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Solving a Rubik's Cube: An Analysis of Problem Solving Strategies

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Abstract

Stumbling upon a difficult problem to solve is inevitable. A difficult problem is defined as a problem in which the solution is not easily discovered for a typical person. These problems can range from real-life situations, such as quitting an addiction or finding a cure for cancer, to recreational puzzles such as a game of chess or a Rubik's Cube. Due to their complexity, and in some cases lack of information on the subject, it is easy to get frustrated and not try when one encounters such a problem. For this reason, it is imperative to have a skill set that one can utilize in these situations. By having a process to follow, it will be easier to avoid the trap of frustration and continue working towards a solution. Many scholars have researched this topic, including mathematicians George Polya and Wayne Wickelgren. Their work reflects several problem solving skills that are invaluable to mathematical problems.

Using Polya's and Wickelgren's research as a foundation, I have done an analysis of various techniques one might use to approach a difficult problem. These techniques are, by nature, based in mathematical reasoning, but made more concrete through their application to solving a 2x2x2 Rubik's Cube. Some of these techniques include starting small, drawing figures, and measuring progress. Many of these techniques can be explained simplistically, but others require in-depth analysis using techniques such as theoretical probability and deduction. It is important to note that the aim of this project is not to find a new solution set to the 2x2x2 Rubik's Cube, but to find a set of skills one may need to utilize in order to determine a solution. The results of this project include a suggested list of problem solving skills that are able to be applied to a variety of problems.

Keywords: problem solving, Rubik's Cube, puzzles, critical thinking

Introduction

At some point, every person will face a situation that they do not know how to approach. It is for this reason that many scholars have studied problem solving skills. One mathematician, George Polya, writes about the importance of problem solving, deeming it "a fundamental human activity. In fact, the greater part of our conscious thinking is concerned with problems. When we do not indulge in mere musing or daydreaming, our thoughts are directed toward some end; we seek means, we seek to solve a problem" (Polya, 1971, 221-222). Given the commonality of problems, it is crucial to be able to know how to approach them. Without any direction, frustration can emerge and inhibit one's ability to think clearly, persevere, and discover a solution.

Noting the significance of problem solving skills, Ben Rohrig, a high school physics teacher, began implementing a Rubik's Cube as a learning tool in his classroom. He got this idea when his son attempted to solve the cube, and he followed suit. After solving the cube, he realized that his "students face similar frustrations when attempting to solve the tasks [he gives] them in class. But the difficulty of the task and the fact that [he] succeeded made all the frustration worth it" (Rohrig, 2010, p. 54). He later goes on to describe how the process of solving a Rubik's Cube can provide various benefits to the solver. One particular benefit is that "*It provides students with a framework for solving problems*. … The sequential reasoning needed to solve the cube is applicable to many other types of problems. … By breaking problems into steps, even the most daunting ones can be solved" (Rohrig, 2010, p. 55-56). Although the relationship between a Rubik's Cube and physics is seemingly nonexistent, the cube is able to teach people how they think about problems and how they approach them. For this reason, the cube provides the perfect analogy for problem solving. It provides a concrete, well-known example of a difficult problem that we can apply strategies to and analyze the process. As such, we will be using the 2x2x2 Rubik's Cube, commonly referred to as the Mini Cube, as our problem to be solved. An analysis of various techniques one might use to approach a difficult problem was conducted using the Cube as a demonstration. Due to its nature, these techniques are based in mathematical reasoning, but are relevant skills to a various problems. It is important to note that the outcomes of this process are a set of skills one may need to utilize in order to determine a solution, not a new solution set to the 2x2x2 Rubik's Cube. The results include a suggested list of problem solving skills that are able to be applied to a variety of problems.

Literature Review

The works of George Polya outline a general problem solving method that will serve as a framework for this literature review. This outline describes problem solving as four distinct phases: understanding the problem, making a plan, carrying out the plan, and looking back (Polya, 1971: 5-6). The specifics of these steps may vary based on the problem at hand, but the overall process generally remains the same. For if a solver attempts to discover a problem while aimlessly going through random operations, it will be impossible to understand how to get to the solution. For this reason, a plan is needed. However, a plan cannot be created until the specifics of the problem are analyzed and understood.

Understanding the problem in its entirety requires more than simply looking at the problem. True understanding means that "we have to see clearly what is required" (Polya, 1971:5). This includes being able to define what the solution entails, as well as the problem's allowable operations. However, it may be impossible to clearly define the objectives if there is no knowledge of the relevant background material the problem might involve. A person may be able to research general problem-solving methods to help them with difficult problems, but these strategies "will not substitute for lack of the relevant knowledge" (Wickelgren, 1995:25). For example, one cannot expect to discover the cure for a disease without understanding the chemistry and anatomy of the human body. Similarly, one cannot expect to solve a problem in calculus two without having taken, and understood, precalculus. Therefore, a person should only take on problems in which they have the background knowledge to understand it. If the solver *does* understand the background of the problem, then they should try to understand the *details* of the problem. This includes examining the givens, the allowable actions to reach the solution, and the appearance of the desired solution. However, practical problems, such as the Rubik's Cube, "are usually far from being perfectly stated," requiring more in depth consideration of what the details of the problem entail (Polya, 1971, p. 98). Perhaps the most difficult application of understanding the details of a practical problem is interpreting what the givens consist of. Typically, "In some kinds of puzzles the givens consist of the materials" (Wickelgren, 1995, p. 11). In relation to the Rubik's Cube, this may be the specific configuration the cube holds at a specific point in time, as well as the size of the cube--a Mini Cube, a standard Rubik's Cube, a 4x4x4 Cube, etc. Determining what a solution will look like is another important detail to investigate because it gives the solver a sense of what they are striving for. In order to be a complete solution, however, it needs to be able to satisfy the four following criteria:

> "A *solution* to a problem contains all four of the following parts. (a) Complete specification of the givens; that is, a unique given state from which the goal can be derived via a sequence of allowable operations. (b) Complete specification of the set of operations to be used. (c) Complete specification of the goals. (d) An ordered succession or sequence of problem states, starting with the given state and terminating with a goal state, such that each successive state is obtained from the preceding state by means of an allowable action (operation applied to one or more expressions in the preceding state)" (Wickelgren, 1995, p. 16).

Applying this framework to a Rubik's Cube, a solution would consist of some given configuration of the cube, followed by a sequence of different configurations of the cube--each of which was obtained by an allowable operation from the previous configuration--and ending with a configuration that satisfies the goal of having each of the six sides having only one color. After the solver can clearly define their desired solution and "Having understood the problem, [they] shall be in a better position to judge which particular points may be the most essential" (Polya, 1971, p. 76). These points should be the main ones to remember while following through the rest of the problem solving process so that the solver has something to revisit when they get stuck. Once the solver can clearly define the problem and their objectives, they can begin to make their plan.

Making a plan is the most complex aspect of problem solving. Since getting started is so difficult, it is useful for solvers to have a general idea of a handful of techniques that aim at determining starting points for difficult problems. One such technique is *connecting* all the knowledge obtained through understanding the problem. To do this, the principal parts of the problem-- the hypothesis, the givens, and the desired solution-- should always be at the forefront of thought. An attempt to connect all these aspects is generally a good starting point in determining a plan of attack because it may give meaning to why each allowable operation acts as it does. This technique, along with thinking about related problems that one has seen before, helps give the solver an idea of what they are up against. A true understanding of the problem at hand will typically provide some spark of connection to another similar problem. Many determined starting points are decided "based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts" (Polya, 1971, p. 9). These similar problems can connect to the current one in terms of either method or results (Polya, 1971, p. 42). For example, if the solver does not need to find a similar solution state, it is more pertinent to mimic the method, or the process, that they took to solve a different problem: how they thought about the problem, why they decided on the course of action they used, and how they determined that their solution set was yielding the desired results. However, if the unknown, or the solution, resembles that of a similar problem, it may help to examine the results. Here, if they can determine that they are the same, then they have a much simpler problem at hand in which they already know how to solve. On the other hand, if the solver determines that the solutions are not exactly equal, then they can go back and alter their methods to adapt them to the new problem. It is very common for the solver to guess at techniques that could be helpful to their problem. When these guesses "occur to us after we have attentively considered and really understood a problem in which we are genuinely interested… [they] usually contain at least a fragment of the truth although, of course, they very seldom show the whole truth. Yet there is a chance to extract the whole truth if we examine such a guess appropriately" (Polya, 1971, p. 99). In essence, if there is logical reasoning and knowledge behind a certain guess, then it typically merits some sort of experimentation. Another important technique to consider is starting with the desired solution. However, "People have a bias to start at the beginning, which they take to mean the givens. This bias is often inappropriate in problem solving, since the goal is frequently a better beginning point than the givens" (Wickelgren, 1995, p. 23). This is because the givens are frequently a jumble of information that is hard to decipher. By starting with a desired solution and working backwards, the solver may get a better idea of how everything is connected. Another useful technique is drawing pictures. This tool is ideal when there are many details to consider because "we cannot imagine all of them simultaneously, but they are all together on the paper. A detail pictured in our imagination may be forgotten; but the detail traced on the paper remains" (Polya, 1971, p. 103). Especially in mathematics, any form of drawing can help the solver see more clearly how all the givens fit together. One particular type of drawing that can be very helpful for puzzles is a state-action tree. This type of figure helps represent the sequences of states that a particular problem may take on. Depending on how the solver defines a solution set, the tree can appear as "*either* a sequence of actions or a sequence of states (terminating with the achievement of the goal), [though] it is very useful to represent *both* the possible sequences of actions and the possible sequences of states in a common diagram" (Wickelgren, 1995, p. 17). These types of diagrams are very useful in determining sequentially how a problem can change form, but they are typically very time consuming. This is due to the fact that the solver is often left "with a search among an extremely large number of alternative action sequences. In these cases, we must 'prune the tree' so that there are not so many possible action sequences to investigate" (Wickelgren, 1995, p. 19). However, simply drawing out a few states can help the solver better visualize how the problem reacts to certain operations. After considering all these techniques, and possibly a few others, the solver should determine which ones are most applicable to their problem and choose the ones that they would like to utilize. Once the solver has decided on their methods, they can begin to carry out the plan they created and work towards a solution.

While carrying out the plan, it is important for the solver to keep track of the results of their methods. If it appears that one particular method is not yielding the desired results, the solver should "Consider [the] problem from various sides. Emphasize different parts, examine different details, examine the same details repeatedly but in different ways, combine the details differently, approach them from different sides" (Polya, 1971, p. 34). By continuously evaluating the situation, the solver should be able to see which techniques are working and which are not. Perhaps, the solver may also discover new techniques that they had not yet considered. However, it is not enough to simply look at "isolated facts, we must combine these facts, and their combination must be well adapted to the problem at hand" (Polya, 1971, p. 157). The solver must find ways to connect all the data that they collect in a way that pushes them toward a solution. They must fully consider all information they come across while carrying out their plan or risk ignoring pieces of information that may lead to a solution. Further, the solver should "*Draw inferences from explicitly and implicitly presented information that satisfy one or both of the following two criteria: (a) the inferences have frequently been made in the past from the same type of information; (b) the inferences are concerned* *with properties (variables, terms, expressions, and so on) that appear in the goal, the givens, or inferences from the goal and the givens*" (Wickelgren, 1995, p. 23). While carrying out the plan, the solver should adapt their techniques and their process based on educated guesses. These inferences may lead the solver to a completed solution if applied properly. However, the solver must be careful when using these inferences as they may not always be based in sound reasoning. Therefore, as the solver continues on, they should keep track of their work and determine if their inferences are in fact correct and lead to the desired solution. If the solver perseveres and continues to adapt, then they should eventually reach their goal.

The final step of problem solving is to look back on the work that was carried out. Although typically ignored, this step is important in terms of problem solving accuracy and efficiency. For "the most frequent fault is carelessness, lack of patience in checking each step. Failure to *check the result* at all is very frequent; the student is glad to get an answer, throws down his pencil, and is not shocked by the most unlikely results" (Polya, 1971, p. 95). The solver should always be aware of the reasonability of their solution and the practicality of their process in finding it because it gives credibility to their work. Furthermore, the solver should also find ways to make their solution set more efficient. To do this, the solver should "Consider the details of the solution and try to make them as simple as [they] can; survey more extensive parts of the solution and try to make them shorter; try to see the whole solution at a glance" (Polya, 1971, p. 36). If the solver is able to find a new way to carry out the plan, or simply combine several of their steps into one, they will be able to save a lot of time when attempting to solve a similar problem in the future. In doing this, the solver will then become a master of their difficult problem.

One critical aspect of problem solving that does not neatly fit into the four stages outlined above is that of the solver's mindset. It would be incorrect to classify problem solving as "a purely 'intellectual affair'; determination and emotions play an important role. Lukewarm determination and sleepy consent to do a little something may be enough for a routine problem in the classroom. But, to solve a serious scientific problem, will power is needed that can outlast years of toil and bitter disappointments" (Polya, 1971, p. 93). It is very difficult to continue with a problem that the solver is either uninterested in or is not making any progress with. In both of these situations, motivation tends to be lacking due to disappointment. This can be somewhat remedied, however, through the solver continuously measuring their progress. For if the solver sees that they are accomplishing something, it should give them hope that they will eventually reach a solution. Polya (1971) describes this phenomenon in the following manner: "If the signs are rare or indistinct, I become more hesitant. And if for a long time they fail to appear altogether, I may lose courage, turn back, and try another road. On the other hand, if the signs become frequent as I proceed, if they multiply, my hesitation fades, my spirits rise, and I move with increasing confidence" (p. 185). Measuring progress can occur at any stage in the problem solving process. Some examples include: "understanding clearly the nature of the unknown… Clearly, disposing the various data so that we can easily recall any one… Visualizing vividly the condition as a whole… and separating the condition into appropriate parts… When we have found a figure that we can easily imagine, or a random notation that we can easily retain, we can reasonably believe that we have made some progress" (Polya, 1971, p. 183). If the solver feels themselves becoming frustrated, or if they feel stuck, they should begin to look for signs of progress to keep up their moral and persevere through the problem.

Solving the Cube

As discussed above, it is important to have a good understanding of the goal you wish to achieve before attempting to solve the problem at hand. Therefore, before attempting to solve a scrambled cube, I examined what my end goal was for the Mini Cube: the properties of a fully solved cube. I noted the following:

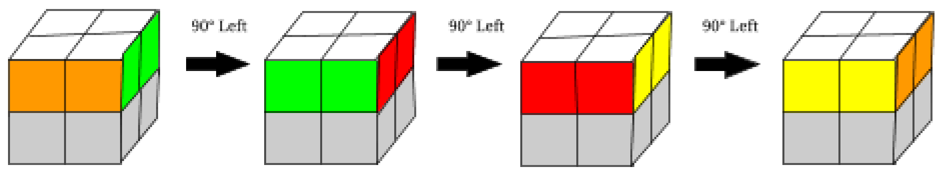
- The Mini Cube has six sides.
- Each side is a different color: white, red, orange, yellow, green, and blue.
- Placing the cube white-face-up, blue-face-down and rotating the cube 90° to the left, the sides were in the following sequence: orange, green, red, then yellow, with the blue face on the bottom.
- There are three pairs of opposite sides. On the cube in my possession these pairs were the following: white opposite blue, red opposite orange, and yellow opposite green.
- Each side is composed of four blocks.
- The allowable operations of a Rubik's Cube are the different ways to twist the cube. This includes rotating each face either clockwise or counterclockwise as many times as desired.

It is important to note that some Mini Cubes have a different arrangement of colors -- commonly white opposite yellow, red opposite orange, and green opposite blue. However, this does not change the *methods* for solving a scrambled cube; it only changes the *appearance* of a solved cube.

Once I determined all the relevant properties of a solved Mini Cube, and therefore all the relevant properties of a desired solution, I scrambled the cube. To truly imitate a scrambled cube, I used as many random twists as possible.

Instead of attempting to solve the whole cube at once, I decided to try to solve one face completely first. By doing this, I hoped to get a better feel for how the cube functioned, as well as gain hope and confidence in my ability to solve the rest of the cube. Before trying to solve the face, however, I needed to define what a solved face entailed. I determined that a completely solved face would have the four like-colored blocks on the same face, as well as two blocks of the same color on each adjacent face. Figure 1 below illustrates what the top half of a cube would look like with the white face completely solved.

Figure 1: White Face Completely Solved

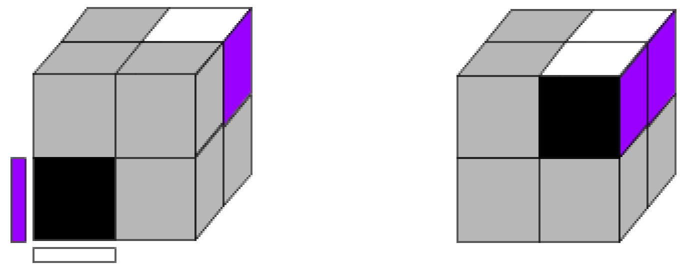


The figure above shows what the top half of a 2x2x2 Rubik's Cube would look like with the white face completely solved. Each successive cube is obtained by rotating the cube 90° to the left. The bottom half of the cube is grayed out to emphasize that it could have any number of configurations.

To solve the white face, I first pick out a block with a white side to be my starting block. Once that first block is chosen, I look at the other two colors on the block. Of the two, I pick one of the colors and try to find the other block with both a white side and the chosen color. For the sake of convenience, I will designate the color purple as the chosen color. Purple is not one of the colors on a Rubik's Cube, so it is meant to emphasize that it can denote any of the colors adjacent to white. The color black will represent the third color on the chosen block. The block of note needs to be on the bottom half of the cube, so if it is not already there, I twist one of the faces of the cube so that the block is on the bottom. From there, there are three possible cases: (1) the white side of the block is face down, (2) the white side of the block is on one of the side faces and the purple side is face down, and (3) both the white side of the block and the purple side are facing out. For the first case, to get the the block in the correct place, I rotate the bottom face until the chosen block is diagonal from the starting block (See Figure 2a below). From there, I twist the right face up 180° so that the

Figure 2: Case 1 for Solving the White Face

2a) Blocks placed diagonally 2b) Blocks correctly positioned

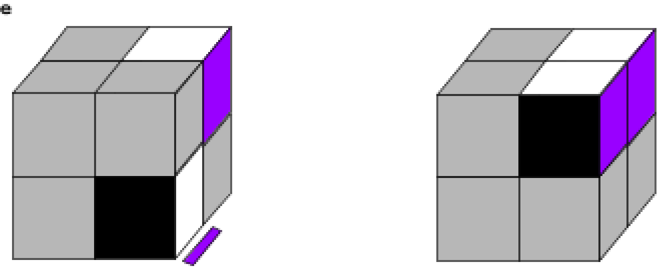


The figure above shows the two phases for placing the second block in solving the white face. The chosen block is denoted by the colors white, purple, and black. The purple side represents the chosen color, while the black side represents the color that the two blocks do not have in common.

white faces of the two blocks are next to each other (See Figure 2b). For the second case, I rotate the bottom face until the starting block's purple side is on the same face as the chosen block's white side (See Figure 3a). From there, I twist the front face counter clockwise 90° so that the white faces

Figure 3: Case 2 for Solving the White Face

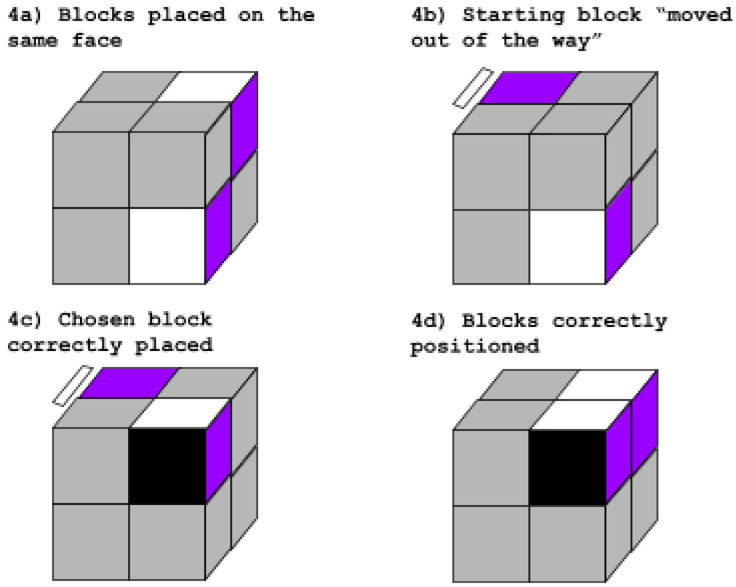
3a) Blocks placed on the same face 3b) Blocks correctly positioned



The figure above shows the two phases for placing the second block in solving the white face. The chosen block is denoted by the colors white, purple, and black. The purple side represents the chosen color, while the black side represents the color that the two blocks do not have in common.

of the two blocks are next to each other (See Figure 3b). For the third case, I rotate the bottom face until the starting block's purple side is on the same face as the chosen block's purple side (See Figure 4a). From there, I twist the back face counter clockwise 90° so that the starting block is

Figure 4: Case 3 for Solving the White Face



The figure above shows the two phases for placing the second block in solving the white face. The chosen block is denoted by the colors white, purple, and black. The purple side represents the chosen color, while the black side represents the color that the two blocks do not have in common.

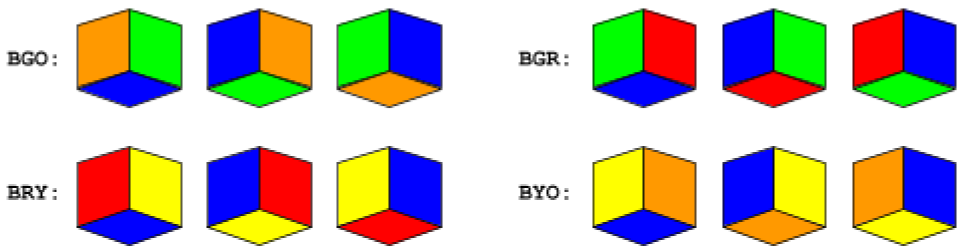
"moved out of the way" (See Figure 4b). Next, I twist the right face up 90° so that the white side is face up (See Figure 4c). Lastly, I twist the back face clockwise 90° so that the starting block is put back in its original position (See Figure 4d). The reasoning behind Case 3's moves is that, ideally, I want to simply rotate the right face up 90° so that the white side is face up. However, by rotating the right face up, the starting block is twisted out of position. So, by rotating the back face, the starting block is put in a position such that it will not be affected by rotating the right face up. After putting the chosen block into its position, the starting block can then be put back into its original spot. Once the first two blocks are in place, I start to place the other two blocks with white sides. The maneuvers to place the third block are exactly the same as those previously discussed. However, placing the last block is different because only the Case 3 will yield a correctly solved face. The first case does not work because no matter which face I twist, one of the previously placed blocks will end up misplaced. The block directly above the chosen block will be on the face that is twisted up, so it will be rotated into a position that is not correct. So, when the final block to be placed has its white side face down, the block should be placed in the spot directly underneath where it needs to be. For our purposes, let us say that the block is in the bottom right corner of the cube. I then twist a face that does not contain the block, say the back face, 90° so that it is "moved out of the way". Next, I rotate the right face up 180° so that its white side is face up. I twist the original face--i.e. The back face--back into position, returning the block to the bottom half of the cube, but with its white side facing out. From there, I follow the steps of Case 3 to put the block into its correct position. Case 2 does not work because it too will cause previously placed blocks to be misplaced, specifically, the cube that is on the face being rotated. To avoid this, I choose the colored side facing out to be my chosen color and follow the steps for Case 3. After placing the last block, the top half of the cube should resemble that shown in Figure 1.

After successfully solving the white face several times, I was feeling very confident. However, I was still unable to determine the steps for solving the bottom face. It was difficult to visualize a series of twists that would allow me to end up with the white face still being completely solved while accomplishing certain goals on the bottom face. After aimlessly attempting to solve the bottom half, I decided that I should try a new strategy.

I recalled that both Polya and Wickelgren praised the use of figures when solving problems, so I determined that I would utilize this technique. I decided to create a state-action tree representing how each possible twist could affect the cube's configuration. To see some examples of my tree's branches, see Appendix A. I started off with a solved cube and drew out what it looked like, rotating the cube 90° to the left three times, then up once so as to capture every side of the cube. From there, I mapped out what each possible twist would do to the cube. There were twelve branches that I ended up drawing: rotating the front face clockwise and counterclockwise, the back face clockwise and counterclockwise, the left face up and down, the right face up and down, the top face clockwise and counterclockwise, and the bottom face clockwise and counterclockwise. From there, I drew the twelve branches coming off each of the first branches. Although it was helpful to have a map of different twists and configurations, the process was tedious and time consuming. To make the process shorter, I tried to look for similar branches so that I could "prune the tree". There were several branches that mapped to an earlier one, so I was able to ignore these and refer back to the first branch of its type. More notably, I realized that each of my twelve branches stemming from their parent mapped to another branch from the same parent. That is when I discovered that each twist was equivalent to another: twisting the front face clockwise was equivalent to rotating the back face counterclockwise and vise versa; rotating the left face up was equivalent to rotating the right face down, and vise versa; and rotating the top face clockwise was equivalent to rotating the bottom face counterclockwise and vise versa. This parity allowed me to cut my drawing needs in half so that I only needed to map out 6 twists instead of twelve. I continued with my state-action tree for a while after this discovery, hoping that it would allow me to reach a solution quicker. Despite my best efforts, however, the tree was becoming exponentially larger, and I was beginning to get discouraged.

In another attempt to cut the amount of work I needed to do, I wanted to only focus on the states where one side is completely solved and the branches stemming from them, instead of mapping out each possible twist on every possible configuration. Since I was able to completely solve one face, I figured that the only beginning states in which to investigate further were the ones that left me lost. With only focusing on a certain type of cube, I decided to estimate how many possibilities for starting states there would be. The first step in determining this number of possibilities is to choose the face which is completely solved, which I chose to be white. If the white face is completely solved, this implies that four blocks are correctly placed and the other four may or may not be. The remaining blocks include: blue-green-orange (BGO), blue-green-red (BGR), blue-red-yellow (BRY), and blue-yellow-orange (BYO). After scrutinizing the cube's properties, I discovered there were three different possibilities to orient the four remaining blocks, as shown in Figure 5. Based on this information, we can determine the total theoretical combinations of the

Figure 5: Possible Orientations of Bottom Blocks



bottom half of the cube when the white face is completely solved. Formula 1 outlines the steps to determining the theoretical possibilities:

Formula 1: Total Theoretical Combinations of Bottom Blocks

1)	Place the blocks:	4! possibilities
2)	Orient the first block:	3 possibilities
3)	Orient the second block:	3 possibilities
4)	Orient the third block:	3 possibilities
5)	Orient the fourth block:	3 possibilities

Total Theoretical Possibilities = 4!*3*3*3*3 = 1,944

Formula 1 shows that there are 1,944 different configurations that the cube could have. To narrow it down, I determined that I could always place at least one block in the correct position, ignoring whether the orientation was correct, by twisting the bottom face. This means that I can narrow down the starting states I would need to research into because I would have more blocks in the correct position, regardless of their orientation. Formula 2 takes this information into account:

Formula 2: Total Theoretical Combinations of Bottom Blocks with one block correctly placed

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1)  Place the blocks:             3! possibilities
2)  Orient the first block:       3  possibilities
3)  Orient the second block:      3  possibilities
4)  Orient the third block:       3  possibilities
5)  Orient the fourth block:      3  possibilities
                                 ________
Total Theoretical Possibilities = 3!*3*3*3*3 = 486
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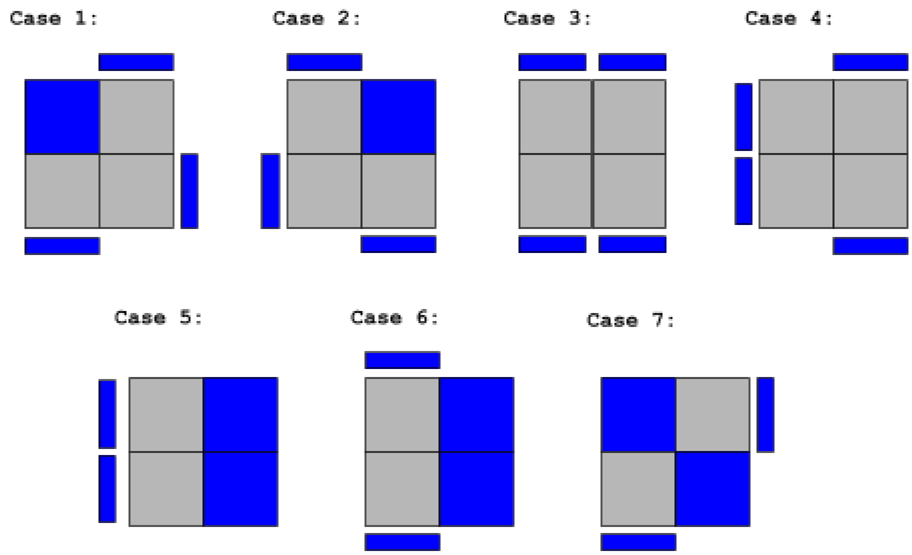
Having 486 possibilities is still a lot to consider, observe, and determine solutions for. So, this begs the questions: is there a way to reduce this number of possibilities? Are there any impossible combinations? I determined that it would be infeasible and impractical to consider every one of these possibilities in addition to determining what would constitute an impossible configuration because these simply represented the starting configurations, not the amount of twists that needed to be used to solve each one. Consequently, I was stuck again.

While playing around with the cube, trying to find inspiration, I ran into an interesting issue. The top face of the cube became so stuck that I was unable to twist any faces besides the top and bottom. I had heard of others taking their cubes apart, greasing it up, and putting it back together in order to make the cube twist more easily, so I decided to give this a try. I got a knife and carefully inserted it into one of the crevices of the cube. The cube burst apart, making it impossible to determine which pieces came from where. I gathered all the pieces, grouping similar shapes together. Next, I applied a thin layer of petroleum jelly on each piece, hoping to help with the ease of twisting the cube's faces. From there, I started to reassemble the cube. The middle piece was the shape of a jack, with six posts. Upright, it looked like an xyz-axis. Six posts were to be attached to the axes, three of which spun easily while the other three were fixed in place. After some trial and error, I realized that the three movable pieces were to go on the three separate planes so that the faces were twistable on all axes. The remaining pieces were the blocks themselves as well as wedge pieces that were to connect the blocks. Two of the wedge pieces were shaped slightly differently than the others, with one side that was wedge-shaped and one side that was square-shaped. Determining the positions for these oddly-shaped pieces also used some trial and error. However, an epiphany came to me the next morning: these special pieces have to be placed on the intersections of the axes with fixed posts so that they do not inhibit the cube's movement. From there, the only step was to place the blocks on the axes such that the assembly resulted in a solved cube. Although this process was its own difficult problem to solve, and it briefly prevented me from solving my original problem, it was a very rewarding experience. It also reminded me of the problem solving technique of "taking apart the problem", which is very useful in many STEM problems. Although taking apart the cube did not give me any new ideas for solving a scrambled cube, I determined that it was a good analogy for a helpful problem solving strategy.

At this point in my process, I decided it was time to look up known solutions to the 2x2x2 cube. Although I really wanted to be able to discover my own set of moves that would deliver a solved cube, I knew that researching into how others have solved the cube was an important skill in solving difficult problems. I reviewed the information booklet for the Mini Cube and documentations of others' experiences in solving the cube. In addition, I interviewed those people that I know who are able to solve the cube. I drew out all the solutions that were presented in these works to try to analyze the similarities and differences between them. All the solutions started with the same first step as I did: solve one side completely. After this step, however, some of the solutions began to differ. The majority of them stated that the next step was to orient the bottom layer--to make all the bottom blocks have their blue side face down. There are a variety of ways to accomplish this. Some solutions suggest memorizing only one algorithm and performing it repeatedly until your goal is reached. Others give two algorithms so that the user does not need to twist the cube as many times. A few, however, described seven possible cases that the bottom layer could be configured into and gave specific algorithms to solve each one of them. Figure 6 illustrates

Figure 6: Possible Configurations of Bottom Blocks



each of these seven different configurations. The main difference in these solutions is the amount of memorization. The most direct method, the one with the least amount of work needed, requires the memorization of seven different sequences, while the most roundabout method only requires the memorization of one. The next step, according to these methods, is to place the bottom layer of blocks into their correct positions such that the cube is completely solved. Most of the solutions give similar algorithms to do this, with the amount of twists needed being all around the same number. The other methods I researched stated that the second two steps should be reversed: place the bottom blocks first, then orient them. Most of these methods did not have specific algorithms aimed at specific configurations, rather they suggested the use of one or two algorithms repeatedly until the desired solution is reached. I determined that the decision of which solution was best would be dependent on the solver's mindset. Personally, I like the most direct method possible when solving a problem so as to minimize my time on it. Therefore, I determined that the best solution for me was to memorize the seven specific algorithms to orient the bottom layer, then an eighth one to place the blocks. Although this requires a decent amount of memorization, it is the most direct way to solve the Mini Cube. However, if a solver does not like to memorize that many steps, they would be better off with memorizing only a handful of algorithms. Here, it would not matter whether they decided to orient the bottom layer first, or place the bottom blocks first, as either method would result in the same amount of memorization. In all, the method any particular solver should take to completely solve the cube depends on that particular solver's memorization preferences.

My Problem Solving Techniques

Analyzing the steps I took along my journey, I determined some general problem solving skills that were utilized. The first, and perhaps the most important, is to understand the problem. First off, this technique includes determining what the solution will look like. In relation to the Mini Cube, a solution is the configuration in which each of the six faces is only one color. Understanding the problem also includes examining the givens and allowable operations. In terms of the cube, the givens can be interpreted as the specific configuration the cube has taken on at a specific point in time. The cube can change states only by twisting one of its six faces either clockwise or counterclockwise any amount of times. These twists are what constitutes the allowable operations.

The second technique can be described as starting small. For the cube, this can mean solving one face completely, regardless of how the rest of the cube is configured. By doing this, the solver can improve their mentality and persevere through the rest of the problem. Accomplishing one aspect of a problem, no matter how small, can help give vision as to how to approach the rest of the problem. By solving one face completely first, the solver is one step closer to a finished cube and can see more clearly how to get to a solved cube.

The third technique is drawing a figure. This approach may not be applicable to every problem one might encounter, but it can be very helpful when applied to the correct problem. Drawing out the problem and the problem states helps the solver visualize what needs to be done.

In the case of solving the cube, mapping the twists of the cube illustrated the different ways the cube could be configured.

The fourth technique is measuring progress. Although this is not necessarily critical to actually finding a solution, it is critical to keeping the solver in a good mindset. In addition, it also helps the solver determine if their plan of action is feasible. I utilized this technique by calculating how many configurations were needed to be examined through mapping, and determining that my current strategy was not efficient, nor was it practical.

The fifth technique is taking apart the problem. To accomplish this, a solver should examine the individual pieces of a problem in order to understand the whole. I stumbled upon this technique when I broke apart the cube and put it back together. Although it did not lead me to a solution, it did represent an important skill that is applicable to many other problems.

The sixth technique, and one of the most important ones, is asking for help. If there are resources out there that can aid in the discovery of a solution, it would be nonsensical not to take advantage of it. This reduces the time it takes to reach a solution and can potentially serve as a basis for determining an even better one. To complete this technique, a solver should research known solution sets and consult experts. In relation to the Mini Cube, this means looking up solution algorithms online and talking to those who know how to solve a Rubik's Cube.

The last technique I utilized was determining which method works best for the solver's mindset. There are typically many different approaches one can take when finding a solution that will all yield the desired results. One kind of solution set is one that is direct and quick but requires a lot of memorization. For example, some algorithms needed to solve the Mini Cube are directed at very specific configurations and require less twists overall to reach the desired solution, but require the memorization of around 8 algorithms. Another kind of solution set is one that is roundabout but easy to remember. For example, some algorithms needed to solve the Mini Cube are able to be applied to all configurations but may need to be applied multiple times before it takes the desired effect. Depending on how the solver prefers to approach problems, they may prefer either kind of solution set. For many, the memorization of only one or two algorithms is simpler, and therefore the preferred method of solving problems. However, for those that prefer to reach a solution in the most efficient way possible, they will prefer to memorize all the algorithms aimed at specific configurations so as to reach a solution with the fewest moves possible.

Other Methods

There are many other problem solving techniques that one may want to consider depending on the problem at hand. Many of these were discussed in the literature review. One very important one, however, that was not discussed was the application of shortcuts. If applicable, shortcuts are able to make the problem solving process much more efficient. The solver must be very cautious when deciding if the use of shortcuts is appropriate, however, because some may be in violation of the allowable operations, the ethical standards of the field, or the background theory behind the accepted solution set. These types of shortcuts are commonly referred to as loopholes. In the case of the Mini Cube, this loophole idea is mostly evident in the commonly suggested solution of taking off the stickers on each block and placing them back on the cube in such a way that results in all six faces being only one color. Consequently, this action is in violation of the allowable operations, which are twisting one of the six faces either clockwise or counterclockwise as many times as the solver wishes. Even though the cube resembles a solved cube in the end, the violation in the process negates the results. Therefore, taking off the stickers and putting them back on is not a viable solution set to the Rubik's Cube.

Another important technique that I did not utilize was relating the problem at hand to a similar one that I had previously solved. This technique was not appropriate for the solving of the

Mini Cube because it was the first type of Rubik's Cube I had attempted to solve. However, when moving on to a 3x3x3, or a 4x4x4, it would make sense to use the methods and algorithms that I have learned for the Mini Cube and apply them to these bigger ones. I do not know if this will yield an appropriate solution, but it would give me a place to start.

Conclusion and Future Work

The process for solving difficult problems will vary depending on the problem at hand. However, all the techniques described above can help in discovering a solution. A specific solver's preferences will also have an effect on the methods utilized, as each person has their own unique strengths. One of the most important things I learned during this process is that asking for help is almost always a necessity. There are few people who are able to approach a difficult problem and never consult another person, and no person should feel that they are inadequate for not being one of these people. Moreover, a solver will typically only reach a solution to a difficult problem if they are interested in it and can find ways to keep their spirits high. These types of problems will always be accompanied by hardships and dead ends, making the ability to deal with frustration a necessity. Therefore, if the solver is willing to persevere, they will make it through, but if they are not, then they face only disappointment.

By successfully solving one difficult problem, consideration should be put into solving the next one. In my case, knowing how to solve the Mini Cube, I would like to expand on what I have learned and try out the standard Rubik's Cube--the 3x3x3. As previously discussed, I would start off with the methods I utilized to solve the Mini Cube, mimicking the algorithms to see if they apply. When this techniques ceases to work, however, I will have to find new ways to approach the problem. Having analyzed the specific problem solving strategies helpful in approaching difficult problems, I am confident that I could persevere through the new problem and find a solution.

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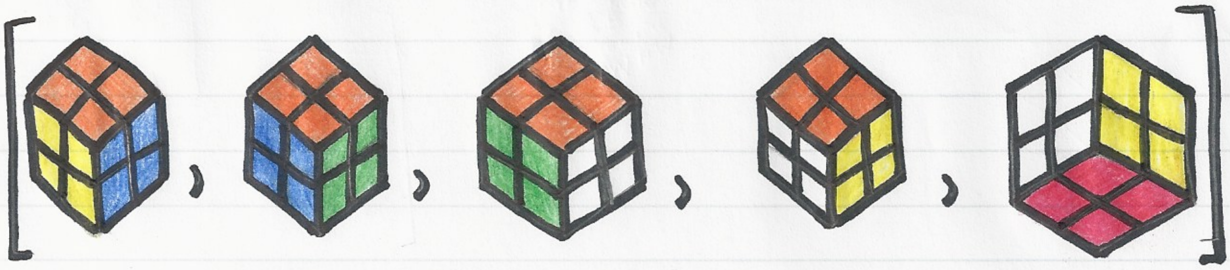
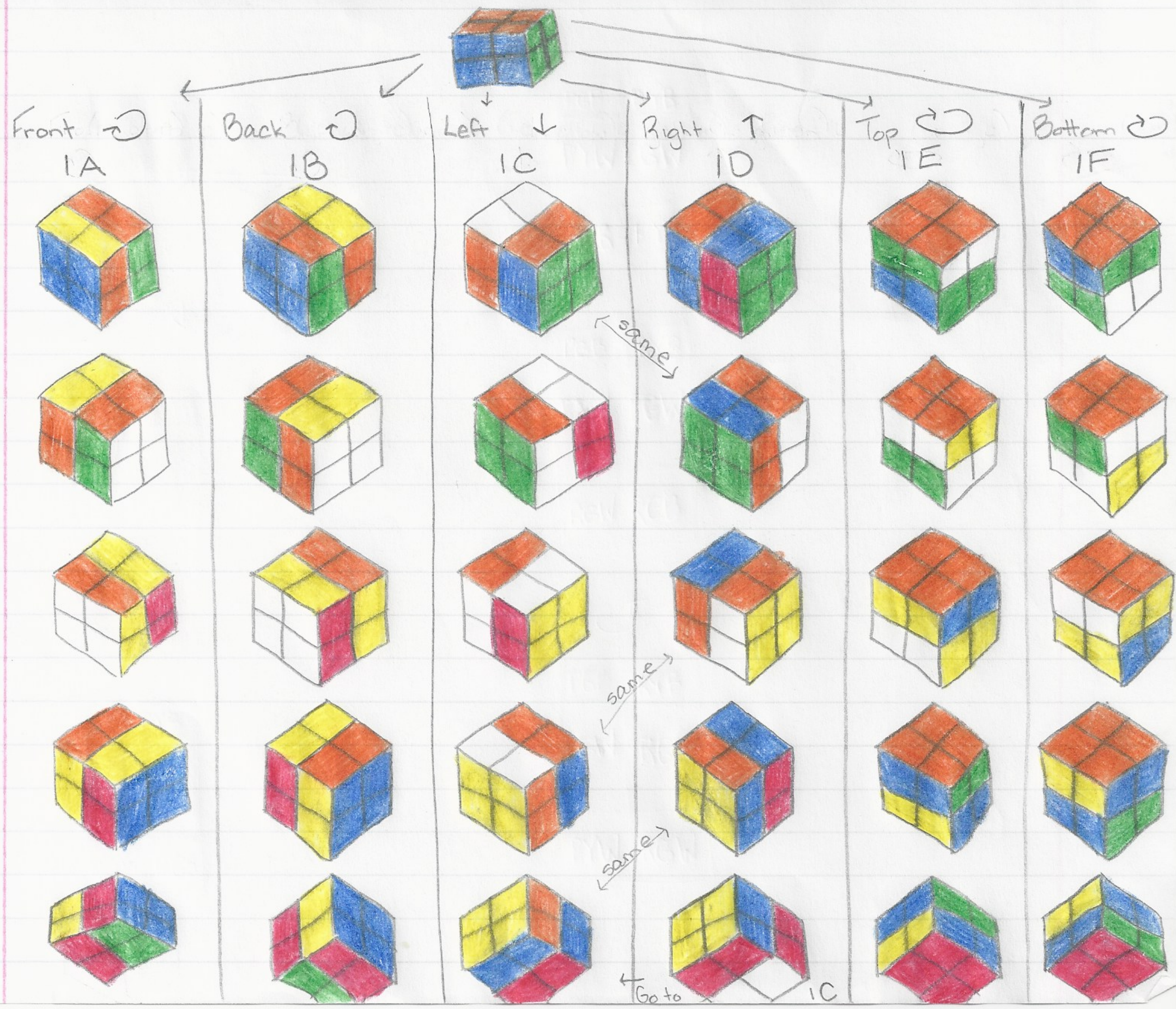
Appendix A: State-Action Tree

Solved State: (State 0)

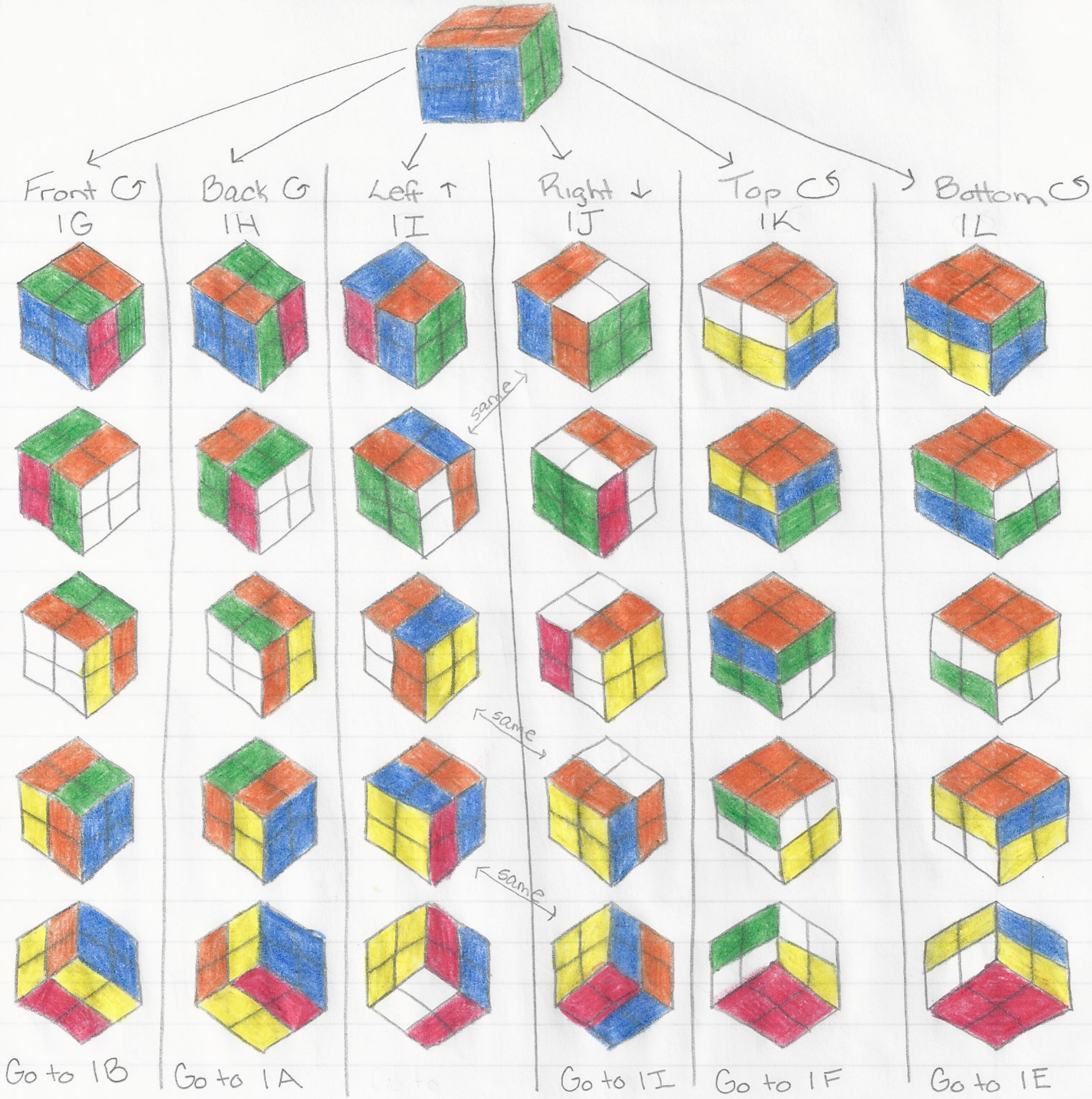


Note: Orange opposite red
White opposite blue
Yellow opposite green

This first diagram illustrates the appearance of a solved Mini Cube. This state will be referred to as State 0 and is the first state in the state action tree.



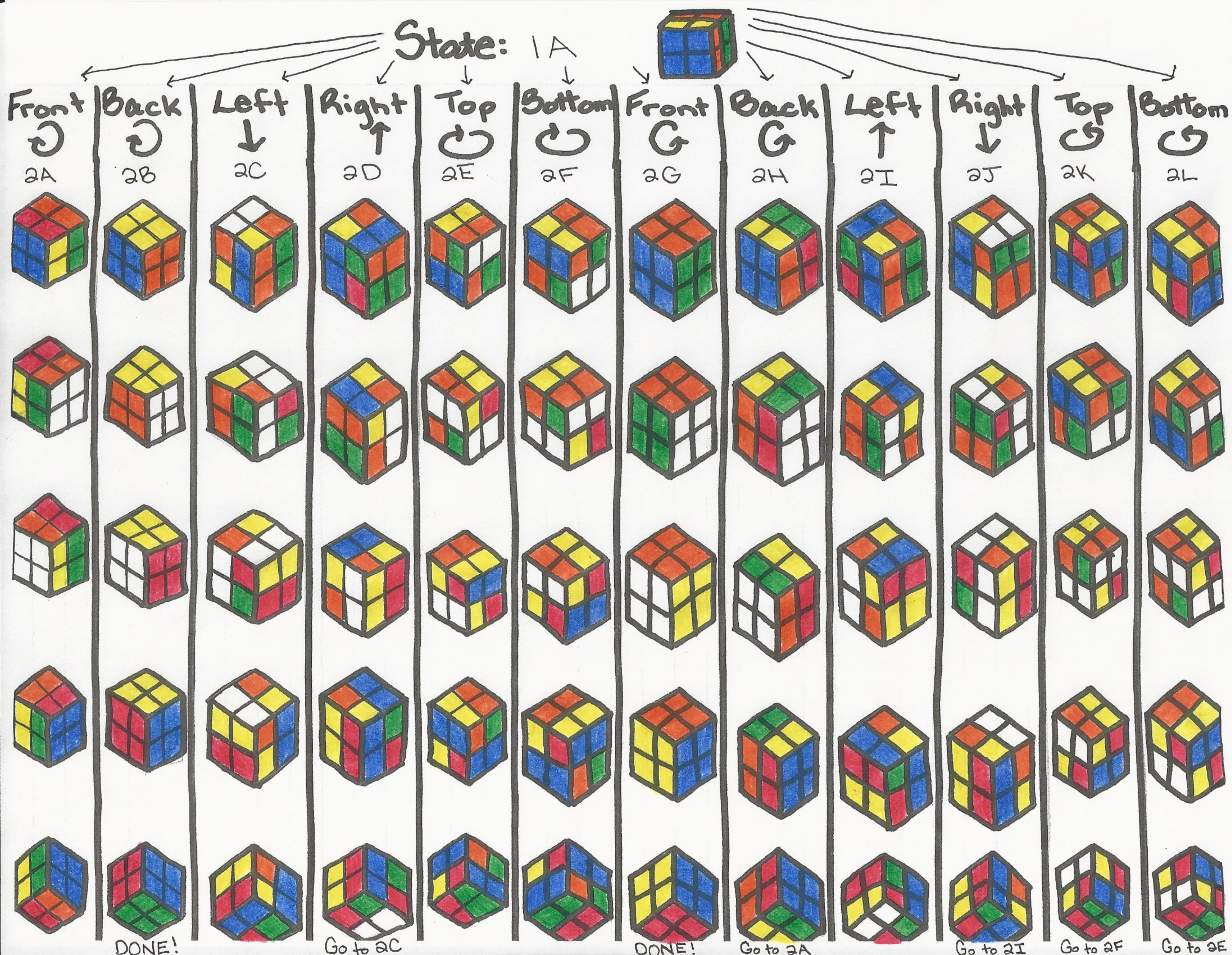
This branch represents the first six branches stemming from State 0. Note that States 1C and 1D are identical configurations, though the vantage points differ. For this reason, 1D can be mapped to 1C, eliminating any branches that would stem from 1D. This is referred to as "pruning the tree".



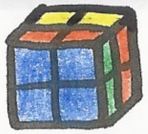
Note: Front ↻ = Back ↺ Back ↻ = Front ↺
Left ↑ = Right ↓ Right ↑ = Left ↓
Top ↻ = Bottom ↺ Top ↺ = Bottom ↻

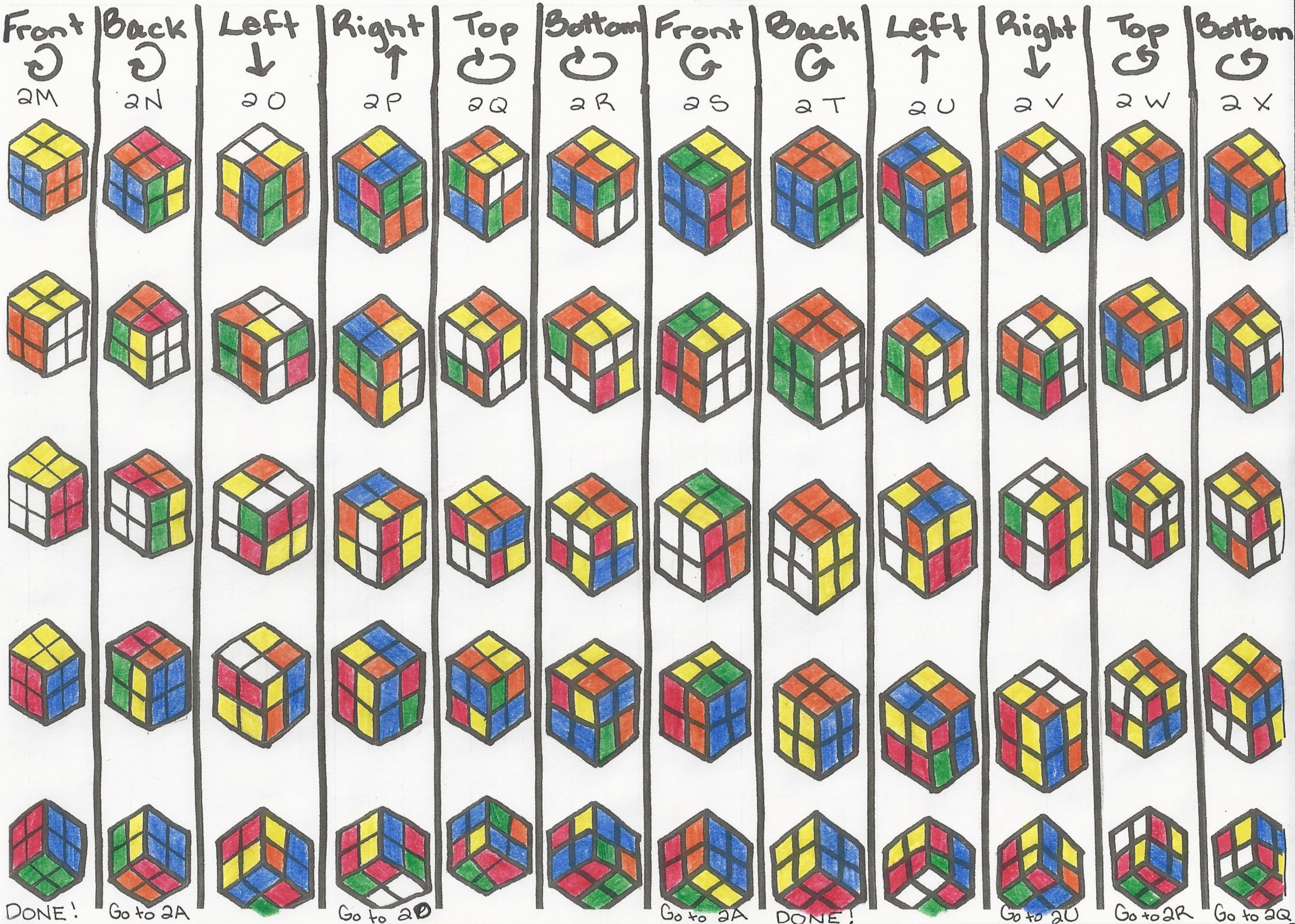
These diagrams represent the other six branches stemming from State 0. Most of these branches are able to be mapped to other States that were already drawn.

The following diagrams represent branches stemming from 1A, 1B, 1C, 1E, 1F, and 1I, then from 2A, 2C, and 2E. All the first stage states have 12 branches stemming from them, representing the 12 possible twists a cube could make. The second stage diagrams only have 6 branches, taking advantage of the parity noted above.


State: 1A
Front 2A
Back 2B
Left 2C
Right 2D
Top 2E
Bottom 2F
Front 2G
Back 2H
Left 2I
Right 2J
Top 2K
Bottom 2L
DONE!
Go to 2C
DONE!
Go to 2A
Go to 2I
Go to 2F
Go to 2E

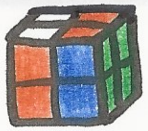
State: 1B



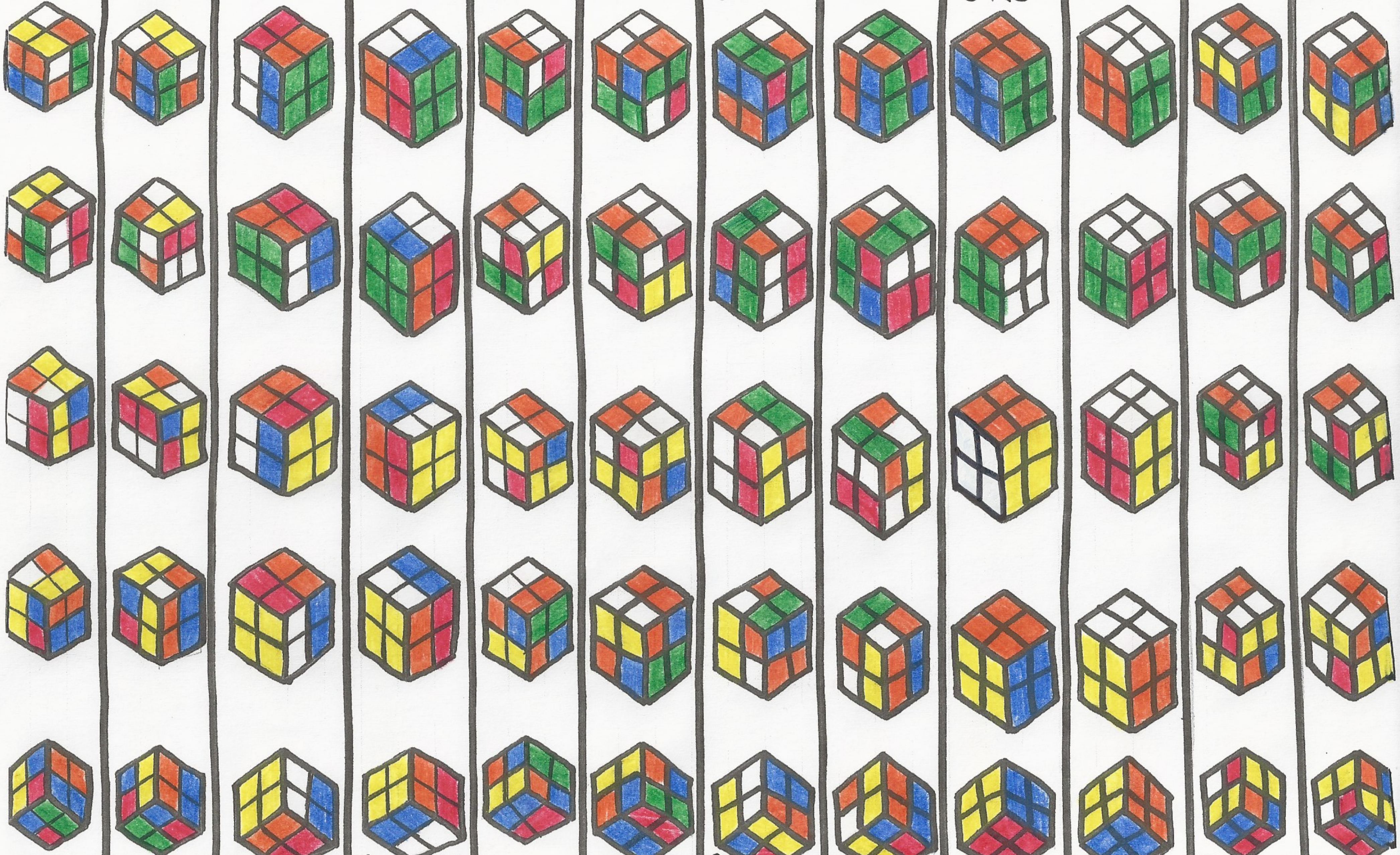


Front
2M
Back
2N
Left
2O
Right
2P
Top
2Q
Bottom
2R
Front
2S
Back
2T
Left
2U
Right
2V
Top
2W
Bottom
2X
DONE!
Go to 2A
Go to 2O
Go to 2A
DONE!
Go to 2U
Go to 2R
Go to 2Q

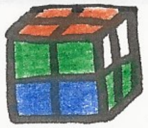
State: 1C



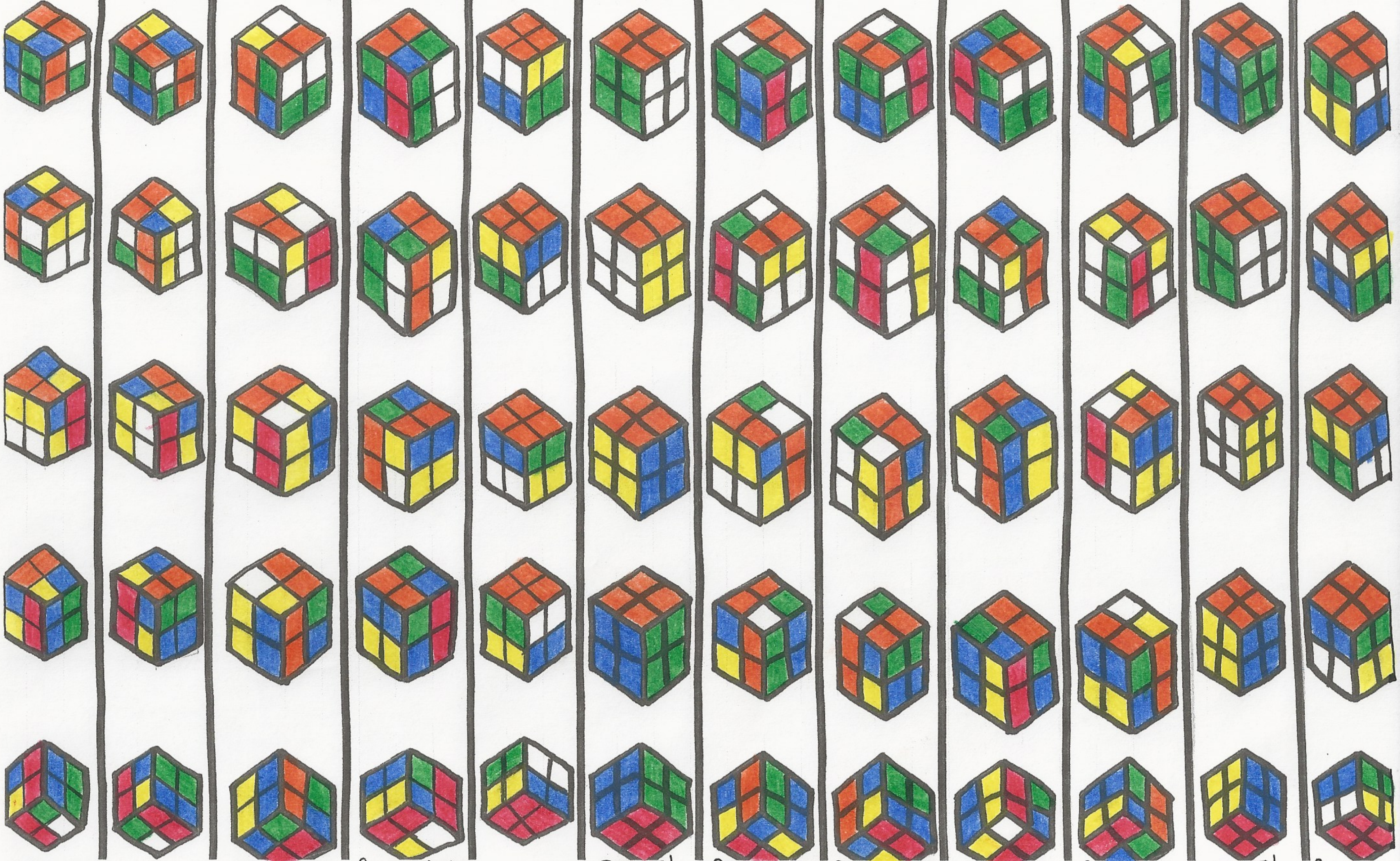
Front ↻ 2Y	Back ↻ 2Z	Left ↓ 2AA	Right ↑ 2AB	Top ↻ 2AC	Bottom ↻ 2AD	Front ↺ 2AE	Back ↺ 2AF	Left ↑ 2AG	Right ↓ 2AH	Top ↺ 2AI	Bottom ↺ 2AJ
			Go to 2AA			Go to 2Z	Go to 2Y	DONE!	DONE!	Go to 2AD	Go to 2AC



State: 1E

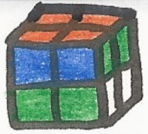


Front	Back	Left ↓	Right ↑	Top	Bottom	Front	Back	Left ↑	Right ↓	Top	Bottom
2AK	2AL	2AM	2AN	2AO	2AP	2AQ	2AR	2AS	2AT	2AU	2AV

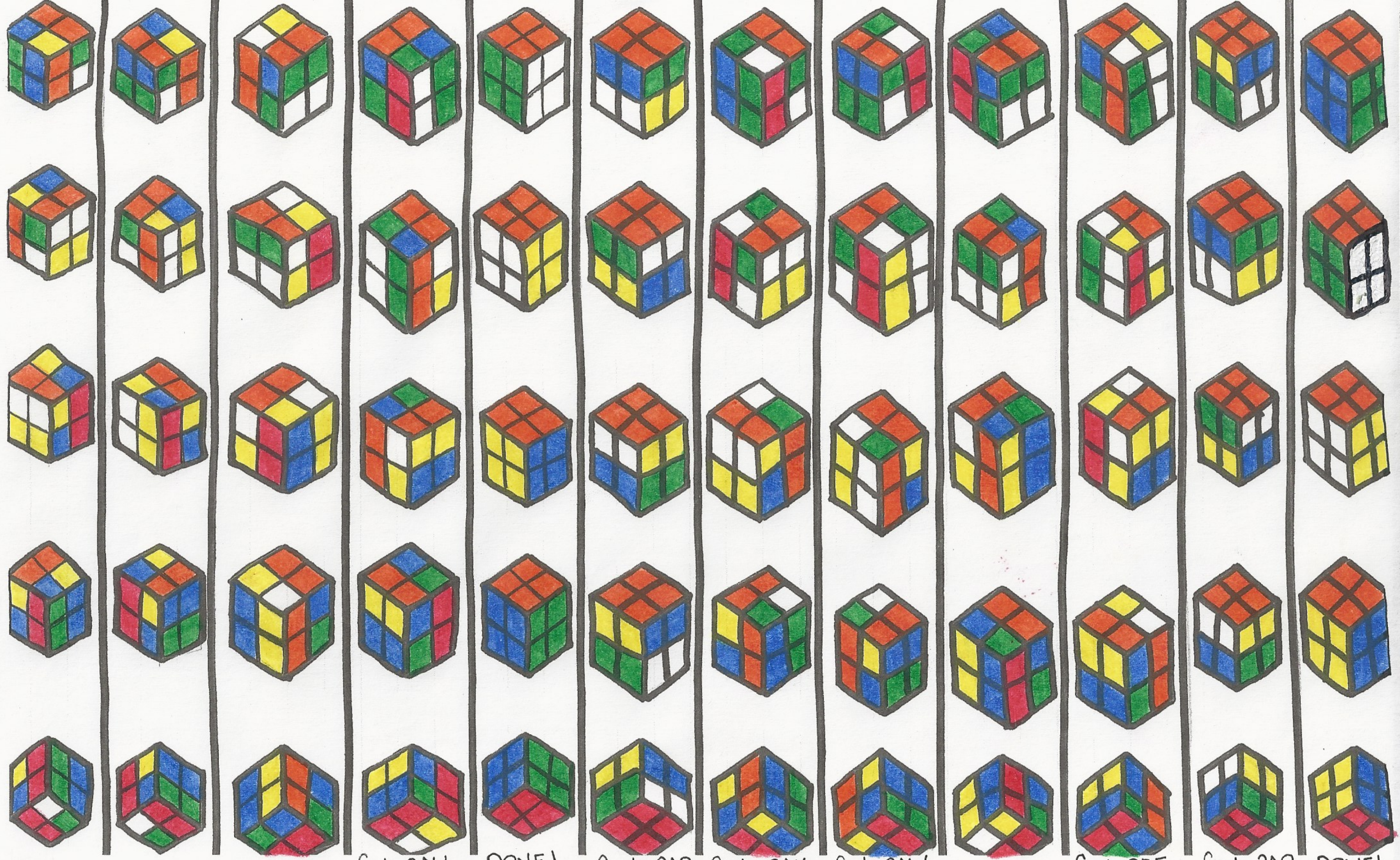


Front	Back	Left ↓	Right ↑	Top	Bottom	Front	Back	Left ↑	Right ↓	Top	Bottom
			Go to 2AM		DONE!	Go to 2AL	Go to 2AK		Go to 2AS	DONE!	Go to 2AO

State: 1F

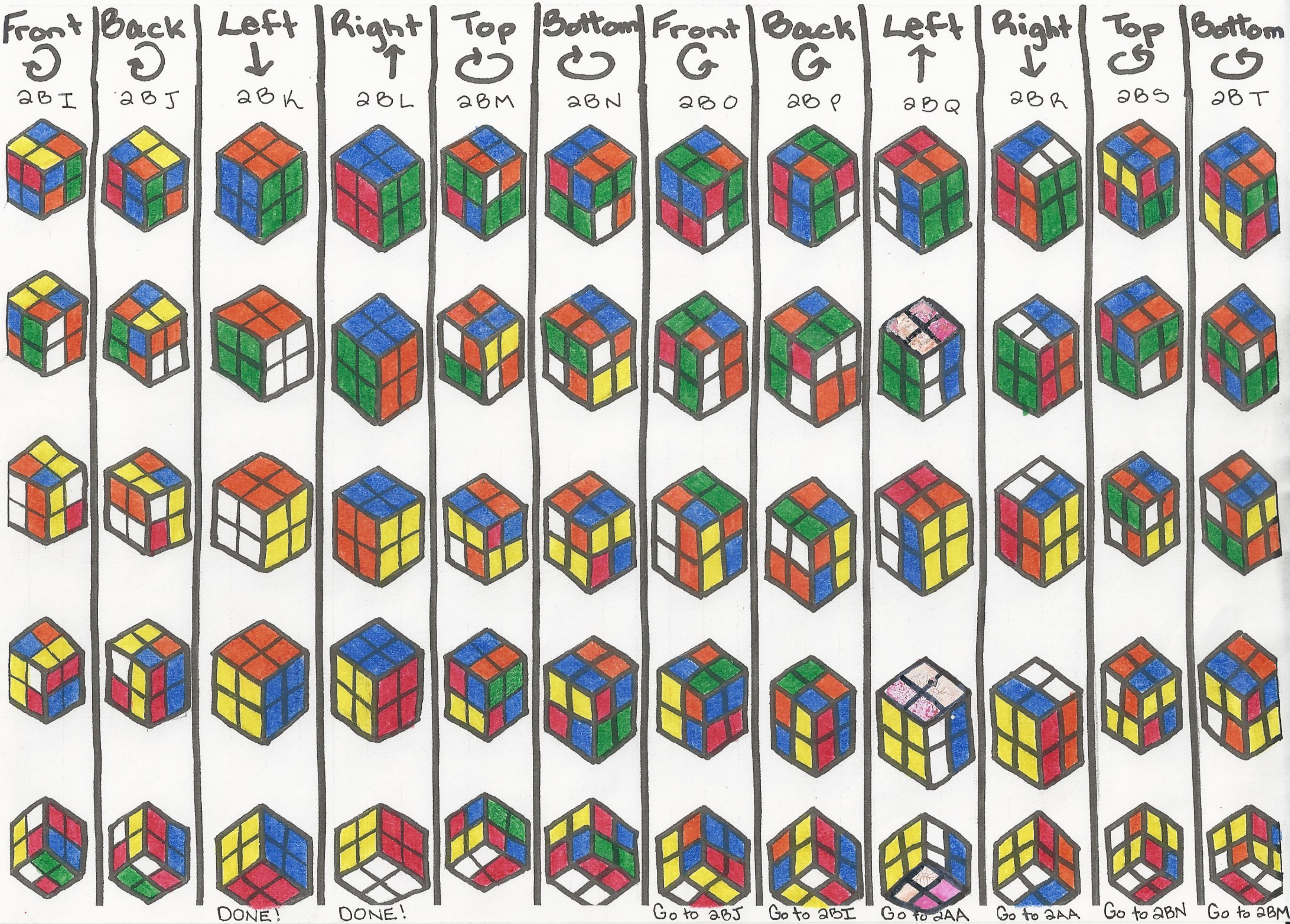


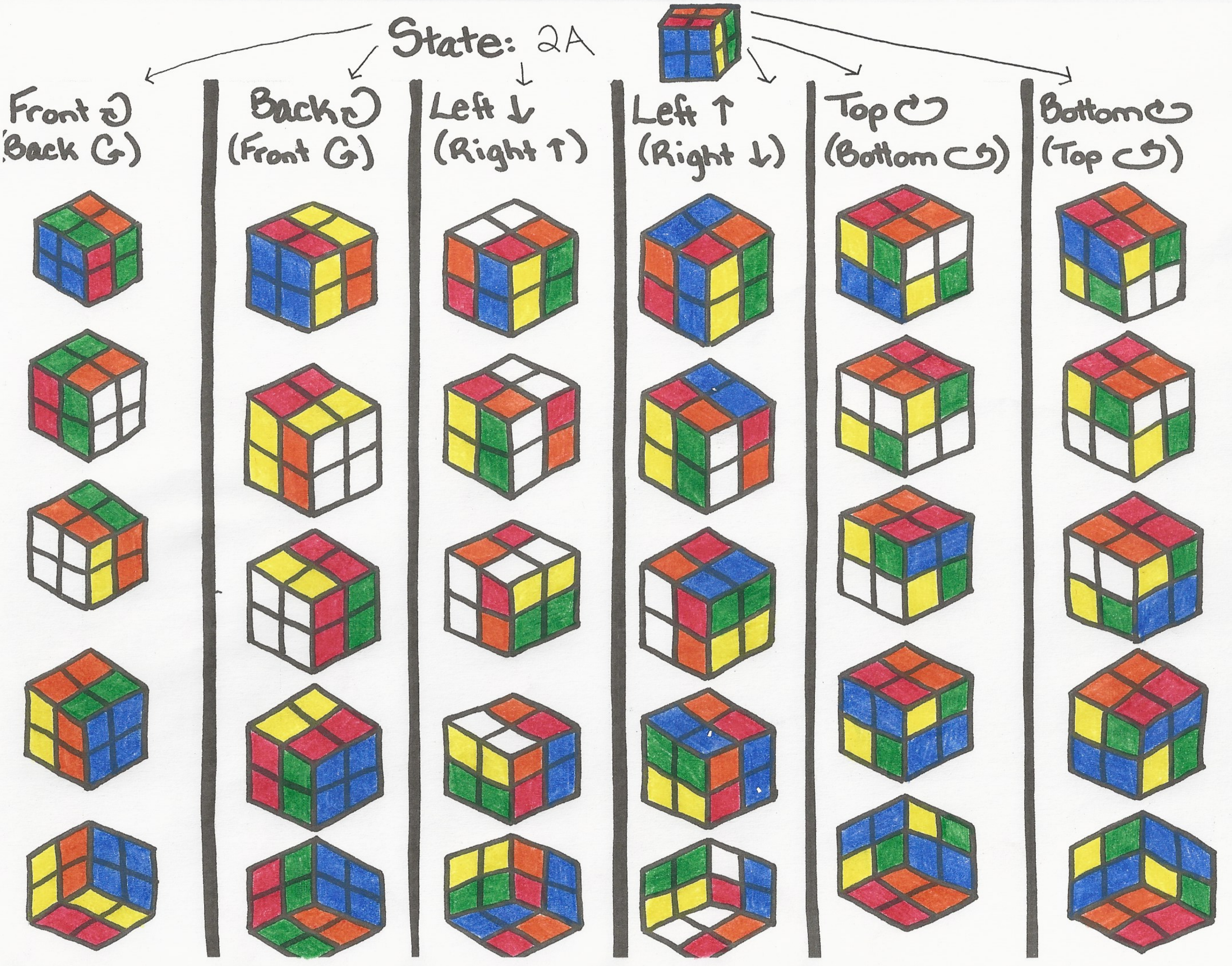
Front	Back	Left	Right	Top	Bottom	Front	Back	Left	Right	Top	Bottom
		↓	↑					↑	↓		
2AW	2AX	2AY	2AZ	2BA	2BB	2BC	2BD	2BE	2BF	2BG	2BH



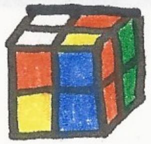
2AW	2AX	2AY	2AZ	2BA	2BB	2BC	2BD	2BE	2BF	2BG	2BH
			Go to 2AY	DONE!	Go to 2AO	Go to 2AX	Go to 2AW		Go to 2BE	Go to 2AO	DONE!

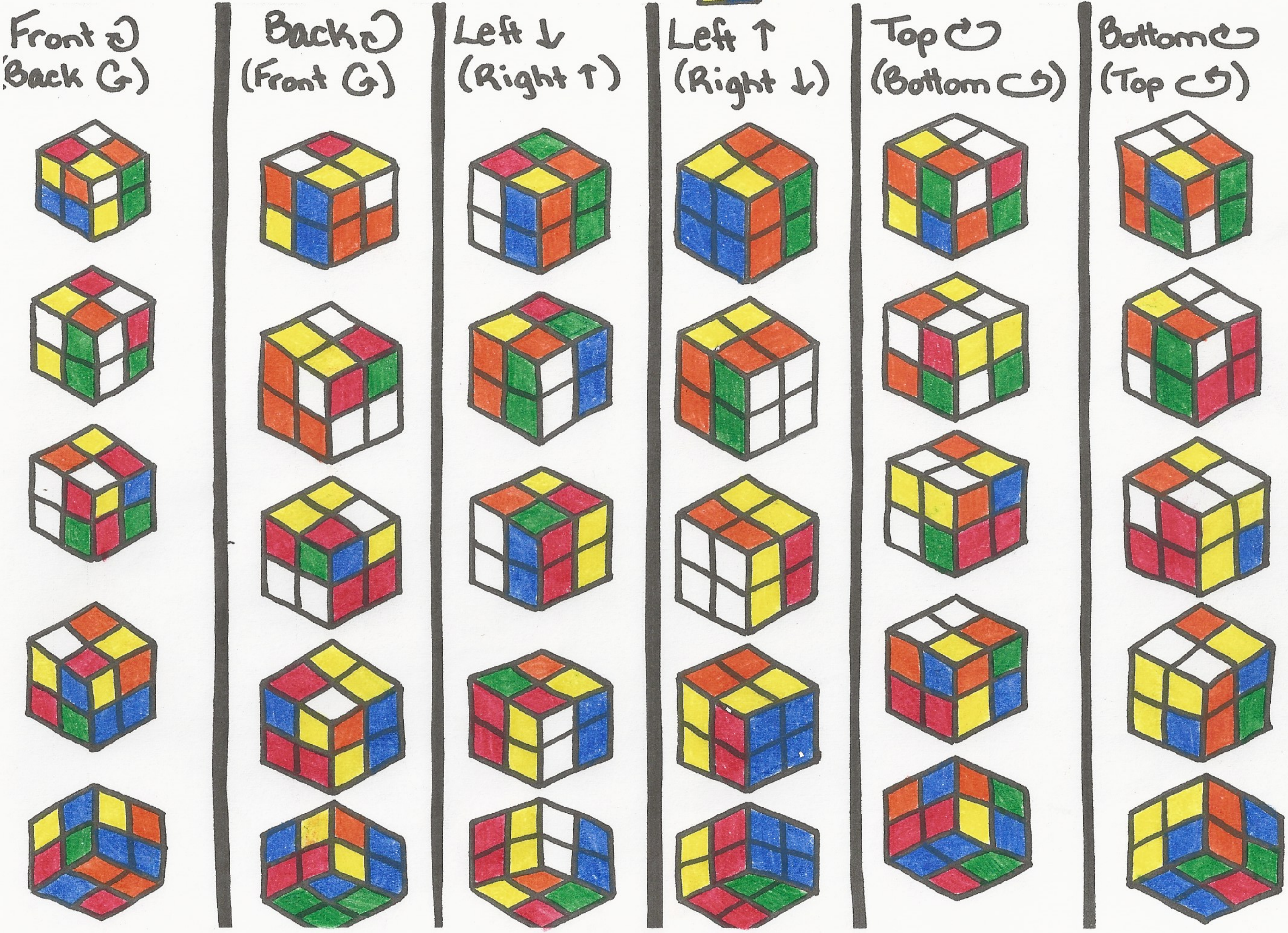
State: 1I


Front
2BI
Back
2BJ
Left
2BK
Right
2BL
Top
2BM
Bottom
2BN
Front
2BO
Back
2BP
Left
2BQ
Right
2BR
Top
2BS
Bottom
2BT
DONE!
DONE!
Go to 2BJ
Go to 2BI
Go to 2AA
Go to 2AA
Go to 2BN
Go to 2BM


State: 2A
Front ↻ (Back ↺)
Back ↻ (Front ↺)
Left ↓ (Right ↑)
Left ↑ (Right ↓)
Top ↻ (Bottom ↺)
Bottom ↻ (Top ↺)

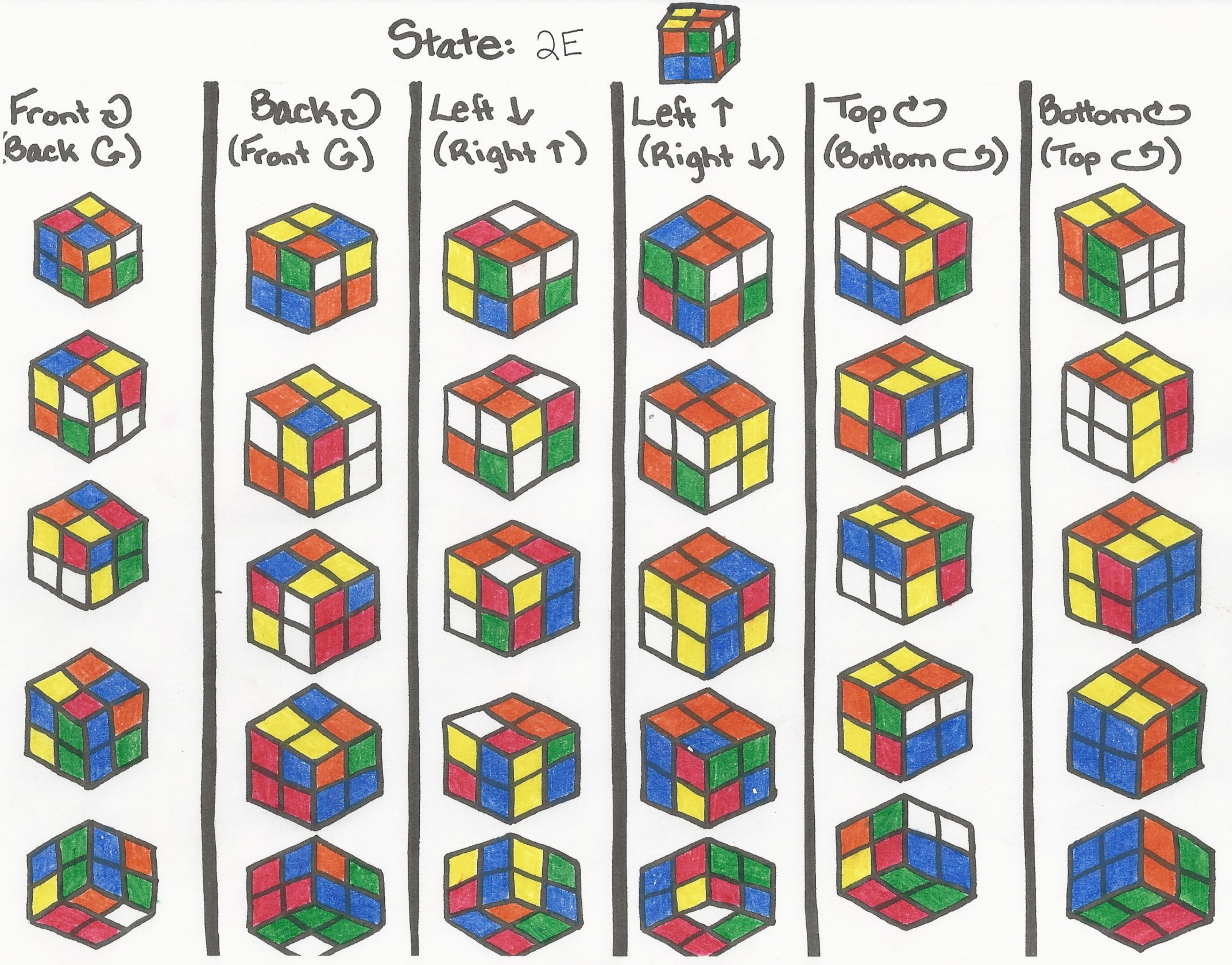
State: 2C





Front ↻
(Back ↺)
Back ↻
(Front ↺)
Left ↓
(Right ↑)
Left ↑
(Right ↓)
Top ↻
(Bottom ↺)
Bottom ↻
(Top ↺)

State: 2E


Front ↻
(Back ↺)
Back ↻
(Front ↺)
Left ↓
(Right ↑)
Left ↑
(Right ↓)
Top ↻
(Bottom ↺)
Bottom ↻
(Top ↺)