up to three period-two solutions. In the case when Equation (10) has three equilibrium solutions and three period-two solutions, the zero equilibrium, the larger positive equilibrium, and one period-two solution are attractors with substantial basins of attraction, which together with the remaining equilibrium and the global stable manifolds of the saddle-point period-two solutions exhaust the first quadrant of initial conditions. This behavior happens when the coefficient a_2 is in some sense dominant to a_1; see [3]. Such behavior is typical for other models in population dynamics such as

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = a_1 x_n + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots,$$

which were also investigated in [3]. In the case of a k-generation population model based on the sigmoid Beverton-Holt difference equation with $k > 2$, one can expect to have attractive period-k solutions as well as chaos.

The first model of the form given in Equation (1), where f is a linear function in both variables (that is, $f(u, v) = cu + dv$ for $c, d, u, v \geq 0$) was considered in [17] and some global dynamics were described in part of the parametric space. Here we will extend the results from [17] to the whole parametric space. In this paper we will then restrict ourselves to the case when $f(u, v)$ is a quadratic polynomial, which will give similar global dynamics to that presented for Equation (10). The corresponding dynamic scenarios will be essentially the same for any polynomial function of the type $f(u, v) = cu^k + du^m$ where $c, d \geq 0$ and m, k are positive integers. Higher values of m and k may only create additional equilibria and period-two solutions but should replicate the global dynamics seen in the quadratic case presented in this paper.

Let the function $F : [0, \infty)^2 \to [0, a)$ be defined as follows:

$$F(u, v) = \frac{af(u, v)}{1 + f(u, v)}. \tag{11}$$

Then Equation (1) becomes $x_{n+1} = F(x_n, x_{n-1})$ for all $n = 0, 1, \ldots,$ where $F(u, v)$ is nondecreasing in both of its arguments.

The following theorem from [2] immediately applies to Equation (1).

Theorem 1 *Let I be a set of real numbers and $F : I \times I \to I$ be a function which is nondecreasing in the first variable and nondecreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = F\left(x_n, x_{n-1}\right), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots, \tag{12}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (12) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. It should be noticed that Theorem 1 is specific for second-order difference equations and does not extend to difference equations of order higher than two. Furthermore, the powerful theory of monotone maps in the plane [14, 15] can be applied to Equation (1) to determine the boundaries of the basins of attraction of the equilibrium solutions and period-two solutions. Finally, when $f(u, v)$ is a polynomial function, all computation needed to determine the local stability of all equilibrium solutions and period-two solutions is reduced to the theory of counting the number of zeros of polynomials in a given interval, as given in [10]. This theory will give more precise results than the global attractivity and global asymptotic stability results in [6, 7]. However, in the case of difference equations of the form

$$x_{n+1} = \frac{a g(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + g(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1,$$

where $a > 0$ and g is nondecreasing in all its arguments, Theorem 1 does not apply for $k > 2$, but the results from [6, 7, 11] can give global dynamics in some regions of the parametric space.

The following theorem from [8] is often useful in determining the global attractivity of a unique positive equilibrium.

Theorem 2 *Let $I \subseteq [0, \infty)$ be some open interval and assume that $F \in C[I \times I, (0, \infty)]$ satisfies the following conditions:*

(i) $F(x, y)$ is nondecreasing in each of its arguments;

(ii) Equation (12) *has a unique positive equilibrium point $\overline{x} \in I$ and the function $F(x, x)$ satisfies the* ***negative feedback condition****:*

$$(x - \overline{x})(F(x, x) - x) < 0 \textit{ for every } x \in I \backslash \{\overline{x}\}.$$

Then every positive solution of Equation (12) *with initial conditions in I converges to $\overline{x}$.*

3.2 Local Stability

In this section we provide general conditions to determine the local stability of equilibrium solutions and period-two solutions.

It is clear that $x_n \leq a$ for all $n \geq 1$. In light of Theorem 1, since all solutions are bounded, if there are no singular points on the boundary of the domain of F, it immediately follows that all solutions to Equation (1) converge to an equilibrium or period-two solution.

An equilibrium $\overline{x}$ of Equation (1) satisfies

$$\overline{x}(1 + f(\overline{x}, \overline{x})) = af(\overline{x}, \overline{x}). \tag{13}$$

Clearly $\overline{x}_0 = 0$ is an equilibrium point if and only if $(0, 0)$ is in the domain of f and $f(0, 0) = 0$.

The linearized equation of Equation (1) about an equilibrium $\overline{x}$ is

$$z_{n+1} = F_u(\overline{x}, \overline{x}) z_n + F_v(\overline{x}, \overline{x}) z_{n-1}, \quad n = 0, 1, \ldots .$$

Since f is a nondecreasing function, it follows that $F_u(\overline{x}, \overline{x}), F_v(\overline{x}, \overline{x}) \geq 0$. Therefore, if

$$\lambda(\overline{x}) = F_u(\overline{x}, \overline{x}) + F_v(\overline{x}, \overline{x}) = \frac{a(f_u(\overline{x}, \overline{x}) + f_v(\overline{x}, \overline{x}))}{(1 + f(\overline{x}, \overline{x}))^2}, \tag{14}$$

then in view of Corollary 2 of [11] we may conclude that

$$\overline{x} \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } \lambda(\overline{x}) < 1 \\ \text{nonhyperbolic} & \text{if } \lambda(\overline{x}) = 1 \\ \text{unstable} & \text{if } \lambda(\overline{x}) > 1. \end{cases}$$

Further, Theorem 2.13 of [13] implies that if $\overline{x}$ is unstable, then

$$\overline{x} \text{ is} \begin{cases} \text{a repeller} & \text{if } \delta(\overline{x}) > 1 \\ \text{nonhyperbolic} & \text{if } \delta(\overline{x}) = 1 \\ \text{a saddle point} & \text{if } \delta(\overline{x}) < 1, \end{cases}$$

where

$$\delta(\overline{x}) = F_v(\overline{x}, \overline{x}) - F_u(\overline{x}, \overline{x}) = \frac{a(f_v(\overline{x}, \overline{x}) - f_u(\overline{x}, \overline{x}))}{(1 + f(\overline{x}, \overline{x}))^2}. \tag{15}$$

Let (ϕ, ψ) be a period-two solution of Equation (1). The Jacobian matrix of the corresponding map $T = G^2$, where $G(u, v) = (v, F(v, u))$ and F is given by Equation (11), is given in Theorem 12 of [5]. The linearized equation evaluated at (ϕ, ψ) is

$$\lambda^2 - Tr J_T(\phi, \psi)\lambda + Det J_T(\phi, \psi) = 0,$$

where

$$Tr J_T(\phi, \psi) = D_2F(\psi, \phi) + D_1F(F(\psi, \phi), \psi) \cdot D_1F(\psi, \phi) + D_2F(F(\psi, \phi), \psi)$$

and

$$Det J_T(\phi, \psi) = D_2F(F(\psi, \phi), \psi) \cdot D_2F(\psi, \phi).$$

3.3 Examples

3.3.1 Linear-Linear: $f(u,v) = cu + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1})}{1 + cx_n + dx_{n-1}}, \quad n = 0, 1, \ldots, \tag{16}$$

where $c \geq 0$ and $d > 0$. If $d = 0$, then Equation (16) becomes Equation (2) after a reduction of parameters. Notice that $f_u(u,v) = c$ and $f_v(u,v) = d$. By Equation (13) we know that $\overline{x}_0 = 0$ is always a fixed point and $\overline{x}_+ = \frac{a(c+d)-1}{c+d}$ is a unique positive fixed point for $a(c+d) > 1$.

Since $\lambda(\overline{x}_0) = a(c+d)$, we have that

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } a(c+d) < 1 \\ \text{nonhyperbolic} & \text{if } a(c+d) = 1 \\ \text{unstable} & \text{if } a(c+d) > 1. \end{cases}$$

Further, notice that

$$\lambda(\overline{x}_+) = \frac{a(c+d)}{\left(1 + \left(\frac{a(c+d)-1}{c+d}\right) \cdot (c+d)\right)^2} = \frac{1}{a(c+d)} < 1$$

for all values of parameters for which $\overline{x}_+$ exists. Therefore

$$\overline{x}_+ = \frac{a(c+d)-1}{c+d} \text{ is always locally asymptotically stable.}$$

Note that there is an exchange in stability from $\overline{x}_0$ to $\overline{x}_+$ as the parametric value $a(c+d)$ passes through 1.

We next search for period-two solutions. Suppose there exists such a solution $\{\psi, \phi, \psi, \phi, \ldots\}$ with $\phi \neq \psi$. We must solve the following system:

$$\begin{cases} \psi & = \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} & = \dfrac{a(c\phi + d\psi)}{1 + c\phi + d\psi} \\ \phi & = \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} & = \dfrac{a(c\psi + d\phi)}{1 + c\psi + d\phi} \end{cases}. \tag{17}$$

Notice that

$$\psi - \phi = \frac{a(d-c)(\psi - \phi)}{(1 + c\phi + d\psi)(1 + c\psi + d\phi)},$$

whence we deduce that $d > c$ and $(1 + c\phi + d\psi)(1 + d\psi + d\phi) = a(d - c)$. Now

$$\psi + \phi = \frac{a\left((c+d)(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi)\right)}{a(d-c)},$$

or equivalently,

$$2c(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi) = 0.$$

Since $\psi + \phi > 0$, it must be the case that $c = 0$, and then $2d^2\psi\phi = 0$ so that one of either ϕ or ψ equals zero. Without loss of generality assume $\phi = 0$. But then $\psi = \frac{ad\psi}{1+d\psi}$, and hence $\psi = \frac{ad-1}{d} = \overline{x}_+$. Thus the only non-equilibrium solution of System (17) is the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \ldots\}$, which exists for $ad > 1$ and $c = 0$.

Theorem 3 *(a) If $a(c + d) \leq 1$, $\overline{x}_0 = 0$ is a global attractor of all solutions.*
(b) If $c = 0$ and $ad > 1$, then there exists a minimal period-two solution $\left\{\frac{ad-1}{d}, 0, \frac{ad-1}{d}, 0, \ldots\right\}$. $\overline{x}_+$ is a global attractor of all solutions with positive initial conditions. Any solution with exactly one initial condition equal to zero will converge to the period-two solution.
(c) If $c > 0$ and $a(c + d) > 1$, $\overline{x}_+$ is a global attractor of all nonzero solutions.

Proof. (a) If $a(c + d) \leq 1$, $\overline{x}_0 = 0$ is the only equilibrium, and no period-two solutions exist. By Theorem 1 all solutions must converge to zero.
(b) Suppose $c = 0$ and $ad > 1$, and consider $I = (0, \infty)$. Notice that

$$F(x, x) = \frac{adx}{1 + dx} \gtrless x \iff \overline{x}_+ \gtrless x,$$

and therefore by Theorem 2 we have that all solutions with initial conditions in I converge to $\overline{x}_+$.

Now suppose one initial condition is zero, so without loss of generality assume $x_{-1} = 0$ and $x_0 > 0$. Then $x_1 = 0$ and

$$x_2 = \frac{adx_0}{1 + dx_0} \gtrless x_0 \iff \frac{ad-1}{d} = \overline{x}_+ \gtrless x_0.$$

Further, one can show $x_2 \lesseqgtr \overline{x}_+ \iff x_0 \lesseqgtr \overline{x}_+$. By induction, $\lim\limits_{k\to\infty} x_{2k} = \overline{x}_+$ and $x_{2k-1} = 0$ for all $k = 0, 1, \ldots$. Thus all solutions with exactly one initial condition equal to zero will converge to the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \ldots\}$.

(c) When $c > 0$ and $a(c + d) > 1$, $\overline{x}_+$ is locally asymptotically stable while $\overline{x}_0$ is unstable. As in the proof of (b) we can employ Theorem 2 to show that all solutions with positive initial conditions must converge to $\overline{x}_+$. Since $c > 0$ and $d > 0$, if $x_0 + x_{-1} > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ (and also $x_2 > 0$), so the solution eventually has consecutive positive terms and must converge to $\overline{x}_+$. □

3.3.2 Translated Linear-Linear: $f(u, v) = cu + dv + k$

We briefly consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1} + k)}{1 + cx_n + dx_{n-1} + k}, \quad n = 0, 1, \ldots, \tag{18}$$

where $c \geq 0$, $d \geq 0$, $c+d > 0$, and $k > 0$. (It is clear that Equation (18) reduces to Equation (16) in the event $k = 0$.) We notice in this example $f(0, 0) = k > 0$, so the origin cannot be an equilibrium. More specifically, an equilibrium of Equation (18) must satisfy

$$(c + d)\overline{x}^2 + (k + 1 - a(c + d))\,\overline{x} - ak = 0$$

Since $c + d > 0$ and $ak > 0$ by Descartes' Rule of Signs it must be the case that there exists a unique positive equilibrium $\overline{x}_+$.

Theorem 4 *Consider Equation* (18) *such that $c + d > 0$ and $k > 0$. The unique positive equilibrium $\overline{x}_+$ is a global attractor.*

Proof. The result follows from an application of Theorem 1.4.8 of [12]. □

3.3.3 Quadratic-Linear: $f(u,v) = cu^2 + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n^2 + dx_{n-1})}{1 + cx_n^2 + dx_{n-1}}, \quad n = 0, 1, \ldots. \tag{19}$$

Remark 1 For the analysis that follows, we will consider Equation (19) with $c > 0$ and $d > 0$. Notice that when $c = 0$ Equation (19) is a special case of Equation (16), and the global dynamics for this case is discussed in Theorem 3. When $d = 0$, Equation (19) is essentially Equation (8), the dynamics of which may be seen in (9).

An equilibrium of (19) satisfies

$$c\overline{x}^3 + d\overline{x}^2 + \overline{x} = ac\overline{x}^2 + ad\overline{x}$$

so that all nonzero equilibria satisfy

$$c\overline{x}^2 + (d - ac)\overline{x} + (1 - ad) = 0, \tag{20}$$

whence we easily deduce the possible solutions

$$\overline{x}_{\pm} = \frac{ac - d \pm \sqrt{(d - ac)^2 + 4c(ad - 1)}}{2c},$$

which are real if and only if $R = (d - ac)^2 + 4c(ad - 1)$ satisfies $R \geq 0$.
Notice that

$$R \geq 0 \iff d^2 - 2acd + a^2c^2 + 4acd - 4c \geq 0 \iff (ac + d)^2 \geq 4c. \tag{21}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x} + d)}{(1 + c\overline{x}^2 + d\overline{x})^2}.$$

Lemma 1 *Equation* (19) *always has the zero equilibrium* $\overline{x}_0 = 0$*, and*

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } ad < 1 \\ \text{nonhyperbolic} & \text{if } ad = 1 \\ \text{a repeller} & \text{if } ad > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = \delta(\overline{x}_0) = ad$. □

Lemma 2 *Assume* $c > 0$ *and* $d > 0$*.*

(1) Suppose either

(a) $d \geq ac$ *and* $1 \geq ad$*, or*

(b) $d < ac$, $1 > ad$*, and* $R < 0$*.*

Then Equation (19) *has no positive equilibria.*

(2) Suppose either

(a) $1 < ad$*, or*

(b) $d < ac$ *and* $1 = ad$*.*

Then Equation (19) *has the positive equilibrium* $\overline{x}_+$*, and it is locally asymptotically stable.*

(3) Suppose $d < ac$, $1 > ad$*, and* $R = 0$*. Then Equation* (19) *has the positive equilibrium* $\overline{x}_\pm$*, and it is nonhyperbolic of stable type.*

(4) Suppose $d < ac$, $1 > ad$*, and* $R > 0$*. Then Equation* (19) *has the two positive equilibria* $\overline{x}_+$*, which is locally asymptotically stable, and* $\overline{x}_-$*, which is a saddle point.*

Proof. The positivity of solutions of Equation (20) follows from Descartes' Rule of Signs. Using Equation (14), notice that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x} + d)}{(1 + c\overline{x}^2 + d\overline{x})^2} = \frac{a(2c\overline{x} + d)}{(a(c\overline{x} + d))^2} = \frac{2c\overline{x} + d}{a(c\overline{x} + d)^2} = \frac{1}{a(c\overline{x} + d)} + \frac{c\overline{x}}{a(c\overline{x} + d)^2}.$$

Further, for the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned}\lambda(\overline{x}_+) \leq 1 &\iff \frac{c\overline{x}_+}{a(c\overline{x}_+ + d)^2} \leq \frac{a(c\overline{x}_+ + d) - 1}{a(c\overline{x}_+ + d)} \\ &\iff c\overline{x}_+ \leq (c\overline{x}_+ + d)\left(a(c\overline{x}_+ + d) - 1\right) = (c\overline{x}_+ + d)(c\overline{x}_+^2 + d\overline{x}_+) \\ &\iff c \leq (c\overline{x}_+ + d)^2 \\ &\iff 4c \leq (2c\overline{x}_+ + 2d)^2 = (ac + d + \sqrt{R})^2,\end{aligned}$$

which is immediately true by Inequality (21). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (19) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $acy^2 - (ac - d)y - d = 0$, which has characteristic values $y_1 = 1$ and $y_2 = -\frac{d}{ac}$, where $-1 < y_2 < 0$ since $ac > d$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type.

When $\overline{x}_-$ exists,

$$\begin{aligned}\lambda(\overline{x}_-) > 1 &\iff 4c > (ac + d - \sqrt{R})^2 \\ &\iff 4c + (ac + d)\sqrt{R} > (ac + d)^2 \\ &\iff (ac + d)\sqrt{R} > (ac + d)^2 - 4c = R \\ &\iff (ac + d)^2 > R = (ac + d)^2 - 4c,\end{aligned}$$

which is of course true since $c > 0$. To show more specifically that $\overline{x}_-$ is a saddle point when $R > 0$, we must show that $\delta(\overline{x}_-) < 1$, where δ is defined by Equation (15). Notice

$$\delta(\overline{x}_-) = \frac{a(d - 2c\overline{x}_-)}{(1 + c\overline{x}_-^2 + d\overline{x}_-)^2} = \frac{a(d - 2c\overline{x}_-)}{(a(c\overline{x}_- + d))^2} = \frac{4(d - 2c\overline{x}_-)}{a(2c\overline{x}_- + 2d)^2} = \frac{4\left(2d - ac + \sqrt{R}\right)}{a(ac + d - \sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) < 1 &\iff 4\left(2d - ac + \sqrt{R}\right) < a\left(ac + d - \sqrt{R}\right)^2 \\ &\iff (2 + a(ac + d))\sqrt{R} < a(ac + d)^2 - 4d.\end{aligned}$$

The right-hand side of the latter inequality is positive since $a(ac+d)^2 - 4d > 4ac - 4d = 4(ac-d) > 0$ by assumption. But then

$$\begin{aligned}
\delta(\overline{x}_-) < 1 &\iff (2+a(ac+d))^2\left((ac+d)^2 - 4c\right) < \left(a(ac+d)^2 - 4d\right)^2 \\
&\iff 3a^3c^2d + 6a^2cd^2 + 3ad^3 - 3a^2c^2 - 2acd - 3d^2 - 4c < 0 \\
&\iff (ad-1)\left(3d^2 + 3a^2c^2 + 2c(3ad+2)\right) < 0,
\end{aligned}$$

which is automatically true since the latter factor is strictly positive and $ad < 1$. Thus indeed $\overline{x}_-$ is a saddle point when it exists for $R > 0$. □

Lemma 3 *There exist no minimal period-two solutions to Equation* (19) *if* $c > 0$ *and* $d > 0$.

Proof. Suppose there exist $\phi, \psi > 0$ with $\phi \neq \psi$ such that

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} &= \dfrac{a(c\phi^2 + d\psi)}{1 + c\phi^2 + d\psi} \\[2ex] \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} &= \dfrac{a(c\psi^2 + d\phi)}{1 + c\psi^2 + d\phi} \end{cases}. \tag{22}$$

From System (22) we notice that

$$\begin{aligned}
\psi - \phi &= \frac{a(c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) - a(c\psi^2 + d\phi)(1 + c\phi^2 + d\psi)}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)} \\
&= \frac{a(\psi - \phi)(d - c(\psi + \phi))}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)},
\end{aligned}$$

whence it immediately follows that $(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) = a(d - c(\psi + \phi))$. But then

$$\begin{aligned}
\psi + \phi &= \frac{a(c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) + a(c\psi^2 + d\phi)(1 + c\phi^2 + d\psi)}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)} \\
&= \frac{2(c\phi^2 + d\psi)(c\psi^2 + d\phi) + c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}.
\end{aligned}$$

Thus we have that necessarily

$$2\phi\psi = \frac{2a^2(c\phi^2 + d\psi)(c\psi^2 + d\phi)}{a(d - c(\psi + \phi))} = a\left((\psi + \phi) - \frac{c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}\right) > 0$$

since both $\psi, \phi > 0$. But this implies that

$$\begin{aligned}
&(\psi+\phi)(d-c(\psi+\phi)) > c(\psi^2+\phi^2)+d(\psi+\phi)\\
\Longleftrightarrow\ & d(\psi+\phi)-c(\psi+\phi)^2 > c(\psi^2+\phi^2)+d(\psi+\phi)\\
\Longleftrightarrow\ & 0 > c(\psi^2+\phi^2)+c(\psi+\phi)^2,
\end{aligned}$$

a clear contradiction since $c > 0$.

Now suppose there exists a period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi \neq \psi$ but $\phi\psi = 0$. Suppose without loss of generality that $\phi = 0$. Now

$$\begin{cases} \psi &= \dfrac{af(0,\psi)}{1+f(0,\psi)} &= \dfrac{ad\psi}{1+d\psi} \\ \\ 0 &= \dfrac{af(\psi,0)}{1+f(\psi,0)} &= \dfrac{ac\psi^2}{1+c\psi^2} \end{cases},$$

which immediately leads to the contradiction $\psi = \phi = 0$ for $c > 0$. Thus Equation (19) has no minimal period-two solutions. □

Theorem 5 *Assume $c > 0$ and $d > 0$.*

(1) Suppose either

(a) $d \geq ac$ and $1 \geq ad$, or

(b) $d < ac$, $1 > ad$, and $R < 0$.

Then $\overline{x}_0$ is a global attractor of all solutions.

(2) Suppose either

(a) $1 < ad$, or

(b) $d < ac$ and $1 = ad$.

Then $\overline{x}_+$ is a global attractor of all nonzero solutions.

(3) Suppose $d < ac$, $1 > ad$, and $R = 0$. Then the system corresponding to Equation (19) has the equilibria $E_0 = (0,0)$, which is locally asymptotically stable, and $E = (\overline{x}_\pm, \overline{x}_\pm)$, which is nonhyperbolic of stable type. Then there exists a

continuous curve $\mathcal{C}$ passing through E such that $\mathcal{C}$ is the graph of a decreasing function. The set of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of two disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$, where

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{C}\},$$

$$\mathcal{B}(E) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{C}\} \cup \mathcal{C}.$$

(4) Suppose $d < ac$, $1 > ad$, and $R > 0$. Then the system corresponding to Equation (19) *has the equilibria $E_0 = (0, 0)$, which is locally asymptotically stable, $E_1 = (\overline{x}_-, \overline{x}_-)$, which is a saddle point, and $E_2 = (\overline{x}_+, \overline{x}_+)$, which is locally asymptotically stable. Then there exist two continuous curves $\mathcal{W}^s(E_1)$ and $\mathcal{W}^u(E_1)$, both passing through E_1, such that $\mathcal{W}^s(E_1)$ is the graph of a decreasing function and $\mathcal{W}^u(E_1)$ is the graph of an increasing function. The set of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of three disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_1) \cup \mathcal{B}(E_2)$, where $\mathcal{B}(E_1) = \mathcal{W}^s(E_1)$,*

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{W}^s(E_1)\}, \text{ and}$$

$$\mathcal{B}(E_2) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{W}^s(E_1)\}.$$

Proof. (1) The proof in this case follows from Theorem 1 as well as Lemmas 1, 2, and 3 since $\overline{x}_0 = 0$ is the sole equilibrium of Equation (19).

(2) The proof used to show that all solutions with positive initial conditions converge to $\overline{x}_+$ follows from an application of Theorem 2 (as used above in the proof of Theorem 3). Notice that $x_1 = F(x_0, x_{-1}) > 0$ if either $x_0 > 0$ or $x_{-1} > 0$ (and similar for x_2), so $I = (0, \infty)$ is an attracting and invariant interval. Thus all nonzero solutions must converge to $\overline{x}_+$.

(3) The proof follows from an application of Theorems 1-4 of [15] applied to the *cooperative* second iterate of the map corresponding to Equation (19). The proof is completely analogous to the proof of Theorem 5 in [3], so we omit the details.

(4) The proof follows from an immediate application of Theorem 5 in [3]. □

3.3.4 Linear-Quadratic: $f(u,v) = cu + dv^2$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1}^2)}{1 + cx_n + dx_{n-1}^2}, \quad n = 0, 1, \dots. \tag{23}$$

Remark 2 For the analysis that follows, we will consider Equation (23) with $c > 0$ and $d > 0$. Notice that when $d = 0$ Equation (23) becomes Equation (2) after a reduction of parameters. When $c = 0$, Equation (23) is a two-parameter version of Equation (8) with delay.

An equilibrium of (23) satisfies

$$d\overline{x}^3 + c\overline{x}^2 + \overline{x} = ac\overline{x} + ad\overline{x}^2$$

so that all nonzero equilibria satisfy

$$d\overline{x}^2 + (c - ad)\overline{x} + (1 - ac) = 0, \tag{24}$$

whence we easily deduce the possible solutions

$$\overline{x}_{\pm} = \frac{ad - c \pm \sqrt{(c - ad)^2 + 4d(ac - 1)}}{2d},$$

which are real if and only if $R = (c - ad)^2 + 4d(ac - 1) \geq 0$.

Notice that

$$R \geq 0 \iff c^2 - 2acd + a^2d^2 + 4acd - 4d \geq 0 \iff (ad + c)^2 \geq 4d. \tag{25}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(c + 2d\overline{x})}{(1 + c\overline{x} + d\overline{x}^2)^2}.$$

Lemma 4 *Equation* (23) *always has the zero equilibrium* $\overline{x}_0 = 0$, *and*

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } ac < 1 \\ \text{nonhyperbolic} & \text{if } ac = 1 \\ \text{unstable} & \text{if } ac > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = ac$. □

Lemma 5 *Assume* $c > 0$ *and* $d > 0$.

(1) Suppose either

(a) $c \geq ad$ *and* $1 \geq ac$, *or*

(b) $c < ad$, $1 > ac$, *and* $R < 0$.

Then Equation (23) *has no positive equilibria.*

(2) Suppose either

(a) $1 < ac$, *or*

(b) $c < ad$ *and* $1 = ac$.

Then Equation (23) *has the positive equilibrium* $\overline{x}_+$, *and it is locally asymptotically stable.*

(3) Suppose $c < ad$, $1 > ac$, *and* $R = 0$. *Then Equation* (23) *has the positive equilibrium* $\overline{x}_\pm$, *and it is nonhyperbolic of stable type.*

(4) Suppose $c < ad$, $1 > ac$, *and* $R > 0$. *Then Equation* (19) *has the two positive equilibria* $\overline{x}_+$, *which is locally asymptotically stable, and* $\overline{x}_-$, *which is unstable.*

Let $K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d$.

(i) If $K < 0$, *then* $\overline{x}_-$ *is a saddle point.*

(ii) If $K > 0$, *then* $\overline{x}_-$ *is a repeller.*

(iii) If $K = 0$, *then* $\overline{x}_-$ *is nonhyperbolic of unstable type.*

Proof. The positivity of solutions of Equation (24) follows from Descartes' Rule of Signs. Much of the local stability analysis is symmetric to the considerations in the proof of Lemma 2. Notice that

$$\lambda(\overline{x}) = \frac{a(c+2d\overline{x})}{(1+c\overline{x}+d\overline{x}^2)^2} = \frac{a(c+2d\overline{x})}{(a(c+d\overline{x}))^2} = \frac{c+2d\overline{x}}{a(c+d\overline{x})^2} = \frac{1}{a(c+d\overline{x})} + \frac{d\overline{x}}{a(c+d\overline{x})^2}.$$

For the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned}
\lambda(\overline{x}_+) \leq 1 &\iff \frac{d\overline{x}_+}{a(c+d\overline{x}_+)^2} \leq \frac{a(c+d\overline{x}_+)-1}{a(c+d\overline{x}_+)} \\
&\iff d\overline{x}_+ \leq (c+d\overline{x}_+)\left(a(c+d\overline{x}_+)-1\right) = (c+d\overline{x}_+)(c\overline{x}_+ + d\overline{x}_+^2) \\
&\iff d \leq (c+d\overline{x}_+)^2 \\
&\iff 4d \leq (2c+2d\overline{x}_+)^2 = (ad+c+\sqrt{R})^2,
\end{aligned}$$

which is immediately true by Inequality (25). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (19) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $ady^2 - cy + c - ad = 0$, which has characteristic values $y_1 = 1$ and $y_2 = \frac{c-ad}{ad}$, where $-1 < y_2 < 0$ since $ad > c$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type.

When $\overline{x}_-$ exists,

$$\begin{aligned}
\lambda(\overline{x}_-) > 1 &\iff 4d > (ad+c-\sqrt{R})^2 \\
&\iff 4d + (ad+c)\sqrt{R} > (ad+c)^2 \\
&\iff (ad+c)\sqrt{R} > (ad+c)^2 - 4d = R \\
&\iff (ad+c)^2 > R = (ad+c)^2 - 4d
\end{aligned}$$

which is of course true since $d > 0$. To more specifically classify $\overline{x}_-$, we must calculate $\delta(\overline{x}_-)$. Notice

$$\delta(\overline{x}_-) = \frac{a(2d\overline{x}_- - c)}{(1+c\overline{x}_- + d\overline{x}_-^2)^2} = \frac{a(2d\overline{x}_- - c)}{(a(c+d\overline{x}_-))^2} = \frac{4(2d\overline{x}_- - c)}{a(2c+2d\overline{x}_-)^2} = \frac{4\left(ad-2c-\sqrt{R}\right)}{a(ad+c-\sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff 4\left(ad - 2c - \sqrt{R}\right) \gtrless a\left(ad + c - \sqrt{R}\right)^2 \\ &\iff (a(ad+c)-2)\sqrt{R} \gtrless a(ad+c)^2 - 4ad + 4c = aR + 4c.\end{aligned}$$

Notice that $R > 0$ automatically implies $a(ad+c) > 2$, as

$$\begin{aligned}0 &< (ad+c)^2 - 4d \\ &= a^2d^2 + 2acd + c^2 - 4d \\ &< a^2d^2 + 2acd + a^2d^2 - 4d \\ &= 2d\,(a(ad+c)-2)\end{aligned}$$

since $c < ad$. Therefore we may square both sides to obtain

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff (a(ad+c)-2)^2 R \gtrless (aR+4c)^2 \\ &\iff R\left(a^2(ad+c)^2 - 4a(ad+c) + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R\left(a^2R - 4ac + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R(1-3ac) - 4c^2 \gtrless 0 \\ &\iff a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d \gtrless 0.\end{aligned}$$

Thus if

$$K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d, \tag{26}$$

$K < 0$ implies $\overline{x}_-$ is a saddle point and $K > 0$ implies it is a repeller. If $K = 0$, $\overline{x}_-$ is nonhyperbolic, and we expect in such case it to be nonhyperbolic of unstable type. Indeed one can show that in the event $K = 0$, the characteristic equation of the linearization of Equation (23) about $\overline{x}_-$, $y^2 = F_u(\overline{x}_-, \overline{x}_-)y + F_v(\overline{x}_-, \overline{x}_-)$, has roots $y_1 = -1$ and $y_2 = F_u(\overline{x}_-, \overline{x}_-) + 1 > 1$, which immediately shows the desired result. $\square$

The investigation of the existence of periodic solutions of Equation (23) is an interesting one that involves a thorough analysis of potential parametric cases. This analysis will reveal the potential for the existence of several nonzero periodic solutions. The juxtaposition of Equation (19) with Equation (23) illustrates an interesting phenomenon in which, loosely speaking, the dominance of the delay term x_{n-1}^2 contributes to the possibility of periodic solutions arising.

A minimal period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi, \psi > 0$ and $\phi \neq \psi$ must satisfy

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} = \dfrac{a(c\phi + d\psi^2)}{1+c\phi+d\psi^2} \\ \\ \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} = \dfrac{a(c\psi + d\phi^2)}{1+c\psi+d\phi^2} \end{cases}. \tag{27}$$

Eliminating either ψ or ϕ from System (27) we obtain

$$\left(d\phi^2 + (c-ad)\phi + (1-ac)\right) h(\phi) = 0,$$

or

$$\left(d\psi^2 + (c-ad)\psi + (1-ac)\right) h(\psi) = 0,$$

where

$$\begin{aligned} h(x) = {} & -d^3x^6 + d^2(c+2ad)x^5 - d(c^2+2d+3acd+a^2d^2)x^4 \\ & + d(c+3ac^2+2ad+3a^2cd)x^3 - (c^2+ac^3+d+2acd+3a^2c^2d+a^3cd^2)x^2 \\ & + ac(1+ac)(2c+ad)x - a^2c^2(1+ac). \end{aligned} \tag{28}$$

Since $dx^2+(c-ad)x+(1-ac) \neq 0$ for any x that is not a solution of the equilibrium equation (24), minimal period-two solutions must be the solutions of the equation

$$h(x) = 0. \tag{29}$$

Lemma 6 *Any real solutions of Equation* (29) *are positive numbers for* $c, d > 0$, *and there exist up to 3 minimal period-two solutions of Equation* (23)*. Furthermore, let* K *be as defined in Equation* (26)*, and define the following expressions:*

$$J = 4a^5cd^4 - 8a^4c^2d^3 + 12a^3c^3d^2 - 24a^3cd^3 - 8a^2c^4d + 28a^2c^2d^2 - a^2d^3 + 4ac^5 + 4ac^3d + 32acd^2 + 4c^4 + 8c^2d + 4d^2$$

$$\Delta_1 = 6d^6$$

$$\Delta_2 = d^{10}\left(8a^2d^2 - 16acd - 7c^2 - 24d\right)$$

$$\Delta_3 = -2d^{12}\left(8a^5cd^5 + 13a^4c^2d^4 + 10a^3c^3d^3 - 44a^3cd^4 + 4a^2c^4d^2 - 34a^2c^2d^3 - 4a^2d^4 - 19ac^5d + 14ac^3d^2 + 44acd^3 + 6c^6 + 7c^4d + 5c^2d^2 + 16d^3\right)$$

$$\Delta_4 = c^2d^{13}\left(-16a^9cd^8 - 12a^8c^2d^7 + 24a^7c^3d^6 + 152a^7cd^7 - 68a^6c^4d^5 + 80a^6c^2d^6 + 8a^6d^7 + 48a^5c^5d^4 - 164a^5c^3d^5 - 464a^5cd^6 - 60a^4c^6d^3 + 20a^4c^4d^4 - 180a^4c^2d^5 - 64a^4d^6 + 56a^3c^7d^2 - 332a^3c^5d^3 + 388a^3c^3d^4 + 488a^3cd^5 - 48a^2c^8d + 272a^2c^6d^2 + 255a^2c^4d^3 + 152a^2c^2d^4 + 136a^2d^5 + 24ac^9 + 8ac^7d + 124ac^5d^2 + 180ac^3d^3 - 152acd^4 + 24c^8 + 68c^6d + 32c^4d^2 - 44c^2d^3 - 32d^4\right)$$

$$\Delta_5 = 2c^4d^{13}J\left(3a^8c^2d^6 + 2a^7cd^6 - 18a^6c^2d^5 - a^6d^6 + 6a^5c^5d^3 + 10a^5c^3d^4 - 8a^5cd^5 - 10a^4c^4d^3 + 44a^4c^2d^4 + 6a^4d^5 + 54a^3c^5d^2 - 25a^3c^3d^3 - 6a^3cd^4 + 3a^2c^8 - 8a^2c^6d + 35a^2c^4d^2 - 39a^2c^2d^3 - 9a^2d^4 + 6ac^7 + 2ac^5d + 4ac^3d^2 + 14acd^3 + 3c^6 + 10c^4d + 11c^2d^2 + 4d^3\right)$$

$$\Delta_6 = a^2c^6d^{14}(ac+1)KJ^2.$$

(1) If $\Delta_i > 0$ *for all* $2 \le i \le 6$ *then Equation* (29) *has six real roots. Consequently, Equation* (23) *has three minimal period-two solutions.*

(2) If $\Delta_j \le 0$ *for some* $2 \le j \le 5$ *and* $\Delta_i > 0$ *for* $i \ne j$*, then Equation* (29) *has two distinct real roots and two pairs of conjugate imaginary roots. Consequently,*

Equation (23) *has one minimal period-two solution.*

(3) If $\Delta_i \leq 0$, $\Delta_{i+1} \geq 0$ *(such that at least one of these is strict) for some* $2 \leq i \leq 4$, *and if* $\Delta_6 < 0$, *then Equation* (29) *has three pairs of conjugate imaginary roots. Consequently, Equation* (23) *has no minimal period-two solutions.*

Proof. The proof of the first statement follows from Descartes' Rule of Signs. Let disc(h) denote the 12×12 discrimination matrix as defined in [10]:

$$\operatorname{disc}(h) = \begin{bmatrix} a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{bmatrix}.$$

Here a_k equals the coefficient of the degree-k term of h as defined in Equation (28); that is, $a_6 = -d^3$, $a_5 = d^2(c + 2ad)$, $a_4 = -d(c^2 + 2d + 3acd + a^2d^2)$, $a_3 = d(c + 3ac^2 + 2ad + 3a^2cd)$, $a_2 = -(c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)$, $a_1 = ac(1 + ac)(2c + ad)$, and $a_0 = -a^2c^2(1 + ac)$. Let Δ_k denote the determinant of the submatrix of disc(h) formed by its first $2k$ rows and $2k$ columns for $k = 1, 2, \ldots, 6$. Then the values of Δ_k are listed above, and the veracity of the statements above may now be verified by employing Theorem 1 of [10]. Notice that $\Delta_1 > 0$ for all $d > 0$. $\square$

Remark 3 The parametric conditions discussed above do not exhaust all of the parametric space but cover a substantial region of parameters for which Equation (23) possesses hyperbolic dynamics.

We will use the sufficient conditions provided in Lemmas 4, 5, and 6 to realize some global dynamic scenarios provided in [3]. We will not investigate the dynamics of Equation (23) when it has one or no positive fixed point since in such cases the dynamics should be similar to the dynamics of Equation (19) discussed in Theorem 5. The following theorem relies on results from [3] and summarizes potential hyperbolic dynamic scenarios for Equation (23) in the event it possesses three fixed points and zero, one, or three pairs of hyperbolic period-two points. See also the statement and proof of Theorem 11 in [3].

Theorem 6 *Assume $0 < c < ad$ and $ac < 1$ such that $R > 0$.*

(i) If $\Delta_i > 0$ for all $2 \leq i \leq 6$ then Eq. (23) *has three equilibria $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, where $\overline{x}_0$ and $\overline{x}_+$ are locally asymptotically stable and $\overline{x}_-$ is a repeller, and three minimal period-two solutions $\{\phi_1, \psi_1\}$, $\{\phi_2, \psi_2\}$, and $\{\phi_3, \psi_3\}$. Here $(\phi_1, \psi_1) \prec_{ne} (\phi_2, \psi_2) \prec_{ne} (\phi_3, \psi_3)$, $\{\phi_1, \psi_1\}$ and $\{\phi_3, \psi_3\}$ are saddle points, and $\{\phi_2, \psi_2\}$ is locally asymptotically stable. The global behavior of Eq.* (23) *is described by Theorem 8 of [3]. For example, this happens for $a = 1$, $c = \frac{389}{2176}$, and $d = \frac{249}{64}$.*

(ii) If $\Delta_j \leq 0$ for some $2 \leq j \leq 5$ and $\Delta_i > 0$ for $i \neq j$, then Eq. (23) *has three equilibria $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, where $\overline{x}_0$ and $\overline{x}_+$ are locally asymptotically stable and $\overline{x}_-$ is a repeller, and one period-two solution $\{\phi_1, \psi_1\}$, which is a saddle point. The global behavior of Eq.* (23) *is described by Theorem 7 of [3]. For example, this happens for $a = 1$, $c = \frac{1}{5}$, and $d = \frac{237}{64}$.*

(iii) If $\Delta_i \leq 0$ and $\Delta_{i+1} \geq 0$ (such that at least one of these is strict) for some $2 \leq i \leq 4$, and if $\Delta_6 < 0$, then Eq. (23) *has three equilibria $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, where $\overline{x}_0$ and $\overline{x}_+$ are locally asymptotically stable and $\overline{x}_-$ is a saddle point, and no period-two solution. The global behavior of Eq.* (23) *is described by Theorem 5 of*

[3]. For example, this happens for $a = 1$, $c = \frac{493}{1024}$, *and* $d = \frac{157}{48}$.

Equation (23) exhibits global dynamics similar to that of Equation (10), which was investigated in [3]. Therefore, we pose the following conjecture.

Conjecture 1 *There exists a topological conjugation between the maps in Equations* (10) *and* (23).

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MANUSCRIPT 4

Higher-Order and Transcendental-Type Generalized Beverton-Holt Equations

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Abstract

We investigate generalized Beverton-Holt difference equations of order k of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + f(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1,$$

where f is a function nondecreasing in all arguments, $a > 0$, and $x_0, x_{-1}, \ldots, x_{1-k} \geq 0$ such that the solution is defined. We will discuss several interesting examples of such equations involving transcendental functions and present some general theory. In particular, we will analyze the global dynamics of the class of difference equations for which $f(x, \ldots, x)$ is chosen to be a concave function. Moreover, we give sufficient conditions to guarantee this equation has a unique positive and globally attracting fixed point.

4.1 Introduction and Preliminaries

Consider the following order-k difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + f(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1, \tag{1}$$

where f is a continuous function nondecreasing in all arguments, the parameter a is a positive real number, and the initial conditions $x_0, x_{-1}, \ldots, x_{1-k}$ are arbitrary nonnegative numbers such that the solution is defined. We assume f is never identically equal to the zero function.

Equation (1) is a generalization of the first-order Beverton-Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \tag{2}$$

where $a > 0$ and $x_{-1}, x_0 \geq 0$. Global dynamics are known and may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a \leq 1 \\ a - 1 & \text{if } a > 1 \text{ and } x_0 > 0. \end{cases}$$

Many variations of Equation (2) have been studied. The form of the model actually predates its use by Beverton and Holt; see [12]. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics.

For instance, the so-called Monod system of differential equations is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma} N \frac{rS}{a+S}, \quad \frac{dN}{dt} = N \frac{rS}{a+S}, \tag{3}$$

where $N(t)$ is the concentration of bacteria at time t, $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, k is a half-saturation constant, and the constant γ is called the growth yield; see [12]. Both Equation (2) and System (3) contain the function $f(x) = rx/(a+x)$ known as the Monod function, Michaelis-Menten function, Beverton-Holt function, or Holling function of the first kind; see [3, 9]. Some global dynamic scenarios of several two-generation models using this function were investigated in [2].

The Beverton-Holt function is an increasing and concave function and we will prove some global attractivity results for general difference equations with a transition function that is increasing and concave along the diagonal. More precisely, we will prove some global attractivity results for Equation (1), where $f(x, \ldots, x)$ is an increasing and concave function.

The following theorem from [1] applies to Equation (1) when $k = 2$.

Theorem 1 *Let I be a set of real numbers and $F : I \times I \to I$ be a function which is nondecreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = F\left(x_n, x_{n-1}\right), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots, \tag{4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (4) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. Notice that Theorem 1 does not apply if $k > 2$, but the results from [5, 7, 10] can give global dynamics in some regions of the parametric space. In the case $k > 2$, Equation (1) may have periodic solutions of different periods and even chaos; see [6].

The following theorem from [8] applies to the kth-order Equation (1) and will be instrumental in establishing our main result.

Theorem 2 *Consider the equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n+1-k}), \quad x_0, x_{-1}, \ldots, x_{1-k} \in I, \quad n = 0, 1, \ldots, \tag{5}$$

where $I \subseteq [0, \infty)$ is some open interval, and assume that $F \in C[I^k, (0, \infty)]$ satisfies the following conditions:

(i) F is nondecreasing in each of its arguments;

(ii) Equation (5) has a unique positive equilibrium point $\overline{x} \in I$ and the function F satisfies the ***negative feedback condition****:*

$$(x - \overline{x})(F(x, \ldots, x) - x) < 0 \text{ for every } x \in I \backslash \{\overline{x}\}.$$

Then every positive solution of Equation (5) with initial conditions $x_0, x_{-1}, \ldots, x_{1-k}$ in I converges to $\overline{x}$.

4.2 General Stability Results and Global Attractivity

Let the function $F : [0,\infty)^k \to [0,a)$ be defined as follows:

$$F(u_1,\ldots,u_k) = \frac{af(u_1,\ldots,u_k)}{1+f(u_1,\ldots,u_k)}. \tag{6}$$

Using Equation (6), Equation (1) may be rewritten as $x_{n+1} = F(x_n, x_{n-1},\ldots,x_{n+1-k})$ for all $n = 0,1,\ldots,$ where F is a nondecreasing function in all its variables. It is clear that $0 \leq x_n < a$ for all $n \geq 1$.

It will be useful to examine the multivariable functions f and F along the diagonal. For convenience, make the following definitions:

$$g(x) = f(x,\ldots,x) \tag{7}$$

$$G(x) = F(x,\ldots,x). \tag{8}$$

An equilibrium $\overline{x}$ of Equation (1) satisfies

$$\overline{x}\,(1+g(\overline{x})) = ag(\overline{x}). \tag{9}$$

Clearly $\overline{x}_0 = 0$ is an equilibrium point if and only if $g(0) = f(0,\ldots,0) = 0$.

4.2.1 Local Stability of an Equilibrium

The linearized equation of Equation (1) about an equilibrium $\overline{x}$ is

$$z_{n+1} = F_{u_1}(\overline{x},\ldots,\overline{x})z_n + \ldots + F_{u_k}(\overline{x},\ldots,\overline{x})z_{n+1-k}, \quad n = 0,1,\ldots.$$

Set

$$\lambda(\overline{x})_k = \sum_{i=1}^{k} F_{u_i}(\overline{x},\ldots,\overline{x}) = \frac{a\sum_{i=1}^{k} f_{u_i}(\overline{x},\ldots,\overline{x})}{(1+f(\overline{x},\ldots,\overline{x}))^2}. \tag{10}$$

In view of Corollary 2 in [10] we have the following result:

Theorem 3 *Let $\overline{x}$ be an equilibrium of Equation* (1)*. Then*

$$\overline{x} \textit{ is} \begin{cases} \textit{locally asymptotically stable} & \textit{if } \lambda(\overline{x})_k < 1 \\ \textit{nonhyperbolic} & \textit{if } \lambda(\overline{x})_k = 1 \\ \textit{unstable} & \textit{if } \lambda(\overline{x})_k > 1. \end{cases}$$

4.2.2 Existence and Global Attractivity of a Unique Positive Equilibrium

We will now establish several sufficient conditions under which Equation (1) will have a unique positive fixed point. Recall the definitions of G and g given in Equations (7) and (8).

Lemma 1 *Suppose G is twice differentiable and satisfies the following three conditions:*
(i) $G(0) = 0$,
(ii) $G'(0) > 1$, and
(iii) $G''(x) < 0$ for all $x \in (0, a)$.
Then Equation (1) *has a unique positive equilibrium.*

Remark 1 Notice that $G(0) = 0$ if and only if $g(0) = 0$. If indeed $G(0) = g(0) = 0$ then $G'(0) = ag'(0)$. Further, since $x \geq 0$, we interpret derivatives at zero in the right-handed sense.

Proof. First we will show that there exists a positive equilibrium for Equation (1). First, let $H(x) = G(x) - x$. Notice that $H(0) = 0$ and $H(a) < 0$, as

$$H(a) = G(a) - a = F(a, \ldots, a) - a = \frac{af(a, \ldots, a)}{1 + f(a, \ldots, a)} - a < a - a = 0.$$

Also, $H'(0) = G'(0) - 1 > 0$ by assumption (ii) and hence H is increasing at $x = 0$; by continuity of H', for any sufficiently small $\delta > 0$ it must be the case that $H(\delta) > 0$. But since $H(\delta) > 0$ and $H(a) < 0$, by the Intermediate Value Theorem there exists some point $p \in (\delta, a)$ such that $H(p) = 0$. But this immediately implies that $G(p) = p$, and hence p is a fixed point of Equation (1), as required.

Next we will show this fixed point is unique. Suppose there are two fixed points $p_1, p_2 > 0$ of Equation (1) such that $p_1 < p_2$. Since $G''(x) < 0$ for all

$x \in (0, a)$, the function is *strictly* concave on this interval; that is, for all $t \in (0,1)$ and all $x, y \in (0, \infty)$ with $x \neq y$,

$$G(tx + (1-t)y) > tG(x) + (1-t)G(y). \tag{11}$$

Let $b \in (0, p_1)$ be arbitrary and set $t = \frac{p_2 - p_1}{p_2 - b}$. Notice that $t \in (0,1)$ since $0 < b < p_1 < p_2$. By Inequality (11), if $x = b$ and $y = p_2$, we obtain the following:

$$\begin{aligned}
& G\left(t\, b + (1-t)p_2\right) > t\, G\left(b\right) + (1-t)G(p_2) \\
\iff & G\left(\left(\frac{p_2 - p_1}{p_2 - b}\right) b + \left(1 - \frac{p_2 - p_1}{p_2 - b}\right) p_2\right) > \left(\frac{p_2 - p_1}{p_2 - b}\right) G(b) + \left(1 - \frac{p_2 - p_1}{p_2 - b}\right) p_2 \\
\iff & p_1 = G(p_1) > \frac{G(b)\left(p_2 - p_1\right) + p_2\left(p_1 - b\right)}{p_2 - b} \\
\iff & p_1(p_2 - b) - p_2(p_1 - b) > G(b)(p_2 - p_1) \\
\iff & b > G(b).
\end{aligned}$$

Therefore for each $b \in (0, p_1)$, $H(b) = G(b) - b < 0$. However, this contradicts our initial claim that $H(\delta) > 0$ for $\delta > 0$ small enough and hence we have a contradiction. $\square$

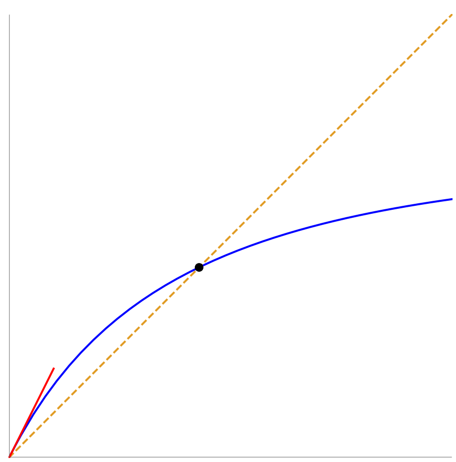

Figure 6. Illustration of Lemma 1, where $G'(0) > 1$.

Lemma 2 *Suppose G is twice differentiable and satisfies the following three conditions:*

(i) $G(0) = 0$*,*

(ii) $G'(0) \leq 1$*, and*

(iii) $G''(x) < 0$ *for all* $x \in (0, a)$*.*

Then there exists no positive fixed point for Equation (1)*.*

Proof. If $H(x) = G(x) - x$, then $H'(x) = G'(x) - 1$ and $H''(x) = G''(x)$, so in particular $H''(x) < 0$ for all $x \in (0, a)$. For any $x \in (0, a]$ we may apply the Mean Value Theorem to H' over $[0, x]$ to conclude that there exists some $c \in (0, x)$ such that

$$\frac{H'(x) - H'(0)}{x - 0} = H''(c).$$

But since $H''(c) < 0$, we have that $H'(x) < H'(0) \leq 0$ and hence H is strictly decreasing for all $x \in (0, a)$. But since $H(0) = 0$, we have that $H(x) < 0$ (and hence $G(x) < x$) for all $x \in (0, a)$, and therefore in this case there exist no positive fixed points for G. $\square$

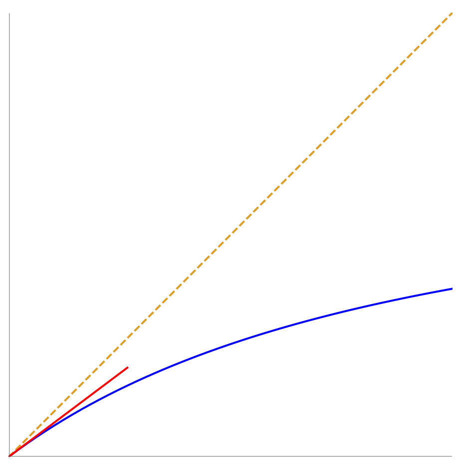

Figure 7. Illustration of Lemma 2, where $G'(0) \leq 1$.

Theorem 4 *Under the hypotheses of Lemma 1, the unique positive equilibrium of* (1) *is a global attractor of all solutions with positive initial conditions.*

Proof. By Lemma 1, Equation (1) has a unique positive fixed point p. Now $H(x) = G(x) - x$ is continuous and has only one positive root (at $x = p$) such that it does not change sign on $(0,p)$ or (p,a); in particular, $H(x)>0$ for $x\in(0,p)$ and $H(x)<0$ for $x\in(p,a)$. If $I=(0,a)$, we have that indeed $(x-p)(G(x)-x)<0$ for all $x\in I\backslash\{p\}$. By Theorem 2, we have that every positive solution with initial conditions in I converges to p. Since $(0,a)$ is an attracting, invariant interval for all solutions with positive initial conditions, the proof is complete. □

Remark 2 If $(0,a)$ is an attracting interval for *all* nonzero solutions, including those with initial conditions that are not all necessarily positive, then the results of Theorem 4 (and later Corollary 1) will give a complete classification of global dynamics for any choice of initial conditions.

Theorem 5 *Under the hypotheses of Lemma 2, the zero equilibrium is a global attractor of all solutions.*

Proof. By Lemma 2, Equation (1) has only the zero equilibrium in the invariant interval $[0,a]$. But then the kth-order extension of Theorems 1.4.8 and A.0.1 of [11] will apply to this equation. Since this interval is attracting, all solutions must converge to the zero equilibrium. □

Corollary 1 *Suppose $g(x)$ is a strictly concave function on $(0,a)$.*

(1) Under hypotheses (i) and (ii) of Lemma 1, the unique positive equilibrium of Equation (1) is a global attractor of all solutions with positive initial conditions. (2) Under hypotheses (i) and (ii) of Lemma 2, the zero equilibrium is a global attractor of all solutions.

Proof. Since $g(x)=f(x,\ldots,x)$ is strictly concave for $0<x<a$, $g''(x)<0$. An immediate computation yields

$$G''(x)=\frac{a\left[g''(x)\,(1+g(x))-2\,(g(x))^2\right]}{(1+g(x))^3}<0.$$

Thus condition (iii) is satisfied for Lemmas 1 and 2, and the proof follows from an application of Theorems 4 and 5. □

Remark 3 Corollary 1 shows that $g(x)$ being concave is a sufficient but not necessary condition for $G(x)$ to be concave. For the case $k = 2$, consider $f(u,v) = pu^2 + qv$. If $a = 1, p = 1, q = 2$, then

$$g''(x) = \frac{d^2}{dx^2}\left(f(x,x)\right) = 2 > 0 \text{ yet } G''(x) = \frac{d^2}{dx^2}\left(F(x,x)\right) = \frac{-6}{(1+x)^4} < 0,$$

so for these values $G(x)$ is concave even though $g(x)$ is convex.

Despite the utility of the above results, there certainly exist scenarios in which neither the function $g(x)$ nor $G(x)$ is concave on the interval $(0,a)$. In such situations it is useful to have the following theorem, which provides a sufficient condition to guarantee the existence (or nonexistence) of a unique positive fixed point that is a global attractor of positive solutions..

Theorem 6 *Let $g(x) > 0$ for all $x > 0$. If*

$$xg'(x) < g(x)\left(g(x) + 1\right) \tag{12}$$

for all $x \in (0,a)$, then Equation (1) has at most one positive fixed point.
(1) If $G(0) = g(0) = 0$ and $G'(0) = ag'(0) > 1$, then Equation (1) has precisely one positive fixed point, and it is a global attractor of all solutions with positive initial conditions.
(2) If $G(0) = g(0) = 0$ and $G'(0) = ag'(0) \leq 1$, then Equation (1) has only the zero equilibrium, and it is a global attractor of all solutions.

Proof. Solve Equation (9) for a to find that

$$a = \frac{x}{g(x)} + x.$$

Set $u(x) = \frac{x}{g(x)} + x$. If $u(x)$ is an injective (or monotone) function, then it intersects the line $y = a$ at most once. Setting $u'(x) > 0$ and rearranging will establish the main claim.

To prove the remaining claims, suppose $u'(x) > 0$. By l'Hôpital's Rule,

$$\lim_{x\to 0^+} u(x) = \lim_{x\to 0^+} \frac{x}{g(x)} + x = \lim_{x\to 0^+} \frac{1}{g'(x)} = \frac{1}{g'(0)}. \tag{13}$$

Now $\lim\limits_{x\to 0^+} u(x) < a$ implies there exists exactly one positive fixed point of Equation (1), so Equation (13) establishes the hypothesis of (1). As in the proof of Theorem 4, the global attractivity of the unique fixed point will again follow from Theorem 2.

If $\lim\limits_{x\to 0^+} u(x) \geq a$, then Equation (1) has only the zero equilibrium since u is increasing, and Equation (13) establishes the hypothesis of (2). Again we may employ the order-k generalization of Theorems 1.4.8 and A.0.1 of [11] to obtain the global attractivity of the zero equilibrium, and the proof is complete. □

Remark 4 In some cases the veracity of Inequality (12) of Theorem 6 may imply the concavity condition required by Theorems 4 or 5 or Corollary 1, but the hypotheses of the latter results may be easier to verify.

4.3 Examples

In most of the provided examples we will focus on equations of second order for concision. However, all results can be generalized to corresponding equations of any order.

4.3.1 Exponential: $f(u,v) = p(1-e^{-u}) + q(1-e^{-v})$

We consider the equation

$$x_{n+1} = \frac{a\left(p(1-e^{-x_n}) + q(1-e^{-x_{n-1}})\right)}{1 + p(1-e^{-x_n}) + q(1-e^{-x_{n-1}})}, \quad n = 0, 1, \ldots, \tag{14}$$

where $p, q > 0$. An equilibrium $\overline{x}$ of Equation (14) satisfies the following:

$$\overline{x} = \frac{a(p+q)(1-e^{-\overline{x}})}{1+(p+q)(1-e^{-\overline{x}})}.$$

In particular, $\overline{x}_0 = 0$ has $\lambda(\overline{x}_0) = a(p+q)$, so

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } a(p+q) < 1 \\ \text{nonhyperbolic} & \text{if } a(p+q) = 1 \\ \text{unstable} & \text{if } a(p+q) > 1. \end{cases}$$

The following results give the global dynamics of Equation (14).

Theorem 7 *(1) If $a(p+q) > 1$, then there exists a unique positive equilibrium $\overline{x}_+$, and it is a global attractor of all nonzero solutions.*
(2) If $a(p+q) \leq 1$, then $\overline{x}_0 = 0$ is a global attractor of all solutions.

Proof. Notice $G(0) = 0$, $G'(0) = ag'(0) = a(p+q)$, and $g''(x) = \frac{d^2}{dx^2}\left(f(x,x)\right) = -e^{-x}(p+q) < 0$. Moreover, if $x_{-1}+x_0 > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ since $p, q > 0$, and so also must x_2 be positive. Thus all solutions enter the attracting, invariant interval $(0, a)$. In view of Remark 2, the result follows by a direct application of Corollary 1. □

We may also consider the kth-order equation

$$x_{n+1} = \frac{a\sum_{i=0}^{k-1} p_i(1-e^{-x_{n-i}})}{1+\sum_{i=0}^{k-1} p_i(1-e^{-x_{n-i}})}, \quad n = 0, 1, \ldots, \tag{15}$$

where $p_i \geq 0$ for $i = 0, \ldots, k-1$. We can establish global results for Equation (15) by immediately applying Corollary 1.

Theorem 8 *(1) If $a\sum_{i=0}^{k-1} p_i > 1$, then there exists a unique positive equilibrium $\overline{x}_+$, and it is a global attractor of all solutions with positive initial conditions.*
(2) If $a\sum_{i=0}^{k-1} p_i \leq 1$, then $\overline{x}_0 = 0$ is a global attractor of all solutions.

However, notice that we cannot necessarily establish global dynamics for all values of the nonnegative parameters and initial conditions. Equation (15) may have a variety of periodic solutions in which some of the entries in the periodic cycle equal zero. However, the above result captures the substantial global dynamics for all solutions with positive initial conditions.

4.3.2 Inverse Tangent: $f(u,v) = p\arctan(u) + q\arctan(v)$

We next consider the equation

$$x_{n+1} = \frac{a\left(p\arctan(x_n) + q\arctan(x_{n-1})\right)}{1 + p\arctan(x_n) + q\arctan(x_{n-1})}, \quad n = 0, 1, \ldots, \tag{16}$$

where $p, q > 0$. An equilibrium $\overline{x}$ of Equation (16) satisfies the following:

$$\overline{x} = \frac{a(p+q)\arctan(\overline{x})}{1 + (p+q)\arctan(\overline{x})}.$$

Again $\overline{x}_0 = 0$ has $\lambda(\overline{x}_0) = a(p+q)$, so

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } a(p+q) < 1 \\ \text{nonhyperbolic} & \text{if } a(p+q) = 1 \\ \text{unstable} & \text{if } a(p+q) > 1. \end{cases}$$

Notice that, as in the previous second-order example, $G(0) = 0$, $G'(0) = a(p+q)$, and $g''(x) = \frac{d^2}{dx^2}\left(f(x,x)\right) = -\frac{2x(p+q)}{(1+x^2)^2} < 0$. It is clear that the global dynamics of Equation (16) are described exactly by Theorem 7.

Remark 5 There are a wealth of other functions f such that $g(x)$ is concave and Corollary 1 applies to Equation (1). Second-order examples include the logarithmic function $f_1(u,v) = \log\left((1+u)^p(1+v)^q\right)$ as well as the shifted sigmoid function $f_2(u,v) = \frac{p}{1+e^{-u}} + \frac{q}{1+e^{-v}} - \frac{p+q}{2} = \frac{p}{2}\tanh(\frac{u}{2}) + \frac{q}{2}\tanh(\frac{v}{2})$.

4.3.3 Trigonometric: $f(u,v) = p\left(u + \sin(u)\right) + q\left(v + \sin(v)\right)$

Consider the equation

$$x_{n+1} = \frac{a\left(p\left(x_n + \sin(x_n)\right) + q\left(x_{n-1} + \sin(x_{n-1})\right)\right)}{1 + p\left(x_n + \sin(x_n)\right) + q\left(x_{n-1} + \sin(x_{n-1})\right)}, \quad n = 0, 1, \ldots, \tag{17}$$

where $p, q > 0$. Notice that $f_u(u, v) = p(1 + \cos(u)) \geq 0$ and $f_v(u, v) = q(1 + \cos(v)) \geq 0$. The second-order difference equation $x_{n+1} = \frac{1}{2} f(x_n, x_{n-1})$ for $p = q = 1$ was investigated in Example 1 of [4].

The applicability of Corollary 1 is limited by the fact that $g(x) = (p + q)(\sin(x) + x)$ is strictly concave only when $\sin(x) > 0$, and therefore global results can only be obtained for $a \leq \pi$. Using the full strength of Theorems 4 and 5 will also have limitations for any choice of $a > 0$; the interval $[0, a]$ is always invariant for Equation (17), but a larger value of a would prescribe the need for a larger interval over which $G(x)$ should be concave. Instead we may consider applying Theorem 6.

Theorem 9 *Suppose that, for all $x \in (0, a)$,*

$$x \cos(x) < (p + q)(\sin(x) + x)^2 + \sin(x). \tag{18}$$

(1) If $2a(p + q) > 1$, then Equation (17) has precisely one positive fixed point, and it is a global attractor of all solutions with positive initial conditions.
(2) If $2a(p + q) \leq 1$, then Equation (17) has only the zero equilibrium, and it is a global attractor of all solutions.

Remark 6 Verifying Inequality (18) in general appears to be difficult, although for specific values of p and q this hypothesis should be able to be easily checked. For example, if $p = q = 1$, this condition is immediately satisfied and leads to a global exchange of stability result as a passes through the critical value $\frac{1}{4}$. In general, *Mathematica* verifies this inequality should hold for all $x > 0$ when approximately $p + q > 0.2015$. For p and q smaller than this threshold, multiple equilibria or even interior periodic solutions may exist.

4.3.4 Order-k Linear

Consider the equation

$$x_{n+1} = \frac{a \sum_{i=0}^{k-1} c_i x_{n-i}}{1 + \sum_{i=0}^{k-1} c_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{19}$$

where $c_i \geq 0$ for $i = 1, \ldots, k-1$. An equilibrium $\overline{x}$ of Equation (19) satisfies the following:

$$\overline{x} = \frac{a \sum_{i=0}^{k-1} c_i \overline{x}}{1 + \sum_{i=0}^{k-1} c_i \overline{x}}.$$

Using Equation (10) we see that $\overline{x}_0 = 0$ has $\lambda(\overline{x}_0)_k = a \sum_{i=0}^{k-1} c_i$, so

$$\overline{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a \sum_{i=0}^{k-1} c_i < 1 \\ \text{nonhyperbolic} & \text{if } a \sum_{i=0}^{k-1} c_i = 1 \\ \text{unstable} & \text{if } a \sum_{i=0}^{k-1} c_i > 1. \end{cases}$$

If $a \sum_{i=0}^{k-1} c_i > 1$, then Equation (19) has the unique positive equilibrium

$$\overline{x}_+ = \frac{a \left(\sum_{i=0}^{k-1} c_i \right) - 1}{\sum_{i=0}^{k-1} c_i}.$$

Since

$$\lambda(\overline{x}_+)_k = \frac{1}{a \sum_{i=0}^{k-1} c_i} < 1,$$

we have that $\overline{x}_+$ is locally asymptotically stable whenever it exists.

Theorem 10 *(1) If $a \sum_{i=0}^{k-1} c_i \leq 1$, then $\overline{x}_0$ is a global attractor of all solutions.*

(2) If $a \sum_{i=0}^{k-1} c_i > 1$, then $\overline{x}_+$ is a global attractor of all solutions with positive initial conditions.

Proof. (1) The proof is the same as that of Theorem 5.

(2) Notice that

$$\begin{aligned}
x_{n+1} - \overline{x}_+ &= \frac{a \sum\limits_{i=0}^{k-1} c_i x_{n-i}}{1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}} - \frac{a \sum\limits_{j=0}^{k-1} c_j - 1}{\sum\limits_{j=0}^{k-1} c_j} \\
&= \frac{a \sum\limits_{j=0}^{k-1} c_j \sum\limits_{i=0}^{k-1} c_i x_{n-i} - \left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right)\left(a \sum\limits_{j=0}^{k-1} c_j - 1\right)}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j} \\
&= \frac{\sum\limits_{i=0}^{k-1} c_i x_{n-i} - \left(a \sum\limits_{j=0}^{k-1} c_j - 1\right)}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j} \\
&= \frac{\sum\limits_{i=0}^{k-1} c_i \left(x_{n-i} - \overline{x}_+\right) + \sum\limits_{j=0}^{k-1} c_j \left(\overline{x}_+ - a\right) + 1}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j} \\
&= \frac{\sum\limits_{i=0}^{k-1} c_i \left(x_{n-i} - \overline{x}_+\right)}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j}.
\end{aligned}$$

Make the substitution $y_n = x_n - \overline{x}_+$ to obtain

$$y_{n+1} = \sum_{l=0}^{k-1} \left(\frac{c_l}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j} \cdot y_{n-l} \right).$$

Let

$$g_l = \frac{c_l}{\left(1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}\right) \sum\limits_{j=0}^{k-1} c_j}$$

to see that

$$\sum_{l=0}^{k-1} |g_l| = \frac{1}{1 + \sum\limits_{i=0}^{k-1} c_i x_{n-i}} \le \frac{1}{1+M} < 1$$

for some $M > 0$ so long as $\sum_{i=0}^{k-1} c_i x_{n-i} > 0$. The latter is true by assumption since $c_i > 0$ for at least one i and the initial conditions satisfy $x_{1-j} > 0$ for each $j = 1, \ldots, k$. By Theorem 1 of [10], $\lim_{n\to\infty} y_n = 0$, and hence $\lim_{n\to\infty} x_n = \overline{x}_+$. $\square$

Remark 7 Theorem 10 is proven using the powerful linearization technique discussed in [10]. However, we could also use Theorems 4 and 5 to arrive at the same result.

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