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Bifurcations of Some Planar Discrete Dynamical Systems with Applications

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BIFURCATIONS OF SOME PLANAR DISCRETE DYNAMICAL SYSTEMS WITH APPLICATIONS

BY

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ABSTRACT

The focus of this thesis is on some contemporary problems in the field of difference equations and discrete dynamical systems. The problems that I worked on range global attractivity results to all types of bifurcations for systems of difference equations in the plane.

The major goal was to investigate the impact of nonlinear perturbation and the introduction of quadratic terms on linear fractional difference equations such as the Beverton-Holt as well as the Sigmoid Beverton-Holt Model with delay that describes the growth or decay of single species.

The first Manuscript was on the study of the following equation:

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2} \tag{1}$$

Which was an open problem suggested by Dr. Kulenović. It is a a perturbation of the linear fractional difference equation:

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}$$

The solutions of Equation(1) undergo very interesting dynamics as I showed that the variation of the parameter p can cause the exhibition of the Naimark-Sacker bifurcation. I compute the direction of the Naimark-Sacker bifurcation for the difference equation (1) and I provide an asymptotic approximation of the closed invariant curve which comes to existence as the unique positive equilibrium point loses its stability. Moreover tools and global stability result to provide a region of the parameter where local stability implies global stability of the equilibrium.

In my second Manuscript, I considered the difference equation:

$$x_{n+1} = \frac{x_n}{Cx_{n-1}^2 + Dx_n + F} \tag{2}$$

where C, D and F are positive numbers and the initial conditions x_{-1} and x_0 are non-negative numbers. Equation (2) which is also a non-linear perturbation of the Beverton Holt model, belongs to the category of difference equations with a unique positive equilibrium that exhibit the Naimark-Sacker bifurcation. The investigation of the dynamics of such equation is very challenging as it depends on more than one parameter. However I give a method for proving that its dynamics undergoes the Naimark-Sacker bifurcation. Moreover I compute the direction of the Neimark-Sacker bifurcation for this difference equation and provide the asymptotic approximation of the invariant closed curve. Furthermore I give the necessary and sufficient conditions for global asymptotic stability of the zero equilibrium as well as sufficient conditions for global asymptotic stability of the positive equilibrium.

The following theorem is the major result that I relied on to prove global asymptotic stability of the equilibria in my first two Manuscripts:

Theorem 1 *Let I be a compact interval of the real numbers and assume that $f : I^3 \to I$ is a continuous function satisfying the following properties:*

1. *$f(x, y, z)$ is non-decreasing in x and non-increasing in y and z*
2. *The system* $\begin{cases} f(M, m, m) = M \\ f(m, M, M) = m \end{cases}$ *has a unique solution $M = m$ in I.*

Then the equation $x_{n+1} = f(x_n, x_{n-1}, x_{n-2})$ has a unique equilibrium $\bar{x}$ in I and *every solution of it that enters I must converge to $\bar{x}$. In addition, $\bar{x}$ is globally asymptotically stable.*

As of my third manuscript, I focused on providing some possible scenarios for general discrete competitive dynamical systems in the plane. I applied the results achieved to a class of second order difference equations of the form:

$$x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \dots$$

where the function $f(x, y)$ is decreasing in the variable x and increasing in the variable y. In my proofs I relied on a collection of well established theorems and results. Furthermore I illustrate my results with an application to equation:

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + dx_n + f}, \quad n = 0, 1, \dots \tag{3}$$

With initial conditions x_{-1} and x_0 arbitrary nonnegative numbers and parameters $c,\ d,\ f$ >0. Equation. (3) is a special case of:

$$x_{n+1} = \frac{Cx_{n-1}^2 + Dx_n + F}{cx_{n-1}^2 + dx_n + f}, \quad n = 0, 1, \dots$$

which of great interest to the field of difference equation and special cases of it were considered by different scholars. It also turns out to be a non-linear perturbation of the Sigmoid Beverton-Holt model. I characterize completely the global bifurcations and dynamics of equation.(3) with the basins of attraction of all its equilibria and periodic solutions. Moreover I provide techniques to investigates cases that are not covered by the established theorems in the theory of competitive maps.

Finally in my fourth manuscript I considered extending some existing theorems and proving some new global stability results, namely for difference equations that are of the form

$$x_{n+1} = f(x_n, x_{n-1})$$

where $f(x, y)$ is either increasing in the first and decreasing in the second variable, or decreasing in both variables. In addition I illustrate my results with examples and applications. I also provide a new proof for Pielou's equation (a mathematical model in population dynamics).

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PREFACE

This is a dissertation in Manuscript format

Manuscript 1 of this thesis was published as:

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Manuscript 4 of this thesis will be submitted as:

T. Khyat and M. R. S. Kulenović, Global Attractivity Results for Second Order Difference Equations)

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Asymptotic Approximation of the Invariant Curve of Certain Difference Equation

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1.1 Introduction and Preliminaries

In this paper I consider the difference equation

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{4}$$

where the parameter p is positive number and the initial conditions x_{-1} and x_0 are positive numbers. Notice that if $x_{-1}, x_0 \neq 0$ in equation (5) then $x_n > 0, n \geq 1$, and so without loss of generality we can assume that $x_{-1} > 0, x_0 > 0$. This implies that our results are global. Clearly equation (27) has the unique equilibrium point $\bar{x} = p + 1$. Linear fractional version of equation (27)

$$x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{5}$$

was considered in [3], where it was proved that the unique equilibrium $\bar{x} = p + 1$ of equation (5) is globally asymptotically stable. Introduction of quadratic terms into equation (5) changes the local stability analysis and consequently the global dynamics as well. In particular, quadratic terms introduces the possibility of Naimark-Sacker bifurcation and the existence of locally stable periodic solution, see [6] for several similar examples.

The linearized equation of equation (5) at the equilibrium point $\bar{x} = p + 1$ is

$$z_{n+1} = \frac{2}{p+1} z_n - \frac{2}{p+1} z_{n-1}, \quad n = 0, 1, \ldots,$$

with the characteristic equation

$$\lambda^2 - \frac{2}{p+1}\lambda + \frac{2}{p+1} = 0,$$

and the characteristic roots

$$\lambda_{\pm} = \frac{1 \pm i\sqrt{2p+1}}{p+1}.$$

Since

$$|\lambda_{\pm}| = \sqrt{\frac{2}{p+1}}$$

it is clear that that the equilibrium point $\bar{x} = p + 1$ is asymptotically stable if $p > 1$, non-hyperbolic if $p = 1$ and unstable if $p < 1$. In all cases the eigenvalues are complex conjugate numbers which indicates the presence of the Naimark-Sacker bifurcation at $p = 1$. We will prove that indeed the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $p \geq \sqrt{2}$ and that the Naimark-Sacker bifurcation takes the place at $p = 1$. Our tool in proving global asymptotic stability of equation (5) is the result in [3, 5]. We conjecture that the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $a > 1$. Furthermore, we give some numeric values of parameter a with corresponding periodic solutions. Our bifurcation diagram indicates a complicated behavior and possible chaos for the values $p < 1$.

Now, for the sake of completeness I give the basic facts about the Naimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle appears. In the discrete setting, the Naimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation.

The Naimark-Sacker bifurcation occurs for a discrete system depending on a parameter, λ say, with a fixed point whose Jacobian has a pair of complex conjugate $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which cross the unit circle transversally at $\lambda = \lambda_0$.

The following result is referred as the Naimark-Sacker bifurcation Theorem [1, 4, 7, 8, 11].

Theorem 2 (Naimark-Sacker bifurcation) *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, \mathbf{0})$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with* $|\mu(\lambda_0)| = 1$*;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) < 0$ *at $\lambda = \lambda_0$ (transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$. (nonresonance condition).*

Then there is a smooth λ-dependent change of coordinate bringing F into the form

$$F(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth function $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} \tilde{r} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{6}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}) \tag{7}$$

where $\mathbf{A}$ is the Jacobian matrix of $\mathbf{F}$ evaluated at the fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here I denote $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be the eigenvectors of A associated with μ satisfying

$$\mathbf{A}\mathbf{q} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{p}\mathbf{q} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{aligned} \mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20}, \\ \mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11}, \\ \mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02}. \end{aligned} \tag{8}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}z^2 + 2\mathbf{K}_{11}z\bar{z} + \mathbf{K}_{02}\bar{z}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}z^3 + 3\mathbf{g}_{21}z^2\bar{z} + 3\mathbf{g}_{12}z\bar{z}^2 + \mathbf{g}_{03}\bar{z}^3) + O(|z|^4), \end{aligned} \tag{9}$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

The following result from [9] gives an approximate expression for the invariant curve from Theorem 5.

Corollary 1 *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficiently small parameter. If $\bar{\mathbf{x}}$ is a fixed point of F then the invariant curve $\Gamma(\lambda)$ from Theorem 5 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "*Re*" represents the real parts of the complex numbers.

The second section of the paper gives global asymptotic stability result for the values of parameter $p \geq \sqrt{2}$ and the third section gives the reduction to the normal form and computation of the coefficients of the Naimark-Sacker bifurcation and the asymptotic approximation of the invariant curve. My computational method is based on the computational algorithm developed in [9] rather than more often used computational algorithm in [10]. The advantage of the computational algorithm of [9] lies in the fact that this algorithm computes also the approximate equation of the invariant curve in the Naimark-Sacker theorem, which is not provided by Wan's algorithm. Here I give numeric and visual evidence that the approximate equation of the invariant curve is accurate. See Figure 4.

1.2 Global Asymptotic Stability

I use the method of embedding of equation (27) into higher order difference equation to which we apply one of global attractivity results [2] . By substituting

$$x_n = p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$$

in equation (27) we get:

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{x_{n-1}}{x_{n-2}^2}\right)^2.$$

Now by substituting for x_{n-1} in the term $\frac{x_{n-1}}{x_{n-2}^2}$ of the last equation we we obtain

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2} \right)^2. \tag{10}$$

From equation (10) we observe that $p<x_n<p+(1+\frac{1}{p}+\frac{1}{p^2})^2$ for $n \geq 4$.

Also from (27) and (10) we have:

$$\begin{cases} x_{n+1} - p = \left(\frac{x_n}{x_{n-1}} \right)^2 \\ x_{n+1} - p = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2} \right)^2 \end{cases}.$$

Consequently

$$\left(\frac{x_n}{x_{n-1}} \right)^2 = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2} \right)^2,$$

which implies:

$$x_{n+1} = p + \frac{px_n}{x_{n-1}^2} + \frac{x_n}{x_{n-2}^2}. \tag{11}$$

Replacing x_n in (11) by $p + \left(\frac{x_{n-1}}{x_{n-2}} \right)^2$ we obtain the equation

$$x_{n+1} = p + \frac{p^2}{x_{n-1}^2} + \frac{p + x_n}{x_{n-2}^2}. \tag{12}$$

Observe now that every solution of equation (27) is also a solution of equation (12), with initial values x_{-2}, x_{-1} and $x_0 = p + (\frac{x_0}{x_{-1}})^2$.

Observe also that it is of the form $x_{n+1} = f(x_n, x_{n-1}, x_{n-2})$ where :

$$f(u, v, w) = p + \frac{p^2}{v^2} + \frac{p+u}{w^2}$$

.

Theorem 3 *If $p \geq \sqrt{2}$ then the equilibrium $\bar{x} = p + 1$ of equation (27) is globally asymptotically stable.*

Proof. First I show that every interval I of the form $[p, \mathcal{U}]$ where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$ with $p > 1$ is invariant for the function f.

Let $\mathcal{U} > p$ then $I = [p,\mathcal{U}]$ is invariant if and only if for all $u, v, w \in I, f(u, v, w) \in I$ that is:

$$p \leq p + \frac{p^2}{v^2} + \frac{p+u}{w^2} \leq \mathcal{U}.$$

As $p \leq u, v, w \leq \mathcal{U}$ we have that: $p \leq f(u, v, w) \leq p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2}$. We also know that if $\mathcal{U}$ satisfies: $p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2} \leq \mathcal{U}$ then we have

$$f(u, v, w) \leq \mathcal{U}.$$

It follows that given $p > 1$ such $\mathcal{U}$ exists and therefore I is invariant for f where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$. In the following we may assume $p > 1$ and $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$, so I is invariant for f.

Next, I prove that I is an attracting interval, that is every solution of equation (11) must enter the interval I. Observe that given the initial values x_{-2}, x_{-1} and x_0 for equation (11), we have $x_n > p$ for $n \geq 1$.

Now if $x_3 \leq \mathcal{U}$ then $x_n \in [p,\mathcal{U}]$ for all $n \geq 3$. Otherwise, from equation (12) given that x_{n-2} , $x_{n-3} > p$ we have

$$x_n < p + 1 + \frac{1}{p} + \frac{x_{n-1}}{p^2},$$

that is if we set $A = p + 1 + \frac{1}{p}$

$$x_n < A + \frac{x_{n-1}}{p^2}.$$

Thus by induction we can conclude that

$$x_n < A\frac{1 - (\frac{1}{p^2})^{n-3}}{1 - \frac{1}{p^2}} + \frac{x_3}{(p^2)^{n-3}}. \tag{13}$$

It is straightforward to check that when $x_3 > \mathcal{U}$ the right hand side of (13) is a decreasing sequence that converges to $\frac{A}{1-\frac{1}{p^2}}$. This limit is in fact $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$. It follows that there must exist $k > 3$ such that: $p < x_k < \mathcal{U}$. Otherwise x_n must converge to $\mathcal{U}$ which is impossible.

Thus we have $x_{k-1}, x_{k-2} > p$ and $x_k \leq \mathcal{U}$, hence $x_{k+1} \in [a, \mathcal{U}]$. Now it follows by induction that $x_n \in [p, \mathcal{U}]$ for $n \geq k$. Consequently every solution of equation (11) must enter the interval $[p, \mathcal{U}]$.

Now we check the conditions of Theorem A.0.5 [3], see also [5]:

$$\begin{cases} f(M,m,m) = M \\ f(m,M,M) = m \end{cases} \Leftrightarrow \begin{cases} M = p + \frac{p^2+p+M}{m^2} \\ m = p + \frac{p^2+p+m}{M^2} \end{cases}.$$

From the second equation we get

$$M^2 = \frac{p^2+p+m}{m-p}. \tag{14}$$

On the other hand the system is equivalent to:

$$\begin{cases} (M-p)m^2 = p^2+p+M \\ (m-p)M^2 = p^2+p+m \end{cases} \Leftrightarrow \begin{cases} Mm^2 = pm^2+p^2+p+M \\ mM^2 = pM^2+p^2+p+m. \end{cases}$$

By subtracting the second equation from the first we obtain:

$$Mm(m-M) = p(m-M)(m+M) - (m-M)$$

and given that $m \neq M$ we have:

$$Mm = p(m+M) - 1$$

which implies:

$$M = \frac{pm-1}{m-p}. \tag{15}$$

Equations (14) and (15) yield

$$\frac{(pm-1)^2}{(m-p)^2} = \frac{p^2+p+m}{m-p},$$

which implies:

$$(pm-1)^2 = (p^2+p+m)(m-p).$$

This leads to the following quadratic equation:

$$m^2(p^2-1) - m(p^2+2p) + p^2(p+1) + 1 = 0,$$

which discriminant is

$$\Delta = (p^2+2p)^2 - 4(p^2-1)(p^2(p+1)+1)$$

and

$$\Delta = -4p^5 - 3p^4 + 8p^3 + 4p^2 + 4 = (\sqrt{2} - p)(4p^4 + (3+4\sqrt{2})p^3 + 3\sqrt{2}p^2 + 2p + 2\sqrt{2}).$$

It is clear that when $p > \sqrt{2}$ there is no real solutions and when $p = \sqrt{2}$ there is one unique solution $m = p + 1 = M$. Consequently if $p \geq \sqrt{2}$ the conditions of Theorem A.0.5 [3] or Theorem 1 [5] are fully satisfied and therefore every solution must converge to the unique equilibrium $(p+1)$. □

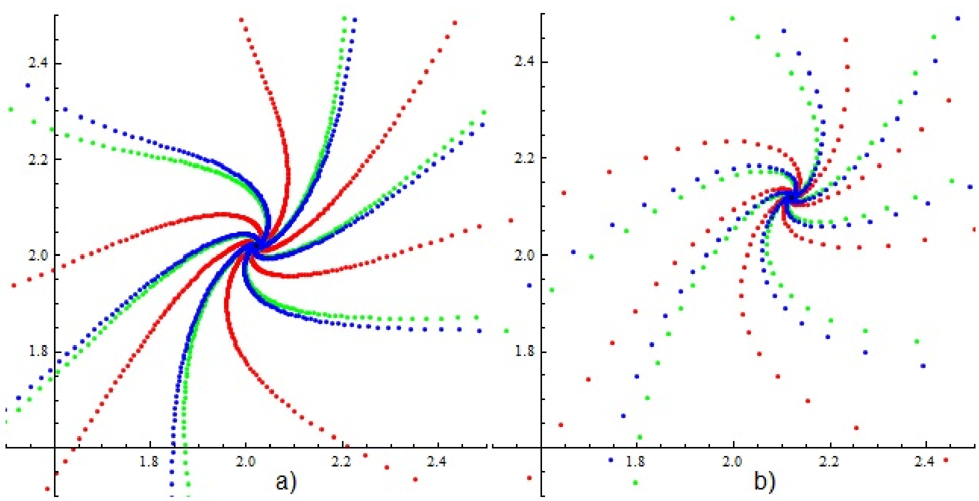

Figure 1. a) Phase diagrams when $n = 10,000$ and a) $p = 1.02$ b) $p = 1.12$

Conjecture 1 *The equilibrium point* $\bar{x} = p + 1$ *of equation* (5) *is globally asymptotically stable if* $p > 1$.

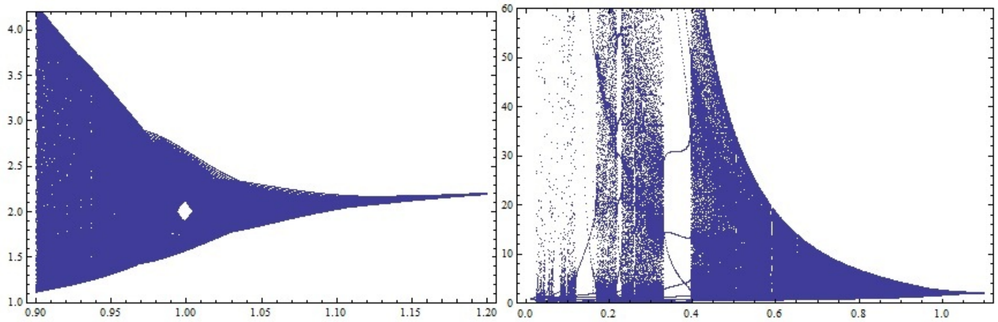

Figure 2. Bifurcation diagrams in the $(p - x)$ plane.

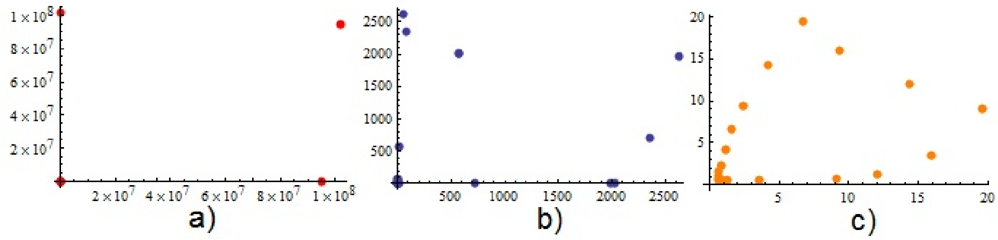

Figure 3. Periodic orbit for a) $p = 0.01$ b) $p = 0.15$ c) $p = 0.5901$ (See Table 1) for numeric values.

p	Period of the sol.	Solution
0.01	7	$\{0.877631, 0.01, 0.0101298, 1.03613, 10462.3, 1.01959 \times 10^8, 9.49713 \times 10^7, 0.877631\}$
0.15	20	$\{574.846, 2023.71, 12.5435, 0.150038, 0.150143, 1.1514, 58.9583, 2622.2, 1978.22, 0.719138, 0.15, 0.193507, 1.81422, 88.0493, 2355.59, 715.88, 0.242359, 0.15, 0.533058, 12.7789\}$
0.5901	19	$\{0.804816, 0.597988, 1.14217, 4.23826, 14.3595, 12.0691, 1.29653, 0.60164, 0.805431, 2.38228, 9.33854, 15.9565, 3.50965, 0.638479, 0.623195, 1.5428, 6.71883, 19.5558, 9.06166\}$

Table 1. Periodic solutions for some values of p.

1.3 Reduction to the normal form

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{(p + y_n + 1)^2}{(p + y_{n-1} + 1)^2} - 1, \quad n = 0, 1, \ldots. \tag{16}$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write equation (27) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{(p+v_n+1)^2}{(p+u_n+1)^2} - 1. \end{aligned} \tag{17}$$

Let $\mathbf{F}$ be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{(p+v+1)^2}{(p+u+1)^2} - 1 \end{pmatrix}. \tag{18}$$

Then $\mathbf{F}$ has the unique fixed point $(0,0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0,0)$ is given by

$$Jac_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}.$$

It is easy to see that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{19}$$

where

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(p+v+1)^2}{(p+u+1)^2} + \frac{2u}{p+1} - \frac{2v}{p+1} - 1 \end{pmatrix}.$$

The eigenvalues of $Jac_{\mathbf{F}}(0,0)$ are $\mu(p)$ and $\overline{\mu(p)}$ where

$$\mu(p) = \frac{1+i\sqrt{2p+1}}{p+1}, \quad |\mu(p)| = \sqrt{\frac{2}{p+1}}.$$

One can prove that for $p = p_0 = 1$ we obtain $\mu(p_0) = 1$ and

$$\mu(p_0) = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^2(p_0) = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^3(p_0) = -1, \quad \mu^4(p_0) = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$$

which implies that $\mu^k(p_0) \neq 1$ for $k = 1, 2, 3, 4$. Furthermore, we obtain

$$\frac{d}{dp}|\mu(p)| = -\frac{1}{\sqrt{2}}\left(\frac{1}{p+1}\right)^{3/2}, \quad \left.\frac{d|\mu(p)|}{dp}\right|_{p=p_0} = -\frac{1}{4} < 0.$$

The eigenvectors of $Jac_{\mathbf{F}}(0,0)$ corresponding to $\mu(p)$ and $\overline{\mu(p)}$ are $\mathbf{q}(p)$ and $\overline{\mathbf{q}(p)}$, where

$$\mathbf{q}(p) = \left(\frac{1-i\sqrt{2p+1}}{p+1}, 1\right)^T.$$

Substituting $p = p_0 = 1$ into (39) we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{20}$$

where

$$\mathbf{A} = Jac_{\mathbf{F}}(0,0)|_{p=1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{(v+2)^2}{(u+2)^2} + u - v - 1 \end{pmatrix}.$$

Hence, for $p = p_0$ system (37) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{21}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$, where $\mathbf{q} = \mathbf{q}(p_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right)\bar{z} + \frac{1}{2}\left(1 - i\sqrt{3}\right)z \\ \bar{z} + z \end{pmatrix}.$$

By using this, we have

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{(\bar{z}+z+2)^2}{\left(\frac{1}{2}\left(1+i\sqrt{3}\right)\bar{z}+\frac{1}{2}\left(1-i\sqrt{3}\right)z+2\right)^2} + \frac{1}{2}\left(-1 + i\sqrt{3}\right)\bar{z} - \frac{1}{2}\left(1 + i\sqrt{3}\right)z - 1 \end{pmatrix}. \tag{22}$$

Thus we obtain that

$$\begin{aligned} \mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \frac{1}{4}i\left(\sqrt{3} + 5i\right) \end{pmatrix}, \\ \mathbf{g}_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \mathbf{g}_{02} &= \frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ -\frac{1}{4}i\left(\sqrt{3} - 5i\right) \end{pmatrix}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{4} \\ \frac{5}{8} - \frac{i\sqrt{3}}{8} \end{pmatrix}, \\ \mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}. \end{aligned} \tag{24}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$ we have that

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20} z^2 + \mathbf{K}_{11} z\bar{z} + \frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ -\frac{i\sqrt{3}}{8} \end{pmatrix}. \quad (25)$$

It is easy to see that $\mathbf{pA} = \mu \mathbf{p}$ and $\mathbf{pq} = 1$ where

$$\mathbf{p} = \left(\frac{i}{\sqrt{3}}, \frac{1}{6}\left(3 - i\sqrt{3}\right)\right)$$

and

$$a(p_0) = \frac{1}{2} Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}) = -\frac{1}{16} < 0.$$

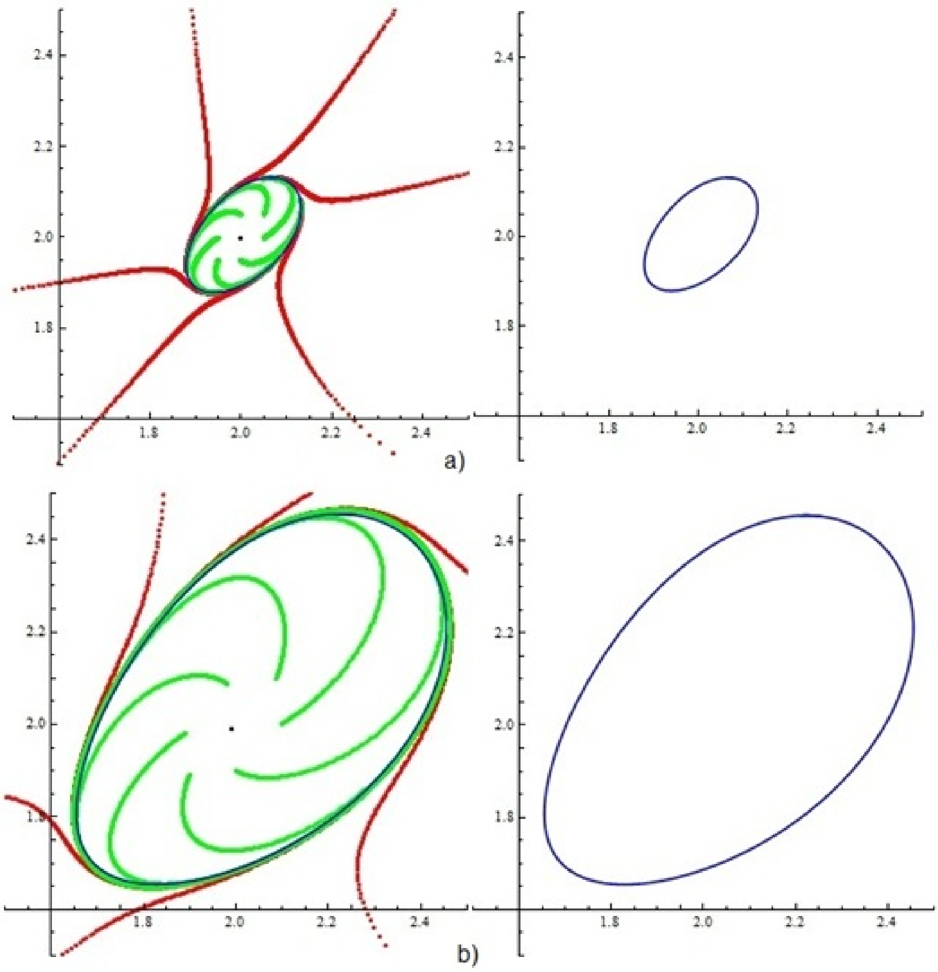

Figure 4. Trajectories with 10000 iterates and invariant curve given by (26) for a) $p = 0.999$ b) $p = 0.99$.

Thus I prove the following result:

Theorem 4 *There is a neighborhood U of the equilibrium point $\bar{x} = p + 1$ and a $\rho > 0$ such that for $|p - 1| < \rho$ and $x_0, x_{-1} \in U$, the ω-limit set of solution of equation (27), with initial condition x_0, x_{-1} is the equilibrium point $\bar{x}$ if $p > 1$ and belongs to a closed invariant C^1 curve $\Gamma(p)$ encircling the equilibrium point $\bar{x}$ if $p < 1$. Furthermore, $\Gamma(1) = 0$ and the invariant curve $\Gamma(p)$ can be approximated by the parametric equation*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} p + 1 + 2\sqrt{1-p}\left(\sqrt{3}\sin\theta + \cos\theta\right) - (p-1)\left(\sqrt{3}\sin 2\theta - 2\cos 2\theta + 4\right) \\ p + 1 + 4\sqrt{1-p}\cos\theta - \frac{1}{2}(p-1)\left(\sqrt{3}\sin 2\theta + 5\cos 2\theta + 8\right) \end{pmatrix}. \tag{26}$$

Proof. The proof follows from above discussion and Theorem 5 and Corollary 2. □

Figure 1 shows convergence to the equilibrium for the values of p slightly larger than 1 which visually confirms Conjecture. Figure 2 shows the bifurcation diagrams on the parametric interval $[0, 1.1]$ and $[0.90, 1.29]$ indicating that equation (26) might have some solutions which do not converge to the equilibrium. Figure 3 shows some periodic solutions for different values of p.

Figure 4 shows 10000 points of the solution and the invariant curve given with the approximate equation (26). Resemblance is remarkable.

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MANUSCRIPT 2

The Invariant Curve Caused by Naimark-Sacker Bifurcation of a Certain Difference Equation and Stability

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2.1 Introduction and Preliminaries

In this paper I consider the difference equation

$$x_{n+1} = \frac{x_n}{Cx_{n-1}^2 + Dx_n + F}, \quad n = 0, 1, \dots, \tag{27}$$

where the parameters C, D and F are positive numbers and the initial conditions x_{-1} and x_0 are positive numbers.

Equation (27) can be considered as a nonlinear perturbation of the Beverton-Holt difference equation

$$x_{n+1} = \frac{x_n}{Dx_n + F}, \quad n = 0, 1, \dots, \tag{28}$$

which is a major mathematical model in population dynamics see [1, 13]. Furthermore, it is similar in appearance to the linear fractional equation of the form

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + Dx_n + F}, \quad n = 0, 1, \dots, \tag{29}$$

which was considered in [5]. Both equations (28) and (29) exhibit a global asymptotic stability of either zero or positive equilibrium solutions and exchange of stability bifurcation. As we will see in this paper the introduction of quadratic term will substantially change dynamics and will introduce the existence of a locally stable periodic solution and possibly chaos. I will show that local asymptotic stability of the zero equilibrium will also implies its global asymptotic stability. In the case of the positive equilibrium solution I will show that such statement is true in some subspace of the parametric region of local asymptotic stability and I pose the conjecture that the same property holds in the complete region of local asymptotic stability. Our tool in proving global asymptotic stability of the positive equilibrium solution consists of embedding considered equation into higher order equation and using global attractivity results for maps with invariant boxes, see [3, 5, 7]. Related rational difference equations which exhibit similar behavior were considered in [4, 8].

Now, for the sake of completeness I give the basic facts about the Neimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle is generated.

In the discrete setting, the Neimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation. The Neimark-Sacker bifurcation occurs for a discrete system in the plane depending on a parameter, λ say, with a fixed point whose Jacobian matrix has a pair of complex conjugate eigenvalues $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which crosses the unit circle transversally at $\lambda = \lambda_0$. In this case the periodic solution, which is in general, of unknown period appears and is locally stable. In this paper we use Murakami computational approach, see [12] to find an asymptotic formula for an invariant locally attracting curve in the phase plane, which represents a periodic solution.

The following result is referred as the Neimark-Sacker bifurcation theorem, see [2, 6, 9, 11, 15].

Theorem 5 (Neimark-Sacker bifurcation) *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, \mathbf{0})$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with* $|\mu(\lambda_0)| = 1$*;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) > 0$ *at* $\lambda = \lambda_0$ *(transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for* $k = 1, 2, 3, 4$ *(nonresonance condition).*

Then there is a smooth λ*-dependent change of coordinate bringing* F *into the form*

$$F(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth functions $a(\lambda)$, $b(\lambda)$, *and* $\omega(\lambda)$ *so that in polar coordinates the function* $\mathcal{F}(\lambda, x)$ *is given by*

$$\mathcal{F} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{30}$$

If $a(\lambda_0) < 0$, *then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that for* $|\lambda - \lambda_0| < \delta$ *and* $x_0 \in U$, *then* ω*-limit set of* x_0 *is the origin if* $\lambda < \lambda_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $\lambda > \lambda_0$. *Furthermore,* $\Gamma(\lambda_0) = 0$.

If $a(\lambda_0) > 0$, *then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that for* $|\lambda - \lambda_0| < \delta$ *and* $x_0 \in U$, *then* α*-limit set of* x_0 *is the origin if* $\lambda > \lambda_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $\lambda < \lambda_0$. *Furthermore,* $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}) \tag{31}$$

where $\mathbf{A}$ is the Jacobian matrix of $\mathbf{F}$ evaluated at the fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here we denote $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be the eigenvectors of A associated with μ satisfying

$$\mathbf{A}\mathbf{q} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{p}\mathbf{q} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{aligned} \mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\ \mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11} \quad . \\ \mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} \end{aligned} \tag{32}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}z^2 + 2\mathbf{K}_{11}z\bar{z} + \mathbf{K}_{02}\bar{z}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}z^3 + 3\mathbf{g}_{21}z^2\bar{z} + 3\mathbf{g}_{12}z\bar{z}^2 + \mathbf{g}_{03}\bar{z}^3) + O(|z|^4), \end{aligned} \tag{33}$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

The next result of Murakami [12] gives an approximate formula for the periodic solution.

Corollary 2 *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficient small parameter. If $\bar{\mathbf{x}}$ is a fixed point of F then the invariant curve $\Gamma(\lambda)$ from Theorem 5 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\bigg|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "*Re*" represents the real parts of those complex numbers.

2.2 Local and global stability

The equilibrium solutions of Equation (27) is the positive solution of the equation $C\bar{x}^2 + D\bar{x} + F - 1 = 0$, that is

$$\bar{x} = \frac{\sqrt{D^2 + 4C(1-F)} - D}{2C}, \quad 0 < F < 1$$

and the origin $\bar{x}_0 = 0$. The linearized equation associated with Equation (27) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = pz_n + qz_{n-1}$$

where

$$p = f_u(\bar{x}, \bar{x}) \text{ and } q = f_v(\bar{x}, \bar{x}).$$

Now the following results hold:

Lemma 1 *For the equilibrium point $\bar{x}_0$ the following holds:*

(i) If $F > 1$ the equilibrium point $\bar{x}_0$ is locally asymptotically stable.

(ii) If $F < 1$ the equilibrium point $\bar{x}_0$ is a saddle point.

(ii) If $F = 1$ the equilibrium point $\bar{x}_0$ is non-hyperbolic.

(iv) If $F \geq 1$ the equilibrium point $\bar{x}_0$ is globally asymptotically stable.

The proof of part (iv) follows from the fact that every solution $\{x_n\}$ of Equation (27) satisfies

$$x_{n+1} = \frac{x_n}{Cx_{n-1}^2 + Dx_n + F} \leq x_n, \quad n = 0, 1, \ldots$$

which shows that $\{x_n\}$ is non-increasing sequence and so convergent. Consequently $\lim_{n\to\infty} x_n = 0$. The proofs of parts $(i) - (iii)$ are immediate.

Lemma 2 *The positive equilibrium $\bar{x}$ satisfies the following:*

(i) If $F < \frac{1}{2}$ and $C < \frac{2D^2}{(1-2F)^2}$ or $\frac{1}{2} \leq F < 1$, the equilibrium point $\bar{x}$ is locally asymptotically stable.

(ii) If $F < \frac{1}{2}$ and $C > \frac{2D^2}{(1-2F)^2}$, the equilibrium point $\bar{x}$ is a repeller.

(iii) If $F < \frac{1}{2}$ and $C = \frac{2D^2}{(1-2F)^2}$, the equilibrium point $\bar{x}$ is non-hyperbolic.

Proof. One can see that

$$p = f_u(\bar{x}, \bar{x}) = \frac{-D\sqrt{4C(1-F)+D^2} + 2C + D^2}{2C},$$

and

$$q = f_v(\bar{x}, \bar{x}) = -\frac{\left(D - \sqrt{4C(1-F)+D^2}\right)^2}{2C} < 0,$$

$$q - p - 1 = \frac{3D\left(\sqrt{4C(1-F)+D^2} - D\right) + 4C(F-2)}{2C} < 0,$$

$$q + p - 1 = \frac{D\left(\sqrt{4C(1-F)+D^2} - D\right) + 4C(F-1)}{2C} < 0,$$

$$q + 1 = \frac{D\sqrt{4C(1-F)+D^2} + 2CF - C - D^2}{C}.$$

The rest of the proof follows from Theorem 2.13 [6]. □

Now I give a global asymptotic stability result for the positive equilibrium solution. I will show that local asymptotic stability of the positive equilibrium will also imply its global asymptotic stability in substantial subregion of the parametric space.

Theorem 6 *Assume that $F < 1$ and*

$$C \leq \frac{3D^2}{4(1-F)}. \tag{34}$$

Then the positive equilibrium of Equation (27) is globally asymptotically stable.

Proof. Clearly we can consider solutions of Equation (27) which are positive, that is for which $x_0 > 0$. The substitution $y_n = \frac{D}{x_n}$ transforms Equation (27) into the equation

$$y_{n+1} = 1 + \left(F + \frac{C}{D^2 y_{n-1}^2}\right) y_n = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{35}$$

One can easily show that Equation (35) has a unique equilibrium $\bar{y} = \dfrac{D}{\bar{x}}$. I will show that $\bar{y}$ is globally asymptotically stable when $F<1$ and $C \leq \dfrac{3D^2}{4(1-F)}$. Our major tool is global asymptotic stability result in [5], more precisely Theorem 1.4.5 [5]. Now I will check the assumptions of this theorem.

1. Clearly $f(x, y)$ is non-decreasing in x and non-increasing in y.

2. There exists an interval I such that $f : I \times I \to I$. Indeed $I = \left[\dfrac{1}{1-F}\ ,\ U\right]$ where $U \geq \frac{D^2}{(1-F)(D^2 - C(1-F))}$.

   If $x, y \in I$ then

$$f(x, y) = 1 + \left(F + \frac{C}{D^2 y^2}\right) x \geq 1 + Fx \geq 1 + \frac{F}{1-F} = \frac{1}{1-F}.$$

   On the other hand for any $U \geq \frac{D^2}{(1-F)(D^2 - C(1-F))}$ we have $f(U, \dfrac{1}{1-F}) \leq U$. Therefore $f(x, y) \in I$, which shows that I is an invariant interval for f.

Next, consider the system of equations

$$\begin{cases} f(M, m) & = M \\ f(m, M) & = m \end{cases} \Leftrightarrow \begin{cases} M & = 1 + \left(F + \frac{C}{D^2 m^2}\right) M \\ m & = 1 + \left(F + \frac{C}{D^2 M^2}\right) m, \end{cases}$$

which is equivalent to :

$$\begin{cases} Mm^2(1-F)D^2 - CM & = m^2 D^2 \\ mM^2(1-F)D^2 - Cm & = M^2 D^2, \end{cases}$$

and show that $M = m$.

Subtracting the second equation from the first we get:

$$(1-F)D^2(m-M)Mm + C(m-M) = D^2(m-M)(m+M).$$

If $M \neq m$ then we have: $Mm(1-F)D^2 + C = D^2(M+m) \Leftrightarrow M = \dfrac{D^2m - C}{(m(1-F)-1)D^2}$, which implies

$$\frac{D^2m - C}{(m(1-F)-1)D^2} = \frac{1}{1-\left(F + \frac{C}{D^2m^2}\right)}.$$

Thus m satisfies the following quadratic equation:

$$D^2((1-F)C - D^2)m^2 + CD^2m - C^2 = 0,$$

with discriminant $\Delta = C^2D^2(4C(1-F) - 3D^2)$. Clearly for $C < \dfrac{3D^2}{4(1-F)}$ there are no real solutions and for $C = \frac{3D^2}{4(1-F)}$ there is the unique solution $m = \dfrac{2c}{D^2} = \bar{y}$.

Consequently, all conditions of Theorem 1.4.5 [5] are satisfied and every solution of Equation (35) which enters the interval I must converge to the unique equilibrium $\bar{y}$.

To show that $\bar{y}$ is globally asymptotically stable, it is sufficient to show that every solution of Equation (35) must enter I. Observe that by Equation (35)

$$y_{n+1} \geq 1 + \left(F + \frac{C}{D^2U^2}\right) y_n, \quad n = 0, 1, \ldots$$

and so by the result on difference inequalities, see [10]

$$y_n \geq \frac{1}{1-A} - \varepsilon, \quad A = F + \frac{C}{D^2U^2}, \varepsilon > 0$$

Since U can be chosen to be large there exists $N \geq 0$ such that $y_n \geq \dfrac{1}{1-F}$ for all $n \geq N$. Furthermore as we can choose $U \geq \frac{D^2}{(1-F)(D^2 - C(1-F))}$ to be as large as we wish, I can conclude that every solution of Equation (35) must enter and remain in I.

Thus I conclude that the unique positive equilibrium $\bar{x}$ of Eq.(27) is globally asymptotically stable for $C \leq \dfrac{3D^2}{4(1-F)}$.

□

Based on my simulation I state the following

Conjecture 2 *The equilibrium point $\bar{x}$ of equation (27) is globally asymptotically stable if it is locally asymptotically stable.*

2.3 Reduction to the normal form

In this section I bring the system that corresponds to Equation (27) to the normal form which can be used for computation of relevant coefficients of Neimark-Sacker bifurcation.

Assume that $0 < F < \frac{1}{2}$. If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{\bar{x} + y_n}{C\left(\bar{x} + y_{n-1}\right)^2 + D\left(\bar{x} + y_n\right) + F} - \bar{x}, \quad n = 0, 1, \ldots . \tag{36}$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write Equation (27) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{\bar{x} + v_n}{C\left(\bar{x} + u_n\right)^2 + D\left(\bar{x} + v_n\right) + F} - \bar{x}. \end{aligned} \tag{37}$$

Let $\mathbf{F}$ be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{\bar{x}+v}{C(\bar{x}+u)^2 + D(\bar{x}+v) + F} - \bar{x} \end{pmatrix}. \tag{38}$$

Then $\mathbf{F}$ has the unique fixed point $(0, 0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0, 0)$ is given by

$$Jac_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2C\bar{x}^2}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} & \frac{C\bar{x}^2 + F}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} \end{pmatrix}.$$

A straightforward calculation shows that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2C\bar{x}^2}{(C\bar{x}^2+D\bar{x}+F)^2} & \frac{C\bar{x}^2+F}{(C\bar{x}^2+D\bar{x}+F)^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{39}$$

where

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v+\bar{x}}{C(u+\bar{x})^2+F+D(v+\bar{x})} + \frac{\bar{x}\left(2Cu\bar{x}-(F+\bar{x}(D+C\bar{x}))^2\right)-v\left(C\bar{x}^2+F\right)}{(F+\bar{x}(D+C\bar{x}))^2} \end{pmatrix}.$$

The eigenvalues of $Jac_{\mathbf{F}}(0,0)$ are $\mu(C)$ and $\overline{\mu(C)}$ where

$$\mu(C) = \frac{\sqrt{2}\sqrt{\Delta}+2C+D^2-D\sqrt{4C(1-F)+D^2}}{4C}$$

where

$$\Delta = 2C^2(8F-7)-2CD^2(F+2)+D^4+(6CD-D^3)\sqrt{4C(1-F)+D^2}.$$

One can prove that for $C = C_0 = \frac{2D^2}{(1-2F)^2}$ we obtain $|\mu(C_0)| = 1$ and

$$\begin{aligned}
\mu(C_0) &= \frac{1}{4}\left(2F+1+i\sqrt{(3-2F)(2F+5)}\right), \\
\mu^2(C_0) &= \frac{1}{8}\left(4F^2+4F-7\right)+\frac{1}{8}i\sqrt{(3-2F)(2F+5)}(2F+1), \\
\mu^3(C_0) &= \frac{1}{16}\left(8F^3+12F^2-18F-11\right)+\frac{1}{16}i\sqrt{(3-2F)(2F+5)}\left(4F^2+4F-3\right), \\
\mu^4(C_0) &= \frac{F^4}{2}+F^3-\frac{5F^2}{4}+\frac{1}{32}i\sqrt{(3-2F)(2F+5)}\left(8F^3+12F^2-10F-7\right)-\frac{7F}{4}+\frac{17}{32}.
\end{aligned}$$

One can see that $\mu^k(C_0) \neq 1$ for $k = 1,2,3,4$ and

$$|\mu(C)|^2 = \frac{-D\sqrt{4C(1-F)+D^2}-2CF+2C+D^2}{C}.$$

Furthermore, we get

$$\frac{d}{dC}|\mu(C)| = \frac{D\sqrt{\frac{-D\sqrt{4C(1-F)+D^2}-2CF+2C+D^2}{C}}}{2C\sqrt{4C(1-F)+D^2}},$$

and

$$\left.\frac{d|\mu(C)|}{dC}\right|_{C=C_0} = \frac{(2F-1)^3}{4D^2(2F-3)} > 0.$$

The eigenvectors corresponding to $\mu(C)$ and $\overline{\mu(C)}$ are $\mathbf{q}(C)$ and $\overline{\mathbf{q}(C)}$, where

$$\mathbf{q} = \mathbf{q}(C_0) = \left(\frac{1}{4}\left(2F - i\sqrt{(3-2F)(2F+5)} + 1\right), 1\right)^T.$$

Substituting $C = C_0$ into (39) we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{40}$$

where

$$\mathbf{A} = Jac_{\mathbf{F}}(0,0)|_{C=C_0} = \begin{pmatrix} 0 & 1 \\ -1 & F+\frac{1}{2} \end{pmatrix}$$

and

$$\mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \dfrac{(2F-1)\left(2u(Du-2F+1)+\left(4F^2-1\right)v\right)}{2\left(4F^2(Dv+1)-4F(D(u+v)+1)+D(2u(Du+1)+v)+1\right)} - \left(F+\frac{1}{2}\right)v + u \end{pmatrix}.$$

Hence, for $C = C_0$ system (37) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{41}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$.

Let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{4}\left(2Fz + i\sqrt{(3-2F)(2F+5)}(\bar{z}-z) + 2F\bar{z} + z + \bar{z}\right) \\ \bar{z} + z \end{pmatrix}.$$

By using this, one can see that

$$\begin{aligned}
\mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{D\left(2F\left(-2F-3i\sqrt{(3-2F)(2F+5)}+2\right)+i\sqrt{(3-2F)(2F+5)}+11\right)}{8F-4} \end{pmatrix}, \\
\mathbf{g}_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ D \end{pmatrix}, \\
\mathbf{g}_{02} &= \frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{D\left(2F\left(-2F+3i\sqrt{(3-2F)(2F+5)}+2\right)-i\sqrt{(3-2F)(2F+5)}+11\right)}{8F-4} \end{pmatrix}
\end{aligned} \tag{42}$$

and

$$\mathbf{K}_{20} = (\mu^2 I - A)^{-1}\mathbf{g}_{20} =$$

$$= \begin{pmatrix} -\frac{8\left(-4iDF^2+6DF\sqrt{(3-2F)(2F+5)}-D\sqrt{(3-2F)(2F+5)}+4iDF+11iD\right)}{(2F-1)(4F^2-9)\left(-4iF^2+2\sqrt{(3-2F)(2F+5)}F+\sqrt{(3-2F)(2F+5)}-4iF+7i\right)} \\ \frac{\left(\sqrt{(3-2F)(2F+5)}-2iF-i\right)^2\left(-4iDF^2+6DF\sqrt{(3-2F)(2F+5)}-D\sqrt{(3-2F)(2F+5)}+4iDF+11iD\right)}{2(2F-1)(4F^2-9)\left(-4iF^2+2F\sqrt{(3-2F)(2F+5)}+\sqrt{(3-2F)(2F+5)}-4iF+7i\right)} \end{pmatrix},$$

$$\mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} \frac{2D}{3-2F} \\ \frac{2D}{3-2F} \end{pmatrix},$$

$$\mathbf{K}_{02} = (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}. \tag{43}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$ we have that

$$\begin{aligned} \mathbf{g}_{21} &= \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20}z^2 + \mathbf{K}_{11}z\bar{z} + \frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\Bigg|_{z=0} \\ &= \begin{pmatrix} 0 \\ \frac{4D^2\left(2F\left(6F+i\sqrt{(3-2F)(2F+5)}-16\right)+3i\sqrt{(3-2F)(2F+5)}+1\right)}{(1-2F)^2(4F^2-9)} \end{pmatrix}. \end{aligned} \tag{44}$$

Next we have that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$ where

$$\mathbf{p} = \left(\frac{2i}{\sqrt{(3-2F)(2F+5)}}, -\frac{(3-2F)(2F+5) - i(2F+1)\sqrt{(3-2F)(2F+5)}}{2\left(4F^2+4F-15\right)}\right).$$

One can see that

$$a(C_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}) = \frac{4D^2}{(1-2F)^2(2F-3)} < 0.$$

Theorem 7 *Let* $0 < F < \frac{1}{2}$ *and*

$$\bar{x} = \frac{\sqrt{D^2 + 4C(1-F)} - D}{2C}.$$

Then there is a neighborhood U *of the equilibrium point* $\bar{x}$ *and a* $\rho > 0$ *such that for*

$$\left| C - \frac{2D^2}{(1-2F)^2} \right| < \rho$$

and $x_0, x_{-1} \in U$*, the* ω*-limit set of solution of Equation* (27)*, with initial condition* x_0, x_{-1} *is equilibrium point* $\bar{x}$ *if*

$$C < \frac{2D^2}{(1-2F)^2}$$

and belongs to a closed invariant C^1 curve $\Gamma(C)$ encircling the equilibrium point $\bar{x}$ if

$$C > \frac{2D^2}{(1-2F)^2}.$$

Furthermore, $\Gamma(C_0) = 0$ and invariant curve $\Gamma(C)$ can be approximated by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right)$$

where

$$\rho_0 = \frac{(1-2F)^{3/2}\sqrt{C(1-2F)^2 - 2D^2}}{4D^2}.$$

Proof. The proof follows from above discussion and Theorem 5 and Corollary 2. See Figure 6 for a graphical illustration. □

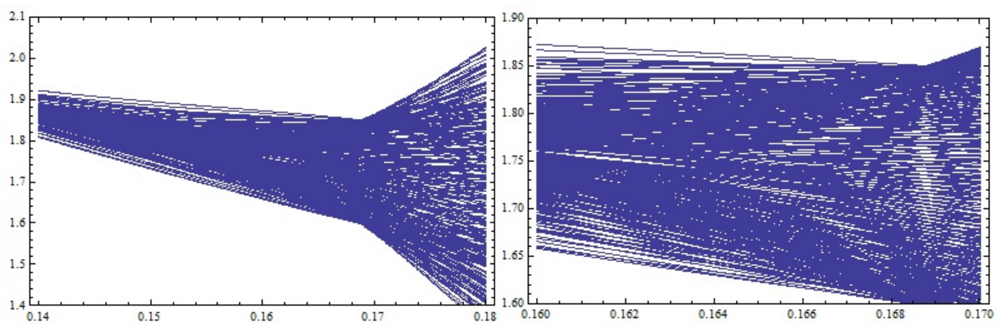

Figure 5. Bifurcation diagrams in (C, x) plane for $D = 0.11$, $F = 0.31$ and $C = 0.168$.

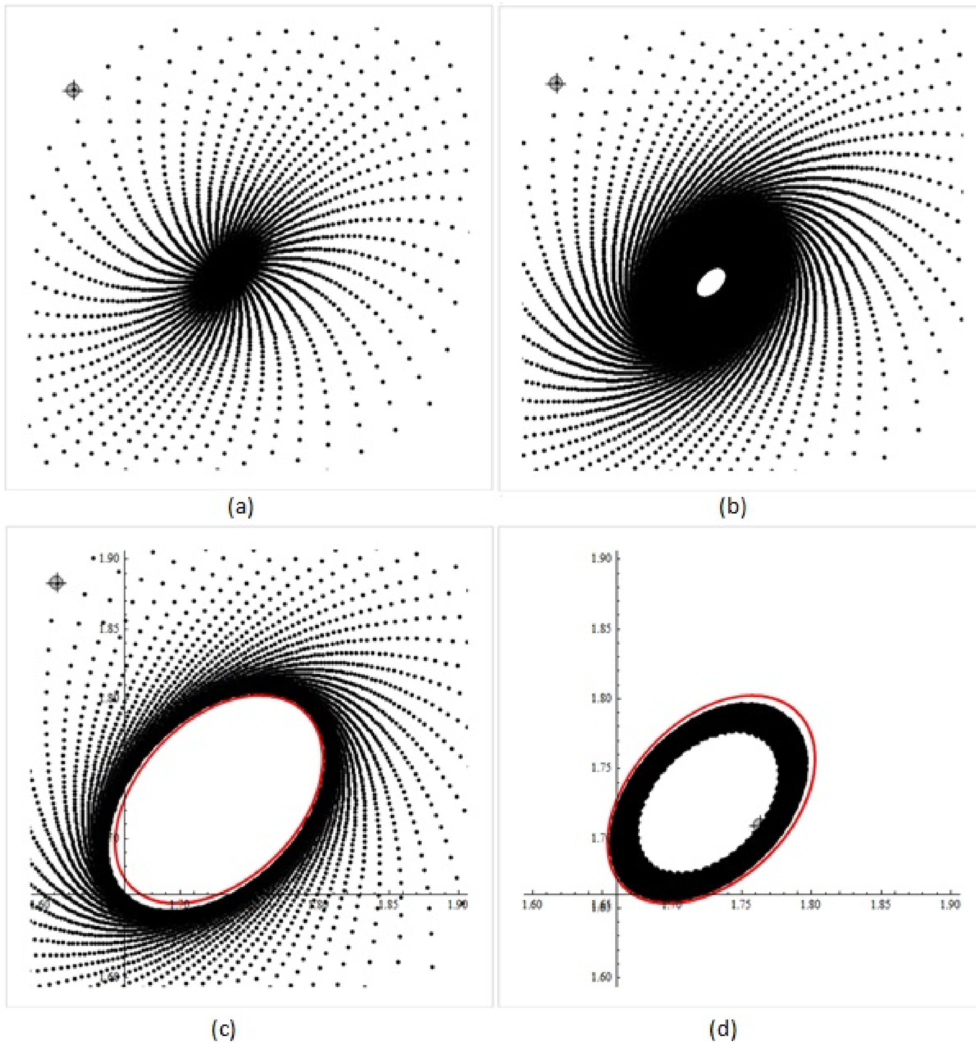

Figure 6. (a) Trajectory for $D = 0.11$, $F = 0.31$ and $C = 0.166$ where $C_0 = 0.16759$ b) Trajectory for $D = 0.11$, $F = 0.31$ and $C = C_0 = 0.16759$. (c)-(d) Trajectories and invariant curve (red) for $D = 0.11$, $F = 0.31$ and $C = 0.168$.

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MANUSCRIPT 3

Global Dynamics for Competitive Maps in the Plane

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3.1 Introduction

Consider the second-order quadratic fractional difference equation

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + dx_n + f}, \quad n = 0, 1, \ldots \tag{45}$$

where the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers and the parameters satisfy that $c, d, f{>}0$. Notice that Equation (45) is a special case of the equation

$$x_{n+1} = \frac{Cx_{n-1}^2 + Dx_{n-1} + F}{cx_n^2 + dx_n + f}, \quad n = 0, 1, \ldots \tag{46}$$

where the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers and the parameters satisfy $C, D, F, c, d, f \geq 0$, $C + D + F > 0$, $c + d + f > 0, c + D > 0$, and $C + d > 0$. For Equation (45) I will define precisely the basins of attraction of the equilibrium points and period-two solutions. My investigation of the global character of Equation (45) will depend on the theory of competitive systems.

Both Equations (45) and (46) are special cases of the general second-order quadratic fractional difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \ldots, \tag{47}$$

where all parameters are nonnegative numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that the solution is well-defined. Many special cases of Equation (47) have been studied in [1, 2, 11, 13] etc.

The first systematic study of global dynamics of a special quadratic fractional case of Equation (47) where $A = C = D = a = c = d = 0$ was performed in [1, 2]. Another special case of Equation (47)

$$x_{n+1} = \frac{x_{n-1}^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots$$

was given in [9] and uses the theory of monotone maps in the plane. Indeed, in [9] the unique coexistence of a unique locally asymptotically stable equilibrium point and a locally asymptotically stable minimal period-two solution was obtained. Equation (45), on the other hand, can have as many as three fixed points and up to three period-two solutions and its dynamics is similar to the dynamics of

$$x_{n+1} = \frac{x_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots$$

investigated in [18]. The possible dynamic scenarios for Equation (45) will be our motivation for getting the corresponding results for the general second order difference equation in section 3.3.

Many other interesting special cases of Equation (47) are studied in [11, 13, 14, 19, 20], which reveal the potential for rich dynamical behaviors that include the Allee effect, period-doubling bifurcation, Neimark-Sacker bifurcation, and chaos.

Equation (45) has an interesting special case when $d = 0$;

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots \tag{48}$$

the well-known sigmoid Beverton-Holt equation whose interesting dynmaics is given in [18]. Thus Equation (45) can be considered as a perturbation of Equation (48).

3.2 Preliminaries

In this section I provide some basic facts about competitive maps and systems of difference equations in the plane.

Definition 1 *Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T : R \to R$ be a map (i.e., a continuous function). Set $T(x, y) = (f(x, y), g(x, y))$. The map T is* competitive *if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. If both f and g are nondecreasing in x and y, we say that T is* cooperative*. If T is competitive (cooperative), the* *associated system of difference equations*

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in R \tag{49}$$

is said to be competitive (cooperative). The map T and associated difference equations system are said to be strongly competitive *(*strongly cooperative*) if the adjectives non-decreasing and non-increasing are replaced by increasing and decreasing.*

First I provide some theorems from [16, 17] used in [9] that will be of particular importance in my investigation of the global dynamics of Equation (45).

Theorem 8 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 10 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 9 *For the curve $\mathcal{C}$ of Theorem 8 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 8 reduces just to $|\lambda| < 1$. This follows from a change of variables that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 10 *(A) Assume the hypotheses of Theorem 8, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 8. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\},$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic

points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ *except for* $\overline{\mathrm{x}}$*, and the following statements are true.*

(iii) For every $x \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_2(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_4(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x,\ \text{and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected, can consist of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 11 *In addition to the hypotheses of part (B) of Theorem 10, suppose that* $\mu > 1$ *and that the eigenspace* E^μ *associated with* μ *is not a coordinate axis. If the curve* $\mathcal{C}$ *of Theorem 8 has endpoints in* $\partial\mathcal{R}$*, then* $\mathcal{C}$ *is the stable set* $\mathcal{W}^s(\overline{x})$ *of* $\overline{x}$*, and the unstable set* $\mathcal{W}^u(\overline{x})$ *of* $\overline{x}$ *is a curve in* $\mathcal{R}$ *that is tangential to* E^μ *at* $\overline{x}$ *and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of* $\mathcal{W}^u(\overline{x})$ *in* $\mathcal{R}$ *are fixed points of* T*.*

The following result is for strictly order preserving maps [23]. The result is stated for a partial order $\preceq$ in $\mathbb{R}^n$, but it also holds in Banach spaces.

Theorem 12 (Order Interval Trichotomy of Dancer and Hess, [23]) *Let $u_1 \preceq u_2$ be distinct fixed points of a strictly order preserving map $T : A \to A$, where $A \subset \mathbb{R}^n$, and let $I = [\![u_1, u_2]\!] \subset A$. Then at least one of the following holds.*

(a) *T has a fixed point in I distinct from u_1 and u_2.*

(b) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_1 to u_2 and satisfying $x_n \preceq x_{n+1}$.*

(c) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_2 to u_1 and satisfying $x_{n+1} \preceq x_n$.*

Corollary 3 *If a and b are stable fixed points, then there exists a third fixed point in $[\![a, b]\!]$.*

The following result is a direct consequence of Theorem 29.

Corollary 4 *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

A simple consequence of this result is the following

Corollary 5 *If monotone map T has exactly three fixed points $a \preceq b \preceq c$, where b is stable, then the interior of $[\![a, c]\!]$ is a subset of the basin of attraction of b.*

The following theorem from [5] applies to Equation (45):

Theorem 13 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad x_{-1}, x_0 \in I, \ n = 0, 1, ... \tag{50}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *of even and odd terms of the solution do exactly one of the following:*

(i) Eventually they are both monotonically increasing.

(ii) Eventually they are both monotonically decreasing.

(iii) One of them is monotonically increasing and the other is monotonically decreasing.

The consequence of Theorem 28 is that every bounded solution of Equation (50) converges to either an equilibrium, a period-two solution, or to the point on the boundary, so we try to determine the basins of attraction of these solutions.

Remark 1 We say that $f(u,v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (50) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (50) is a strictly competitive map on $I \times I$ (see [17]).

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation(50) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array} \quad , \qquad n = 0, 1, \ldots.$$

Let $T(u,v) = (v, f(v,u))$. The second iterate T^2 is given by

$$T^2(u,v) = (f(v,u),\ f(f(v,u),v))$$

and it is strictly competitive on $I \times I$, see [17].

Remark 2 The characteristic equation of Equation (50) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{51}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 8-11 depends on the nonexistence of a minimal period-two solution.

3.3 Main Results

In this section I present some global dynamics scenarios for competitive system (49) which will be applied to Equation (45).

Theorem 14 *Consider the competitive map T generated by the system* (49) *on a set $\mathcal{R}$ with a non-empty interior.*

(a) Assume that T has seven fixed points $E_1, \ldots, E_7$ such that five belongs to the west and south boundaries of the region $\mathcal{R}$ and two fixed points are interior points. Moreover assume that E_1 and E_2 belong to the west boundary, E_3 is South-west corner of the region $\mathcal{R}$ and E_4 and E_5 are on the south boundary of $\mathcal{R}$, such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$. Moreover assume that $E_6 \preceq_{ne} E_7$ and that $E_6 \notin [\![E_2, E_4]\!]$ and $E_7 \in [\![E_1, E_5]\!]$. Finally assume that E_1, E_3, E_5 are locally asymptotically stable, E_6 is a repeller and E_2, E_4 are saddle points. If E_7 is either a saddle point or non-hyperbolic point of stable type and T has no period-two solutions then all solutions which start between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converge to E_3, all solutions which start between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_7)$ converge to E_1 and all solutions which start between the stable manifolds $\mathcal{W}^s(E_4)$ and $\mathcal{W}^s(E_7)$ converge to E_5.

(b) Assume that T has exactly six fixed points $E_1, \ldots, E_6$, where the points have the same configuration and points $E_1, \ldots, E_5$ have the same local character as in part (a), while E_6 is non-hyperbolic point of unstable type. If T has no period-two solutions then there exist two increasing continuous curves C_1 and C_2, $C_2 \preceq_{se} C_1$ emanating from E_6 such that all solutions which start between C_1 and C_2 converge to E_6. Furthermore, all solutions which start between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converge to E_1, and all

solutions which start above $\mathcal{W}^s(E_2) \cup C_2$ converge to E_1 and all solutions which start below $\mathcal{W}^s(E_4) \cup C_1$ converge to E_5.

(c) *Assume that T has exactly nine fixed points $E_1, \ldots, E_9$, where the points $E_1, \ldots, E_7$ have the same configuration and the same local character as in (a). Assume that the fixed points E_8, E_9 are saddle points such that $E_8 \in [\![E_1, E_7]\!]$ and $E_9 \in [\![E_7, E_5]\!]$. Assume that T has no period-two solutions then all solutions which start between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converge to E_3, all solutions which start above $\mathcal{W}^s(E_2) \cup \mathcal{W}^s(E_8)$ converges to E_1, and all solutions which start below $\mathcal{W}^s(E_4) \cup \mathcal{W}^s(E_9)$ converge to E_5. Finally, all solutions which start between the stable manifolds $\mathcal{W}^s(E_8)$ and $\mathcal{W}^s(E_9)$ converge to E_7.*

Proof.

(a) The existence of the global stable and unstable manifolds of the saddle fixed points is guaranteed by Theorems 8 - 11. In any case all stable manifolds $\mathcal{W}^s(E_2), \mathcal{W}^s(E_4)$ and $\mathcal{W}^s(E_7)$ has an end point at E_6 and $\mathcal{W}^s(E_7)$ has another end point at (∞, ∞).

In view of Theorem 10 every solution which starts between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ eventually enters $int[\![E_2, E_4]\!]$ and so it converges to E_3.

If the initial point (x_0, y_0) is above $\mathcal{W}^s(E_2) \cup \mathcal{W}^s(E_7)$ one can find the point (x_l, y_l) on the y-axis and a point $(x_u, y_u) \in \mathcal{W}^s(E_2) \cup \mathcal{W}^s(E_7)$, such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$. This will imply that $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u))$, and so $T^n((x_0, y_0)) \in int[\![E_1, E_7]\!]$ eventually.

Now, in view of Corollary 7 $T^n((x_0, y_0)) \to E_1$ as $n \to \infty$.

In a similar way the case when the initial point (x_0, y_0) is below $\mathcal{W}^s(E_4) \cup \mathcal{W}^s(E_7)$ can be handled.

(b) The existence of the global stable and unstable manifolds of the saddle fixed points is guaranteed by Theorems 8 - 11. Both global stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ have an end point at E_6. The existence of curves C_1 and C_2 follows from Theorem 8. The proof that the region between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ eventually enters $int[\![E_2, E_4]\!]$ and so it converges to E_3 is the same as in part (a).

In a similar way as in the proof of part (a) we can show that if the initial point (x_0, y_0) is above $\mathcal{W}^s(E_2) \cup C_2$ it will eventually enter $int[\![E_1, E_6]\!]$ and so it will converge to E_1. In a similar way we can show that if the initial point (x_0, y_0) is below $\mathcal{W}^s(E_4) \cup C_1$ it will eventually enter $int[\![E_6, E_5]\!]$ and so it will converge to E_5.

Finally, if the initial point (x_0, y_0) is between C_1 and C_2 then one can find the point $(x_l, y_l) \in C_2$ and a point $(x_u, y_u) \in C_1$, such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$. This will imply that $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u))$, and so $T^n((x_0, y_0)) \to E_6$ as $T^n((x_u, y_u)) \to E_6, T^n((x_l, y_l)) \to E_6$.

(c) The proof that the region between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ is the basin of attraction of E_3 is same as in part (a) and will be omitted. The proof that all solutions which start above $\mathcal{W}^s(E_2) \cup \mathcal{W}^s(E_8)$ converges to E_1 and all solutions which start below $\mathcal{W}^s(E_4) \cup \mathcal{W}^s(E_9)$ converge to E_5 is same as in part (a) and so will be ommitted.

If the initial point (x_0, y_0) is between $\mathcal{W}^s(E_8)$ and $\mathcal{W}^s(E_9)$ then one can find the point $(x_l, y_l) \in \mathcal{W}^s(E_8)$ and a point $(x_u, y_u) \in \mathcal{W}^s(E_9)$, such that

$(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$. This will imply that $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u))$, and so $T^n((x_0, y_0)) \in int[\![E_8, E_9]\!]$ as $T^n((x_u, y_u)) \to E_9, T^n((x_l, y_l)) \to E_8$ as $n \to \infty$. Now in view of Corollary 8 $T^n((x_0, y_0)) \to E_7$ as $n \to \infty$.

□

In the case of Equation (50) we have the following results which are direct applications of Theorem 14.

Theorem 15 *Consider Equation* (50) *and assume that f is decreasing in first and increasing in the second variable on the set* $(a, b)^2$.

(a) *Assume that Equation* (50) *has three equilibrium points $E_0 \preceq_{ne} E_- \preceq_{ne} E_+$, where E_0 is locally asymptotically stable, E_- is repeller and E_+ is either saddle point or a non-hyperbolic point of stable type. Furthermore assume that Equation* (50) *has two minimal period-two solutions $\{P_1, Q_1\}, \{P_2, Q_2\}$ such that $P_2 \preceq_{se} P_1 \preceq_{se} E_0 \preceq_{se} Q_1 \preceq_{se} Q_2$ and $E_-, E_+ \in [\![P_2, Q_2]\!] \setminus [\![P_1, Q_1]\!]$. If $\{P_1, Q_1\}$ is a saddle point and $\{P_2, Q_2\}$ is a locally asymptotically stable, then every solution which starts between the stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ converges to E_0 while every solution which starts off $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1) \cup \mathcal{W}^s(E_+)$ converges to the periodic solution $\{P_2, Q_2\}$.*

(b) *Assume that Equation* (50) *has two equilibrium points $E_0 \preceq_{ne} E$, where E_0 is locally asymptotically stable and E_+ is a non-hyperbolic point of unstable type. Furthermore assume that Equation* (50) *has two minimal period-two solutions $\{P_1, Q_1\}, \{P_2, Q_2\}$ such that $P_2 \preceq_{se} P_1 \preceq_{se} E_0 \preceq_{se} Q_1 \preceq_{se} Q_2$ and $E_-, E_+ \in [\![P_2, Q_2]\!] \setminus [\![P_1, Q_1]\!]$. If $\{P_1, Q_1\}$ is a saddle point and $\{P_2, Q_2\}$ is a locally asymptotically stable, then every solution which starts between the stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ converges to E_0 while every solution* *which starts above $\mathcal{W}^s(P_1) \cup C_2$ and below $\mathcal{W}^s(Q_1) \cup C_1$ converges to the periodic solution $\{P_2, Q_2\}$. Finally every solution which starts between C_1 and C_2 converges to E.*

(c) *Assume that Equation* (50) *has three equilibrium points $E_0 \preceq_{ne} E_- \preceq_{ne} E_+$, where E_0 and E_+ are locally asymptotically stable and E_- is repeller. Furthermore assume that Equation* (50) *has three minimal period-two solutions $\{P_1, Q_1\}, \{P_2, Q_2\}, \{P_3, Q_3\}$ such that $P_2 \preceq_{se} P_1 \preceq_{se} E_0 \preceq_{se} Q_1 \preceq_{se} Q_2$ and $E_-, E_+ \in [\![P_2, Q_2]\!] \setminus [\![P_1, Q_1]\!]$ and $P_2 \preceq_{se} P_3 \preceq_{se} E_+ \preceq_{se} Q_3 \preceq_{se} Q_2$.*

If $\{P_1, Q_1\}$ and $\{P_3, Q_3\}$ are saddle points and $\{P_2, Q_2\}$ is a locally asymptotically stable, then every solution which starts between the stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ converges to E_0 and every solution which starts between the stable manifolds $\mathcal{W}^s(P_3)$ and $\mathcal{W}^s(Q_3)$ converges to E_7 while every solution which starts off in the complement of the basins of attractions of E_0, E_7, $\{P_1, Q_1\}$ and $\{P_3, Q_3\}$ converges to the periodic solution $\{P_2, Q_2\}$.

Proof. In view of Remark 1 the second iterate T^2 of the map T associated with Equation (50) is strictly competitive and does not have any period-two points.

(a) By noticing that the period-two points of T are the fixed points of T^2, two period-two solutions $\{P_1, Q_1\}, \{P_2, Q_2\}$ become four fixed points of T^2. Applying Theorem 14 part (a) to T^2 we complete the proof.

(b) In view of Remark 1 the second iterate T^2 of the map T associated with Equation (50) is strictly competitive and has six equilibrium points $E_1 = P_2, E_2 = P_1, E_3 = E_0, E_4 = Q_, E_5 = Q_2$ and $E_6 = E$. Applying Theorem 14 part (b) to T^2 we conclude that $\lim\limits_{n\to\infty} T^{2n}((x_0, y_0)) = E_0$ for every (x_0, y_0) between the stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$. Furthermore, we also have that $\lim\limits_{n\to\infty} T^{2n+1}((x_0, y_0)) = \lim\limits_{n\to\infty} T\left(T^{2n}\left((x_0, y_0)\right)\right) =$

$T\left(\lim_{n\to\infty} T^{2n}\left((x_0, y_0)\right)\right) = T(E_2) = E_2$, where we utilize continuity of the map T. Consequently $\lim_{n\to\infty} T^n((x_0, y_0)) = E_0$. The proof of other cases is similar.

(c) By noticing that the period-two points of T are the fixed points of T^2, three period-two solutions $\{P_1, Q_1\}, \{P_2, Q_2\}, \{P_3, Q_3\}$ become six fixed points of T^2. Applying Theorem 14 part (c) to T^2 we complete the proof.

□

3.4 Case Study: Equation (45)

3.4.1 Local stability analysis for Equilibria

An equilibrium point for of Eq.(1) must satisfy:

$$\bar{x} = \frac{\bar{x}^2}{c\bar{x}^2 + d\bar{x} + f}$$

i.e.

$$\bar{x} = 0 \quad \text{or} \quad c\bar{x}^2 + (d-1)\bar{x} + f = 0$$

Therefore Eq.(1) corresponds to the following:

1. The unique equilibrium $E_0 = (0, 0)$ if: $d \geq 1$ or ($d<1$ and $(d-1)^2 - 4fc < 0$)
2. Two equilibrium points E_0 and $E^* = (\frac{1-d}{2c}, \frac{1-d}{2c})$ if: $d<1$ and $(d-1)^2 - 4fc = 0$
3. Three equilibrium points E_0 and $E_\pm = (\frac{(1-d)\pm\sqrt{(1-d)^2-4fc}}{2c}, \frac{(1-d)\pm\sqrt{(1-d)^2-4fc}}{2c})$ otherwise.

If we denote $F(u, v) = \dfrac{v^2}{cv^2 + du + f}$ then Eq.(1) has the following linearized equation:

$$z_{n+1} = pz_n + qz_{n-1},$$

Where

$$p = \frac{\partial F}{\partial u}(\bar{x}, \bar{x}) = \frac{d\bar{x}^2}{(c\bar{x}^2 + d\bar{x} + f)^2}, \quad q = \frac{\partial F}{\partial v}(\bar{x}, \bar{x}) = \frac{2\bar{x}(d\bar{x} + f)}{(c\bar{x}^2 + d\bar{x} + f)^2}$$

If $\bar{x} = 0$ then clearly $p = q = 0$

If $\bar{x} \neq 0$ then by the equilibrium equation:

$$p = -d \quad \text{and} \quad q = 2(d + \frac{f}{\bar{x}}) = 2(1 - c\bar{x})$$

Proposition 1

Given that $c>0$, $d>0$, $f>0$

1. *The equilibrium E_0 is locally asymptotically stable for all values of parameter.*
2. *If $d<1$ and $(d-1)^2 - 4fc = 0$ then: The positive equlibrium point E^* is non-hyperbolic of unstable type.*
3. *If $d<1$ and $(d-1)^2 - 4fc>0$ then the equilibrium point E_- is a repeller while the stability of E_+ is subject to the following conditions:*
   (a) *E_+ is locally asymptotically stable if: $(d-1)^2 - 4fc>4d^2$*
   (b) *E_+ is non-hyperbolic of stable type if $(d-1)^2 - 4fc = 4d^2$*
   (c) *$E+$ is a saddle point if $(d-1)^2 - 4fc<4d^2$*

Proof.

1. Since $p = q = 0$ for $\bar{x} = 0$ then E_0 corresponds to the unique eigenvalue $\lambda = 0$, thus E_0 is locally asymptotically stable for all values of c ,d and f.
2. As $p = -d$ and $q = 2(1 - c\bar{x})$ then the chacteristic equation is given by:

$$\lambda^2 + d\lambda - 2(1 - x\bar{x}) = 0$$

and given that: $\bar{x}^* = \frac{1-d}{2c}$ we have:

$$\lambda^2 + d\lambda - (d+1) == 0$$

which corresponds to: $\lambda_+ = 1$ and $\lambda_- = -(d+1)$. The latter indicates clearly that E^* is non-hyperbolic of unstable type.

3. The roots of the characteristic equation are: $\lambda_+ = \frac{-d+\sqrt{d^2+8(1-c\bar{x})}}{2}>0$ and $\lambda_- = \frac{-d-\sqrt{d^2+8(1-c\bar{x})}}{2}<0$

For E_-:

Since $\bar{x}_-<\frac{1-d}{2c}$ one can easily check that: $\sqrt{d^2+8(1-c\bar{x}_-)} > (2+d)$ which implies $\lambda_+>1$.

On the other hand one can use the fact that $\bar{x}_-<\frac{1+d}{2c}$ to show that $\lambda_-<-1$.

As of E_+:

Since $\bar{x}_+>\frac{1-d}{2c}$ one can similarly show that $\lambda_+<1$.

Moreover a simple algebraic verification shows the following:

i $|\lambda_-|<1$ for $4d^2<(1-d)^2-4fc$

ii $|\lambda_-|=1$ for $4d^2=(1-d)^2-4fc$

iii $|\lambda_-|>1$ for $4d^2>(1-d)^2-4fc$

Consequently we conclude that: E_- is a repeller whenever it exists while:

i E_+ is locally asymptotically stable: for $4d^2<(1-d)^2-4fc$

ii E_+ is non-hyperbolic of stable type: for $4d^2=(1-d)^2-4fc$

iii E_+ is a saddle point: for $4d^2>(1-d)^2-4fc$

□

3.4.2 Local stability analysis of minimal period two solutions

Here I present the results about the existence and the stability of minimal period two solutions of Eq.(1)

Theorem 16

Given that $c>0$, $d>0$, $f>0$

1. *If* $4fc>1$ *then Eq.(1) has no minimal period two solutions.*

2. *If* $4fc=1$ *then Eq.(1) has a minimal period two solution:*
   $\{P_x=(\frac{1}{2c},0),P_y=(0,\frac{1}{2c})\}$

3. *If* $(d-1)^2-4d^2\leq 4fc<1$ *then Eq.(1) has two minimal period two solutions:*

   $$\left\{P_x^1\left(\tfrac{1-\sqrt{1-4cf}}{2c},0\right),P_y^1\left(0,\tfrac{1-\sqrt{1-4cf}}{2c}\right)\right\}\ \textit{and}\ \left\{P_x^2\left(\tfrac{1+\sqrt{1-4cf}}{2c},0\right),P_y^2\left(0,\tfrac{1+\sqrt{1-4cf}}{2c}\right)\right\}.$$

4. *If* $4fc<(d-1)^2-4d^2$ *then Eq.(1) has three minimal period two solutions:*
   $\left\{P_x^1,P_y^1\right\},\left\{P_x^2,P_y^2\right\}$ *and* $\left\{P^i\mp,P_{\pm}^i\right\}$ *where:*
   $P_{\mp}^i=(\frac{1+d-\sqrt{(1-d)^2-4d^2-4fc}}{2c},\frac{1+d+\sqrt{(1-d)^2-4d^2-4fc}}{2c})$
   and:
   $P_{\pm}^i=(\frac{1+d+\sqrt{(1-d)^2-4d^2-4fc}}{2c},\frac{1+d-\sqrt{(1-d)^2-4d^2-4fc}}{2c})$

Proof. For the sake of obtaining minimal period two solutions we must seek the ordered pairs (ϕ,ψ) that satisfy the following system of equations:

$$\begin{cases}\phi &= \dfrac{\phi^2}{c\phi^2+d\psi+f}\\[2em] \psi &= \dfrac{\psi^2}{c\psi^2+d\phi+f}\end{cases}$$

It follows that:

If $\psi=0$ the first equation becomes $c\phi^2-\phi+f=0\ldots(*)$

If $\phi = 0$ then the second equation turns into $c\psi^2 - \psi + f = 0$

If $\phi \neq 0$ and $\psi \neq 0$ then the system is equivalent to:

$$\begin{cases} c\phi^2 + d\psi - \phi + f &= 0 \\ c\psi^2 + d\phi - \psi + f &= 0 \end{cases} \quad \Rightarrow \quad c(\phi^2 - \psi^2) + d(\psi - \phi) + (\psi - \phi) = 0$$

Now given that $\phi \neq \psi$ we get $\phi = \frac{1+d}{c} - \psi$ which implies:

$$c\psi^2 - (1+d)\psi + \frac{1+d}{c} + f = 0 \ldots (**)$$

thus for solution of the form $(0, \psi)$ or $(\phi, 0)$ by equation(*) we must have $1-4fc \geq 0$ and for solutions of the form (ϕ, ψ), $\phi \neq 0$, $\psi \neq 0$ by equation (**) we must have $(1-d)^2 - 4d^2 - 4fc>0$. Consequently:

1. If $4fc>1$ then $1-4fc<0$ and $(1-d)^2-4d^2-4fc<0$.It follows that equations (*) and (**) have no real solutions, thus Eq.(1) has no minimal period two solutions.

2. If $4fc = 1$ equation (*) has the unique solution $\frac{1}{2c}$ and equation (**) has no real solutions. It follows that $\{P_x, P_y\}$ is the unique minimal period two solution of Eq.(1).

3. If $(1-d)^2 - 4d^2 \leq 4fc<1$: the quadratic equation (*) has two solutions $\frac{1\pm\sqrt{1-4fc}}{2c}$ while equation (**) has one unique solution which is an equilibria $\phi = \psi = \bar{x}_+$. Therefore Eq.(1) has two minimal period two solutions:

$$\left\{P_x^1\left(\frac{1-\sqrt{1-4cf}}{2c}, 0\right), P_y^1\left(0, \frac{1-\sqrt{1-4cf}}{2c}\right)\right\} \text{ and } \left\{P_x^2\left(\frac{1+\sqrt{1-4cf}}{2c}, 0\right), P_y^2\left(0, \frac{1+\sqrt{1-4cf}}{2c}\right)\right\}.$$

4. If $4fc<(1-d)^2-4d^2$ Then equation (*) has the solutions: $\frac{1\pm\sqrt{1-4fc}}{2c}$ while equation(**) has the solutions:$\frac{1+d\pm\sqrt{(1-d)^2-4d^2-4fc}}{2c}$.

   Therefore Eq.(1) has three minimal period two solutions:
   $\left\{P_x^1, P_y^1\right\}, \left\{P_x^2, P_y^2\right\}$ together with:
   $\left\{P_{\mp}^i = (\frac{1+d-\sqrt{(1-d)^2-4d^2-4fc}}{2c}, \frac{1+d+\sqrt{(1-d)^2-4d^2-4fc}}{2c})\right\}$
   $\left\{P_{\pm}^i = (\frac{1+d+\sqrt{(1-d)^2-4d^2-4fc}}{2c}, \frac{1+d-\sqrt{(1-d)^2-4d^2-4fc}}{2c})\right\}$.

$\square$

Now consider the following substitution: $u_n = x_{n-1}$ and $v_n = x_n$ then the behavior of the solutions of Eq.(1) can be investigated by the following two dimensional system:

$$\left\{ \begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = \dfrac{u_n^2}{cu_n^2 + dv_n + f} \end{array} \right. . \tag{52}$$

which corresponds to the following map:

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ h(u,v) \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{u^2}{cu^2+dv+f} \end{pmatrix}. \tag{53}$$

The second iteration of the map T is given by:

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} v \\ h(u,v) \end{pmatrix} = \begin{pmatrix} h(u,v) \\ h(v,h(u,v)) \end{pmatrix} = \begin{pmatrix} G(u,v) \\ H(u,v) \end{pmatrix},$$

where:

$$H(u,v) = \frac{v^2}{cv^2 + dh(u,v) + f}$$

clearly the map T^2 is competitive and its Jacobian matrix is given by:

$$J_{T^2}(u,v) = \begin{pmatrix} \dfrac{\partial G}{\partial u}(u,v) & \dfrac{\partial G}{\partial v}(u,v) \\ \dfrac{\partial H}{\partial u}(u,v) & \dfrac{\partial H}{\partial v}(u,v) \end{pmatrix},$$

Where:

$$\begin{aligned}
\frac{\partial G}{\partial u}(u,v) &= \frac{2u(f+dv)}{(cu^2+f+dv)^2} \\
\frac{\partial G}{\partial v}(u,v) &= -\frac{du^2}{(cu^2+f+dv)^2} \\
\frac{\partial H}{\partial u}(u,v) &= -\frac{2duv^2(f+dv)}{(du^2+(cu^2+f+dv)(cv^2+f))^2} \\
\frac{\partial H}{\partial v}(u,v) &= \frac{v\left(\frac{d\left(2cu^2+2f+3dv\right)u^2}{(cu^2+f+dv)^2}+2f\right)}{\left(\frac{du^2}{cu^2+f+dv}+cv^2+f\right)^2}
\end{aligned}$$

The following theorem describes the local stability of minimal period two solutions of Eq.(1) whenever they exist.

Theorem 17

1. *The minimal period two solutions* $\{P_x, P_y\}$ *are non-hyperbolic of stable type.*

2. *The minimal period two solutions* $\{P_x^1, P_y^1\}$ *are saddle points while* $\{P_x^2, P_y^2\}$ *are locally asymptotically stable.*

3. *The minimal period two solutions* $\{P_{\mp}^i, P_{\pm}^i\}$ *are saddle points.*

Proof.

1. The minimal period solutions $\{P_x, P_y\}$ exist when $4fc = 1$, thus the Jacobian matrix of the second iterate of the map T at P_x and P_y is the following:

$$J_{T^2}\left(P_x\right) = \begin{pmatrix} 1 & -d \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}\left(P_y\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

both correspond to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$ therefore $\{P_x, P_y\}$ are non-hyperbolic of stable type.

2. (a) For $\{P_x^1, P_y^1\}$; the Jacobian matrix is of the form:

$$J_{T^2}\left(P_x^1\right) = \begin{pmatrix} 1+\sqrt{1-4cf} & -d \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}\left(P_y^1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1+\sqrt{1-4cf} \end{pmatrix}.$$

Both with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1+\sqrt{1-4cf}$. Clearly $\{P_x^1, P_y^1\}$ are saddle points.

(b) As of $\{P_x^2, P_y^2\}$; the corresponding Jacobian matrix is given by:

$$J_{T^2}\left(P_x^2\right) = \begin{pmatrix} 1-\sqrt{1-4cf} & -d \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}\left(P_y^2\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1-\sqrt{1-4cf} \end{pmatrix}.$$

Both with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1 - \sqrt{1-4cf} < 1$. Clearly $\{P_x^2, P_y^2\}$ are locally asymptotically stable.

3. Now at the interior period two solutions $\{P_\mp^i, P_\pm^i\}$

$$J_{T^2}\left(P_\mp^i\right) = \begin{pmatrix} \frac{-dc+\sqrt{-c^2(d(3d+2)+4cf-1)}}{c}+1 & -d \\ -\frac{d\left(-dc+c+\sqrt{-c^2(d(3d+2)+4cf-1)}\right)}{c} & \frac{(d-1)dc-\sqrt{-c^2(d(3d+2)+4cf-1)}}{c}+1 \end{pmatrix}$$

and

$$J_{T^2}\left(P_\pm^i\right) = \begin{pmatrix} \frac{-dc-\sqrt{-c^2(d(3d+2)+4cf-1)}}{c}+1 & -d \\ \frac{d\left(c(d-1)+\sqrt{-c^2(d(3d+2)+4cf-1)}\right)}{c} & \frac{(d^2-d)c+\sqrt{-c^2(d(3d+2)+4cf-1)}}{c}+1 \end{pmatrix}$$

Observe that:

$p = TrJ_{T^2}\left(P^i_{\mp}\right) = TrJ_{T^2}\left(P^i_{\pm}\right) = 2 + d(d-2)$

$q = DetJ_{T^2}\left(P^i_{\mp}\right) = DetTrJ_{T^2}\left(P^i_{\pm}\right) = 4(d^2 + cf)$

Moreover:

$$|p|>|1+q| \Leftrightarrow (d-1)^2 - 4d^2 > 4cf \quad \text{(condition of existence)}$$

Consequently; the interior period two solutions $\{P^i_{\mp}, P^i_{\pm}\}$ are saddle points whenever they exist.

$\square$

Remark 3 *Observe that the interior period two solutions* $\{P^i_{\mp}, P^i_{\pm}\}$ *exist if and only if:*

1. $d<\frac{1}{3}$ *; since* $(d-1)^2 - 4d^2 > 4cf \Rightarrow (d-1)^2 - 4d^2 > 0 \Rightarrow (1-3d)(1+d) > 0$

2. *There are 3 equilibrium points* E_0, E_-, E_+ *where* E_+ *is L.A.S*

3.4.3 Global dynamics of Eq.(1)

In this section, I present the global dynamics for all the possible values of parameters of Eq.(1). First I provide the following three figures that describe all possible bifurcations produced by different values taken by $(4fc)$.

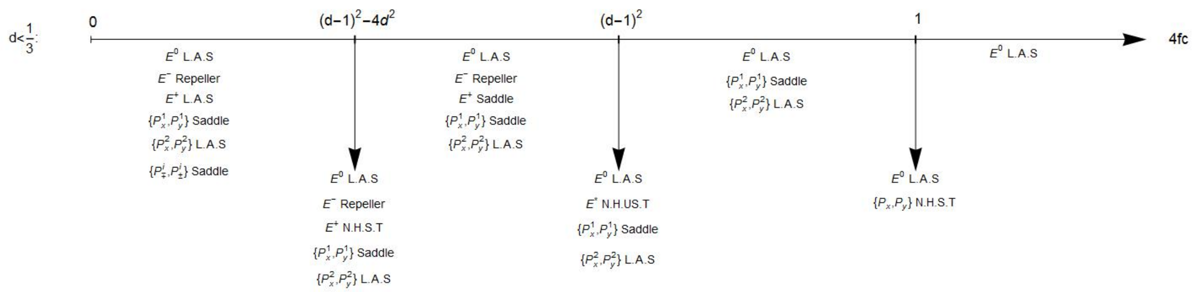

Figure 7. Bifurcation for: $d<\frac{1}{3}$

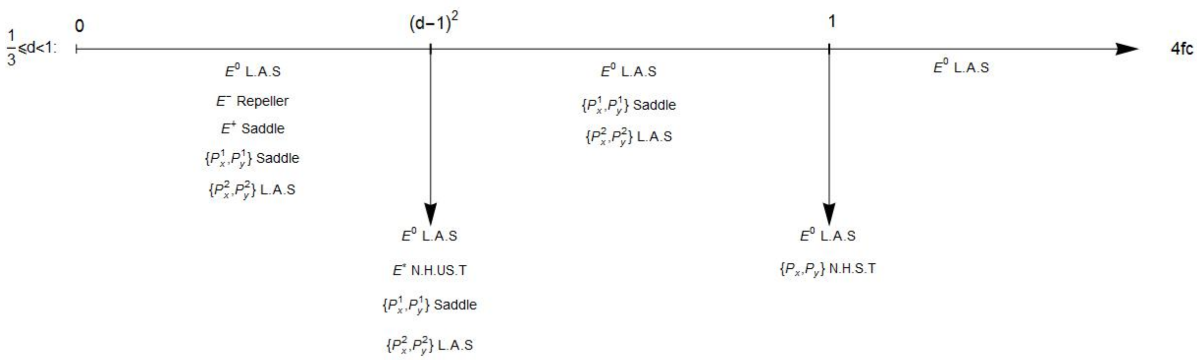

Figure 8. Bifurcation for: $\frac{1}{3} \leq d<1$

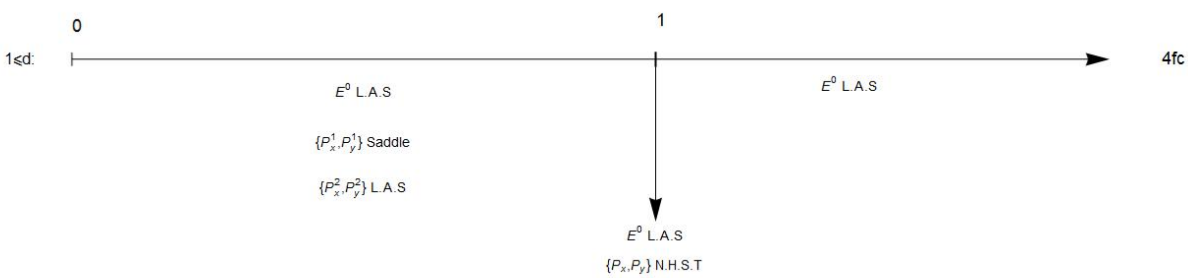

Figure 9. Bifurcation for: $1 \leq d$

Theorem 18

If $4fc>1$, then the equilibrium E_0 is globally asymptotically stable. see Figure 10,

Proof. First observe that every solution of Eq.(1) is bounded; as for $x_{n-1} \neq 0$

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + dx_n + f} = \frac{1}{c + \frac{dx_n}{x_{n-1}^2} + \frac{f}{x_{n-1}^2}} < \frac{1}{c}$$

Moreover, by theorem 28 subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ are eventually monotonic.

Now as $4fc>1$ there are no minimal period two solutions we conclude that both x_{2n} and x_{2n+1} must converge to the unique equilibrium $\bar{x} = 0$. $\square$

Theorem 19 *Let $\mathcal{B}(.)$ denote the basin of attraction.*

1. *The sets $[0, +\infty) \times \{0\}$ and $\{0\} \times [0, +\infty)$ are invariant by T^2.*
2. *If $4fc = 1$ then:*
   (a) $(\frac{1}{2c}, +\infty) \times \{0\} \subset \mathcal{B}(P_x)$
   (b) $\{0\} \times (\frac{1}{2c}, +\infty) \subset \mathcal{B}(P_y)$
   (c) $\{0\} \times (0, \frac{1}{2c}) \cup (0, \frac{1}{2c}) \times \{0\} \subset \mathcal{B}(E_0)$
3. *If $4fc<1$ then:*
   (a) $(\frac{1-\sqrt{1-4cf}}{2c}, +\infty) \times \{0\} \subset \mathcal{B}(P_x^2)$
   (b) $\{0\} \times (\frac{1-\sqrt{1-4cf}}{2c}, +\infty) \subset \mathcal{B}\left(P_y^2\right)$
   (c) $(0, \frac{1-\sqrt{1-4cf}}{2c}) \times \{0\} \cup \{0\} \times (0, \frac{1-\sqrt{1-4cf}}{2c}) \subset \mathcal{B}(E_0)$

Proof.

1. let $\alpha \geq 0$, then: $T^2(\alpha, 0) = (\frac{\alpha^2}{c\alpha^2+f}, 0)$ and $T^2(0, \alpha) = (0, \frac{\alpha^2}{c\alpha^2+f})$ which implies that: The sets $[0, +\infty) \times \{0\}$ and $\{0\} \times [0, +\infty)$ are invariant by T^2.

2. First recall that every solution of Eq.(1) is bounded and by theorem 6 every solution must either converge to an equilibrium or a minimal period two solution. It follows that every solution generated by T^2 must converge to an equilibrium. Now consider $\{s_n\}_{n=1}^{\infty}$ the solution with initial $s_1 = (x_1, 0) \in (\frac{1}{2c}, +\infty) \times \{0\}$ then:

   $T^2(x_n, 0) = (x_{n+1}, 0) = (\frac{x_n^2}{cx_n^2+f}, 0)$ and observe that: $x_{n+1} - xn = \frac{-x_n(cx_n^2-x_n+f)}{cx_n^2+f}$

   (a) If $4fc = 1$ one can easily show that x_n is monotone decreasing thus:

      i. If $x_n > \frac{1}{2c}$ then $s_n = (x_n, 0) \to (\frac{1}{2c}, 0)$

      ii. If $x_n < \frac{1}{2c}$ then $s_n = (x_n, 0) \to (0, 0)$

   (b) If $4fc<1$ one can easily show that x_b is monotone decreasing in: $(0, \frac{1-\sqrt{1-4fc}}{2c}) \cup (\frac{1+\sqrt{1-4fc}}{2c}, +\infty)$ and monotone increasing in $(\frac{1-\sqrt{1-4fc}}{2c}, \frac{1+\sqrt{1-4fc}}{2c})$ thus:

      i. If $x_n \in (\frac{1-\sqrt{1-4fc}}{2c}, +\infty)$ then $s_n = (x_n, 0) \to (\frac{1+\sqrt{1-4fc}}{2c}, 0)$.

      ii. If $x_n \in (0, \frac{1-\sqrt{1-4fc}}{2c})$ then $s_n = (x_n, 0) \to (0, 0)$.

   The remaining part of the proof follows similarly by considering solutions of the form:

   $s_n = (0, x_n)$.

□

Theorem 20

If $4fc = 1$ Then Eq.(1) has:

- *One Unique equilibrium point E_0 locally asymptotically stable.*

- *One minimal period-two solution* $\{P_x, P_y\} = \{(\frac{1}{2c}, 0), (0, \frac{1}{2c})\}$ *non-hyperbolic of stable type.*

There exist two invariant curves $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *which are graphs of strictly increasing continuous functions of the first coordinate on an interval with endpoints in* P_x *and* P_y *respectively. Basins of attraction of the minimal period-two solutions are*

$$\mathcal{B}(P_x) = \mathcal{C}_1 \cup \mathcal{W}^+(\mathcal{C}_1) \quad \text{and} \quad \mathcal{B}(P_y) = \mathcal{C}_2 \cup \mathcal{W}^-(\mathcal{C}_2),$$

while the basin of attraction of the equilibrium point E_0 *is the region between curves* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *i.e.*

$$\mathcal{B}(E_0) = \mathcal{W}^-(\mathcal{C}_1) \cap \mathcal{W}^+(\mathcal{C}_2).$$

See Figure 11.

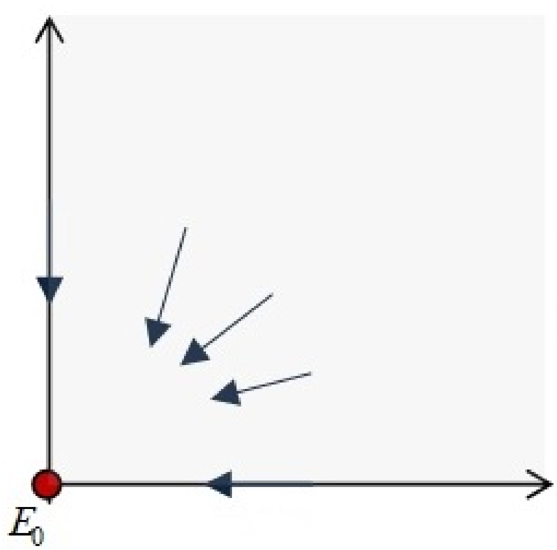

Figure 10. One equilibrium; $4fc>1$

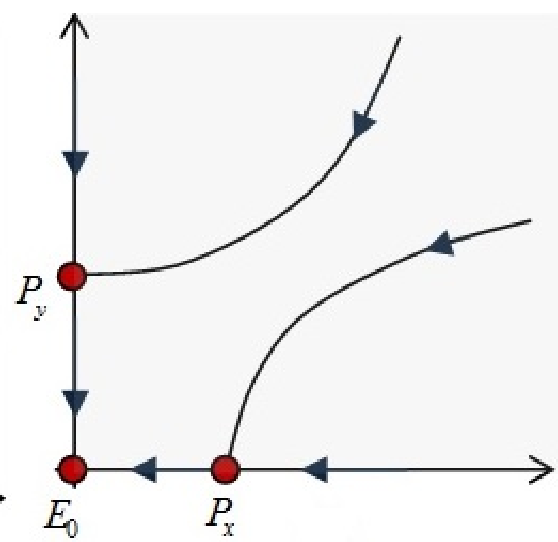

Figure 11. One equilibrium, one P-2; $4fc = 1$

Proof. The Jacobian matrix of the second iterate of the map at P_x is given by:

$$J_{T^2}(P_x) = \begin{pmatrix} 1 & -d \\ 0 & 0 \end{pmatrix}$$

with two eigenvalues $\lambda_1 = 0$ associated with the eigenvector $\begin{pmatrix} d \\ 1 \end{pmatrix}$ and $\lambda_2 = 1$ which corresponds the the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Observe that the eigenvector associated with λ_1 is not parallel to the x-axis and the map T^2 is strongly competitive. It follows by theorems 8-11 and 15; that there exists an invariant curve $\mathcal{C}_1$ through the point P_x which a subset of $\mathcal{W}^s(P_x)$. Moreover $\mathcal{C}_1$ is the graph of a strictly increasing continuous that separates the first quadrant into two connected subregions: an upper one $\mathcal{W}^-(\mathcal{C}_1)$ and a lower one $\mathcal{W}^+(\mathcal{C}_1)$ where:

$$\mathcal{B}(P_x) = \mathcal{C}_1 \cup \mathcal{W}^+(\mathcal{C}_1)$$

. As of the Jacobian matrix of P_y : $J_{T^2}(P_y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, it has two eigenvectors that are parallel to the coordinate axis; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponding to $\lambda_1 = 0$ and $\lambda_2 = 1$ respectively. By Hartman Grobman theorem [22], we know that there exist a C^1 curve $\mathcal{C}$ through P_y that is tangential at P_y to the eigenspace associated with $\lambda = 0$ such that $T^2(\mathcal{C}) \subset \mathcal{C}$.

Claim 1

The stable manifold at P_y is a linearly strongly ordered curve in the northeast ordering, where it is given for δ positive and small enough as : $\mathcal{W}^s_{loc}(P_y) = \{(t, \phi(t)) : 0 \le t \le \delta\}$

Proof.(of Claim 1)

First Recall that $(0, \frac{1}{2c}) \times (0, \frac{1}{2c}) \subset \mathcal{B}(E_0)$.

Now let $u_0>0$:

Since T^2 is strongly competitive we have: $T^2(P_y) \ll_{se} T^2\left(u_0, \frac{1}{2c}\right)$ and that implies: $T^2\left(u_0, \frac{1}{2c}\right) \in int\left(Q_4\left(P_y\right)\right)$. So, there exists a ball $B_\varepsilon\left(T^2\left(u_0, \frac{1}{2c}\right)\right)$such that: $B_\varepsilon\left(T^2\left(u_0, \frac{1}{2c}\right)\right) \subset int\left(Q_4\left(P_y\right)\right)$. Since the map T^2 is continuous on $\mathbb{R}^2_+$, there exists a ball $B_{\delta_1}\left(\left(u_0, \frac{1}{2c}\right)\right)$ such that:

$$T^2\left(B_{\delta_1}\left(\left(u_0, \frac{1}{2c}\right)\right)\right) \subset B_{\varepsilon}\left(T^2\left(u_0, \frac{1}{2c}\right)\right) \subset int\left(Q_4\left(P_y\right)\right),$$

which implies $T^{2n}(u,v) \to (0,0)$ when $n \to \infty$ for all points $(u,v) \in B_{\delta_1}\left(\left(u_0, \frac{1}{2c}\right)\right)$. It follows that: $\mathcal{W}^s_{loc}(P_y) \cap int\left(Q_4\left(P_y\right)\right) = \emptyset$ Now observe that $\phi'(0) = 0$ as its curve must be tangential to the horizontal eigenspace. Moreover $\phi'' \geq 0$ in a small neighborhood of $t = 0$ otherwise:

$\phi'' \leq 0 \Rightarrow$ there exists $\delta{>}0$ such that $\phi'(t)$ is decreasing in $(0, \delta) \Rightarrow \phi'(t) \leq 0$ in $(0, \delta)$

$\Rightarrow \phi(t) \leq \frac{1}{2c}$ in $(0, \delta)$ which contradicts the fact that $\mathcal{W}^s_{loc}(P_y) \cap int\left(Q_4\left(P_y\right)\right) = \emptyset$.

Therefore for sufficiently small δ_1: $\mathcal{W}^s_{loc}(P_y) = \{(t, \phi(t)) : 0 \leq t \leq \delta_1\}$ is linearly ordered in the northeast ordering and as T^2 is competitive: $\mathcal{W}^s_{loc}(P_y) \cap \mathbb{R}^2_+$ can be extended to an unbounded curve (global stable manifold) $\mathcal{C}_2$, see [16, 17].

□

Hence the curve $\mathcal{C}_2$ separates the region into two connected components an upper subregion $\mathcal{W}^-\left(\mathcal{C}_2\right)$ and a lower subregion $\mathcal{W}^+\left(\mathcal{C}_2\right)$.

Clearly the basin of attraction of P_y is $\mathcal{B}\left(P_y\right) = \mathcal{C}_2 \cup \mathcal{W}^-\left(\mathcal{C}_2\right)$, and finally the basin of attraction of the zero equilibrium E_0 is:

$$\mathcal{B}\left(E_0\right) = \mathcal{W}^+\left(\mathcal{C}_2\right) \cap \mathcal{W}^-(\mathcal{C}_1)$$

□

Theorem 21

If $(d \geq 1$ *and* $4fc{<}1)$ *or* $(d{<}1$ *and* $(d-1)^2{<}4fc{<}1)$, *then Eq.(1) has:*

- *One unique equilibrium point E_0 which is locally asymptotically stable.*

- *Two minimal period two solutions $\left\{P_x^1, P_y^1\right\}$ which are saddle points and $\left\{P_x^2, P_y^2\right\}$ which are Locally asymptotically stable.*

There exist global stable manifolds $\mathcal{W}^s\left(P_x^1\right)$ and $\mathcal{W}^s\left(P_y^1\right)$ which are basins of attractions of $\left\{P_x^1, P_y^1\right\}$ and the unstable manifolds have the following form:

$$\mathcal{W}^u\left(P_x^1\right) = \{(x,0) : x \in A\}, \mathcal{W}^u\left(P_y^1\right) = \{(0,y) : y \in A\}.$$

where $A = (0, \frac{1+\sqrt{1-4cf}}{2c}) \setminus \{\frac{1-\sqrt{1-4cf}}{2c}\}$

The basin of attraction of the equilibrium point $E_0 = (0,0)$ is the region between the global stable sets

$$\mathcal{B}(E_0) = \mathcal{W}^-\left(P_x^1\right) \cap \mathcal{W}^+\left(P_y^1\right).$$

Basin of attraction of the minimal period-two solutions $\left\{P_x^2, P_y^2\right\}$ is given with the following

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+\left(P_x^1\right),\ \mathcal{B}\left(P_y^2\right) = \mathcal{W}^-\left(P_y^1\right).$$

See Figure 12.

Proof. Recall that:

$$J_{T^2}\left(P_x^1\right) = \begin{pmatrix} 1+\sqrt{1-4cf} & -d \\ 0 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 0$ with eigenvector $\begin{pmatrix} \frac{d}{\sqrt{1-4\text{fc}}+1} \\ 1 \end{pmatrix}$ and $\lambda_2 = \sqrt{1-4\text{fc}}+1$ associated with the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus there exists a local stable manifold at P_x^1 that is linearly strongly ordered in the north east ordering with P_x^1 as an endpoint. As T^2 is competitive the local stable manifold can be extended to a curve $\mathcal{W}^s\left(P_x^1\right)$ which separates the region into two connected components $\mathcal{W}^+\left(P_x^1\right)$ and $\mathcal{W}^-\left(P_x^1\right)$. on the other hand

$$J_{T^2}\left(P_y^1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1+\sqrt{1-4cf} \end{pmatrix}.$$

with eigenvalues $\lambda_1 = 0$ with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = \sqrt{1-4\text{fc}} + 1$ associated with the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

By Theorem 19 we know that $(0, \frac{1-\sqrt{1-4fc}}{2c}) \times (0, \frac{1-\sqrt{1-4fc}}{2c}) \subset \mathcal{B}(E_0)$, thus we know that the local stable manifold at P_y^1 is tangential the horizontal Eigenspace but cannot enter the box $(0, \frac{1-\sqrt{1-4fc}}{2c}) \times (0, \frac{1-\sqrt{1-4fc}}{2c})$. I conclude that the local stable manifold at P_y^1 is a linearly strongly ordered curve (in the northeast ordering) with P_y^1 as an endpoint. Similarly we conclude its extension to a global stable manifold $\mathcal{W}^s\left(P_y^1\right)$ which separates the region into two connected components $\mathcal{W}^+\left(P_y^1\right)$ and $\mathcal{W}^-\left(P_y^1\right)$.

Finally by the uniqueness of the stable manifold of the saddle point P_x^1 we know that no solution in $\mathcal{W}^+(P_x^1)$ will converge to P_x^1, on the other hand all solutions are bounded and we know that by monotonicity of the map T every solution must converge to an equilibrium. It follows that $\mathcal{B}(P_x^2) = \mathcal{W}^+(P_x^1)$ and analogously $\mathcal{B}(P_y^2) = \mathcal{W}^+(P_y^1)$

□

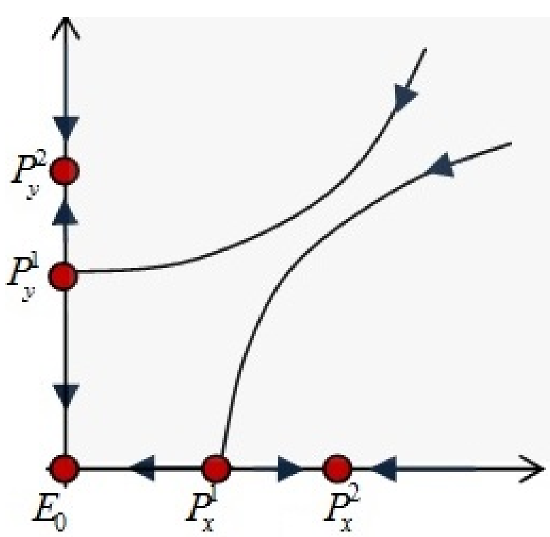

Figure 12. One equilibrium, two P-2

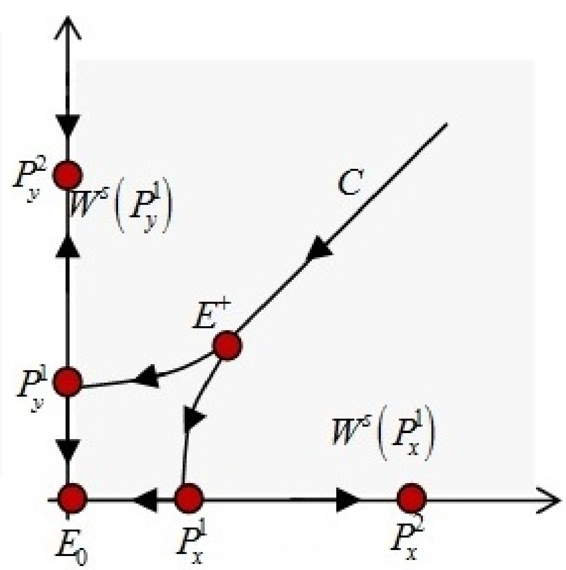

Figure 13. Two equilibriums, two P-2

Theorem 22

If $d<1$ and $(d-1)^2 = 4fc<1$, the Eq.(1) has:

- E_0 *is locally asymptotically stable,*
- $E^* = \left(\frac{1-d}{2c}, \frac{1-d}{2c}\right)$ *is a non-hyperbolic point of unstable type,*

  and two minimal period-two solutions:
- $\{P_x^1, P_y^1\} = \left\{\left(\frac{1-\sqrt{1-4fc}}{2c}, 0\right), \left(0, \frac{1-\sqrt{1-4fc}}{2c}\right)\right\}$ *are saddle points,*
- $\{P_x^2, P_y^2\} = \left\{\left(\frac{1+\sqrt{1-4fc}}{2c}, 0\right), \left(0, \frac{1+\sqrt{1-4fc}}{2c}\right)\right\}$ *are locally asymptotically stable.*
- *There exist global stable manifolds* $\mathcal{W}^s\left(P_x^1\right)$ *and* $\mathcal{W}^s\left(P_y^1\right)$ *which are the basins of attraction of the periodic solutions* $\{P_x^1, P_y^1\}$ *and which are tangential at the equilibrium point* E^*.
- *There exists a global stable manifold* $\mathcal{W}^s\left(E^*\right)$ *contained in* $\mathcal{Q}_1(E^*)$ *which is the basin of attraction of the equilibrium* E^*.

  The Basin of attraction of equilibrium point E_0 *is the region between those stable manifolds i.e.*

$$\mathcal{B}\left(E_0\right) = \mathcal{W}^-\left(P_x^1\right) \cap \mathcal{W}^+\left(P_y^1\right).$$

  The basins of attraction of P_x^2 *and* P_y^2 *are given by:*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+\left(S_x\right), \text{ where } S_x = \mathcal{W}^s\left(P_x^1\right) \cup \mathcal{W}^s\left(E^*\right),$$

$$\mathcal{B}\left(P_y^2\right) = \mathcal{W}^-\left(S_y\right), \text{ where } S_y = \mathcal{W}^s\left(P_y^1\right) \cup \mathcal{W}^s\left(E^*\right).$$

See Figure 13.

Proof.

The existence and orientation of the global stable manifold at P_x^1 can be determined as in theorem 21. However in general the orientation can be determined by studying the curvature of the local curves given by:
$\mathcal{W}_{loc}^s\left(P_y^1\right) = \{(t, \phi_1(t) : 0 \leq t \leq \delta_1\}$ and $\mathcal{W}_{loc}^s\left(P_x^1\right) = \{(\phi_2(t), t) : 0 \leq t \leq \delta_2\}$ for δ_1 and δ_2 small enough, where if $f(x,y)$ and $g(x,y)$ are the coordinate functions of T^2:

$$\begin{aligned}
\phi_1(f(t,\phi_1(t))) =& g(t,\phi_1(t)), \quad \phi_1(0) = \frac{1-\sqrt{1-4fc}}{2c}, \quad \phi_1'(0) = 0. \\
\phi_2(g(\phi_2(t),t)) =& f(\phi_2(t),t), \quad \phi_2(0) = \frac{1-\sqrt{1-4fc}}{2c}, \quad \phi_2'(0) = 0
\end{aligned}$$

This is useful when the local curve is tangent parallel to the axis at the fixed point which is the case here for P_y^1.
By differentiating both sides of the equation above we get:

$$\phi_1''(0) = \frac{g_{xx}(0,\phi_1(0))}{(f_x(0,\phi_1(0)))^2 - g_y(0,\phi_1(0))} = \frac{d(1-\sqrt{1-4fc})}{f(d+2cf-d\sqrt{1-4fc})} > 0$$

which confirms the argument used in theorem 21. I conclude the existence of curves $\mathcal{C}_1$ and $\mathcal{C}_2$ (global stable manifolds) which are linearly ordered in the northeast ordering. Furthermore the curves cannot intersect the interior of the sets $\mathcal{Q}_2(E^*)\cap\mathbb{R}^2$ and $\mathcal{Q}_4(E^*)\cap\mathbb{R}^2$, as the monotonicity of T^2 forces the latter sets to be invariant. Thus $T^{-2n}(P) \to E^*$ for all $P \in \mathcal{C}_l, \quad l = 1,2$, therefore $\mathcal{C}_1$ and $\mathcal{C}_2$ are also center manifolds of E^*.
On the other hand by letting $T^2(f(x,y), g(x,y)$ the center manifold $\phi(x)$ must satisfy:

$$\phi(f(x,\phi(x))) = g(x,\phi(x))$$

By using a Taylor expansion substitution we can approximate the center manifold by:

$$\phi(x) = x - \frac{c\left(x - \frac{1-d}{2c}\right)^2}{d+2} - \frac{2c^2\left(x - \frac{1-d}{2c}\right)^3}{(d+2)^3} + O\left(|x - \tfrac{1-d}{2c}|^4\right)$$

The dynamics on the center manifold are given by the reduced difference equation $u_{n+1} = f(u_n, \phi(u_n))$ which has the following asymptotic representation:

$$u_{n+1} = \frac{1-d}{2c} + \left(u_n - \frac{1-d}{2c}\right) - \frac{2c\left(u_n - \frac{1-d}{2c}\right)^2}{d+2} + \frac{2c^2 d\left(u_n - \frac{1-d}{2c}\right)^3}{(d+2)^3} - ... + O\left(|u_n - \frac{1-d}{2c}|^5\right)$$

Clearly $\bar{u} = \frac{1-d}{2c}$ is a semi-stable fixed point for the latter scalar difference equation, it follows that E^* is a semi=stable fixed point for T^2, furthermore the coefficient of the lowest nonlinear term in the reduced map is negative thus by [21] the local basin of attraction of the equilibrium E^* is a one dimensional curve. I conclude that there is a unique center manifold curve $\mathcal{U}$ which satisfies $T^2(\mathcal{U}) \subset \mathcal{U}$. Moreover $\mathcal{U}$ is tangential to the eigenspace associated with $\lambda = 1$ namely $\mathcal{S}pan\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. It follows that $\mathcal{U}$ is contained in $\mathcal{Q}_1(E^*)$ and is linearly ordered in the northeast ordering and therefore can be extended to an unbounded curve $\mathcal{C}$ (The global stable manifold).

Now for all point $q \in \mathcal{W}^-(P_x^1) \cap \mathcal{W}^+\left(P_y^1\right)$ there exist $q_x \in \mathcal{W}^s(P_x^1)$ and $q_y \in \mathcal{W}^s\left(P_y^1\right)$ such that:

$q_y \preceq_{se} q \preceq_{se} q_x$ which implies that: $T^{2n}(q_y) \preceq_{se} T^{2n}(q) \preceq_{se} T^{2n}(q_x)$, but we know that:

$$T^{2n}(q_y) \to P_y^1 \text{ and } T^{2n}(q_x) \to P_x^1$$

Consequently there exist N such that:

$$P_y^1 \preceq_{se} T^{2N}(q) \preceq_{se} P_x^1$$

It follows by theorem 19 that $q \in \mathcal{B}(E_0)$. As of the basins of attractions of P_x^2 and P_y^2 The proof is analogous to the one given in theorem 21. □

Theorem 23

If $d<1$ *and* $(d-1)^2-4d^2<4fc<(d-1)^2$*, then Eq.(1) has:*

Three equilibrium points

- E_0 *: locally asymptotically stable,*
- E_- *: repeller,*
- E_+ *: is a saddle point*

and two minimal period-two solutions:

- $\left\{P_x^1, P_y^1\right\}$ *: saddle points,*
- $\left\{P_x^2, P_y^2\right\}$ *: locally asymptotically stable.*
- *There exist global stable manifolds* $\mathcal{W}^s\left(P_x^1\right)$ *and* $\mathcal{W}^s\left(P_y^1\right)$ *which are the basins of attraction of the periodic solutions* $\{P_x^1, P_y^1\}$ *and which are tangential at the equilibrium point* E_-*.*
- *There exists a global stable manifold* $\mathcal{W}^s\left(E_+\right)$ *which is an unbounded curve of an increasing function contained in* $\mathcal{Q}_1(E_+)\cup\mathcal{Q}_3(E_+)$ *with an endpoint at* E_-*.* $\mathcal{W}^s\left(E_+\right)$ *is the basin of attraction of the equilibrium* E_+*.*
- *There exist a global unstable manifold* $\mathcal{W}^u\left(E_+\right)$ *which is a curve of a decreasing function contained in* $\mathcal{Q}_2(E_+)\cup\mathcal{Q}_4(E_+)$ *with endpoints* P_x^2 *and* P_y^2*.*

- *The Basin of attraction of equilibrium point* E_0 *is the region between those stable manifolds i.e.*

$$\mathcal{B}\left(E_0\right)=\mathcal{W}^-\left(P_x^1\right)\cap\mathcal{W}^+\left(P_y^1\right).$$

- *The basins of attraction of P_x^2 and P_y^2 are given by:*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+\left(S_x\right), \text{ where } S_x = \mathcal{W}^s\left(P_x^1\right) \cup \mathcal{W}^s\left(E_+\right),$$

$$\mathcal{B}\left(P_y^2\right) = \mathcal{W}^-\left(S_y\right), \text{ where } S_y = \mathcal{W}^s\left(P_y^1\right) \cup \mathcal{W}^s\left(E_+\right).$$

See Figure 14.

Proof.

The existence of $\mathcal{W}^s(P_x^1)$ and $\mathcal{W}^s(P_y^1)$ as well as the basin of attraction of E_0 were discussed in theorem 22.

The existence of the stable $\mathcal{W}^s\left(E_+\right)$ and the unstable manifold $\mathcal{W}^u\left(E_+\right)$ follows from theorem8-11 and 15. The basins of attraction of P_x^2 and P_y^2 were discussed in theorem 21. $\square$

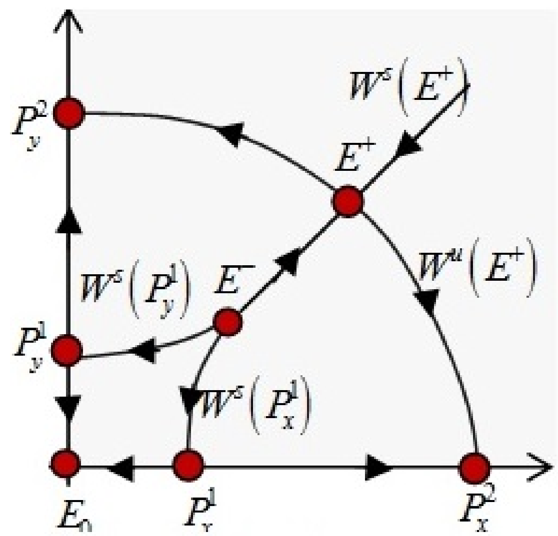

Figure 14. Three equilibriums, two P-2

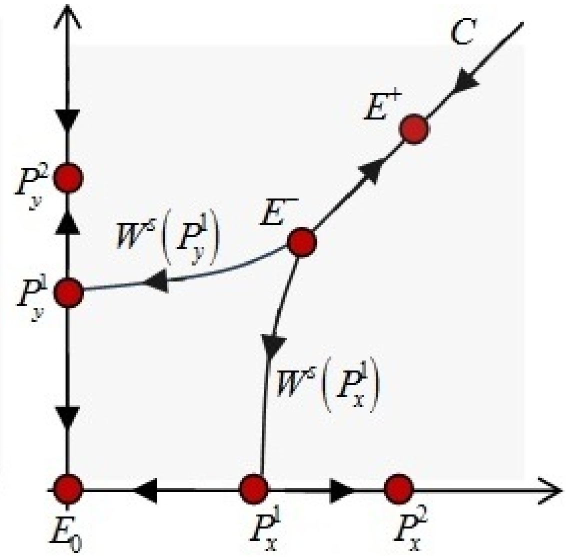

Figure 15. Three equilibriums, two P-2

Theorem 24

If $d<1$ and $0<(d-1)^2-4d^2=4fc$, then Eq.(1) has:

Three equilibrium points

- E_0 *: locally asymptotically stable,*
- $E_- = \left(\frac{1-3d}{2c}, \frac{1-3d}{2c}\right)$ *: repeller,*
- $E_+ = \left(\frac{1+d}{2c}, \frac{1+d}{2c}\right)$ *: non-hyperbolic point of stable type*

and two minimal period-two solutions:

- $\{P_x^1, P_y^1\}$ *: saddle points and*
- $\{P_x^2, P_y^2\}$ *: locally asymptotically stable.*
- *There exist global stable manifolds* $\mathcal{W}^s\left(P_x^1\right)$ *and* $\mathcal{W}^s\left(P_y^1\right)$ *which are the basins of attraction of the periodic solutions* $\{P_x^1, P_y^1\}$ *and which are tangential at the equilibrium point* E_-.

- *There exists a global stable manifold* $\mathcal{W}^s(E_+)$ *which is an unbounded curve of an increasing function contained in* $\mathcal{Q}_1(E_+) \cup \mathcal{Q}_3(E_+)$ *with an endpoint at* E_-. $\mathcal{W}^s(E_+)$ *is the basin of attraction of the equilibrium* E_+.

- *The Basin of attraction of equilibrium point* E_0 *is the region between those stable manifolds i.e.*

$$\mathcal{B}(E_0) = \mathcal{W}^-\left(P_x^1\right) \cap \mathcal{W}^+\left(P_y^1\right).$$

- *The basins of attraction of* P_x^2 *and* P_y^2 *are given by:*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+(S_x), \ \textit{where } S_x = \mathcal{W}^s\left(P_x^1\right) \cup \mathcal{W}^s(E_+),$$

$$\mathcal{B}\left(P_y^2\right) = \mathcal{W}^-(S_y), \ \textit{where } S_y = \mathcal{W}^s\left(P_y^1\right) \cup \mathcal{W}^s(E_+).$$

See Figure 15.

Proof.

The existence and orientation of the global stable manifold $\mathcal{W}^s(E_+)$ follows from theorems 8-11 and 15. The remaining of the proof is analogous to the discussions in theorems 21 and 22. $\square$

Theorem 25

If $d<1$ *and* $4fc<(d-1)^2-4fc$ *then Eq.(1) has:*

Three equilibrium points:

- E_0 *is locally asymptotically stable,*

- E_- *is repeller,*

- E_+ *is locally asymptotically stable,* *and three minimal period-two solutions:*

- $\left\{P_x^1, P_y^1\right\}$ *are saddle points,*

- $\left\{P_x^2, P_y^2\right\}$ *are locally asymptotically stable,*

- $\left\{P_{\mp}^i, P_{\pm}^i\right\}$ *are saddle points.*

- *There exist global stable manifolds* $\mathcal{W}^s\left(P_x^1\right)$ *and* $\mathcal{W}^s\left(P_y^1\right)$ *which are the basins of attraction of the periodic solutions* $\{P_x^1, P_y^1\}$ *and which are tangential at the equilibrium point* E_-.

- *The Basin of attraction of equilibrium point* E_0 *is the region between those stable manifolds i.e.*

$$\mathcal{B}\left(E_0\right) = \mathcal{W}^-\left(P_x^1\right) \cap \mathcal{W}^+\left(P_y^1\right).$$

- *There exists a global stable manifold* $\mathcal{W}^s\left(P_{\mp}^i\right)$ *which is an unbounded curve of an increasing function contained in* $\mathcal{Q}_1((P_{\mp}^i) \cup \mathcal{Q}_3((P_{\mp}^i)$ *with an endpoint at* E_-. $\mathcal{W}^s\left(P_{\mp}^i\right)$ *is the basin of attraction of the equilibrium* $P_{\mp}^i$.

- *There exists a global stable manifold* $\mathcal{W}^s\left(P_{\pm}^i\right)$ *which is an unbounded curve of an increasing function contained in* $\mathcal{Q}_1((P_{\pm}^i) \cup \mathcal{Q}_3((P_{\pm}^i)$ *with an endpoint at* E_-. $\mathcal{W}^s\left(P_{\pm}^i\right)$ *is the basin of attraction of the equilibrium* $P_{\pm}^i$.

- *The Basin of attraction of equilibrium point* E_+ *is the region between those stable manifolds i.e.*

$$\mathcal{B}\left(E_+\right) = \mathcal{W}^-\left(P_{\pm}^i\right) \cap \mathcal{W}^+\left(P_{\mp}^i\right).$$

- *There exist a global unstable manifold* $\mathcal{W}^u\left(P_{\mp}^i\right)$ *which is a curve of a decreasing function contained in* $\mathcal{Q}_2(P_{\mp}^i) \cup \mathcal{Q}_4(P_{\mp}^i)$ *with endpoints* P_y^2 *and* E_+.

- *There exist a global unstable manifold $\mathcal{W}^u\left(P^i_\pm\right)$ which is a curve of a decreasing function contained in $\mathcal{Q}_2(P^i_\pm) \cup \mathcal{Q}_4(P^i_\pm)$ with endpoints P^2_x and E_+.*

- *The basins of attraction of P^2_x and P^2_y are given by:*

$$\mathcal{B}\left(P^2_x\right) = \mathcal{W}^+\left(S_x\right), \ \textit{where } S_x = \mathcal{W}^s\left(P^1_x\right) \cup \mathcal{W}^s\left(P^i_\pm\right),$$

$$\mathcal{B}\left(P^2_y\right) = \mathcal{W}^-\left(S_y\right), \ \textit{where } S_y = \mathcal{W}^s\left(P^1_y\right) \cup \mathcal{W}^s\left(P^i_\pm\right).$$

See Figure 16.

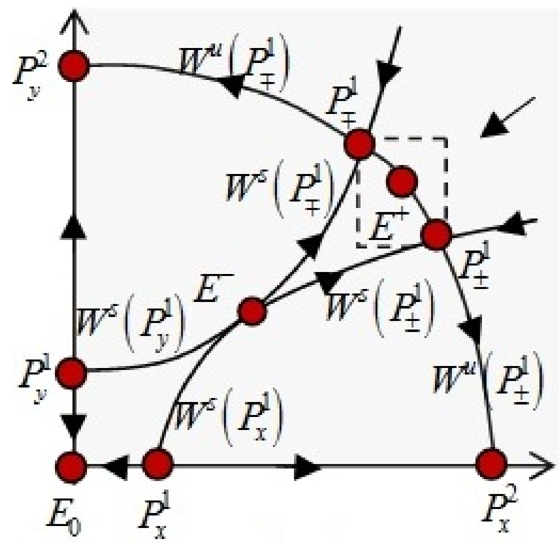

Figure 16. Three equilibriums, three P-2

Proof.

The existence and orientation of the stable manifold $\mathcal{W}^s\left(P^i_\mp\right)$ follows from theorems 8-11 and 15.Moreover it cannot intersect another manifold or the boundary of the region at any point as the latter sets are invariant. Thus it must have an endpoint at E_-. Similarly the existence and orientation of the unstable manifold $\mathcal{W}^u\left(P^i_\mp\right)$ is given by theorem [theorem part 1].

On the other hand $\mathcal{W}^u\left(P^i_\mp\right) \cap [\![(P^i_\mp, E_+]\!] \neq \emptyset$ and $[\![(P^i_\mp, E_+]\!]$ is invariant. Thus $\mathcal{W}^u\left(P^i_\mp\right)$ cannot leave the latter set and must end at E_+. Analogous arguments and conclusions also hold for $\mathcal{W}^s\left(P^i_{\pm}\right)$ and $\mathcal{W}^u\left(P^i_{\pm}\right)$. In addition we know that for $p_{\mp} \in \mathcal{W}^s\left(P^i_{\mp}\right)$ and $p_{\pm} \in \mathcal{W}^s\left(P^i_{\pm}\right)$:

$$T^{2n}(p_{\mp}) \to P^i_{\mp} \quad \text{and} \quad T^{2n}(p_{\pm}) \to P^i_{\pm}$$

.

Furthermore for all $p \in \mathcal{W}^-\left(P^i_{\pm}\right) \cap \mathcal{W}^+\left(P^i_{\mp}\right)$. there exist $p_{\mp} \in \mathcal{W}^s\left(P^i_{\mp}\right)$ and $p_{\pm} \in \mathcal{W}^s\left(P^i_{\pm}\right)$
such that:

$$p_{\mp} \preceq_{se} p \preceq_{se} p_{\pm} \Rightarrow T^{2n}(p_{\mp}) \preceq_{se} T^{2n}(p) \preceq_{se} T^{2n}(p_{\pm}) \text{ for all n} \geq 0$$

. It follows that there exists $N{>}0$ such that $T^{2N}(p) = q \in [\![P^i_{\mp}, P^i_{\pm}]\!]$. Thus there exist $q_{\mp} \in \mathcal{W}^u\left(P^i_{\mp}\right)$ and $q_{\pm} \in \mathcal{W}^u\left(P^i_{\pm}\right)$ such that:

$$q_{\mp} \preceq_{se} q \preceq_{se} q_{\pm} \Rightarrow T^{2n}(q_{\mp}) \preceq_{se} T^{2n}(q) \preceq_{se} T^{2n}(q_{\pm}) \text{ for all n} \geq 0$$

. where

$$T^{2n}(q_{\mp}) \to E_{+} \quad \text{and} \quad T^{2n}(q_{\pm}) \to E_{+}$$

which implies that $T^{2n}(q) \to E_{+} \Rightarrow T^{2n}(p) \to E_{+}$ I conclude that:

$$\mathcal{B}\left(E_{+}\right) = \mathcal{W}^-\left(P^i_{\pm}\right) \cap \mathcal{W}^+\left(P^i_{\mp}\right).$$

As of $\mathcal{W}^s(P^1_x)$, $\mathcal{W}^s(P^1_y)$, $\mathcal{B}(E_0)$, $\mathcal{B}(P^2_x)$ and $\mathcal{B}(P^2_y)$ the proof is analogous to the discussion in theorems 21 and 22. $\square$

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MANUSCRIPT 4

Global Attractivity Results for Second Order Difference Equations

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4.1 Introduction and Preliminaries

The following results were obtained first in [13, 14] and were extended to the case of higher order difference equations and systems in [14, 17, 21, 23, 24].

Theorem 26 *Let $[a, b]$ be a compact interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, M) = m \quad \textit{and} \quad f(M, m) = M, \tag{54}$$

then $m = M$.

Then

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{55}$$

has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(55) converges to $\overline{x}$.

Theorem 27 *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non-increasing in both variables;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \quad \textit{and} \quad f(M, M) = m, \tag{56}$$

then $m = M$.

Then Eq.(55) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(55) converges to $\overline{x}$.

Similar results has been proved for other two cases of coordinate-wise monotone function f, see [14]. These results have been very useful in proving attractivity results for equilibrium or periodic solutions of Eq.(55) as well as for higher order difference equations and systems of difference equations, see [6, 9, 14, 15]. Theorems 26 and 27 have attracted considerable attention of the leading specialists in difference equations and discrete dynamical systems and have been generalized and extended to the case of maps in $\mathbf{R}^n$, see [17], and maps in Banach space with the cone, see [21] and [23, 24], as well as in the case of monotone mappings in partially ordered complete metric spaces, see [4, 2].

The global behavior of solutions of Equation (55) in the case where f is either increasing in both variables or decreasing in the first and increasing in the second variable is well described by the following result from [1, 5].

Theorem 28 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is either non-increasing in the first variable and non-decreasing in the second variable or non-decreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (55) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually both monotonic.*

The consequence of Theorem 28 is that every bounded solution of (55) converges to either an equilibrium or a period-two solution or to the point on the boundary where equation is not defined, see [3, 10]. Thus the most important question becomes determining the basins of attraction of these solutions. The answer to this question follows from an application of theory of monotone maps in the plane, which was developed in [18, 19, 23].

The global behavior of solutions of Equation (55) in the case where f is either decreasing in both variables or increasing in the first and decreasing in the second variable is much more complicated and it can range from global asymptotic stability of the unique equilibrium as in the cases of difference equations

$$x_{n+1} = a + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \quad a > 0, x_{-1}, x_0 > 0, \tag{57}$$

see [14],

$$x_{n+1} = \frac{a + x_n}{A + x_{n-1}}, \quad n = 0, 1, \ldots, \quad a, A > 0, x_{-1}, x_0 \geq 0, \tag{58}$$

[20], and

$$x_{n+1} = \frac{a}{x_n} + \frac{A}{x_{n-1}}, \quad n = 0, 1, \ldots, \quad a, A > 0, x_{-1}, x_0 > 0, \tag{59}$$

[14] to the conservative chaos as in the case of Lyness' difference equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \quad A > 0, x_{-1}, x_0 > 0, \tag{60}$$

or the following difference equation

$$x_{n+1} = \frac{a}{x_{n-1}(1 + x_n)}, \quad n = 0, 1, \ldots, \quad a > 0, x_{-1}, x_0 > 0, \tag{61}$$

see [8]. Also such equations may exhibit Neimark-Sacker bifurcation such as

$$x_{n+1} = a + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \ldots, \quad a > 0, x_{-1}, x_0 > 0, \tag{62}$$

see [12] and global convergence to singular zero solution

$$x_{n+1} = \frac{a x_n^2}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots, \quad a > 0, x_{-1}, x_0 > 0. \tag{63}$$

The proofs for Equations (57) and (59) are using the embedding of corresponding equations into higher order difference equations and then higher order version of Theorems 26 and 27. The proof for Equation (59) is based on a construction of the sophisticated Lyapunov function in the region of the parameters where Theorem 26 is not applicable, see [20]. The proofs of global behavior of solutions of Equations (60) and (61) are based on KAM theory and some results from algebraic and projective geometry.

In this paper we show that we can use the theory of monotone maps to improve results of Theorems 26 and 27 in the case where f is either increasing in first and decreasing in second argument or is decreasing in both arguments. In fact, we will give an interesting special result which shows that under certain mild condition the local stability implies global asymptotic stability. Our method will be based on embedding Equation (55) into related monotone two dimensional system of difference equations to which we will apply the global attractivity theorems of monotone systems. This will imply the global asymptotic stability of an equilibrium of the corresponding monotone two dimensional system as well as global asymptotic stability of the equilibrium of Equation (55).

In the following I provide an application of theorem (54) for higher order difference equations.

Example 1

A new proof for Pielou's second order difference equation:

Pielou's equation is a mathematical model in population biology that was introduced by Pielou as a discrete analogue of the logistic equation with delay, see [6, 14] and was investigates by [references] using various method. in the following I will provide an alternative proof of the global stability of the positive equilibrium using The M-m theorem.

Pielou's equation is given by:

$$x_{n+1} = \frac{Ax_n}{1+x_{n-1}}, \quad x_{-1} \geq 0, \quad x_0 > 0 \text{ and } A > 1$$

By using the following change of variable $z_n = \dfrac{1}{x_n}$ we get:

$$z_{n+1} = \frac{z_n}{A} + \frac{z_n}{Az_{n-1}} \tag{64}$$

Now by using the embedding method we can use $z_n = \dfrac{z_n}{A} + \dfrac{z_n}{Az_{n-1}}$ and substitute in equation (64) as follows:

$$z_{n+1} = \frac{z_n}{A} + \frac{1}{Az_{n-1}}\left(\frac{z_n}{A} + \frac{z_n}{Az_{n-1}}\right)$$

$$z_{n+1} = \frac{1}{A^2} + \frac{z_n}{A} + \frac{1}{A^2 z_{n-2}} \tag{65}$$

Eq. (65) is of the form $z_{n+1} = f(z_n, z_{n-1}, z_{n-2})$,and has has a unique fixed point $\bar{z} = \frac{1}{A-1}$. To apply the theorem first we show that $f(x, y, z)$ has an invariant interval $I = \left[\frac{1}{A^2}, U\right]$: where $U \geq \frac{A^2+1}{A(A-1)}$

Let $x, y, z \in I$ and $f(x,y,z) = \frac{1}{A^2} + \frac{x}{A} + \frac{1}{A^2 z} > \frac{1}{A^2}$, then:

$$\frac{1}{A^2} < f(x,y,z) \leq \frac{1}{A^2} + \frac{U}{A} + \frac{A^2}{A^2} = \frac{1}{A^2} + 1 + \frac{U}{A} = \frac{A^2+1}{A^2} + \frac{U}{A} \leq \frac{U(A-1)}{A} + \frac{U}{A} = U$$

$\Rightarrow f(x,y,z) \in I.$

Next is to solve the system:

$$\begin{cases} M &= f(M,m,m) \\ m &= f(m,M,M) \end{cases} \Leftrightarrow \begin{cases} M &= \frac{1}{A^2} + \frac{M}{A} + \frac{1}{A^2 m} \\ m &= \frac{1}{A^2} + \frac{m}{A} + \frac{1}{A^2 M} \end{cases} \Leftrightarrow$$

$$\begin{cases} \frac{Mm(A-1)}{A} &= \frac{1}{A^2} + \frac{m}{A^2} \qquad (*) \\ \frac{mM(A-1)}{A} &= \frac{1}{A^2} + \frac{M}{A^2} \qquad (**) \end{cases}$$

By subtracting Eq.(*) from Eq.(**) we get: $\frac{M}{A^2} - \frac{m}{A^2} = 0 \Rightarrow M = m.$

I conclude that every solution of Eq.(65) with initial conditions in I must converge to the unique equilibrium $\bar{z}$. As I can choose $U \geq \frac{A^2+1}{A(A-1)}$ to be arbitrarily big, I can also conclude that every solution of Eq.(65) with positive initial conditions must converge to the unique positive equilibrium $\bar{z} = \frac{1}{A-1}$.

Consequently is $\bar{x} = A - 1$ is globally asymptotically stable for Pielou's equation.

Remark 4 The connection between the theory of monotone maps and the asymptotic behavior of Equation (55) follows from the fact that if f is strongly increasing in both variables, then a map associated to Equation (55) is a cooperative map on $\mathcal{I}^2$ while the second iterate of a map associated to Equation (55) is a strictly cooperative map on $\mathcal{I}^2$. If f is strongly decreasing in first variable and increasing in the second variable, then a map associated to Equation (55) is a competitive map on $\mathcal{I}^2$ while the second iterate of a map associated to Equation (55) is a strictly competitive map on $\mathcal{I}^2$.

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (55) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array} , \quad n = 0, 1, \ldots .$$

Now a map associated to Equation (55) is $F(u, v) = (v, f(v, u))$. Then F maps $\mathcal{I}^2$ into itself and the second iterate $T := F^2$ is given by

$$T(u, v) = (f(v, u),\ f(f(v, u), v))\,.$$

and it is clearly strictly cooperative on $\mathcal{I}^2$, when f is increasing in both arguments and competitive on $\mathcal{I}^2$, when f is decreasing in first and increasing in second variable. Unfortunately, there is no such a result in the case when f is either decreasing in both variables or increasing in the first and decreasing in the second variable.

Remark 5 The characteristic equation of Equation (55) at an equilibrium point $(\bar{x}, \bar{x})$ is

$$\lambda^2 - p\lambda - q = 0, \tag{66}$$

where $p = f_x(\bar{x}, \bar{x})$ and $q = f_y(\bar{x}, \bar{x})$. If f is increasing in both arguments then Equation (66) has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$. Here $D_i f, i = 1, 2$ denotes the partial derivative with respect to the i-th variable.

The following result is for strictly order preserving maps [7]. The result is stated for a partial order $\preceq$ in $\mathbb{R}^n$, but it also holds in Banach spaces.

Theorem 29 (Order Interval Trichotomy of Dancer and Hess, [7]) *Let $u_1 \preceq u_2$ be distinct fixed points of a strictly order preserving map $T : A \to A$, where $A \subset \mathbb{R}^n$, and let $I = [\![u_1, u_2]\!] \subset A$. Then at least one of the following holds.*

(a) *T has a fixed point in I distinct from u_1 and u_2.*

(b) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_1 to u_2 and satisfying $x_n \preceq x_{n+1}$.*

(c) *There exists an entire orbit $\{x_n\}_{n\in\mathbb{Z}}$ of T in I joining u_2 to u_1 and satisfying $x_{n+1} \preceq x_n$.*

Corollary 6 *If a and b are stable fixed points, then there exists a third fixed point in $[\![a, b]\!]$.*

The following result is a direct consequence of Theorem 29.

Corollary 7 *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

A simple consequence of this result is the following

Corollary 8 *If monotone map T has exactly three fixed points $a \preceq b \preceq c$, where b is stable, then the interior of $[\![a, c]\!]$ is a subset of the basin of attraction of b.*

4.2 Main Results

The next result is an extension and improvement of Theorem 26.

Theorem 30 *Let $[a, b]$ be a compact interval of real numbers and assume that*

$$f : [a, b]^2 \to [a, b]$$

is a continuous function one fixed point $\overline{x} \in [a, b]$ such that:

(a) $f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) System (54) has at most three solutions;

(c) $p - q < 1$, where p and q are defined in Remark 5.

Then $\overline{x}$ is globally asymptotically stable in $[a, b]$.

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$M_i = f(M_{i-1}, m_{i-1}), \quad m_i = f(m_{i-1}, M_{i-1}), \quad i = 0, 1, \ldots. \tag{67}$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \ldots \leq m_i \leq \ldots \leq M_i \leq \ldots \leq M_1 \leq M_0$$

and

$$m_i \leq x_k \leq M_i \quad \text{for } k \geq 2i + 1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then

$$M \geq \limsup_{i\to\infty}\ x_i \geq \liminf_{i\to\infty}\ x_i \geq m \tag{68}$$

and by the continuity of f,

$$m = f(m, M) \quad \text{and} \quad M = f(M, m).$$

Now if $m = M = \bar{x}$ we are done by Theorem 26. Otherwise $m < M$, which means that system (67) has three equilibrium points $(m, M) \preceq_{ne} (\bar{x}, \bar{x}) \preceq_{ne} (M, m)$ and that the ordered interval $[\![(m, M), (M, m)]\!]$ is invariant. Since system (67) is a competitive system the global attractivity of its equilibrium $E(\bar{x}, \bar{x})$, in view of (68) would imply the global attractivity and so global asymptotic stability of $\bar{x}$ as the equilibrium of Equation (55). It is interesting to note that the condition for local stability of $\bar{x}$ as the equilibrium of Equation (55) is different from the the condition for local stability of $E(\bar{x}, \bar{x})$ as the equilibrium of system (67). Indeed, the Jacobian matrix of system (67) evaluated at E is

$$J = \begin{bmatrix} p & q \\ q & p \end{bmatrix},$$

with eigenvalues $\lambda_{\pm} = p \pm q$. In view of the fact that $p > 0, q < 0$ the condition $|\lambda_{\pm}| < 1$ becomes equivalent to the condition

$$p - q < 1. \tag{69}$$

Now in view of Corollary 8 the interior of the ordered interval $[\![(m, M), (M, m)]\!]$ is attracted to E.

The well known condition for local asymptotic stability of the equilibrium $\bar{x}$, under the restrictions $p > 0, q < 0$ is that

$$p + q < 1, -1 < q. \tag{70}$$

Clearly, condition (69) implies (70), which means whenever E is local attractor for system (67) then $\bar{x}$ is local attractor for Equation (55), but converse is not true.

□

The next result is an extension and improvement of Theorem 26.

Theorem 31 *Let $[a, b]$ be a compact interval of real numbers and assume that*

$$f : [a, b]^2 \to [a, b]$$

is a continuous function one fixed point $\overline{x} \in [a, b]$ such that::

(a) $f(x, y)$ is non-increasing in both variables;

(b) System (56) *has at most three solutions;*

(c) $q > -1, q + p > -1$, where p and q are defined in Remark 5.

Then $\overline{x}$ is globally asymptotically stable in$[a, b]$

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$M_i = f(m_{i-1}, m_{i-1}), \quad m_i = f(M_{i-1}, M_{i-1}), \quad i = 0, 1, \ldots. \tag{71}$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \ldots \leq m_i \leq \ldots \leq M_i \leq \ldots \leq M_1 \leq M_0$$

and

$$m_i \leq x_k \leq M_i \quad \text{for } k \geq 2i + 1.$$

Set

$$m = \lim_{i \to \infty} m_i \quad \text{and} \quad M = \lim_{i \to \infty} M_i.$$

Then (68) holds and by the continuity of f,

$$m = f(M, M) \quad \text{and} \quad M = f(m, m).$$

Now if $m = M = \bar{x}$ we are done by Theorem 27. Otherwise $m < M$, which means that system (71) has three equilibrium points $(m, m) \preceq_{ne} (\bar{x}, \bar{x}) \preceq_{ne} (M, M)$ and that the ordered interval $[\![(m,m),(M,M)]\!]$ is invariant. System (71) implies that M_i and m_i are solutions of the following first order difference equation

$$y_{i+1} = G(y_{i-1}), \quad i = 0, 1, \ldots, \tag{72}$$

where

$$G(u) = f(f(u,u), f(u,u))$$

is an increasing function. If the equilibrium $\bar{x}$ is global attractor for Equation (72) then it is also global attractor for Equation (55).

The Jacobian matrix of system (71) evaluated at E is

$$J = \begin{bmatrix} 0 & p+q \\ p+q & 0 \end{bmatrix},$$

with eigenvalues $\lambda_{\pm} = q$. The well known condition for local asymptotic stability of the equilibrium $E(\bar{x}, \bar{x})$ of system (71), under the restrictions $p, q < 0$ is that $q > -1$ and $q + p > -1$, which shows that E is locally stable. Since system (71) is anti-cooperative, using the result for global attractivity of such systems [11] we conclude that E attracts the interior of the box $[\![(m,m),(M,M)]\!]$, which completes the proof.

□

Remark 6 Theorems 26 and 27 were originally applied to the difference equation

$$x_{n+1} = \frac{a + bx_n + cx_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{73}$$

where all parameters and the initial conditions x_{-1}, x_0 are non-negative and such that $A + Bx_n + Cx_{n-1} > 0$ for every n. It is interesting to observe that in the case of Equation (73), the condition (56) has only the equilibrium solution $\bar{x}$ as solution and so in this case the condition (56) is automatically satisfied. Furthermore, the condition (54) can have only one solution $(m, M), m < M$ so the condition (b) of Theorem 30 is automatically satisfied for Equation (73).

4.3 Examples

In this section we give some examples of difference equations where Theorems 30 and 31 apply.

Example 2 Equation

$$x_{n+1} = \frac{1}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{74}$$

where $B, C > 0, x_{-1}, x_0 \geq 0, x_{-1} + x_0 > 0$ was considered in [14], where we proved that every solution of this equation converges to the unique equilibrium $\bar{x} = \frac{1}{B+C}$. We used the method of limiting sequences in [14]. Here we use Remark 6 to prove this result. Indeed, we need to find an invariant and attracting interval $[L, U]$. Choose $0 < L < U$ such that $LU = \frac{1}{B+C}$ and $x_{-1}, x_0 \in [L, U]$ we have that

$$f(x_n, x_{n-1}) = \frac{1}{Bx_n + Cx_{n-1}} \leq \frac{1}{(B+C)L} = U$$

and

$$f(x_n, x_{n-1}) = \frac{1}{Bx_n + Cx_{n-1}} \geq \frac{1}{(B+C)U} = L.$$

Since (56) is automatically satisfied every solution of Equation (74) converges to the unique equilibrium, which is globally asymptotically stable.

Example 3 Equation

$$x_{n+1} = \frac{1}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{75}$$

where $A, B, C > 0, x_{-1}, x_0 \geq 0$ was considered in [14], where we proved that every solution of this equation converges to the unique equilibrium $\bar{x}$. Here we use Remark 6 to prove this result. Indeed, we only need to find an invariant and attracting interval $[L, U]$. Choose $L = 0, U = 1/A$. If $x_{-1}, x_0 \in [L, U]$ we have that

$$L = 0 \leq f(x_n, x_{n-1}) = \frac{1}{A + Bx_n + Cx_{n-1}} \leq \frac{1}{A} = U$$

Since (56) is automatically satisfied the unique equilibrium of Equation (74) is globally asymptotically stable.

Example 4 Equation

$$y_{n+1} = \frac{y_n + p}{y_n + qx_{n-1}}, \quad n = 0, 1, \ldots, \tag{76}$$

where $p, q > 0, y_{-1}, y_0 \geq 0, y_{-1} + y_0 > 0$ was considered in [14], where we proved that every solution of this equation converges to the unique equilibrium $\bar{y}$ for all parameter values except for the parametric region $p < q < 1 + 4p$. Here we will use Theorem 30 to prove global asymptotic stability of the unique equilibrium $\bar{y}$ for $p < q < 1 + 4p$. As we have shown in [14] the interval $[q/p, 1]$ is an invariant and attracting interval for Equation (76) when $q > p$. We only need to check the condition (b) of Theorem 30 to prove our statement. Indeed

$$f_u(\bar{y}, \bar{y}) = \frac{q\bar{y} - p}{\bar{y}^2(q+1)^2}, \quad f_u(\bar{y}, \bar{y}) = -\frac{q(p + \bar{y})}{\bar{y}^2(q+1)^2},$$

where

$$f(u, v) = \frac{u + p}{u + qv}.$$

Now the condition $f_u(\bar{y}, \bar{y}) - f_v(\bar{y}, \bar{y}) < 1$ after simplification yields

$$(q - 1)\bar{y} < 2p, \tag{77}$$

which is clearly satisfied if $q \leq 1$. If $q > 1$ then after straightforward simplifications condition (77) becomes equivalent to

$$p(q + 1)^2(q - 1 - 4p) < 0,$$

which holds. Thus we prove the following result.

Theorem 32 *The unique equilibrium of Equation* (76) *is globally asymptotically stable.*

Finally I introduce the following global stability result that can be useful when the maps are not monotone or continuous:

Corollary 9 *Let $\mathcal{S} \subseteq \mathbb{R}^2$, $f : \mathcal{S} \to \mathbb{R}$ and I a compact interval of the real line. Suppose there exists a continuously differentiable function $G : I \times I \to I$ satisfying the following properties:*

1. *$G(x, y)$ is non-decreasing in x and non-increasing in y*
2. *for all $m, M \in I$, $m \le x, y \le M \Rightarrow G(m, M) \le f(x, y) \le G(M, m)$*
3. *$G(x, y)$ has a unique fixed point $\bar{x}$ in I*
4. $\begin{cases} G(M, m) = M \\ G(m, M) = m \end{cases}$

   (a) *has "a unique solution $M = m = \bar{x}$ in I*

   " or " alternatively:

   (b) *has "three solutions $(\bar{x}, \bar{x})$,(M, m) and (m, M) where $M \neq m$ and $G_x(\bar{x}, \bar{x}) - G_y(\bar{x}, \bar{x})<1$"*

Then:

1. *$\bar{x}$ is also a unique fixed point for $f(x, y)$ in I*
2. *every solution of the difference equation $x_{n+1} = f(x_n, x_{n-1})$ in I must converge to $\bar{x}$.*

The proof of this corollary is similar to the proof of theorems 30 and 54.

Example 5

Consider the equation:

$$x_{n+1} = f(x_n, x_{n-1}) = \frac{a + x_n}{A + Bx_n + x_{n-1}} \tag{78}$$

Consider the function $G(x, y) = \frac{a+x}{A+(B+1)y}$. which is non-decreasing in x and non-increasing in y.

One can show that: $G(x, y)$ has one unique positive fixed point and has an invariant interval: $[0, U]$ for $u \geq \frac{a}{A-1}$ provided that A>1. Furthermore:

for all $M, m \in I$, $m \leq x, y \leq M$ implies :

$$\frac{a + m}{A(B + 1)M} \leq \frac{a + x}{A + Bx + y} \leq \frac{a + M}{A + (B + 1)m}$$

Thus

$$G(m, M) \leq f(x, y) \leq G(M, m)$$

In addition the system $\begin{cases} G(M, m) = M \\ G(m, M) = m \end{cases}$ has one solution since:

$$\begin{cases} M &= \dfrac{a + M}{A + (B + 1)m} \\ m &= \dfrac{a + m}{A(B + 1)M} \end{cases} \Rightarrow \begin{cases} a + M &= AM + (B + 1)Mm \\ a + m &= Am + (B + 1)mM \end{cases}$$ By subtracting one equation from the other we get:

$M - m = (A - 1)(M - m)$ but as A>1 the latter implies $M = m$. I conclude that every solution of Eq.78 in the interval $[0, U]$ must converge to the unique positive equilibrium. On the other hand since we can choose $U \geq \dfrac{a}{A - 1}$ arbitrarily large, I conclude that the positive equilibrium of Eq.78 is globally asymptotically stable.

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