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Beam Position Transient Response in the Controlled Optical Beam Waveguide

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BEAM POSITION TRANSIENT RESPONSE

IN THE CONTROLLED OPTICAL

BEAM WAVEGUIDE

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ABSTRACT

Beam position transients in self-aligning beam waveguides are studied. The control system senses the beam position at each lens and introduces a correction to the transverse position of the preceding lens. The Laplace Transform of the beam position response at some $(k + n)^{th}$ lens as a result of a disturbance of the k^{th} lens is found. The special case of the confocal waveguide using the integrator with gain as a control results in a time function of the beam position at the lenses equal to the product of a decaying exponential and Laguerre Polynomials. The second-order controlled waveguide has been simulated with the results showing that overshoot can be controlled by increasing the damping term in the control. An alternative control structure is presented which isolates the disturbed element and its correction from the remaining controls resulting in the simple exponential decay of the path perturbation using the integrator with gain as the control.

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PREFACE

This thesis is prepared under the MANUSCRIPT THESIS PLAN as set forth in the STATEMENT ON THESIS PREPARATION published by the Graduate School, University of Rhode Island, 5/17/72.

The work is divided into two sections. Section I contains a manuscript covering the research which has been submitted for publication. Section II consists of several Appendices which more fully develop the derivations indicated in the manuscript.

An additional Appendix (C) has been added which outlines an alternative to the form of control developed in the main body of the work. This material was arrived at too late for inclusion in the original manuscript but is of sufficient significance to warrant its inclusion here.

I

TRANSIENTS IN BEAM POSITION CONTROL SYSTEMS FOR OPTICAL WAVEGUIDES

1. Introduction

Beam waveguides provide a low loss medium for the transmission of energy and information at infrared and optical frequencies.[1, 2, 3] The advantage of these wavelengths for communications systems, extreme bandwidth and high information capacity, will be important for future transmission systems owing to the ever-increasing demands placed on links by computer data, video, business and personal transmissions.

An inherent problem involved in a beam waveguide composed of discrete elements (lenses or reflectors) is instability of the transmission path. This is due to random displacements of the elements leading to eventual departure of the beam from the guide axes.[4, 5]

This paper analyzes the performance of linear self-aligning beam waveguides. The linear correction system has been described and shown to be stable in time.[6] An understanding of the transient response of this system is necessary to determine its suitability and practicality.

2. The system

The beam waveguide is composed of discrete guidance elements spaced uniformly along the guide axis. The guidance elements are lenses or are equivalent to a lens of focal length (f). They are separated by distance (d), (Figure 1).

The trajectory of the beam center in the guide is described by ray optics. The transmission matrix for a lens relates the paraxial ray position and slope, leaving an individual lens, to position and slope entering. For a lens whose optical center is not displaced from the guide axis,[7]

$$1)\quad \begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

where the ray positions r_{out} and r_{in} and the ray slopes r'_{out} and r'_{in} are measured relative to the guide axis.

If the optical center of the lens is displaced an amount S, the emergent ray position is modified as follows;

$$2)\quad \begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{f} \end{bmatrix} S$$

The transmission through the basic unit cell of the waveguide is found by relating the ray at the n^{th} reference plane located immediately before the n^{th} lens to the ray at the $(n + 1)^{th}$ reference plane immediately before the next lens;

$$3)\quad \begin{bmatrix} r_{n+1} \\ r'_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_{n} \\ r'_{n} \end{bmatrix} + \begin{bmatrix} \frac{d}{f} \\ \frac{1}{f} \end{bmatrix} S_n$$

3. General description and analysis

To control the beam position at each lens, the system monitors the beam position at each reference plane. This signal is used to correct the alignment of the previous lens. With the system initially at rest, the Laplace Transform S_n, of the total displacement of the

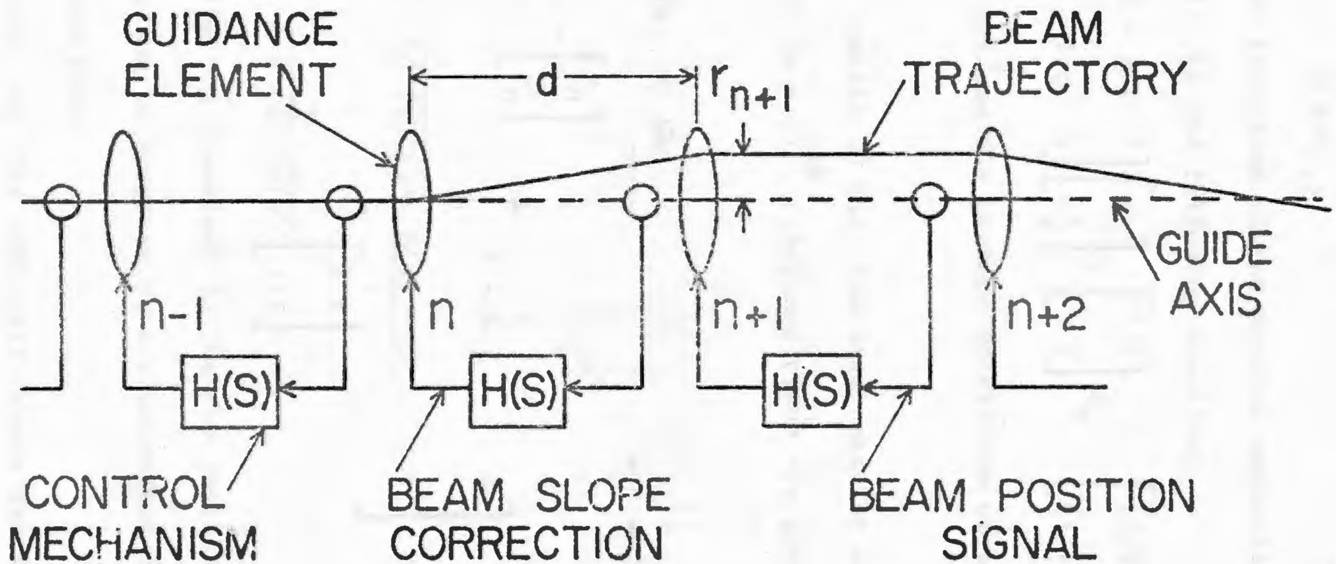

Figure 1. The self-aligning beam waveguide.

n^{th} lens is the sum of the transform of the disturbance d_n and the transform of the correction

4) $S_n = d_n + H(s)r_{n+1}$

where H(s) is the transform of the control mechanism.

This results in the following equation;

5) $$\begin{bmatrix} r_{n+1} \\ r'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - d/f & d \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} r_n \\ r'_n \end{bmatrix} + \begin{bmatrix} d/f \\ 1/f \end{bmatrix} d_n + \begin{bmatrix} (d/f)H(s) \\ (1/f)H(s) \end{bmatrix} r_{n+1}$$

where r_n and r'_n are now the Laplace Transforms of the beam position and slope.

It follows from equation 5 that the beam position and slope transforms at the $(n + 1)^{th}$ reference plane are given by the matrix equation;

6) $R_{n+1} = TR_n + GD_n$

7) where $R_n = \begin{bmatrix} r_n \\ r'_n \end{bmatrix}$

8) $$T = \frac{1}{1 - (d/f)H(s)} \begin{bmatrix} 1 - d/f & d \\ \dfrac{H(s) - 1}{f} & 1 \end{bmatrix}$$

9) and $$G = \frac{1}{1 - (d/f)H(s)} \begin{bmatrix} d/f \\ 1/f \end{bmatrix}$$

To determine the transient in the ray position at some arbitrary reference plane in response to a disturbance in the position of any lens we assume that:

1) Initially (at time t=0), all lenses are aligned on the reference axis and the ray is on axis at all reference planes;

10) $R_n = 0$ for all n at $t=0$

2) We assume an input lens disturbance from the axis at some

k^{th} lens;

11) $$D_n = \begin{cases} D_k(s) & n = k \\ 0 & n \neq k \end{cases}$$

It follows from equation 6 that

12) $$R_{k+n}(s) = T(s)^{n-1}\, G(s)\, D_k(s)$$

or, in matrix form;

13) $R_{k+n}(s) =$

$$\begin{bmatrix} \frac{1 - (d/f)}{1 - (d/f)H(s)} & \frac{d}{1 - (d/f)H(s)} \\ \frac{H(s) - 1}{f(1 - (d/f)H(s))} & \frac{1}{1 - (d/f)H(s)} \end{bmatrix}^{n-1} \begin{bmatrix} \frac{d/f}{1 - (d/f)H(s)} \\ \frac{1/f}{1 - (d/f)H(s)} \end{bmatrix} D_k(s)$$

Solving this relationship for the ray position component of the vector R we have;[8]

14) $r_{k+n}(s) =$

$$(d/f)\, D_k(s) \left(\frac{1}{1 - (d/f)H(s)} \right)^{\frac{n+1}{2}} \left(\frac{(2 - (d/f))\, U_{n-2}(a)}{(1 - (d/f)H(s))^{1/2}} - U_{n-3}(a) \right)$$

where the $U_m(a)$ are the Chebyshev Polynomials of the second kind with

15) $$a = \frac{2 - (d/f)}{2(1 - (d/f)H(s))^{1/2}}$$

Using the recursion relation for $U_{n-3}(a)$, and combining terms we obtain the Laplace relationship between a disturbance at some k^{th} lens and the ray position response at the $(k + n)^{th}$ reference plane;[9]

16) $$r_{k+n}(s) = (d/f)\, D_k(s) \left(\frac{1}{1 - (d/f)H(s)} \right)^{\frac{n+1}{2}} U_{n-1}(a)$$

where $$a = \frac{2 - (d/f)}{2(1 - (d/f)H(s))^{1/2}}$$

In order to find the ray position response as a function of time it is necessary to expand the Chebyshev Polynomial using the expansion summation[10]

17) $$U_m(x) = \sum_{m=0}^{n/2} \binom{n-m}{m} (-1)^m (2x)^{n-2m}$$

The time domain ray position response is

18) $$r_{k+n}(t) = (d/f) \sum_{m=0}^{\frac{n-1}{2}} \binom{n-1-m}{m} (-1)^m (2-(d/f))^{n-1-2m} \left(\frac{D_k(s)}{(1-(d/f)H(s))^{n-m}} \right)$$

In this equation is inserted the particular form of H(s) to be evaluated and the ratio of spatial separation of lenses to focal length, (d/f). The Laplace Transform can be inverted for particular values of d/f and transfer function H(s). This results in a time domain response at any lens expressed in a closed summation.

4. Special cases of the time response

Owing to the form of the time response of the ray position (equation 18) a case of immediate interest is the confocal waveguide where focal length and spatial separation of the lenses are related by

19) $$(d/f) = 2$$

In this case only one term of the summation remains, that term for which n-1-2m=0, and in this case the response of the ray position at an arbitrary $(k+n)^{th}$ lens for a disturbance at the k^{th} lens is

20) $$r_{k+n}(s) = \begin{cases} \dfrac{2\,(-1)^{\frac{n-1}{2}}}{(1-2H(S))^{\frac{n+1}{2}}} D_k(s) & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

If in this case the control function is taken to be an integrator with gain;

21) $$H(s) = \frac{-G}{s}$$

and the input is a unit step disturbance at the k^{th} lens, we find for the time domain response at the $(k+n)^{th}$ reference plane;

$$22)\quad r_{k+n}(t) = \begin{cases} 2\,(-1)^{\frac{n-1}{2}}\, e^{-2Gt}\, L_{\frac{n-1}{2}}(2Gt) & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

where $L_{\frac{n-1}{2}}(2Gt)$ is the Laguerre Polynomial defined by the explicit expansion[11]

$$23)\quad L_m(x) = \sum_{n=0}^{m} \binom{m}{m-n} \frac{1}{n!}\,(-x)^n$$

The step response is of interest because it is identical to the response that results from initial misalignment. It is likely that the control system will not operate continuously but will be turned on periodically and will respond to initial misalignment.

The response given by equation 22 is plotted in Figure 2 out to the 13^{th} lens. The beam position is bounded and decays rapidly to the axis. This result was cross checked by simulating the guide on a digital computer.

The case of the control function

$$24)\quad H(s) = \frac{-\omega_n^2}{2s\,(s - 2\xi\omega_n)}$$

is also of practical interest. It represents a second order control correcting at each lens. The results of simulating a guide employing this control function at each lens are shown in Figures 3 and 4. In both of these figures the response is to a step disturbance at the zeroth lens. The simulations show that undesirable overshoot results when the control system is underdamped. The response at the

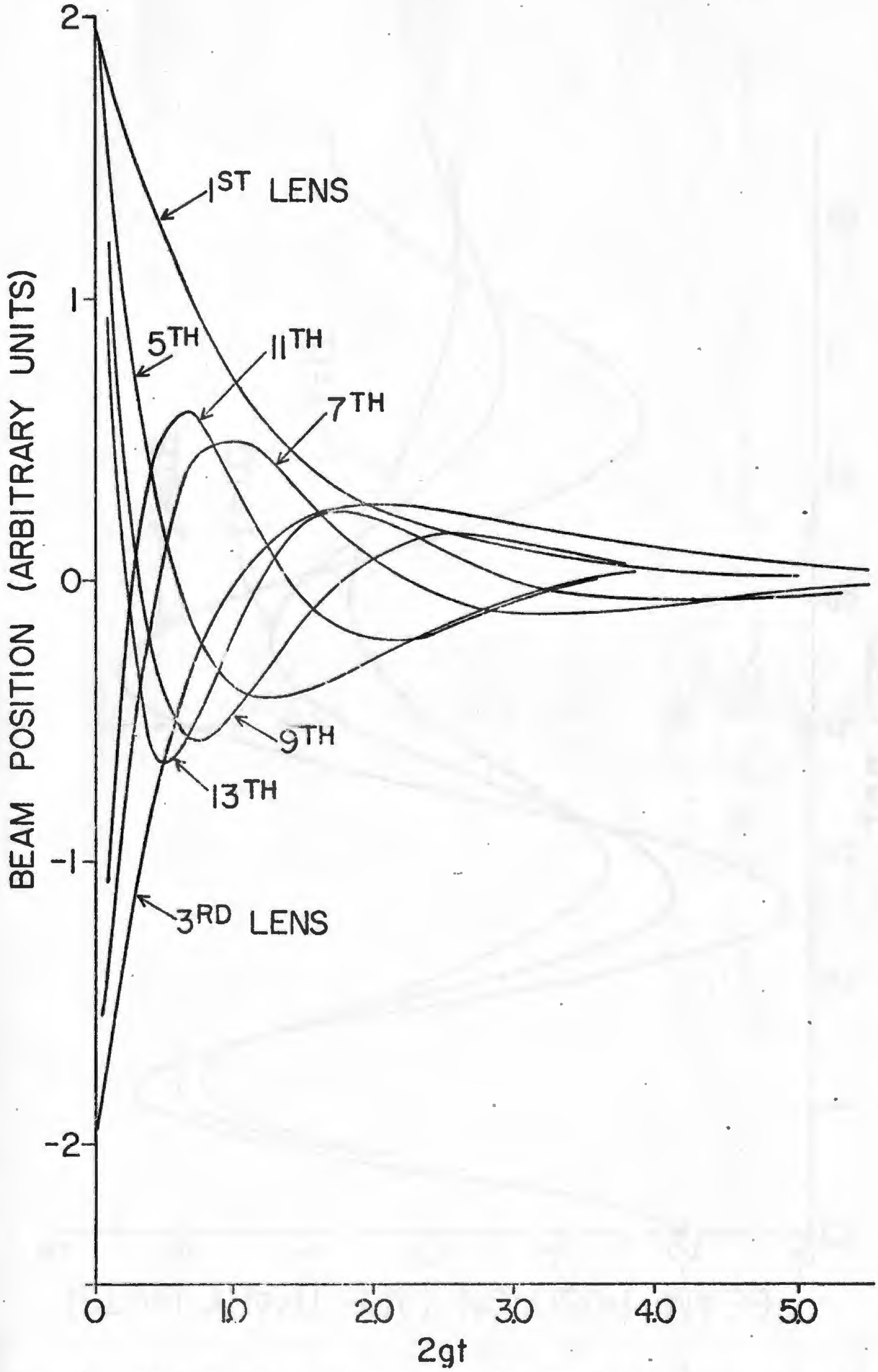

Figure 2. Ray position response at the seven odd lenses (up to the thirteenth) following a unit step disturbance of the zeroth lens for a control function of $H(s) = -G/s$.

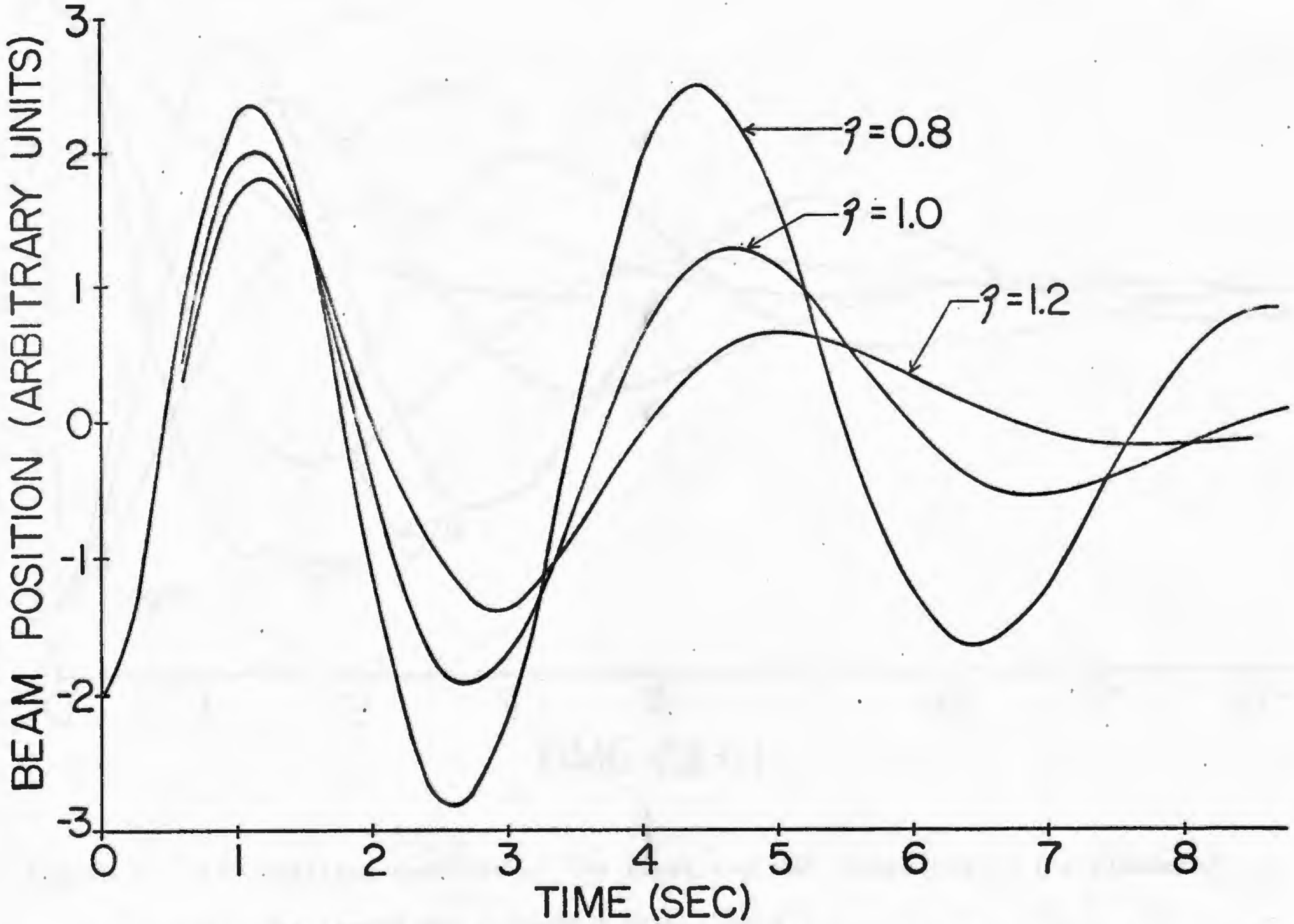

Figure 3. Ray position response at the eleventh lens for the second order control; $\omega_n = 1.0$.

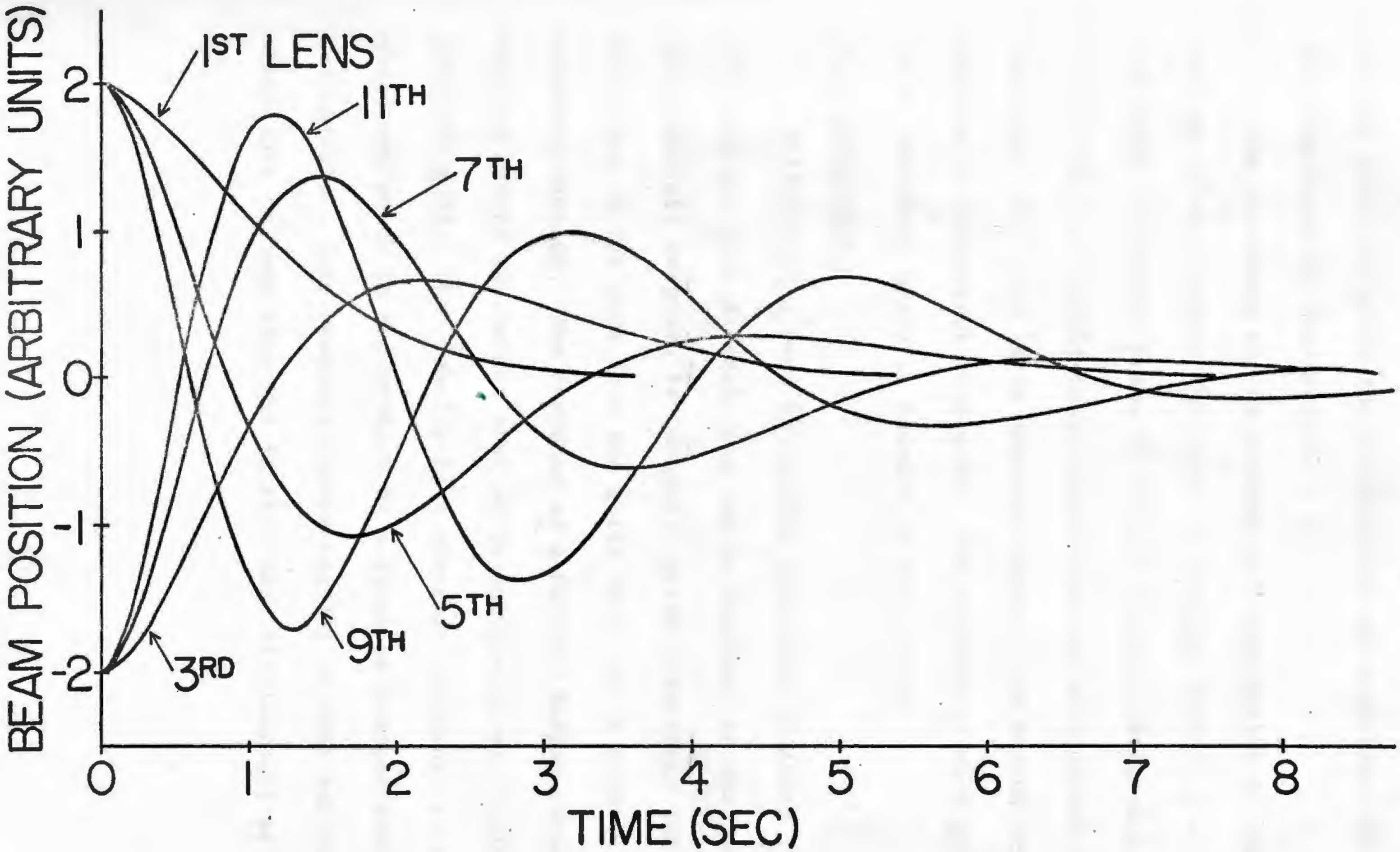

Figure 4. Ray position response at the first six odd lenses (up to the eleventh) for the second order control with $\zeta = 1.2$.

first lens is the classic second order system response. At lenses further down the guide the oscillation and overshoot increase with distance from the disturbance.

The overshoot can be reduced for long guides by increasing the damping of the control function. A damping factor, $\zeta = 1.2$, allows the guide of eleven lenses to correct a step disturbance without having the beam displacement exceed displacement caused by the disturbance. For very large damping factor ζ the second order control becomes an integrator with gain. The integrator with gain results in an overshoot which is bounded at all lenses.

5. Conclusion

Self-aligning beam waveguides employing feedback to stabilize the beam position at each lens can be designed so that overshoot in the transient response is bounded. Large transients and excessive departure of the beam from the guide axis can be eliminated by increasing damping. The response of a heavily damped second-order control system approaches that of an integrator with gain. In a confocal guide, this results in a transient response to a step displacement which is the product of a decaying exponential and Laguerre Polynomials. This response decays rapidly to zero and has an overshoot that is less than the initial beam displacement at each lens.

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(10) National Bureau of Standards, AMS-55, 1966, Equation 22.3.7.

(11) National Bureau of Standards, AMS-55, 1966, Equation 22.3.9.

II

APPENDIX A

INTRODUCTION AND REVIEW

The well recognized "communication explosion" which has been going on for some time will soon reach the point where available transmission equipment will not be capable of handling the ever increasing traffic. Computer data, television and rapidly expanding personal and business transmissions (both audio and video with the implementation of the "picture 'phone" now in service in some areas) are already taxing the ingenuity of communications engineers to provide sufficient bandwidth to accommodate present traffic, and future requirements will present even greater challenges to technology unless a transmission system with vastly increased information carrying capacity is exploited.

Laser carrier transmission appears to fill this need and methods are currently available both for modulation and demodulation of the coherent beam output of the basic device. The literal "missing link" at the present time however, is a suitable and dependable transmission path to assure uninterrupted propagation of the modulated beam over long distances from transmitter to receiver.

Atmospheric transmission is not feasible due to scatter and random disturbances disrupting the beam path (haze, rain, dust, etc.).

Optical fibers are useful for short distances but the high attenuation factor eliminates this technique for applications requiring transmission over long paths without the use of an excessive number of amplifiers or repeaters. An alternative which is not subject to the objections noted is the beam waveguide.

Originally developed for the millimeter and submillimeter wave range and then extended to the optical region of the spectrum, the beam waveguide is based on the principle of periodically restoring the field distribution of the propagated beam of energy by means of physical elements which function as phase transformers. These devices are dielectric elements with a quadratic profile which reset the cross-sectional phase distribution in the beam at regular intervals along the guiding structure. The individual elements are, for optical wavelengths, positive lenses or pairs of reflectors which yield a phase correction identical to a single lens.[1, 2, 3]

In terms of propagation losses, the beam waveguide is a most attractive means of transmitting optical energy. Diffraction losses at the apertures are negligible (10^{-3} db/lens) for lenses of only 10 mm. radius. The iteration loss per lens is therefore determined by inherent reflection and absorption which can be reduced to less than 0.1 db/lens with current technology.[12]

An inherent problem in the implementation of the beam waveguide as a transmission link is the stability of alignment of the phase transforming elements to the propagation axis. If the elements remain in position with the lens optical centers on the transmission axis the beam is perfectly iterated with minimum loss. Misalignment of a single lens results in a beam path which oscillates about the axis with no increase in loss provided the lens apertures are sufficiently large to accommodate the deviations in beam position. More serious lens displacements lead to an increase in overall attenuation of the beam, and the beam path becomes unstable for periodic spatial displacements of the lenses.[4, 13]

An automatic alignment system for the waveguide which corrects lens disturbances along the structure in a series of small movements has been developed and tested. In this control a small (0.05 mm) corrective step is applied at each lens on the waveguide based on the direction of beam deviation from the transmission axis at the following lens. After each lens in the guide has been moved one step, the entire process is repeated. The beam path returns to the axis after a sufficient number of correction cycles have occurred to return all lenses to the axis. A system which operates continuously on each lens is mentioned but no evaluation other than possible increased speed of response is presented.[14]

The continuous servo control has been analyzed in the steady state and shown to provide an asymptotically stable system which returns the propagated beam to the guide axis for initial displacements which do not completely disrupt the transmission path.[6] Such disturbances are to be considered as catastrophic failures and are not subject to correction by any form of control.

Before the controlled beam waveguide can be regarded as a viable alternative to existing transmission methods however, information concerning the transient performance of the system with disturbances in element alignment must be obtained. Some method of stabilization which returns the beam to the waveguide axis in a reasonable period of time with a nominal expenditure of control energy is desirable in terms of maintaining satisfactory iteration of the beam and minimal attenuation of the propagated energy.

This thesis derives the as yet unpublished transient response relationship between beam position and transverse disturbances of the structure elements.

APPENDIX B

DERIVATION OF THE TRANSIENT RESPONSE

For a detailed derivation of the transient response of the controlled beam waveguide we first require an expression showing how the beam path in the guiding structure is affected by transverse lens displacements from the propagation axis. We assume that the beam diameter is sufficiently smaller than the lens apertures to allow us to ignore the effects of diffraction and therefore enable us to use paraxial geometric optics to trace the path of an ideal ray through the waveguide.

The structure is composed of a series of identical thin lenses of focal length (f), separated by distance (d). The effect of a thin lens on a paraxial ray may be expressed in one dimension as[7]

1) $$R_{out} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot R_{in}$$

2) where $$R = \begin{bmatrix} r \\ r' \end{bmatrix}$$

The unprimed term is ray position and the primed term is ray slope relative to the optical axis of the lens. This expression relates ray position and slope at an output reference plane immediately following the lens to the same parameters at an input plane immediately preceding the lens as is shown in Figure B.1.

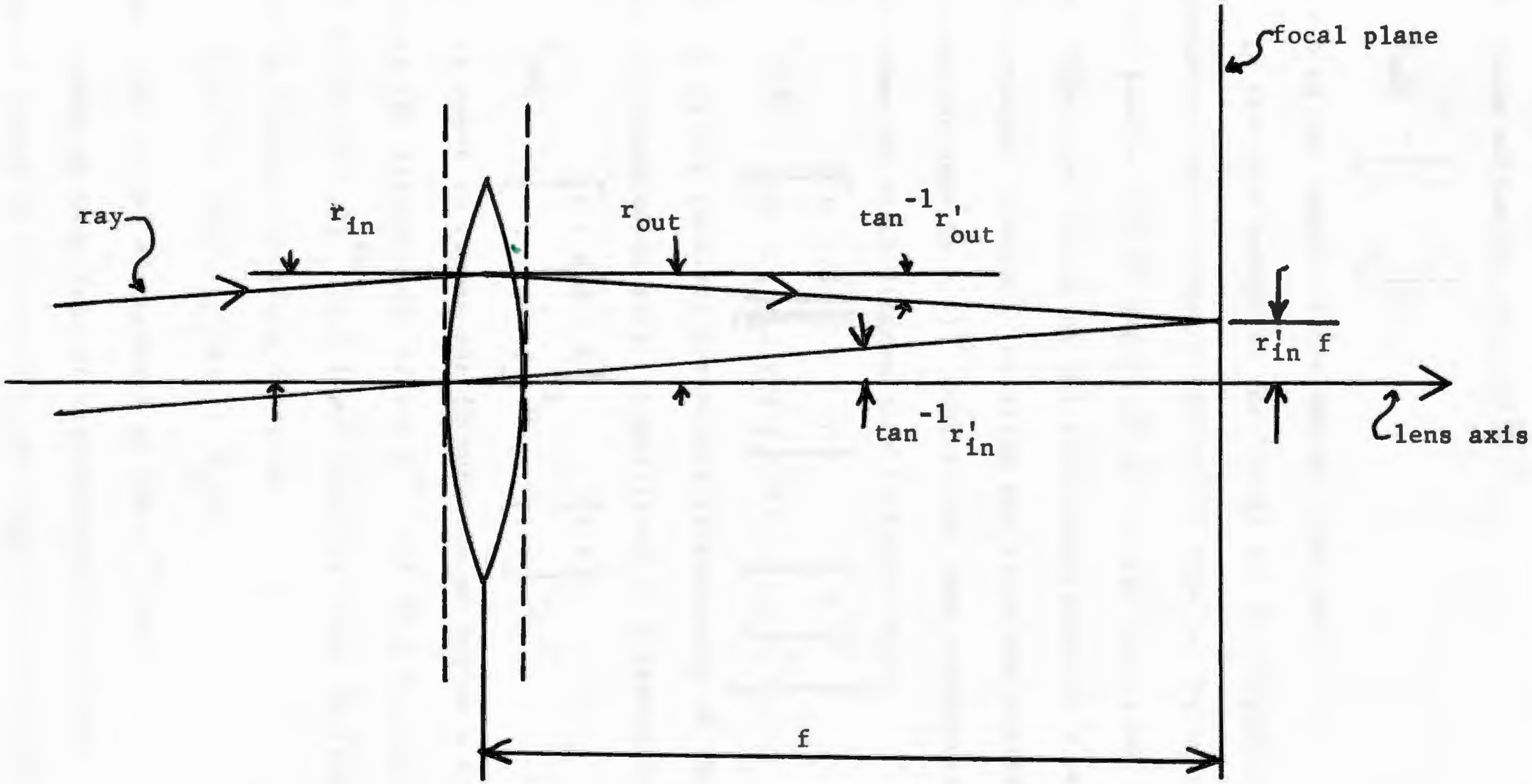

Figure B.1. Transmission of a paraxial ray by a thin lens.

In like manner, the transmission of a straight section of a homogeneous medium is given by[7]

3) $$R_{out} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} R_{in}$$

where L is the length of the medium traversed.

In the beam waveguide the lenses may be displaced from the transmission axis, adding a correction term to the output ray slope and the lenses will be separated by straight sections. Combining these effects we arrive at the ray transmission of a basic unit cell of the guiding structure, relating ray slope and position at the input plane of some $(n + 1)^{th}$ lens to the same parameters at the n^{th} lens (shown in block diagram form in Figure B.2);

4) $$R_{n+1} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} R_n + \begin{bmatrix} 0 \\ 1/f \end{bmatrix} s_n \right]$$

where s_n is the (scalar) transverse displacement of the n^{th} lens from the transmission axis. Simplifying this expression we have;

5) $$R_{n+1} = \begin{bmatrix} 1 - d/f & d \\ -1/f & 1 \end{bmatrix} R_n + \begin{bmatrix} d/f \\ 1/f \end{bmatrix} s_n$$

In order to remove the disturbance we assume a control which corrects the displacement of the n^{th} lens as a function of ray position at the $(n + 1)^{th}$ lens input plane as shown in Figure 1. We now have, in Laplace Transform notation,

6) $$S_n(s) = H(s)\, r_{n+1}(s) + D_n(s)$$

where $D_n(s)$ is the disturbance at the n^{th} lens.

Including this control relationship in equation 5 and simplifying we arrive at the basic input-output expression for the unit cell of the controlled beam waveguide, shown in block diagram in

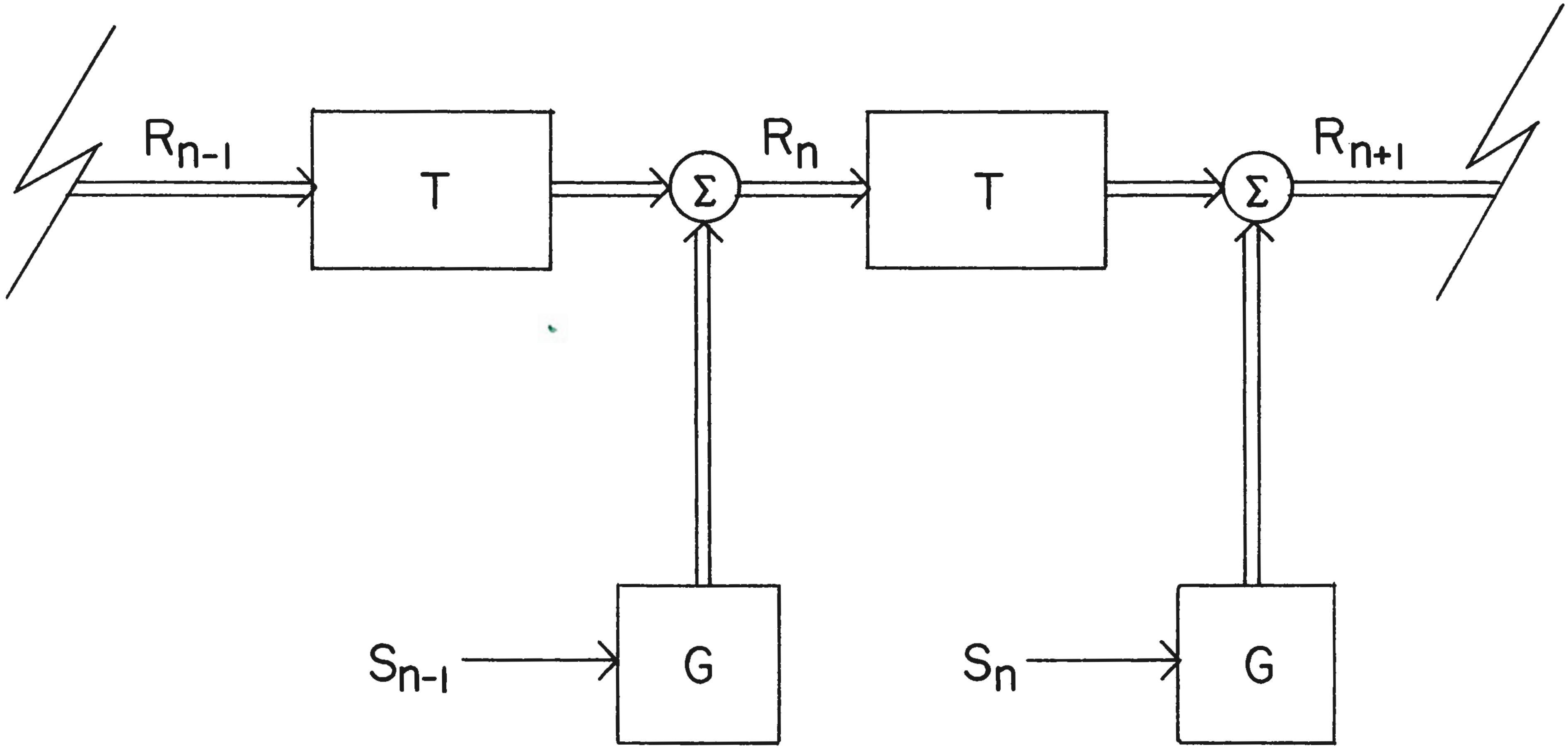

Figure B.2. Block diagram of the uncontrolled beam waveguide.

Figure B.3;

7) $$R_{n+1}(s) = T(s)\,R_n(s) + G(s)\,D_n(s)$$

8) $$\text{with } T(s) = \begin{bmatrix} \dfrac{1 - (d/f)}{1 - (d/f)H(s)} & \dfrac{d}{1 - (d/f)H(s)} \\[2ex] \dfrac{H(s) - 1}{f(1 - (d/f)H(s))} & \dfrac{1}{1 - (d/f)H(s)} \end{bmatrix}$$

9) $$\text{and } G(s) = \begin{bmatrix} \dfrac{d/f}{1 - (d/f)H(s)} \\[2ex] \dfrac{1/f}{1 - (d/f)H(s)} \end{bmatrix}$$

To find the transient beam position at an arbitrary lens entrance plane as a result of a disturbance of some previous lens from the axis we assume that the waveguide elements are initially aligned (all lenses are centered on the reference axis and the beam is on axis throughout the structure) and introduce a transverse displacement of a single lens.

By iteration of equation 7 with the assumed initial conditions;

10) $$R_m(s) = 0 \quad \text{for all m at time} \quad t=0$$

11) $$D_m(s) = \begin{cases} D_k(s) & m = k \\ 0\,, & m \neq k \end{cases}$$

we find the response of beam position at a lens entrance plane n unit cells beyond the element (the k^{th}) which is disturbed;

12) $$R_{k+n}(s) = T(s)^{n-1}\,G(s)\,D_k(s)$$

To simplify this expression we first rewrite the matrix T(s) by extracting a common scalar factor;

13) $$T(s) = \left(\frac{1}{1 - (d/f)H(s)}\right)^{\frac{n-1}{2}} \bar{T}(s)^{n-1}$$

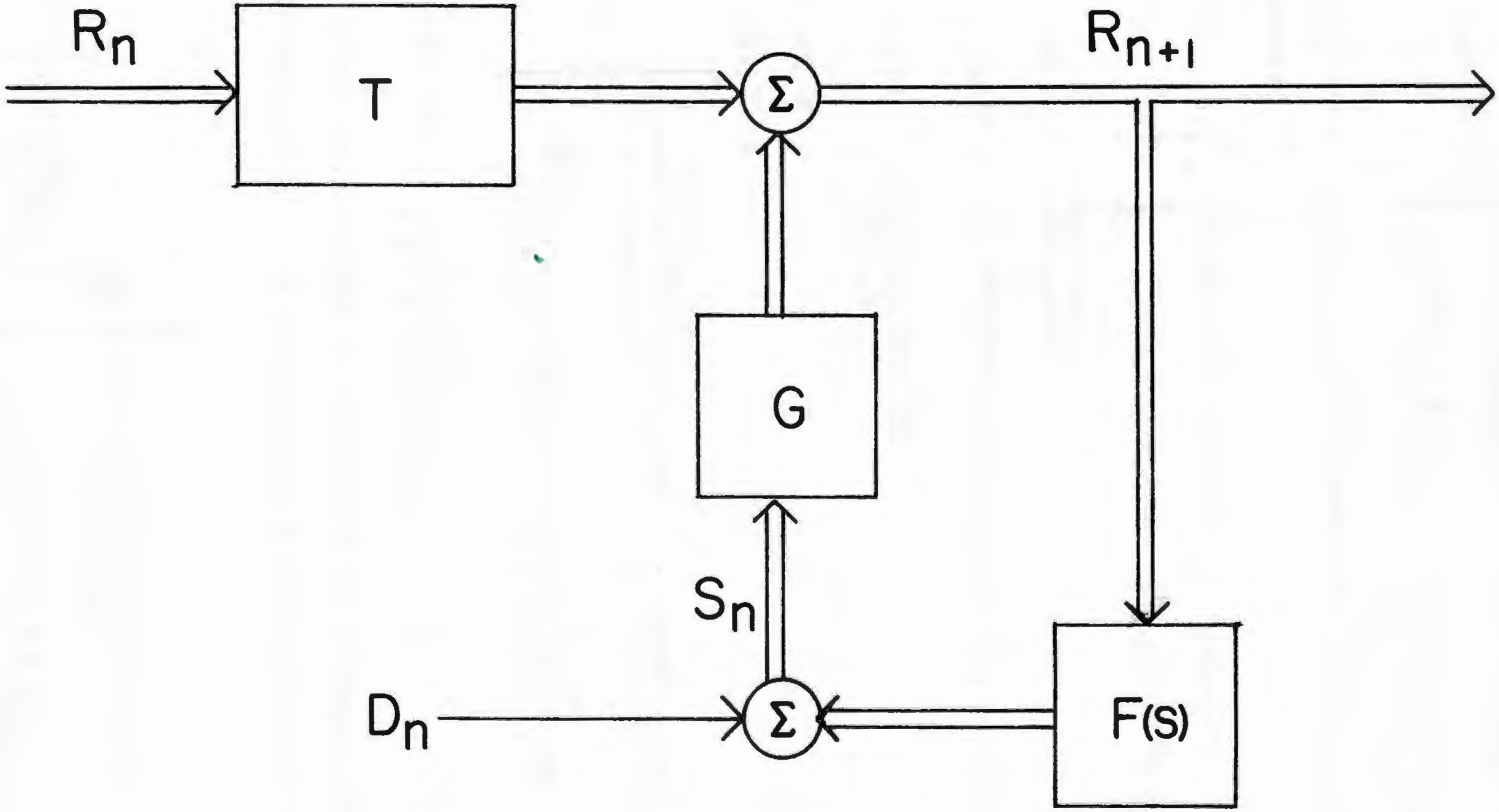

Figure B.3. Block diagram of the basic unit cell of the controlled beam waveguide.

14) where $\overline{T}(s) = \begin{bmatrix} \dfrac{1 - (d/f)}{(1 - (d/f)H(s))^{1/2}} & \dfrac{d}{(1 - (d/f)H(s))^{1/2}} \\ \dfrac{H(s) - 1}{f(1 - (d/f)H(s))^{1/2}} & \dfrac{1}{(1 - (d/f)H(s))^{1/2}} \end{bmatrix}$

and then use a relation for the N^{th} power of a matrix with unity determinant,[8]

$$15)\quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^N = \begin{bmatrix} m_{11}U_{N-1}(a) - U_{N-2}(a) & m_{12}U_{N-1}(a) \\ m_{21}U_{N-1}(a) & m_{22}U_{N-1}(a) - U_{N-2}(a) \end{bmatrix}$$

16) where $a = \dfrac{m_{11} + m_{22}}{2}$

and the $U_m(a)$ are the Chebyshev Polynomials of the second kind;

$$17)\quad U_m(a) = \frac{\sin((m + 1)\cos^{-1}a)}{\sin(\cos^{-1}a)}$$

resulting in

$$18)\quad \overline{T}(s)^{n-1} =$$

$$\begin{bmatrix} \dfrac{(1 - (d/f))U_{n-2}(x)}{(1 - (d/f)H(s))^{1/2}} - U_{n-3}(x) & \dfrac{d}{(1 - (d/f)H(s))^{1/2}} U_{n-2}(x) \\ \dfrac{H(s) - 1}{f(1 - (d/f)H(s))^{1/2}} U_{n-2}(x) & \dfrac{U_{n-2}(x)}{(1 - (d/f)H(s))^{1/2}} - U_{n-3}(x) \end{bmatrix}$$

19) with $x = \dfrac{2 - (d/f)}{2(1 - (d/f)H(s))^{1/2}}$

Using this expansion in equation 12 and performing the indicated matrix multiplication we arrive at the final vector relation;

$$20)\quad R_{k+n}(s) =$$

$$D_k(s)\left(\frac{1}{1 - (d/f)H(s)}\right)^{\frac{n+1}{2}} \begin{bmatrix} (d/f)\,\dfrac{2 - (d/f)}{(1 - (d/f)H(s))^{1/2}}U_{n-2}(x) - U_{n-3}(x) \\ (1/f)\,\dfrac{1 - (d/f)(1 + H(s))}{(1 - (d/f)H(s))^{1/2}}U_{n-2}(x) - U_{n-3}(x) \end{bmatrix}$$

The ray position response relative to the transmission axis at the $(k + n)^{th}$ reference plane is the upper term of this vector;

21) $r_{k+n}(s) =$

$$D_k(s)\left(\frac{1}{1-(d/f)H(s)}\right)^{\frac{n+1}{2}}(d/f)\left(\frac{2-(d/f)}{(1-(d/f)H(s))^{1/2}}U_{n-2}(x) - U_{n-3}(x)\right)$$

Now, by applying the recursion relationship for the Chebyshev Polynomial [15]

22) $U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0$

in equation 21 we reduce the ray position response to a compact expression;

23) $$r_{k+n}(s) = \left(\frac{1}{1-(d/f)H(s)}\right)^{\frac{n+1}{2}}(d/f)\,U_{n-1}(x)\,D_k(s)$$

24) where $$x = \frac{2-(d/f)}{2(1-(d/f)H(s))^{1/2}}$$

An alternate expression results from using the polynomial expansion[10]

25) $$U_m(x) = \sum_{n=0}^{m/2}\begin{pmatrix} m-n \\ n \end{pmatrix}(-1)^n\,(2x)^{m-2n}$$

in equation 23 yielding the closed form summation

26) $r_{k+n}(s) =$

$$(d/f)\sum_{m=0}^{\frac{n-1}{2}}\begin{pmatrix} n-1-m \\ m \end{pmatrix}(-1)^m(2-(d/f))^{n-1-2m}\left(\frac{1}{1-(d/f)H(s)}\right)^{n-m}D_k(s)$$

This equation gives us a closed form summation for the ray position response at a lens entrance plane located n unit cells beyond a transverse disturbance of the k^{th} element in the guiding structure. Using this relation it is now possible to evaluate the response of any desired configuration of guiding structure and control. The pertinent parameters (lens focal length to separation ratio (d/f), and control form H(s)) may be inserted in equation 26 and the inverse Laplace Transform taken to arrive at the time domain expression for response for the structure being tested.

The confocal waveguide ((d/f) = 2) is particularly interesting due to the fact that in this case only one term of the summation in equation 26 remains (that term for which n - 1 - 2m = 0), yielding

$$27)\quad r_{k+n}(s) = \begin{cases} 2(-1)^{\frac{n-1}{2}} D_k(s)\left(\dfrac{1}{1-2H(s)}\right)^{\frac{n+1}{2}} & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

illustrating that the beam passes through the transmission axis at all even numbered lenses after the disturbance in this case.

For this configuration an analytic expression for the time domain response may be found for a control in the form of an ideal integrator with gain

$$28)\quad H(s) = -G/s$$

which, when used in equation 27 and assuming a unit step disturbance at the k^{th} lens, results in

$$29)\quad r_{k+n}(s) = 2(-1)^{\frac{n-1}{2}}\left[\frac{s^{\frac{n-1}{2}}}{(s+2G)^{\frac{n+1}{2}}}\right] \qquad \text{for n odd}$$

or, in the time domain [16]

$$30)\quad r_{k+n}(t) = 2(-1)^{\frac{n-1}{2}} e^{-2Gt} \sum_{m=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{m} \frac{1}{m!}(-2Gt)^m \qquad ;\ n\ \text{odd}$$

In order to reduce this expression to something less formidable we note that the Laguerre Polynomial of integer order is given by [11]

31) $$L_m(x) = \sum_{n=0}^{m} \binom{m}{m-n} \frac{1}{n!} (-1)^n x^n$$

Making the appropriate substitutions in this expansion;

32) $$m = \frac{n-1}{2}$$

33 and $$x = 2Gt$$

we have for equation 30;

34) $$r_{k+n}(t) = 2(-1)^{\frac{n-1}{2}} e^{-2Gt} L_{\frac{n-1}{2}}(2Gt) \quad ; \quad n \text{ odd}$$

the time domain ray position response at any arbitrary lens entrance plane located an odd number (n) of unit cells beyond a lens disturbed by a transverse unit step. This response for the first six odd numbered lenses (up to the thirteenth) is plotted in Figure 2.

Digital computer simulation of the controlled waveguide using the integrator with gain as a correction for the disturbance has been carried out as a test of the analytic results. Disagreements between the computer output and the derived expression were negligible and attributable to digital integration error. The simulation program used is located at the end of this appendix (Listing BL.1--Guidel).

In an effort to determine the form of the time domain response for a class of physically realizable controls rather than ideal cases the waveguide was simulated using a correction function

35) $$H(s) = \frac{-\omega_n^2}{2s\,(s - 2\zeta\omega_n)}$$

a second order system operating on the transverse position of each lens. The simulation program used (Guide2) is listed as BL.2. Two sets of resultant unit step response curves are shown in Figures 3 and 4. Figure 3 shows the response at the eleventh lens after the disturbance for three values of the damping factor, ζ = 0.8, 1.0 and 1.2.

These curves demonstrate that for light damping (ζ = 0.8), the overshoot exceeds the initial disturbance. The response at the first lens is the classic second order system response but as we progress down the waveguide the nature of the response becomes increasingly oscillatory with an overshoot that increases with distance from the disturbance.

By increasing the damping factor to ζ = 1.2, the beam at the eleventh lens is returned to the axis with overshoot remaining less than the initial displacement. Figure 4 shows the response at the odd numbered lenses up to the eleventh for a damping factor of ζ = 1.2. Note that increasing to a very large value yields a control which approximates the integrator with gain resulting in a beam displacement which remains bounded by the magnitude of the initial disturbance at all lenses.

BL.1. Simulation of the confocal waveguide with H(s) = -G/s.

```
Guidel
      dimension g(11),s(100),r(100)
      dimension cor(100),sd1(100)z(25,10,50)
      s(2)=1.0
      write(6,100)
  100 format(' up to what number (odd) lens is the response to run')
      read(5,*)L5
      L5=L5 + 2
      write(6,101)
  101 format(' input initial gain and gain increment for ten runs')
      read(5,*)g(1),dg
   30 format(5x,11(f6.3,2x))
      write(6,102)
  102 format(' enter final time of lens #1 response; 0.0--4.9')
      read(5,*)tfinl
      itpr=10.0 * tfinl
      do 22 m=1,10
      g(m+1)=g(m) + dg
      do 21 k=L,490
      do 10 n=1,L5
      sd1(n+1)=g(m) * r(n+2)
      cor(n+1)= .01 * sd1(n+1) + cor(n+1)
      r(n+2)= 2. * (s(n+1) - cor(n+1)) - r(n)
      k1=k/10
      if(k=1)go to 20
      k2=k - k1*10
      if(k2)21,20,21
   20 k4=k1 + 1
   10 z(n,m,k4)=r(n)
   21 continue
      do 22 n=1,100
      sd1(n)=0.0
      cor(n)=0.0
      r(n)=0.0
   22 continue
      write(6,103)(g(m),m=1,10)
  103 format(' gain=',7x,10(f6.2,2x))
      do 11 n=3,L5,2
      j=n - 2
      write(6,105)j
  105 format(///5x,'time',20x,'response at lens #',i2,' after dist')
      time =0.0
      do 12 k=1,itpr
      write(6,30)(time,(z(n,m,k),m=1,10))
   12 time=time +0.1
   11 itpr=50
      stop
      end
```

BL.2. Simulation of the confocal waveguide with second order control.

```
Guide2
      dimension g(11),r(25),sd1(25),cor(25)
      dimension s(25),z(25,10,50)
      s(2)=1.0
      write(6,101)
  101 format(' up to what number (odd) lens is the response to run')
      read(5,*)L5
      L5=L5 + 2
      write(6,102)
  102 format(' input desired cell control time constant')
      read(5,*)tau
   30 format(5x,11(f6.3,2x))
      write(6,103)
  103 format(' input initial gain and gain increment for ten runs')
      read(5,*)g(1),dg
      x=1. - .01/tau
      do 22 m=1,10
      g(m+1)=G(m) + dg
      y=g(m)/tau
      do 21 k=1,490
      do 12 n=1,L5
      sd1(n+1)=y * r(n+2)
      cor(n+1)= .01 * sd1(n+1) + x * cor(n+1)
      r(n+2)=2. * (s(n+1) - cor(n+1)) - r(n)
      k1=k/10
      if(k=1)go to 20
      k2=k - k1*10
      if(k2)21,20,21
   20 k4=k1 + 1
   12 z(n,m,k4)=r(n)
   21 continue
      do 22 n=1,25
      r(n)=0.0
      cor(n)=0.0
      sd1(n)=0.0
   22 continue
      write(6,106)tau
  106 format(' for cell control time constant; tau=',f6.3)
      write(6,104)(g(m),m=1,10)
  104 format(' gain=',7x,10(f6.3,2x))
      do 11 n=3,L5,2
      j=n - 2
      write(6,105)j
  105 format(///5x,'time',20x,' response at lens #',i2,'after dist')
      time=0.0
      do 11 k=1,50
      write(6,30)(time,(z(n,m,k),m=1,10))
   11 time=time + 0.1
      stop
      end
```

APPENDIX C

NON-ITERATIVE CONTROL IN THE CONFOCAL WAVEGUIDE

The form of the time domain response at lenses well removed from the disturbance (Figures 2, 3 and 4) leaves this scheme of iterated control open to question. As a result of a disturbance all lenses in the structure are given erroneous corrections due to the instantaneous path taken by the beam after a displaced lens. The controls themselves thereby cause a misalignment throughout the guide beyond the disturbance. The consequence of this action is that no lens on the waveguide is returned to the axis until all previous lenses are aligned, yielding a system in which the final alignment propagates down the structure causing the transmitted beam to spend considerable time off the axis at the output (receiver) end of the waveguide.

A more desirable control system would correct a single lens disturbance by moving only that lens back to equilibrium and not waste control effort and time inducing and removing erroneous displacements on those elements of the structure which were in proper alignment originally. This isolated or non-iterative form of control is possible through the use of the beam propagation characteristics of the confocal waveguide.

In the confocal case, the result of a unit transverse displacement of a single lens from the transmission axis is that the instantaneous beam position is found at any lens following the disturbance by solution of the basic unit cell matrix difference equation (B.5) repeated here with the confocal condition ((d/f = 2) included;

$$1)\qquad R_{n+1} = \begin{bmatrix} -1 & d \\ -1/f & 1 \end{bmatrix} R_n + \begin{bmatrix} 2 \\ 1/f \end{bmatrix} S_n$$

The beam vector at some $(k + n)^{th}$ lens entrance plane resulting from a unit disturbance at the k^{th} lens is therefore;

$$2)\qquad R_{k+n} = \begin{bmatrix} -1 & d \\ -1/f & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 2 \\ 1/f \end{bmatrix} S_k$$

yielding the ray position as;

$$3)\qquad r_{k+n} = \begin{cases} 2\,(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

This relation shows that upon displacement of a lens in the confocal waveguide the beam is moved at the next lens in the same direction as the disturbance by twice the magnitude of the lens movement and thereafter the beam takes an oscillatory path through the structure passing through the transmission axis at every even numbered lens after the disturbance and reaching the same magnitude ($2S_k$) above or below the axis at alternate odd numbered lenses.

Taking advantage of this beam path, we arrange the input to the position control for each lens as a function of beam position at the preceding and succeeding lens entrance planes as shown in Figure C.1.

The displacement at any k^{th} lens now will be the sum of the initial disturbance and a correction involving the sum of ray

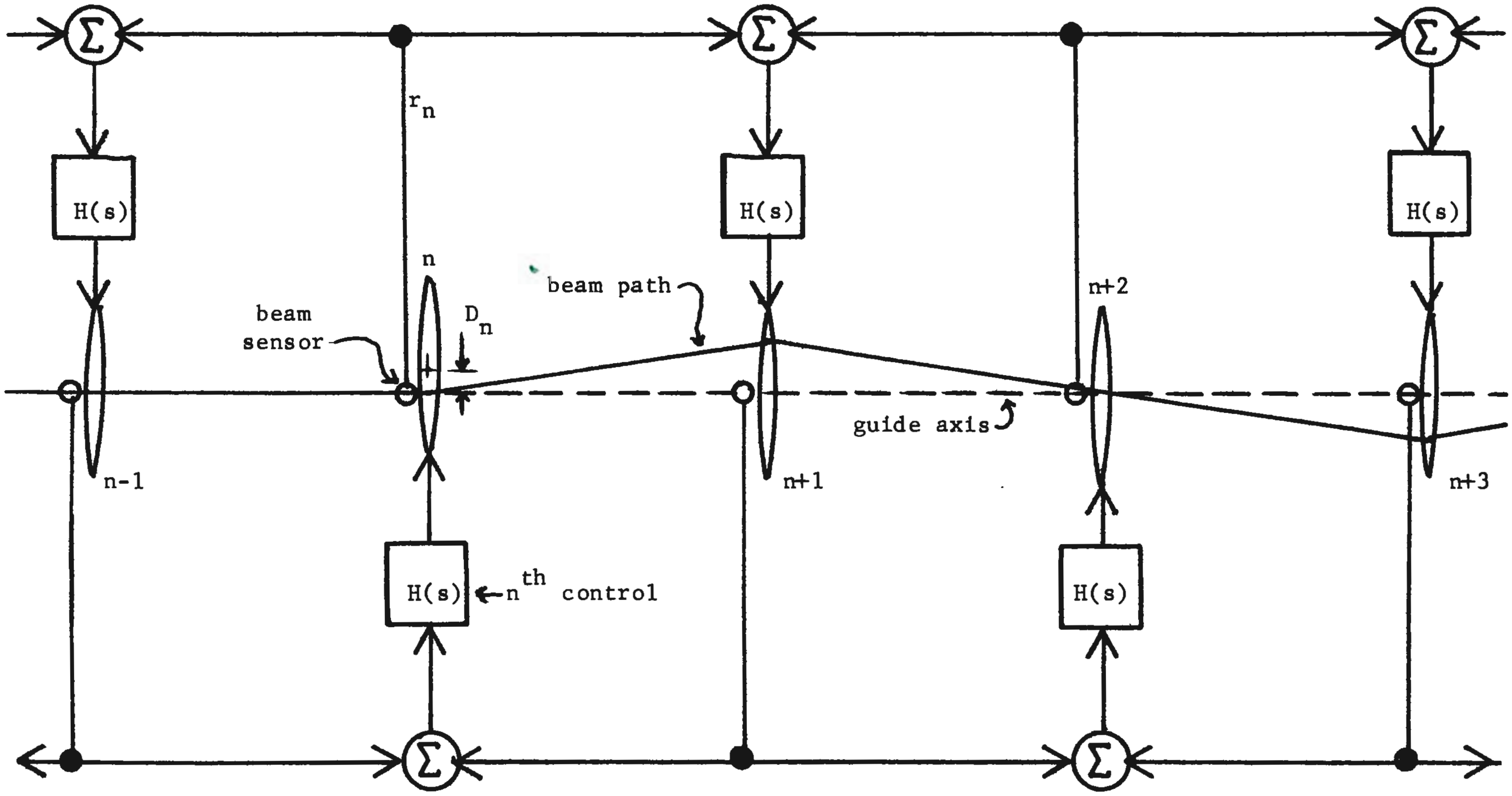

Figure C.1. Confocal waveguide with non-iterative control (n^{th} lens is displaced from the axis).

positions at the $(k - 1)^{th}$ and $(k + 1)^{th}$ lenses;

4) $S_k(s) = D_k(s) + H(s)(r_{k+1}(s) + r_{k-1}(s))$

We now have two possibilities. If there is no disturbance of the k^{th} lens we have from iteration of equation 1;

5) $r_{n+1}(s) = -r_{n-1}(s)$

and therefore the total displacement term in equation 4 goes to zero, meaning that the k^{th} lens will not be moved erroneously by the control.

On the other hand, if the k^{th} lens is disturbed, $D_k(s)$ will have some magnitude and will affect ray position at the $(k + 1)^{th}$ lens.

Tracing the ray from the $(k - 1)^{th}$ lens to the $(k + 1)^{th}$ lens we have;

6) $$R_{k+1}(s) = \begin{bmatrix} -1 & d \\ -1/f & 1 \end{bmatrix}^2 R_{k-1}(s) + \begin{bmatrix} 2 \\ 1/f \end{bmatrix} S_k(s)$$

or, equivalently;

7) $$R_{k+1}(s) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} R_{k-1}(s) + \begin{bmatrix} 2 \\ 1/f \end{bmatrix} S_k(s)$$

Using the value of ray position found in equation 7;

8) $r_{k+1}(s) = 2S_k(s) - r_{k-1}(s)$

in equation 4, we have;

9) $S_k(s) = D_k(s) + H(s)(r_{k-1}(s) + 2S_k(s) - r_{k-1}(s))$

or, upon simplification;

10) $$S_k(s) = D_k(s)\left(\frac{1}{1 - 2H(s)}\right)$$

This correction is applied to the displaced lens resulting in a ray vector at the lens following the disturbance of

$$\text{11)} \quad R_{k+1}(s) = \begin{bmatrix} -1 & d \\ -1/f & 1 \end{bmatrix} R_k(s) + \begin{bmatrix} 2 \\ 1/f \end{bmatrix} \left(\frac{D_k(s)}{1 - 2H(s)} \right)$$

where the perturbation from undisturbed transmission is

$$\text{12)} \quad \bar{r}_{k+1}(s) = \frac{2D_k(s)}{1 - 2H(s)}$$

If we now assume a control in the form of the integrator with gain;

$$\text{13)} \quad H(s) = \frac{-G}{2s}$$

and an input disturbance of a unit step, we have

$$\text{14)} \quad \bar{r}_{k+1}(s) = \frac{2}{s + 2G}$$

immediately yielding the exponential time domain response of the perturbation from undisturbed transmission;

$$\text{15)} \quad \bar{r}_{k+1}(t) = 2\, e^{-2Gt}$$

As is evident from equation 5, this response will be seen at all odd numbered lenses beyond the disturbed element. Therefore, due to the control configuration, we now have a system in which the correction is not iterated by the unit cells following that in which the initial disturbance occurs. The end result of this control scheme is that a transverse lens displacement is isolated from the structure and corrected without unnecessary motion of the other lenses on the waveguide. The control is now able to distinguish between errors in beam position caused by the disturbance of the controlled lens and those errors generated by a lens associated with a previous control.

The confocal waveguide with non-iterative control has been simulated on a digital computer with results confirming the conclusion of isolated control indicated above. The simulation program appears at the end of this appendix as CL.1 (Guide3).

CL.1. Simulation of the non-iterative confocal guide; H(s)= -G/s.

```
Guide3
     dimension r(25),cor(25),sdl(25)
     dimension s(25),g(11),z(25,10,50)
     write(6,103)
 103 format(' up to what number (odd) lens is the response to run')
     read(5,*)L5
     L5=L5 + 2
     L6=L5 + 2
     write(6,104)
 104 format(' input initial gain and gain increment for ten runs')
     read(5,*)g(1),dg
     s(2)=1.0
     s(3)=1.0
     do 10 m=1,10
     g(m+1)=g(m) + dg
     do 20 it=1,490
     do 30 n=1,L5
     sdl(n+1)=g(m) * (r(n) + r(n+2))
     cor(n+1)=.01 * sdl(n+1) + cor(n+1)
     r(n+2)=2.0 * (sdl(n+1) - cor(n+1)) - r(n)
     n1=n + 2
     do 40 k=n1,L6,2
  40 r(k+2)=-r(k)
     it1=it/10
     if(it=1)go to 11
     it2=it - it1*10
     if(it2)20,11,20
  11 it3=it1 + 1
  30 z(n,m,it3)=r(n)
  20 continue
     do 10 n=1,25
     r(n)=0.0
     cor(n)=0.0
     sdl(n)=0.0
  10 continue
     write(6,105)(g(m),m=1,10)
 105 format(' gain=',6x,10(f6.3,2x))
     do 50 n=3,L5
     j=n - 2
     write(6,106)j
 106 format(///5x,'time',20x,'response at lens #',i2,'after dist')
     time=0.0
     do 50 k=1,50
     write(6,107)(time,(z(n,m,k),m=1,10))
 107 format(5x,11(f6.3,2x))
  50 time=time + 0.1
     stop
     end
```

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