PLANAR DIFFERENCE EQUATIONS: ASYMPTOTIC BEHAVIOR OF SOLUTIONS AND 1-1 RESONANT POINTS

William T. Jamieson
University of Rhode Island, william.t.jamieson@gmail.com

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WILLIAM T. JAMIESON

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WILLIAM T. JAMIESON

APPROVED:

Dissertation Committee:

Major Professor Orlando Merino

Mustafa Kulenovic

Richard Vaccaro

Nasser H. Zawia

DEAN OF THE GRADUATE SCHOOL

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ABSTRACT

In order to study the global behavior of difference equations, it is necessary to understand the local behavior in a neighborhood of a equilibrium point of the difference equation. This thesis focuses on two aspects of the local behavior of planar difference equations: the asymptotic behavior of a solution converging to a hyperbolic fixed point, and the local qualitative behavior of a non isolated fixed point whose jacobian matrix has a particular structure.

Manuscript 2 describes how closely a convergent solution $\{x_n\}$ of (real or complex) difference equations $x_{n+1} = J\,x_n + f_n(x_n)$ can be approximated by its linearization $z_{n+1} = J\,z_n$ in a neighborhood of a fixed point; where x_n is a m-vector, J is a constant $m \times m$ matrix and $f_n(y)$ is a vector valued function which is continuous in y for fixed n, and where $f_n(y)$ is small in a sense.

Manuscript 3 describes completely the local qualitative behavior of a real planar map in a neighborhood of a non-isolated fixed point whose jacobian matrix is similar to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, also called a non-isolated *1-1 resonant* fixed point. Theorem 3 gives conditions for four non-conjugate dynamical scenarios to occur.

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PREFACE

This thesis has been written in manuscript form according to the guidelines provided by the Graduate School of the University of Rhode Island. The content of this thesis is made up of two research papers, Manuscripts 2 and 3. Manuscript 2 of this thesis has been accepted pending revisions by the *Journal of Mathematical Analysis and Applications*. Manuscript 3 will be submitted for publication shortly.

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MANUSCRIPT 1

Introduction

The main goal of the study of difference equations is to establish the global dynamical behavior of the orbits of the difference equation. A common strategy used to help understand difference equations in general is to restrict the analysis to classes of difference equations [7], [15]. In particular, since the early 1990s there has been considerable interest in the study of rational difference equations and monotone difference equations. Much work has been done in these areas by Elaydi, Smith, Agarwal, Ladas, Grove, Kulenović, Merino [7], [8], [14], [15], [16] and others cited therein. In order to understand the global dynamical behavior of a difference equation, one must first understand the local dynamical behavior near the fixed points of the difference equation. In difference equations, fixed points are classified by the modulus of the eigenvalues of the jacobian matrix associated with the fixed point. In the plane, there are three scenarios for the modulus of the eigenvalues of a jacobian matrix of the map of a difference equation. I will consider each scenario below:

In the first case when the eigenvalues $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$, the local dynamics of this case are well understood. Phillip Hartman (see [7] for a secondary source) proved for a C^2- diffeomorphism that in a neighborhood of a hyperbolic fixed point, that the map is C^1 conjugate to its linearization at the fixed point. This is the discrete analog to the Hartman-Grobman Theorem [6].

In the second case when $|\lambda_1| = 1$ and $|\lambda_2| = 1$, the dynamics are also understood in some circumstances. If λ_1 and λ_2 are complex conjugates with $\lambda_{1,2} = e^{\pm iq\theta} \neq 1$ for $q = 0, 1, 2, 3,$ or 4 (these exceptions are called *strong resonances*), then techniques from KAM theory can be used to establish the local dynamical behavior, see [16] for an overview or [2] for an application of the theory. A fixed point is called 1-1 resonant if the jacobian of the map at the fixed point is similar to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The local dynamical behavior near a 1-1 resonant fixed point is still unresolved because this type of fixed point has a strong resonance.

In the third case when $|\lambda_1| \neq |\lambda_2| = 1$, the local dynamical behavior has not been completely established. However, the existence of invariant center-stable and center-unstable manifolds has been investigated completely. In 1977 Hirsch, Pugh, and Shub [11] proved that in the neighborhood of a pseudo hyperbolic fixed point (this covers the first and third cases mentioned above), a C^k-system possesses a C^k-unstable manifold, establishing the existence of a center-unstable manifold associated with the fixed point. In the case when the map is invertible, their theorem also establishes a center-stable manifold associated with the fixed point. The local behavior in this case has been established for some subclasses; for example in 2010 Kulenović and Merino [16] described the local dynamical behavior in the case when $\lambda_1 < \lambda_2 = 1$ for monotone systems of difference equations. There is no general theory that can be applied to the remaining configurations of λ_1, λ_2 or for more general classes of functions. Manuscript 3 classifies the local dynamical behavior of near a non-isolated fixed point whose jacobian has both eigenvalues equal to 1.

Even when the local dynamics of a system are understood, as they are in the case when the eigenvalues $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$, there is still more analysis to be done. In this case, if one of the eigenvalues lies within the unit circle in the complex plane, by the discrete version of the Hartman-Grobman Theorem [6] we know that the behavior of the map is conjugate to its linearization. In particular, there exists a stable manifold on which orbits under forward iteration of the map will converge to the fixed point. If we consider a solution that converges to the fixed point, the question still remains, *How close can orbits of the linearization be to orbits from the original map?* This question is important in practice because if you approximate a nonlinear map with its linearization in a neighborhood of the fixed point, it is imperative to understand how much error you are introducing by making the approximation. There have been several papers [17] - [20] studying error estimates between a map and its linearization that have been published in the last 15 years. Manuscript 2 gives a improved bound on the error.

To conclude the introduction, we review some basic notation and definitions related to the content of this dissertation. A continuous function F from a set $D \subset \mathbb{R}^2$ into D is called a *planar map*, or just *map*. The relation

$$(x_{n+1}, y_{n+1}) = F(x_n, y_n), \quad n = 0, 1, 2, \ldots, \quad (x_0, y_0) \in D \tag{1}$$

is a *first order autonomous difference equation in the plane*. A point $(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}) = F(\bar{x}, \bar{y})$ is called a *fixed point* of the map, or *equilibrium point* of the difference equation. A sequence $\{(x_n, y_n)\}$ that satisfies a difference equation is called a *solution* or *orbit* of the difference equation. For differentiable functions F, if all the eigenvalues of the *jacobian* (also called the *characteristic values*) of F at a fixed point lie off the unit circle in $\mathbb{C}$, we say that the fixed point is *hyperbolic*. Otherwise, the fixed point is called *non hyperbolic*. A fixed point of a system of difference equations is called *isolated* if there exists a neighborhood of the fixed point that does not contain any other fixed points. A fixed point is called *non-isolated* if the fixed point is not isolated. If a real analytic system of difference equations has a non-isolated fixed point, one of the eigenvalues of the jacobian at the fixed point must be equal to 1. This can only occur in the second and third cases above. A planar map F is real analytic at a point (u, v) if there exists a neighborhood V of (u, v) such that on V each coordinate function of F admits a convergent power series representation of the form $\sum_{j,k=0}^{\infty} a_{jk}(x-u)^j(y-v)^j$. A function F is said to be real analytic on an open set U if it is real analytic at each point (u, v) in U.

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MANUSCRIPT 2

Asymptotic Behavior Results for Solutions to Some Nonlinear Difference Equations

William T. Jamieson and Orlando Merino

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Abstract

We give asymptotic results for convergent solutions $\{x_n\}$ of (real or complex) difference equations $x_{n+1} = J\,x_n + f_n(x_n)$, where x_n is a m-vector, J is a constant $m \times m$ matrix and $f_n(y)$ is a vector valued function which is continuous in y for fixed n, and where $f_n(y)$ is small in a sense. In addition, we obtain asymptotic results for solutions $\{x_n\}$ of the Poincaré difference equation $x_{n+1} = (A + B_n)\,x_n$ where B_n satisfies $\|B_n\| = O(\eta^n)$ with $\eta \in (0, 1)$. An application and examples illustrate the results.

2.1 Introduction

Consider the difference equation

$$x_{n+1} = J\,x_n + f_n(x_n), \tag{2}$$

where x_n is an m-vector, J is a constant $m \times m$ matrix and $f_n(y)$ is a vector valued function which is continuous in y for fixed n, and where $f_n(y)$ is small in some sense as $(n, \|y\|) \to (\infty, 0)$. Equation (2) can be seen as a non-autonomous perturbation of the linear, constant coefficients equation $x_{n+1} = J\,x_n$. A natural question is, what can be said about the asymptotic bahavior of solutions $\{x_n\}$ to (2) that converge to zero? Equation (2) is very general and it includes as a particular case the (matrix) Poincaré equation

$$x_{n+1} = (A + B_n)\,x_n\,, \tag{3}$$

where x_n is an m-vector, and A, B_n are an $m \times m$ matrices for $n = 1, 2, \ldots$ such that $\|B_n\| \to 0$.

Equation (2) has been studied by O. Perron, C. V. Coffman, and others [1] [2], who obtained asymptotic behavior results for solutions $\{x_n\}$ of (2). Coffman's Theorem 5.1 in [2] is a refinement of results of Perron [1], and it states that if

$\|f_n(y)\|/\|y\| \to 0$ as $(n, y) \to (\infty, 0)$, solutions $x = x_n$ to (2) that converge to zero either have $x_n = 0$ for large n or satisfy

$$\lim_{n\to\infty} |x_n|^{1/n} = |\lambda| \,,$$

where λ is an eigenvalue of J. Although not stated explicitly, Coffman further established (Theorems 5.1 and 8.1 of [2]) that $0 < |\lambda| < 1$ implies the existence of $\tilde{y} \in \mathbb{C}^d$ and $\kappa \in \mathbb{N}$ for which the following asymptotic relation holds:

$$x_n = J^n \, \tilde{y} + o(n^\kappa \, |\lambda|^n) \quad \text{as} \quad n \to \infty \,. \tag{4}$$

The origin of the study of relation (2) can be traced to Poincaré [3], who investigated the non-autonomous scalar linear difference equation

$$\zeta_{n+m} + p_{m-1,n} \, \zeta_{n+m-1} \; + \; \cdots \; + \; p_{1,n} \, \zeta_{n+1} \;\; = \;\; 0 \,, \tag{5}$$

for which the following limits are assumed to exist:

$$\lim_{n\to\infty} p_{m-1,n} = q_{m-1} \,, \qquad \ldots \qquad , \lim_{n\to\infty} p_{1,n} = q_1 \,. \tag{6}$$

Equation (5) has a limiting equation

$$\zeta_{n+m} \; + \; q_{m-1} \, \zeta_{n+m-1} \; + \; \cdots \; + \; q_1 \, \zeta_{n+1} \;\; = \;\; 0. \tag{7}$$

Poincaré proved under the hypothesis that the roots of the characteristic equation of (7) have distinct moduli, that for every solution ζ_n of (5) for which $\zeta_n \neq 0$ for all large n, the ratios ζ_{n+1}/ζ_n approach one of the characteristic roots of (7) [3]. Poincaré assumed the coefficients $p_{\ell,n}$ in (5) to be rational functions of n, but this condition may be dropped if $p_{0,n}$ is required to be nonzero for all n (see the statement and proof of Theorem 2.13.1 in [6]). Poincaré's result was generalized by Perron [1], and by Gelfond et al [4] to systems (2) and (3) (see [2]).

Since the mid 1990s there has been renewed interest in asymptotic results, see the book by Elaydi [5] and work cited therein, and see also [7, 8, 9, 10, 11], Pituk, Pituk et al, Bodine and Lutz, [12, 13, 14, 15, 16, 17, 19], Kalabusić-Kulenović [20], and many others [21– 26].

In [17] R. P. Agarwal and M. Pituk studied the scalar equation (5) when the convergence in (6) is at a geometric rate. In [18], S. Bodine and D. A. Lutz greatly improved the estimates of Agarwal and Pituk, considering (matrix) Poincaré systems (3), again under the assumption of geometric convergence. The following result is Theorem 1 of [18] with changes in notation to match that of this paper.

Theorem BL1 [Bodine-Lutz, 2009] *Suppose that* x_n *is a solution of* (3), *where* A *is a constant and invertible matrix with eigenvalues* $\{\lambda_1, \ldots, \lambda_d\}$ *repeated according to their multiplicity. Suppose that there exists* $\eta \in (0,1)$ *and* $c > 0$ *such that* $\|B_n\| \leq c\,\eta^n$ *for* $n = 0, 1, \ldots$. *For every fixed* $i \in \{1, \ldots, d\}$ *let* $q_i := \max\{\left|\frac{\lambda_j}{\lambda_i}\right| : 1 \leq j \leq d$ *such that* $\left|\frac{\lambda_j}{\lambda_i}\right| < 1\}$ *if at least one such* λ_j *exists, and* $q_i = 0$ *otherwise. Define* $q := \max\{q_i : 1 \leq i \leq d\}$. *Then* (3) *has for* n *sufficiently large a fundamental matrix satisfying*

$$Y_n = [I + O(n^p[q^n + \eta^n])]\, A^n \quad as \quad n \to \infty\,, \tag{8}$$

where p *can be explicitly estimated.*

The formula $p = 4t - 3$ for the parameter p in (8) is given in page 825 of [18], where t is the size of the largest elementary Jordan block of A. A better estimate for p is possible in many cases. For example, p may be taken to be zero when the matrix A is diagonalizable and $q_i \neq \eta$ for all i ([18], page 823). In this paper we give an asymptotic estimate more precise than (8): the parameter q is removed from the asymptotic formula, and an estimate for the exponent p of the polynomial term is obtained which, in many cases, improves upon the one given in [18]. This estimate is obtained with a method of proof that is different than that in [18]. In addition, the matrix A is not required to be invertible. More precisely, Theorem 2 presented in Section 2.4 gives the following asymptotic expansion for a solution x_n of (3) such that $\limsup \|x_n\|^{1/n} = \rho$ and $\tilde{y}$ is a suitable vector:

$$x_n = A^n \, \tilde{y} + O(n^{\beta} \, (\eta \, \rho)^n) \quad \text{as} \quad n \to \infty. \tag{9}$$

According to Theorem 2 the parameter β may be chosen in (9) so that $\beta \le 3\,t-2$, and often $\beta \le 3\,(t-1)$ suffices. Our proof of Theorem 1 gives a procedure for approximating the vector $\tilde{y}$ in (16), see Corollary 2 in Section 2.4.

In order to compare formulas (8) and (9), apply formula (8) to an initial vector $\tilde{x}$ with corresponding solution x_n satisfying $\limsup \|x_n\|^{(1/n)} = \rho$. Then formula (8) becomes

$$x_n = A^n \tilde{x} + O(n^p \rho^n [q^n + \eta^n]) = \begin{cases} A^n \tilde{x} + O(n^p (\eta\rho)^n) & \text{if } q \le \eta \\ A^n \tilde{x} + O(n^p (q\rho)^n) & \text{if } \eta < q \end{cases} \quad \text{as} \quad n \to \infty\,. \tag{10}$$

When $\eta < q$ it can be easily seen by comparing equations (9) and (10) that the asymptotic formula (9) is more accurate than (8). When $\eta \ge q$, the exponential terms in both (9) and (8) are the same. However $\beta \le p$, and thus (9) is a better asymptotic estimate than (8) or in the worst case, equivalent.

To illustrate the situation outlined in the previous paragraph, consider the Poincaré difference system

$$x_{n+1} = (A + B_n) x_n = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + B_n \right) x_n \,, \quad \text{where } \|B_n\| < \left(\tfrac{1}{3}\right)^n.$$

The matrix A has eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{4}$. Suppose that a solution $\{x_n\}$ satisfies $\limsup \|x_n\|^{(1/n)} = \frac{1}{2} = \rho$ (this is the generic case). In the notation of (8) we have $\eta = \frac{1}{3}$, $q = \frac{1}{2}$, and since $q \ne \eta$, we have $p = 0$. Theorem BL1 implies that

$$x_n = A^n \tilde{x} + O((\tfrac{1}{4})^n) \quad \text{as} \quad n \to \infty\,,$$

whereas under the same assumptions Theorem 2 states that there exists a $\tilde{y}$ such that

$$x_n = A^n \tilde{y} + O((\tfrac{1}{6})^n) \quad \text{as} \quad n \to \infty\,. \tag{11}$$

Note that in relation (11) the parameter β of the asymptotic formula (9) is zero since the condition of Theorem 2 that $\lambda \neq \eta\,\rho$ for every eigenvalue λ of A is satisfied.

In [18] Bodine and Lutz also obtain an asymptotic relation for convergent sequences $\{x_n\} \to \bar{x}$ generated by iteration of a map T whose jacobian at $\bar{x}$ is Lipschitz continuous.

Theorem BL2 [Bodine-Lutz, 2009] *Let $T : \mathbb{C}^d \to \mathbb{C}^d$ and $\bar{x} \in \mathbb{C}^d$ such that $T(\bar{x}) = \bar{x}$. Assume the following:* (i) *there exists an open convex set Ω containing $\bar{x}$ such that the jacobian matrix J_x exists for $x \in \Omega$,* (ii) *the eigenvalues $\lambda_1, \ldots, \lambda_d$ of $J_x(\bar{x})$ do not have modulus 0 or 1, and* (iii) *J_x is Lipschitz continuous in a neighborhood of $\bar{x}$. Then, for any solution $\{x_n\}$ of $x_{n+1} = T(x_n)$ for $n \geq n_0$ such that $x_n \neq \bar{x}$ for large n and satisfying $x_n \to \bar{x}$, there exists an eigenvalue of $J_{\bar{x}}$ with modulus $\rho \in (0,1)$ and corresponding linear combinations of characteristic solutions of the limiting system $y_{n+1} = J_{\bar{x}}\, y_n$ such that*

$$x_n = \bar{x} + y_n + O((\rho\,\delta)^n) \qquad \textit{as} \qquad n \to \infty, \tag{12}$$

where δ is an arbitrary number satisfying $\delta > \max\left\{\rho, \max\limits_{|\lambda_j|<\rho} \frac{|\lambda_j|}{\rho}\right\}$.

Our Corollary 5 presented in Section 2.4 gives the following asymptotic relation

$$x_n \;=\; \bar{x} + J_{\bar{x}}^n\, \tilde{y} \;+\; O(\, n^\beta \rho^{2\,n}\,) \quad \text{as} \quad n \to \infty\,. \tag{13}$$

where $\tilde{y}$ is a suitable vector and β is a suitable nonnegative integer that can be estimated. Clearly relation (13) is more precise than Bodine-Lutz's relation (12). The hypotheses of our Corollary 5 require T to be of class C^2, which is a small strengthening of the requirement in Theorem BL2 that J_x be Lipschitz continuous. The hypothesis on J_x in Corollary 5 may be easily modified to match that of Theorem BL2 (see the proof of Theorem 4 in [18] up to (38)), however this is not done here. We also point out that the y_n in (13) is not restricted to be a linear combination of characteristic solutions of the limiting system, as it is the case in (12). Finally, Corollary 5 does not require that $J_{\bar{x}}$ be invertible.

To give an example illustrating the difference between Bodine-Lutz's asymptotic relation (12) and the asymptotic relation (13) from our Corollary 5, consider the map $T(\xi_1, \xi_2) = (\frac{1}{2}\xi_1 + \xi_1\xi_2, \frac{2}{5}\xi_2)$ for which $\bar{x} = (0,0)$ and $J = J_{\bar{x}}$ has eigenvalues 1/2 and 2/5. Suppose a sequence $x_n = T^n(x)$ for which $x_n \to (0,0)$ satisfies $\lim \|x_n\|^{1/n} = 1/2$. Relation (12) of Theorem BL2 states that for $\delta > \max\{\frac{1}{2}, \frac{2/5}{1/2}\} = \frac{4}{5}$,

$$x_n = y_n + O(\,(\tfrac{1}{2}\delta)^n\,) \quad \text{as} \quad n \to \infty\,.$$

Under the same conditions, Corollary 5 gives the following relation

$$x_n = y_n + O(\,(\tfrac{1}{4})^n\,) \quad \text{as} \quad n \to \infty\,. \tag{14}$$

In relation (14) the parameter β of the asymptotic formula (13) is zero since the condition of Corollary 5 that $\lambda \neq \rho^2$ for every eigenvalue λ of A is satisfied.

The rest of this paper is structured as follows. In Section 2.2 we introduce the main hypothesis, and two results of C.V. Coffman are given for easy reference. Section 2.3 presents an auxiliary asymptotic result for powers of matrices. In Section 2.4 the main result Theorem 1 is presented, along with relevant corollaries: Corollaries 2 and 3 give a formula satisfied by $\tilde{y}$, Corollary 4 treats the real valued case, Corollaries 5 and 6 give results for the autonomous vector and scalar cases respectively, Theorem 2 is for the Poincaré systems and Corollary 7 deals with the Poincaré scalar equation. Section 2.5 presents further results for planar systems or for scalar difference equations of second order. These are applied to a planar system studied in [20].

2.2 Main hypothesis and results of C. V. Coffman

In this paper we obtain asymptotic results valid for large classes of problems, including the following:

(i) *Autonomous, nonlinear*: The sequence $\{x_n\}$ satisfies $x_{n+1} = T(x_n)$ for $n \geq 0$ and converges to zero. Here T is a map on a subset of $\mathbb{C}^m$ that is two-times continuously differentiable in a neighborhood of zero, or at least T satisfies $T(x) = J\,x + r(x)$, where J is a given matrix, and $\|r(x)\| \leq c\,\|x\|^\alpha$ for some $c > 0$, $\alpha > 1$, and x in some neighborhood of zero.

(ii) *Linear, non-autonomous*: The sequence $\{x_n\}$ satisfies $x_{n+1} = A\,x_n + B_n\,x_n$, where B_n is a $m \times m$ matrix valued function of $n \in \mathbb{N}$ such that B_n converges to the zero matrix at a geometric rate $\eta < 1$.

The asymptotic relation (4) was established in [2] under highly technical conditions on the f_n (see Theorem B in Section 2), which in particular apply in cases (i) and (ii) above. We shall use a strengthened version of those hypotheses, which while still covering cases (i) and (ii), will be shown in Theorem 1 to be sufficient to obtain a strengthened version of relation (4). The key hypothesis in the main result is the following.

(H) There exist $\eta \in (0,1]$, $\alpha \in [1,\infty)$, $c > 0$, and a neighborhood V of $0 \in \mathbb{C}^m$ such that α and η are not both equal to 1 and

$$\|f_n(x)\| \;\leq\; c\,\eta^n\,\|x\|^\alpha\,, \quad (x \in V\,,\, n \in \mathbb{N})\,. \tag{15}$$

Theorem 1, presented in Section 2.4, states that under hypothesis (H), if $\{x_n\}$ is a solution to (2) for which $x_n \neq 0$ for all n large and $\rho = \lim \|x_n\|^{1/n}$ satisfies

$0 < \rho < 1$, then there exists $\tilde{y} \in \mathbb{C}^m$ and a positive integer β such that

$$x_n \;=\; J^n \, \tilde{y} \;+\; O(\, n^\beta \, \eta^n \, \rho^{\alpha\, n} \,) \quad \text{as} \quad n \to \infty \,. \tag{16}$$

Relation (16) is a significant improvement over Bodine and Lutz's formula (8). Condition (H) is less general than the hypotheses of Coffman's Theorem 8.1 of [2] (see Theorem B in Section 2.2 below), but it is easier to verify, and it is general enough to be applicable to many classes of problems. Furthermore, our proof of Theorem 1 gives a procedure for approximating the vector $\tilde{y}$ in (16); see Corollary 2 in Section 2.4.

For convenience we review some of C. V. Coffman's results from [2]. For the most part, we follow the notation and setup from [2]. Assume that the matrix J in (2) is in block-diagonal form

$$J = \mathrm{diag}(J_1, J_2, \ldots, J_R),$$

where for $1 \le i \le R$, J_i is an $m_i \times m_i$ elementary Jordan block with associated eigenvalue λ_i. Assume that the eigenvalues are given in nondecreasing order of magnitude, and let

$$s_1 \;<\; s_2 \;<\; \cdots \;<\; s_f$$

be the f distinct numbers among the $|\lambda_i|$. Let some integer t, $1 \le t \le f$ be chosen. An integer j, $1 \le j \le g$ will be designated by $p = p(t)$, $q = q(t)$, or $r = r(t)$ according as $|\lambda_j| < s_t$, $|\lambda_j| = s_t$, or $|\lambda_j| > s_t$. Define

$$L_t(y) := \sum_q \sum_{k=1}^{h(q)} |y^{qk}| \,, \quad 1 \le t \le f, \; y \in \mathbb{C}^d$$

For any $x \in \mathbb{R}^d$ denote with $\|x\|_1$ the 1-norm of x, i.e., $\|x\|_1 = \sum_{i=1}^d |x_i|$. Thus

$$\|y\|_1 = L_1(y) + \cdots + L_f(y) \,.$$

The following result is Theorem 5.1 in [2] with hypotheses stated explicitly. While the original result is given in terms of $\|\cdot\|_1$, clearly it can be stated in terms of the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^d$ without any other changes and without affecting the validity of the result.

Theorem A [Coffman] *Let $f_n(y)$ satisfy*

$$\|f_n(y)\|/\|y\| \to 0 \quad \textit{as} \quad (n,y) \to (\infty, 0)\,. \tag{17}$$

Let $y = y_n$ be a solution of (2) defined for all sufficiently large n and such that $y_n \neq 0$ for large n, and $y_n \to 0$ as $n \to \infty$. Then there exists an integer t_0, $1 \le t_0 \le f$, such that $\rho := s_{t_0} \le 1$ satisfies

$$\lim_{n\to\infty} \|y_n\|^{1/n} = \rho \tag{18}$$

and

$$L_j(y_n) = o(L_{t_0}(y_n)) \quad \textit{as} \quad n \to \infty \quad \textit{if} \;\; j \neq t_0\,. \tag{19}$$

The following result is Theorem 8.1 in [2], with hypotheses stated explicitly, and restated here in terms of the Euclidean norm $\|\cdot\|$.

Theorem B [Coffman] *Let t, $1 \le t \le f$, be such that $\rho = s_t$ satisfies* $0 < \rho < 1$. *Let h_* denote the maximum multiplicity of the elementary divisors of J, the corresponding eigenvalues of which have absolute value s_t, and let h_0 be any number satisfying $h_* \le h_0$, For each $n \ge 0$ let $f_n(y)$ be defined and continuous in y for $\|y\| \le$ positive constant Let there exist, for each $n \ge 0$, a scalar function $\phi(n,r)$, defined and continuous in r for $|r| \le$ constant for each n, and satisfying*

$$\|f_n(y)\| \le \phi(n, \|y\|)\,. \tag{20}$$

Suppose that $\phi(n,r)$ has the form

$$\phi(n,r) = \psi(n)\,r, \quad \textit{where} \quad \sum n^{h_0 - 1}\,\psi(n) < \infty\,, \tag{21}$$

or suppose that $\phi(n,r)$ satisfies conditions (i), (ii), and (iii):

$$\begin{array}{lll} \text{(i)} & \phi(n,r) & \textit{is nondecreasing in } r \textit{ for each } n, \\ \text{(ii)} & \phi(n,r) & \textit{is nondecreasing in } n \textit{ for each fixed } r, \\ \text{(iii)} & \multicolumn{2}{l}{\sum(\rho-\delta)^{-n}\, n^{h_0-1}\, \phi(n, \text{constant}(\rho-\delta)^n) < \infty \textit{ for some } \delta \in (0,\rho)\,.} \end{array} \tag{22}$$

Let $y = y_n$ be a solution of (2) satisfying (18) and (19). Then for some j_0, $0 \le j_0 < h^*$ and some constants c_{qk} not all 0, y satisfies

$$\begin{aligned} y_n^{jk} &= o(n^{1-h_0+j_0}\,\rho^n) \quad \textit{if} \quad j \neq q \\ y_n^{qk} &= \textstyle\sum_{i=\ell(q)}^{k(q)} c^{qi} \binom{n}{k-i} \lambda^{n-k+i}(q) + o(n^{k-h_0+j_0}\,\rho^n)\,. \end{aligned} \tag{23}$$

2.3 A Preliminary Lemma

Denote with $\sigma_1(L), \ldots, \sigma_m(L)$ the singular values of $L \in \mathbb{C}^{m\times m}$, in nonincreasing order. Thus $\sigma_1(L) = \|L\| =$ the operator norm of matrix L. The following lemma is probably known, but the authors of this note did not find a reference for it.

Lemma 1. *Let $L \in \mathbb{C}^{m\times m}$ be similar to an elementary Jordan block with main diagonal entries equal to $\mu \neq 0$. Then there exist positive constants C_1 and C_2 such that*

$$C_1 \frac{|\,\mu\,|^n}{n^{m-1}} \;\le\; \sigma_m(L^n) \;\le\; \sigma_1(L^n) \;\le\; C_2\, n^{m-1}\,|\,\mu\,|^n, \quad n = 1, 2, \ldots$$

Proof. Suppose $L = P^{-1}\, J\, P$, where J is an elementary $m\times m$ Jordan block with $\mu \neq 0$ on the diagonal. Then

$$\begin{aligned} \sigma_1(L^n) &\le \sigma_1(P^{-1})\;\sigma_1(J^n)\;\sigma_1(P) \\ &= \sigma_1(P^{-1})\;\sigma_1\Big(\tfrac{1}{n^{m-1}\,|\,\mu\,|^n}\, J^n\Big)\;\sigma_1(P)\, n^{m-1}\,|\,\mu\,|^n, \quad n = 1, 2, \ldots \end{aligned} \tag{24}$$

The matrix J^n is an upper triangular Toeplitz matrix whose j-th super-diagonal has entries equal to $\binom{n}{j}\mu^{n-j}$, $j = 1, 2, \ldots m-1$. Thus the entries a_j on the j-th super-diagonal of the $m \times m$ matrix $\frac{1}{n^{m-1}\,|\,\mu\,|^n}\,J^n$ satisfy, for $n \geq j$,

$$|\,a_j\,| \;=\; \tfrac{1}{n^{m-1}\,|\,\mu\,|^n}\,\tbinom{n}{j}\,\left|\,\mu^{n-j}\,\right| = \tfrac{n(n-1)\ldots(n-j+1)}{n^{m-1}\,j!}\,|\,\mu\,|^{-j} \leq \tfrac{1}{j!}\,|\,\mu\,|^{-j}\,, \quad j = 1, \ldots, m-1\,. \tag{25}$$

Relation (25) implies that $\sigma_1\!\left(\frac{1}{n^{m-1}\,|\,\mu\,|^n}\,J^n\right)$ is bounded as a function of $n \in \mathbb{N}\backslash\{0\}$. This observation together with (24) guarantee the existence of a constant $C_2 > 0$ such that

$$\sigma_1(L^n) \;\leq\; C_2\,n^{m-1}\,|\,\mu\,|^n, \quad n = 1, 2, \ldots \tag{26}$$

The matrix L^{-1} is similar to an elementary Jordan block with main diagonal entries equal to $1/\mu$. Apply relation (26) to L^{-n} to obtain that for some positive constant C,

$$\sigma_1(L^{-n}) \;\leq\; C\,n^{m-1}\,|\,\mu\,|^{-n}, \quad n = 1, 2, \ldots \tag{27}$$

By setting $C_1 := 1/C$ in (27) and from the relation $1/\sigma_m(L^n) = \sigma_1(L^{-n})$ we obtain

$$C_1\,\frac{|\,\mu\,|^n}{n^{m-1}} \;\leq\; \sigma_m(L^n), \quad n = 1, 2, \ldots\,,$$

which completes the proof of the lemma. □

2.4 Results

We begin by giving a corollary to Theorem A.

Corollary 1. *Let $f_n : \mathbb{C}^m \to \mathbb{C}^m$ be continuous for each $n \in \mathbb{N}$ be such that* (H) *is satisfied, and let $J \in \mathbb{C}^{m\times m}$. If $\{x_n\}$ is a sequence that satisfies $x_{n+1} = J\,x_n + f_n(x_n)$ such that $x_n \neq 0$ for all large n, then $\rho := \lim \|x_n\|^{1/n}$ exists and equals the modulus of an eigenvalue of J.*

Proof. Let c, V, α and η be as in condition (H). If either $\alpha > 1$ or $\eta < 1$ are such that (15) holds, it follows that hypothesis (17) of Theorem A is satisfied. Thus the conclusion of Theorem A holds. □

The following is the central result of this paper.

Theorem 1. *Let* $f_n : \mathbb{C}^m \to \mathbb{C}^m$ *be continuous for each* $n \in \mathbb{N}$ *be such that* (H) *is satisfied, and let* $J \in \mathbb{C}^{m\times m}$*. Let* $\{x_n\}$ *be a sequence that satisfies* $x_{n+1} = J\,x_n + f_n(x_n)$ *and* $x_n \neq 0$ *for all large* n*, and let* ρ *be the number guaranteed by Corollary 1.*

Let k *denote one less than the largest geometric multiplicity of the eigenvalues of* J*, and let* k_* *denote one less than the largest geometric multiplicity of the eigenvalues of* J *whose modulus equal* ρ*. If* $0 < \rho < 1$*, then there exists* $\tilde{y} \in \mathbb{R}^m$ *such that with* $\beta = \alpha\,k_* + 2\,k + 1$*,*

$$x_n \;=\; J^n\,\tilde{y} \;+\; O(\,n^\beta\,\eta^n\,\rho^{\alpha\,n}\,) \quad \text{as} \quad n \to \infty\,. \tag{28}$$

Furthermore, if $|\lambda| \neq \eta\,\rho^\alpha$ *for every eigenvalue* λ *of* J*, then* $\beta = \alpha\,k_* + 2\,k$ *is valid in* (28)*.*

Proof. We first verify that the hypotheses of Theorem B are satisfied. To this end, consider first the case where $\alpha > 1$. Then, from (15), statement (17) of Theorem A holds, and with $\phi(n,r) = c\,r^\alpha$, inequality (20) holds. Also, (i) of (22) is clearly valid, while (ii) of (22) becomes

$$\sum_{n=1}^{\infty} n^{h_0-1}c(\rho-\delta)^{(\alpha-1)n} < \infty \qquad \text{for some} \quad \delta \in (0,\rho)\,. \tag{29}$$

Since $\alpha > 1$, (29) is true. Thus (ii) of (22) holds. Now suppose that $\eta < 1$ holds. From (15) it follows that (17) of Theorem A holds, and with
$\phi(n,r) = c\,\rho^n\,r$ inequality (20) is valid. Also, the second statement in (21) is equivalent to $\sum n^{h_0-1}\rho^n < \infty$ which is a true statement. Thus condition (21) of Theorem B is satisfied. It follows that in all cases considered, the conclusion of Theorem B is true. Let h_0, κ and j_0 be as in Theorem B. Note that the exponent $\kappa - h_0 + j_0$ in (23) satisfies

$$1 - h_0 + j_0 \leq -1 \leq \kappa - h_0 + j_0 \leq \kappa(q) - 1 \leq k_*\,. \tag{30}$$

Thus we have from (23) and (30),

$$x_n = o(n^{k_*}\,\rho^n) \quad \text{as} \quad n \to \infty\,. \tag{31}$$

By (15) and (31) there exists $C > 0$ such that

$$\|f_n(x_n)\| < C\, n^{\alpha k_*}\, \eta^n\, \rho^{\alpha n}, \qquad n = 1, 2, \ldots. \tag{32}$$

Henceforth we shall assume, without loss of generality, that J is in block-diagonal form $J = \mathrm{diag}(J_1, J_2, \ldots, J_R)$, where for $1 \le i \le R$, J_i is an $m_i \times m_i$ elementary Jordan block with associated eigenvalue λ_i, arranged in order of nonincreasing magnitude. Define $\varrho_i := |\lambda_i|$ for each $i = 1, 2, \ldots, R$, and define p to be the number of non-zero terms ϱ_i if one such value exists, else set p to be zero. Then

$$\varrho_i \ge \varrho_{i+1} > 0 \quad \text{for} \quad 1 \le i \le p-1, \quad \text{and} \quad \varrho_i = 0 \quad \text{for} \quad p < i \le R.$$

Thus J_i is invertible for $1 \le i \le p$ and nilpotent for $p < i \le R$. Vectors in $x \in \mathbb{C}^m$ will be partitioned as follows

$$x = \left(x^{(1)}, x^{(2)}, \cdots, x^{(R)}\right) \tag{33}$$

where $x^{(i)}$ consists of the entries of x, corresponding to J_i. For each $n \ge 0$ let $y_n \in \mathbb{C}^m$ be given by

$$y_n^{(i)} := \begin{cases} J_i^{-n} x_n^{(i)} & \text{if } \; i = 1, \ldots, p \\ 0 \in \mathbb{C}^{m_i} & \text{if } \; i = p+1, \ldots, R. \end{cases}$$

To begin we assume $i \in \{1, \ldots, p\}$, so in particular $\varrho_i > 0$ and J_i is invertible. From (2),

$$\begin{aligned} \|y_{n+1}^{(i)} - y_n^{(i)}\| &= \|J_i^{-(n+1)} x_{n+1}^{(i)} - J_i^{-n} x_n^{(i)}\| \\ &= \|J_i^{-(n+1)} (J_i x_n^{(i)} + f_n(x_n)^{(i)}) - J_i^{-n} x_n^{(i)}\| \\ &= \|J_i^{-(n+1)} f_n(x_n)^{(i)}\| \\ &\le \|J_i^{-(n+1)}\|\, \|f_n(x_n)\|. \end{aligned} \tag{34}$$

From Lemma 1 and the fact that the set of indices i is finite, there exists a constant $C' > 0$ independent of i such that for $n \geq 0$,

$$\sigma_1(J_i^{-(n+1)}) \leq C' \, (n+1)^{m_i - 1} \, \frac{1}{\varrho_i^{(n+1)}} \, ,$$

which together with the inequalities $m_i - 1 \leq k$ for $i = 1, \ldots, p$, imply that there exists a constant $C_2 > 0$ independent of i such that for $n \geq 0$,

$$\sigma_1(J_i^{-(n+1)}) \leq C_2 \, n^k \, \frac{1}{\varrho_i^n}. \tag{35}$$

From (32), (34), and (35), for $n \geq 0$, we have

$$\|y_{n+1}^{(i)} - y_n^{(i)}\| \leq \left(C_2 \, n^k \, \frac{1}{\varrho_i^n} \right) \left(C \, n^{\alpha k_*} \, \eta^n \, \rho^{\alpha n} \right) = C_2 \, C \, n^{\alpha k_* + k} \left(\frac{\eta \, \rho^\alpha}{\varrho_i} \right)^n . \tag{36}$$

Let $\tilde{C}$ be an arbitrary positive constant, and $\beta = \alpha \, k_* + k + 1$. For $n \geq 0$, define $S_n^{(i)}$ to be the set in $\mathbb{C}^{m_i}$ whose image under J_i^n is a ball with center $x_n^{(i)}$ and radius $\tilde{C} \, n^\beta \, \eta^n \, \rho^{\alpha n}$ respectively. Then,

$$S_n^{(i)} = J_i^{-n} \left(x_n^{(i)} + B(0; \tilde{C} \, n^\beta \, \eta^n \, \rho^{\alpha n}) \right) = y_n^{(i)} + J_i^{-n} B(0; \tilde{C} \, n^\beta \, \eta^n \, \rho^{\alpha n}) \, . \tag{37}$$

Thus $S_n^{(i)}$ is an ellipsoid centered at $y_n^{(i)}$. Let $s_n^{(i)}$ be the radius of the largest ball contained in $S_n^{(i)}$ that is centered at $y_n(i)$. Then

$$\begin{aligned} s_n^{(i)} &= \min_{x \in \partial S_n^{(i)}} \|x - y_n^{(i)}\| &&= \min_{z \in \partial J_i^{-n} B(0\,;\,\tilde{C} n^\beta \, \eta^n \, \rho^{\alpha n})} \|z\| \\ &= \min_{w \in \partial B(0\,;\,\tilde{C} n^\beta \, \eta^n \, \rho^{\alpha n})} \|J_i^{-n} w\| &&= \min_{w \in \partial B(0\,;\,1)} \tilde{C} n^\beta \, \eta^n \, \rho^{\alpha n} \|J_i^{-n} w\|. \end{aligned} \tag{38}$$

The smallest singular value of J_i^{-n} is $\sigma_{m_i}(J_i^{-n})$. If follows from (38) that

$$s_n^{(i)} = \left(\tilde{C} \, n^\beta \, \eta^n \, \rho^{\alpha n} \right) \, \sigma_{m_i} \left(J_i^{-n} \right) . \tag{39}$$

From (39) and from Lemma 1, there exists $C_1 > 0$ (which due to the fact that the set of indices i is finite may be assumed to be independent of i) such that for $n \geq 0$,

$$s_n^{(i)} \geq \left(\tilde{C} \, n^\beta \, \eta^n \, \rho^{\alpha n} \right) \left(C_1 \, \frac{1}{n^{m_i - 1} \, \varrho_i^n} \right) \geq \tilde{C} \, C_1 \, n^{\beta - k} \left(\frac{\eta \, \rho^\alpha}{\varrho_i} \right)^n . \tag{40}$$

We shall need the following constants:

$$
\begin{aligned}
s &:= \max\ \{\ \varrho_\ell/(\eta\,\rho^\alpha)\ :\ 1 \le \ell \le p\ ,\ \varrho_\ell < \eta\rho^\alpha\ \} && (41)\\
t &:= \max\ \{\ \eta\rho^\alpha/\varrho_\ell\ :\ 1 \le \ell \le p\ ,\ \varrho_\ell > \eta\rho^\alpha\ \} && (42)\\
\tau &:= \sum_{\ell=0}^{\infty}(1+\ell)^{\beta-k-1}\, t^\ell && (43)\\
D &:= \max\left\{\ \frac{C_2\, C}{C_1}\,\tau\ ,\ \frac{C_2\, C}{C_1}\frac{1}{1-s}\ ,\ \frac{C_2\, C}{C_1}\ ,\ C\, m\,(\eta\rho^\alpha)^{-m-1}\ \right\}. && (44)
\end{aligned}
$$

Claim 1. *If $\tilde{C} > D$ then for each $i \in \{1, \ldots, p\}$, there exists $z_i \in \mathbb{C}^{m_i}$ such that*

$$
J_i^n z_i \in B(x_n^{(i)}\ ;\ \tilde{C}\, n^\beta\,(\eta\,\rho^\alpha)^n) \quad \textit{for} \quad n \ge 1. \tag{45}
$$

To prove the claim, fix $i \in \{1, \ldots, p\}$ and consider the cases (a) $\varrho_i > \eta\,\rho^\alpha$, (b) $0 < \varrho_i < \eta\,\rho^\alpha$, and (c) $\varrho_i = \eta\,\rho^\alpha$.

(a) Assume i is such that $\varrho_i > \eta\,\rho^\alpha$. From (36) follows that $\{y_n^{(i)}\}$ is a Cauchy sequence and thus converges, say to z_i. Furthermore, by (36) we have for $n \ge 1$

$$
\begin{aligned}
\|y_n^{(i)} - z_i\| &\le \sum_{\ell=n}^{\infty} \|y_{\ell+1}^{(i)} - y_\ell^{(i)}\| \\
&\le \sum_{\ell=n}^{\infty} C_2\, C\, \ell^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^\ell \\
&= C_2\, C\, n^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n \sum_{\ell=0}^{\infty}\left(1+\frac{\ell}{n}\right)^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^\ell .
\end{aligned}
\tag{46}
$$

By (42) and (43), $\sum_{\ell=0}^{\infty}\left(1+\frac{\ell}{n}\right)^{\beta-k-1}\left(\frac{\eta\rho^\alpha}{\varrho_i}\right)^\ell < \sum_{\ell=0}^{\infty}(1+\ell)^{\beta-k-1}\, t^\ell = \tau$ for $n \ge 1$, thus inequality (46) yields

$$
\|y_n^{(i)} - z_i\| < C_2\, C\,\tau\, n^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n, \quad n \ge 1. \tag{47}
$$

For $\tilde{C} > D$, (44) implies

$$
C_2\, C\,\tau\ n^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n < \tilde{C}\, C_1\, n^{\beta-k-1}\left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n, \quad n \ge 1. \tag{48}
$$

Then by (40), (47), and (48),

$$\|y_n^{(i)} - z_i\| < s_n^{(i)}, \quad n \geq 1,$$

which implies that

$$z_i \in S_n^{(i)}, \quad n \geq 1. \tag{49}$$

Then (37) and (49) give (45), which completes the proof of part (a).

(b) Assume i is such that $0 < \varrho_i < \eta\,\rho^\alpha$. From (36) we have for $n \geq 1$

$$\begin{aligned}
\|y_n^{(i)} - y_0^{(i)}\| &\leq \sum_{\ell=1}^{n} \|y_\ell^{(i)} - y_{\ell-1}^{(i)}\| \\
&< \sum_{\ell=1}^{n} C_2\, C\, \ell^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^\ell \\
&= C_2\, C\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n \sum_{\ell=1}^{n} \left(\frac{\ell}{n}\right)^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^{\ell-n} \\
&= C_2\, C\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n \sum_{\ell'=0}^{n-1} \left(1 - \frac{\ell'}{n}\right)^{\beta-k-1} \left(\frac{\varrho_i}{\eta\,\rho^\alpha}\right)^{\ell'} \\
&\leq C_2\, C\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n \sum_{\ell'=0}^{\infty} \left(\frac{\varrho_i}{\eta\,\rho^\alpha}\right)^{\ell'}.
\end{aligned} \tag{50}$$

Then by (41) and (50),

$$\|y_n^{(i)} - y_0^{(i)}\| \leq C_2\, C \frac{1}{1-s}\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n, \quad n \geq 1. \tag{51}$$

For $\tilde{C} > D$, (44) gives

$$\frac{C_2\, C}{1-s}\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n < \tilde{C}\, C_1\, n^{\beta-k-1} \left(\frac{\eta\,\rho^\alpha}{\varrho_i}\right)^n, \quad n \geq 1. \tag{52}$$

Then by (40), (51), and (52),

$$\|y_n^{(i)} - y_0^{(i)}\| < s_n^{(i)}, \quad n \geq 1,$$

which implies that

$$y_0^{(i)} \in S_n^{(i)}, \quad n \geq 1. \tag{53}$$

Set $z_i := y_0^{(i)}$. Then (45) follows from (37) and (53), completing the proof of (b).

(c) If $\varrho_i = \eta\,\rho^\alpha$, then from (36) we have for $n \ge 1$

$$\|y_n^{(i)} - y_0^{(i)}\| \le \sum_{\ell=1}^{n} \|y_\ell^{(i)} - y_{\ell-1}^{(i)}\| \le \sum_{\ell=1}^{n} C_2\,C\,\ell^{\alpha\,k_*+k} < C_2\,C\,n^{\alpha\,k_*+k+1}, \quad n \ge 1. \tag{54}$$

For $\tilde{C} > D$, (44) gives

$$C_2\,C\,n^{\alpha\,k_*+k+1} \;=\; C_2\,C\,n^{\beta-k} \;<\; \tilde{C}\,C_1\,n^{\beta-k}, \quad n \ge 1. \tag{55}$$

Then by (40), (54), and (55),

$$\|y_n^{(i)} - y_0^{(i)}\| < s_n^{(i)}, \quad n \ge 1,$$

which implies that

$$y_0^{(i)} \in S_n^{(i)}, \quad n \ge 1. \tag{56}$$

Set $z_i := y_0^{(i)}$. Finally, (37) and (56) imply (45). This completes the proof of (c) and the proof of Claim 1.

To continue with the proof of the theorem, we assume $i \in \{p+1, \dots, R\}$, so $\varrho_i = 0$.

Claim 2. *If $\tilde{C} > D$ then*

$$0 \in B(\,x_n\,;\,\tilde{C}\,n^\beta\,(\eta\,\rho^\alpha)^n), \quad \text{for} \quad n \ge 1, \quad i = p+1, \dots, R\,. \tag{57}$$

Proof. Assume $i \in \{p+1, \dots, R\}$. By iterating (2) we obtain

$$x_n^{(i)} = J_i^{n-1} x_0^{(i)} + \sum_{j=1}^{n-1} J_i^{n-1-j} f_j(x_j)^{(i)}\,. \tag{58}$$

Since the matrix J_i is nilpotent of order no larger than m, equation (58) becomes

$$x_n^{(i)} = \sum_{j=n-1-m}^{n-1} J_i^{n-1-j} f_j(x_j)^{(i)} \quad \text{for} \quad n \ge m. \tag{59}$$

From (32) and (59), along with the fact that $\|J_i\| = 1$, we have

$$\begin{aligned}
\|x_n^{(i)}\| = \left\| \sum_{j=n-1-m}^{n-1} J_i^{n-1-j} f_j(x_j)^{(i)} \right\| &\le \sum_{j=n-1-m}^{n-1} \|J_i^{n-1-j}\| \|f_j(x_j)^{(i)}\| \\
&= \sum_{j=n-1-m}^{n-1} \|f_j(x_j)^{(i)}\| \\
&\le C \sum_{j=n-1-m}^{n-1} j^{\alpha\, k_*} (\eta\, \rho^\alpha)^j \\
&\le C\, m\, n^{\alpha\, k_*} \, (\eta\, \rho^\alpha)^{n-1-m}.
\end{aligned} \tag{60}$$

Since $\tilde{C} > D$, (44) gives $\tilde{C}_i > C\, m\, (\eta\, \rho^\alpha)^{-m-1}$, which together with (60) implies

$$\|x_n^{(i)}\| < \tilde{C}_i\, n^{\alpha\, k_*} \, (\eta\, \rho^\alpha)^n < \tilde{C}\, n^\beta \, (\eta\, \rho^\alpha)^n. \tag{61}$$

Relation (57) follows from (61) which establishes Claim 2.

To finish the proof of the theorem, choose $\tilde{C} > D$ and set

$$\tilde{y} = (z_1, \ldots, z_p, 0, \ldots, 0)$$

where z_i for each i is given in Claim 1. Then Claims 1 and 2 imply that $J^n \tilde{y} \in B(x_n\, ;\, \tilde{C}\, n^\beta \, (\eta\, \rho^\alpha)^n)$ for $n \ge 1$, which in turn implies (28).

If $|\lambda| \ne \eta\, \rho^\alpha$ for every eigenvalue λ of J, then case (c) in the proof of Theorem 1 does not occur, and β in the conclusion of Theorem 1 may be taken to be $\beta = \alpha\, k_* + 2\, k$.

As a final remark, note that from proof of Claim 1 one can see that the components of the vector $\tilde{y}$ obtained in case (a) are determined uniquely (i.e., if $\varrho_i > \eta\, \rho^\alpha$). If $\varrho_i \le \eta\, \rho^\alpha$ then this is no longer the case, since there exists a neighborhood about $x_0^{(i)}$ consisting of points x such that $\|y_n^{(i)} - x\| < s_n^{(i)}$. Indeed, from (40) it can be seen that in this case, $s_n^{(i)} \to \infty$ as $n \to \infty$, so any point x can be used with an appropriate choice of $\tilde{C}$. $\square$

From the remark at the end of the proof of Theorem 1 we have the following two results.

Corollary 2. *Suppose the matrix J is in Jordan canonical form $J = diag(J_1, \dots, J_R)$, where J_i has associated eigenvalue λ_i so that $\rho_i = |\lambda_i|$ satisfies $\rho_i \geq \rho_{i+1}$, $i = 1, \dots, R-1$. Then the entries $\tilde{y}^{(i)}$ of $\tilde{y}$ corresponding to ϱ_i such that $\varrho_i > \eta\,\rho^\alpha$ are uniquely determined, while the entries of $\tilde{y}$ corresponding to ϱ_i such that $\varrho_i \leq \eta\,\rho^\alpha$ may be set to be any predetermined number. A particular vector $\tilde{y}$ for which the asymptotic formula* (28) *holds is $\tilde{y} = (\tilde{y}^{(1)}, \cdots, \tilde{y}^{(r)})$, where*

$$\tilde{y}^{(i)} = \begin{cases} \lim_{n\to\infty} J_i^{-n}\, x_n^{(i)} & \text{if} \quad \varrho_i > \eta\,\rho^\alpha\,. \\ \\ 0 & \text{if} \quad \varrho_i \leq \eta\,\rho^\alpha\,. \end{cases}$$

Corollary 3. *If every eigenvalue λ of J satisfies $|\lambda| > \eta\,\rho^\alpha$, then the vector $\tilde{y}$ in* (28) *is uniquely determined by the relation*

$$\tilde{y} = \lim_{n\to\infty} J^{-n}\, x_n.$$

For real difference equations, we have the following result.

Corollary 4. *If in Theorem 1 the matrix J has real entries, the functions f_n are $\mathbb{R}^m$-valued, and $\{x_n\} \subset \mathbb{R}^m$, then the vector $\tilde{y}$ in the conclusion may be chosen to be in $\mathbb{R}^m$.*

Proof. The result follows from taking real part of both members of equation (28).

□

In the case of autonomous difference systems with a solution that converges to a fixed point we have the following corollary.

Corollary 5. *Let T be a map on a set $\mathcal{R} \subset \mathbb{R}^m$ with a fixed point $\bar{x}$ in the interior of $\mathcal{R}$, such that T is of class C^2 on a neighborhood of $\bar{x}$. Let $\{x_n\}$ be such that $x_{n+1} = T(x_n)$ for $n = 0, 1, \dots$ with $x_0 \in \mathcal{R}$. If $x_n \to \bar{x}$, then either $x_n = \bar{x}$ for all n large, or the following statements are true.*

(a) *There exists* ρ *with* $0 \le \rho \le 1$ *such that* ρ *is the modulus of an eigenvalue of the jacobian matrix* J *of* T *at* $\bar{x}$ *and such that* $\lim \|x_n - \bar{x}\|^{1/n} = \rho$.

(b) *Let* k *denote one less than the largest geometric multiplicity of the eigenvalues of* J*, and let* k_* *denote one less than the largest geometric multiplicity of* λ *such that* $|\lambda| = \rho$*. If* $0 < \rho < 1$*, then there exists* $\tilde{y} \in \mathbb{R}^m$ *such that with* $\beta = 2\,k_* + 2\,k + 1$,

$$x_n \;=\; \bar{x} + J^n\,\tilde{y} \;+\; O(\,n^{\beta}\,\rho^{2\,n}\,) \quad \textit{as} \quad n \to \infty\,. \tag{62}$$

Furthermore, if $|\lambda| \ne \rho^2$ *for every eigenvalue* λ *of* J *then* $\beta = 2\,k_* + 2\,k$ *is valid in* (62).

Proof. A Taylor expansion of T about $\bar{x}$ gives $T(x) = \bar{x} + J\,(x - \bar{x}) + K(x)$ where $\|K(x)\| \le c\,\|x - \bar{x}\|^2$ for some $c > 0$ and x in a neighborhood of $\bar{x}$. Thus condition (H) with $\alpha = 2$ and $\eta = 1$ is satisfied by the translation of T by $\bar{x}$. The result now follows from Theorem 1. □

If the Taylor expansion of a map at a fixed point contains no nonlinear terms of degree $\ell - 1$ or smaller, then $\|K(x)\| \le c\|x - \bar{x}\|^{\ell}$ for some $c > 0$ and x in a neighborhood of $\bar{x}$. In this case, the asymptotic formula (62) can be made more accurate:

$$x_n \;=\; \bar{x} + J^n\,\tilde{y} \;+\; O(\,n^{\beta}\,\rho^{\ell\,n}\,) \quad \text{as} \quad n \to \infty\,.$$

For scalar autonomous difference equations

$$\zeta_{n+1} = g(\zeta_{n-m+1}, \ldots, \zeta_n)\,, \qquad n = m-1\,,\, m\,,\, \ldots \tag{63}$$

an equilibrium point $\bar{\zeta}$ satisfies $g(\bar{\zeta}, \ldots, \bar{\zeta}) = \bar{\zeta}$. With g_ℓ, $\ell = 1, \ldots, m$ denoting first partial derivatives of g and with $\bar{z} := (\bar{\zeta}, \ldots, \bar{\zeta})$, the linearization of (63) about the equilibrium $\bar{\zeta}$ is the equation

$$\phi_{n+1} = \sum_{\ell=1}^{m} g_\ell(\bar{z})\,\phi_{n-m+\ell}. \tag{64}$$

The *companion matrix associated with* (64) is the matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1(\bar{z}) & g_2(\bar{z}) & g_3(\bar{z}) & \cdots & g_m(\bar{z}) \end{pmatrix}.$$

We have the following result for solutions to (63) that converge to an equilibrium point.

Corollary 6. *Let $g : \mathcal{R} \subset \mathbb{R}^m \to \mathbb{R}$ be of class C^2 in a neighborhood of $(\bar{\zeta}, \ldots, \bar{\zeta}) \in \mathcal{R}$, where $\bar{\zeta}$ is an equilibrium point of* (63)*, and let $\{\zeta_n\}$ be a solution of equation* (63)*. If $\zeta_n \to \bar{\zeta}$, then either $\zeta_n = \bar{\zeta}$ for all n large, or the following statements are true.*

(a) *There exists ρ with $0 \le \rho \le 1$ such that ρ is the modulus of a characteristic value of the linearization* (64) *at $\bar{\zeta}$ and such that $\limsup |\zeta_n - \bar{\zeta}|^{1/n} = \rho$.*

(b) *Let J denote the companion matrix associated with the linearization* (64) *of equation* (63)*. Let k denote one less than the largest geometric multiplicity of the eigenvalues of J, and let k_* denote one less than the largest geometric multiplicity of λ such that $|\lambda| = \rho$. If $0 < \rho < 1$, then there exists a solution $\{\tilde{\phi}_n\}$ to the linearized equation* (64) *such that with $\beta = 2\,k_* + 2\,k + 1$,*

$$\zeta_n \;=\; \bar{\zeta} + \tilde{\phi}_n \;+\; O(\,n^{\beta}\,\rho^{2\,n}\,) \quad as \quad n \to \infty\,. \tag{65}$$

Furthermore, if $|\lambda| \ne \rho^{\alpha}$ for every eigenvalue λ of J, then $\beta = \alpha\,k_* + 2\,k$ is valid in (65).

The lim sup rather than just lim in item (a) of Corollary 6 is necessary, as it was observed by M. Pituk in [12], p. 205. A modification of Pituk's example illustrates the point: consider the equation $\zeta_{n+1} = \frac{1}{2}\,\zeta_{n-1}$, $n = 1, 2, \ldots$. A particular solution that converges to the equilibrium $\bar{\zeta} = 0$ is $\{\zeta_n\}$, where $\zeta_n = (\frac{1}{2})^n$ for n even and $\zeta_n = 0$ for n odd. Thus $\limsup |\zeta_n|^{1/n} = \frac{1}{2}$, while $\lim |\zeta_n|^{1/n}$ does not exist.

The next result gives an asymptotic expression valid for any solution of a Poincaré difference system. Part (a) of Theorem 2 below is Theorem 1 in [12], and it is included here to have a more comprehensive statement.

Theorem 2. *Let $A \in \mathbb{C}^{m\times m}$, and let $B_n \in \mathbb{C}^{m\times m}$ for $n \in \mathbb{N}$, $\eta \in (0,1)$ and $c > 0$ be such that*

$$\|B_n\| < c\,\eta^n \quad \textit{for} \quad n = 1, 2, \ldots . \tag{66}$$

Let $\{x_n\} \subset \mathbb{C}^m$ be a solution to the Poincaré difference system

$$x_{n+1} = (\,A + B_n\,)\,x_n \qquad n = 0, 1, 2, \ldots \tag{67}$$

such that $x_n \neq 0$ for n sufficiently large. Let k denote one less than the largest geometric multiplicity of the eigenvalues of A. Then the following statements are true.

(a) *The limit $\rho = \lim \|x_n\|^{1/n}$ exists and equals the modulus of an eigenvalue of A.*

(b) *If $\rho \neq 0$, then there exists $\tilde{y} \in \mathbb{R}^m$ such that with $\beta = 3k + 1$,*

$$x_n \;=\; A^n\,\tilde{y} \;+\; O(\,n^\beta\,\eta^n\,\rho^n\,) \quad \textit{as} \quad n \to \infty\,. \tag{68}$$

Furthermore, if $|\lambda| \neq \eta\,\rho$ for every eigenvalue λ of A, then $\beta = 3\,k$ is valid in (68)*.*

Proof. Following Perron (as mentioned in [12], page 206), choose a number $\mu > \|A\|$ and set $J := \mu^{-1}\,A$. Then J has eigenvalues $\mu^{-1}\,\lambda_\ell$, where λ_ℓ is an eigenvalue of A. In particular, J is a contraction. Set $y_n = \mu^{-n}\,x_n$, and for $y \in \mathbb{C}^m$ and $n \geq 0$ set $f_n(y) := \mu^{-1}\,B_n\,y$. Thus $y_n \neq 0$ for n sufficiently large. Equation (67) becomes

$$y_{n+1} = J\,y_n + f_n(y_n), \quad n = 0, 1, \ldots .$$

From the definition of f_n and hypothesis (66) follows that

$$\|f_n(y)\| \le \frac{c}{\mu}\,\eta^n\,\|y\|, \quad n = 0, 1, 2, \ldots .$$

Thus (15) holds with $\alpha = 1$. By part (a) of Theorem 1, the limit $\tilde\rho := \lim \|y_n\|^{1/n} = \lim \|\mu^{-n}\,x_n\|^{1/n}$ exists, it is equal to the modulus of an eigenvalue of J, and $\tilde\rho < 1$ since J is a contraction. Thus $\rho := \lim \|x_n\|^{1/n} = \mu\,\tilde\rho$ is equal to the modulus of an eigenvalue of A. Also, $\|y_{n+1}\| \le (\|J\| + c\,\mu^{-1}\,\eta^n)\,\|y_n\|$ for $n = 1, 2, \ldots$, and since $\|J\| < 1$ we have $y_n \to 0$ as $n \to \infty$. If $\rho \ne 0$ is assumed, then $0 < \tilde\rho < 1$, and by part (b) of Theorem 1 there exists of $\tilde y \in \mathbb{C}^m$ such that

$$y_n = J^n\,\tilde y + O\left(n^\beta \eta^n \left(\frac{\rho}{\mu}\right)^n\right), \quad n = 0, 1, 2, \ldots \tag{69}$$

Relation (68) follows from substituting $y_n = \mu^{-n} x_n$ and $J^n = \mu^{-n} A^n$ in (69). $\square$

Since any m-th order scalar difference equation may be formulated as a m-vector first order equation (p. 117 in [5]), Theorem 2 has the following corollary about the scalar Poincaré equation.

Corollary 7. *Let $\eta \in (0,1)$ and $c > 0$ and for $n = 1, 2, \ldots,\ 1 \le \ell \le m$, let $p_{\ell,n}$ and q_ℓ be such that*

$$|\,p_{\ell,n} - q_\ell\,| < c\,\eta^n \quad \textit{for} \quad n = 1, 2, \ldots,\ 1 \le \ell \le m.$$

Let k denote one less than the largest geometric multiplicity of the eigenvalues of the companion matrix J associated with the linearized equation about the origin. Let $\{\zeta_n\}$ be a solution to the scalar equation (5) such that $\zeta_n \ne 0$ for all n large. If $q_1 \ne 0$ then there exists a nonzero characteristic root λ of equation (7) and there exists $\{\tilde\zeta_n\}$ a nontrivial solution to (7) such that, with $\rho := |\lambda|$ and $\beta := 3\,k + 1$,

$$\zeta_n = \tilde\zeta_n + O(n^\beta\,\eta^n\,\rho^n) \quad \textit{as} \quad n \to \infty\,. \tag{70}$$

Furthermore, if $|\lambda| \ne \eta\,\rho$ for every eigenvalue λ of A, then $\beta = 3\,k$ is valid in (70).

Proof. The condition $q_1 \neq 0$ implies that zero is not a characteristic root of the linearized equation. Thus $\rho \neq 0$ in Theorem 2. The result follows. $\square$

2.5 An Application

In this section we present results for smooth difference systems in the plane or second order scalar difference equations and an example. If the characteristic roots at an equilibrium of a system in the plane are a pair of complex conjugate numbers, then there is only one rate at which solutions approach such equilibrium, namely the modulus ρ of the roots. Furthermore, since the eigenvalues of the jacobian matrix J at the equilibrium are complex and have geometric multiplicity one, we have $k_* = k = 0$ in Corollaries 5 (b) and 6 (b). Also, under the given setup there does not exist an eigenvalue whose modulus is ρ^2, thus $\beta = 0$ is valid in the asymptotic relation. These considerations provide a justification for the following two corollaries.

Corollary 8. *Let T be a map on a set $\mathcal{R} \subset \mathbb{R}^2$ with a fixed point $\bar{x}$ in the interior of $\mathcal{R}$ such that T is of class C^2 on a neighborhood of $\bar{x}$. Let $\{x_n\}$ be such that $x_{n+1} = T(x_n)$ for $n = 0, 1, \ldots$ with $x_0 \in \mathcal{R}$. Suppose the eigenvalues of the jacobian matrix J of T at $\bar{x}$ are a complex conjugate pair with modulus ρ. If $x_n \to \bar{x}$, then either $x_n = \bar{x}$ for all n large, or the following statements are true.*

(a) $0 \leq \rho \leq 1$ *and* $\lim \|x_n - \bar{x}\|^{1/n} = \rho$.

(b) *If* $0 < \rho < 1$, *then the limit*

$$\tilde{y} \;=\; \lim_{n\to\infty} J^{-n}\, x_n$$

exists, and

$$x_n \;=\; \bar{x} + J^n\, \tilde{y} \;+\; O\left(\rho^{2n}\right) \qquad \text{as} \qquad n \to \infty\,.$$

Corollary 9. *Let $g : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$,and let $\{\zeta_n\}$ be a solution of $\zeta_{n+1} = g(\zeta_n, \zeta_{n-1})$. Let $\bar{\zeta}$ be an equilibrium point, and assume that g is of class C^2 in a neighborhood of $(\bar{\zeta}, \bar{\zeta})$. Suppose the roots of the characteristic equation of the associated linearized equation at $\bar{\zeta}$ are complex conjugate numbers with modulus ρ. If $\zeta_n \to \bar{\zeta}$, then either $\zeta_n = \bar{\zeta}$ for all n large, or*

$$0 \leq \rho \leq 1 \quad \text{and} \ \limsup |\zeta_n - \bar{\zeta}|^{1/n} = \rho \,.$$

In the latter case, if $0 < \rho < 1$ then the limit

$$\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} 0 & 1 \\ g_1(\bar{\zeta}, \bar{\zeta}) & g_2(\bar{\zeta}, \bar{\zeta}) \end{pmatrix}^{-n+1} \begin{pmatrix} \zeta_{n-1} \\ \zeta_n \end{pmatrix} \tag{71}$$

exists, and the solution $\{\psi_n\}$ to the linearized equation with initial values given by (71) *satisfies*

$$\zeta_n = \bar{\zeta} + \psi_n + O\left(\rho^{2n}\right) \qquad \text{as} \qquad n \to \infty \,.$$

□

Example 1: The difference equation

$$\zeta_{n+1} = \frac{p + q\,\zeta_n}{1 + \zeta_{n-1}}, \quad n = 1, 2, \ldots, \qquad \zeta_0 \geq 0,\ \zeta_1 \geq 0. \tag{72}$$

has been studied in [27], where it was shown that for $p > q > 0$, the unique positive equilibrium is globally asymptotically stable on the positive quadrant.

To simplify calculations, we follow [27] and introduce a transformation in terms of new parameters $u > 1$ and $\alpha > 0$. Substitute

$$\zeta_n = \frac{1}{\alpha}\,\phi_n, \quad p = \frac{u^2 + (\alpha - 1)\,u}{\alpha^2}, \quad q = \frac{1}{\alpha},$$

in (72) to get

$$\phi_{n+1} = \frac{u^2 + (\alpha - 1)\,u + \phi_n}{\alpha + \phi_{n-1}}, \quad n = 1, 2, \ldots, \qquad \phi_0 \geq 0,\ \phi_1 \geq 0. \tag{73}$$

The equilibrium of equation (73) is precisely $\bar{\phi} = u$.

We now verify that the only solution $\{\phi_n\}$ for which $\phi_n = u$ for all large n is the equilibrium solution, i.e., that for which $\phi_0 = \phi_1 = u$. Let N be the first natural number for which $\phi_N = u$. If $N > 0$, then $\phi_{N-1} \neq u$. Taking $n = N$ in (73) together with $\phi_N = \phi_{N+1} = u$ we obtain

$$u = \frac{u^2 + (\alpha - 1)\, u + u}{\alpha + \phi_{N-1}}$$

from which we get $\phi_{N-1} = u$, a contradiction. Thus $\phi_0 = \phi_1 = u$, and $\{\phi_n\}$ is the equilibrium solution.

The linearized equation of (73) at the equilibrium u is

$$\psi_{n+1} = \frac{-u}{u+\alpha}\, \psi_{n-1} + \frac{1}{u+\alpha}\, \psi_n. \tag{74}$$

The characteristic roots of the linearized equation at the equilibrium are the complex conjugate pair

$$\lambda_{\pm} \;=\; \frac{1}{2\,(u+\alpha)} \;\pm\; i\,\frac{\sqrt{4\,u^2 + 4\,\alpha\, u - 1}}{2(\alpha+u)}.$$

Set

$$\rho \;:=\; |\,\lambda_{\pm}\,| \;=\; \sqrt{\frac{u}{u+\alpha}}.$$

Let J be the jacobian matrix of the map associated with equation (73), at the equilibrium. Then J and its inverse are given by

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{u}{u+\alpha} & \frac{1}{u+\alpha} \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} \frac{1}{u} & -\frac{u+\alpha}{u} \\ 1 & 0 \end{pmatrix}. \tag{75}$$

By Corollaries 3 and 9, relation (75), and the Main Theorem in [27] we have the following result.

Proposition 1. *For every solution $\{\phi_n\}$ to equation (73) other than the equilibrium solution,*

$$\limsup |\; \phi_n - u\; |^{1/n} \;=\; \sqrt{\frac{u}{u+\alpha}}\,.$$

Furthermore, the limit

$$\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} \frac{1}{u} & -\frac{u+\alpha}{u} \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \phi_{n-1} \\ \phi_n \end{pmatrix} \tag{76}$$

exists, and the solution $\{\psi_n\}$ *to the linearized equation* (74) *with initial values given by* (76) *satisfies*

$$\phi_n = u + \psi_n + O\left(\left(\frac{u}{u+\alpha}\right)^n\right) \qquad \text{as} \qquad n \to \infty\,.$$

As an illustration of Proposition 1, consider $u = 7/4$ and $\alpha = 1/4$ in (73). The equation is

$$\phi_{n+1} = \frac{\frac{7}{4} + \phi_n}{\frac{1}{4} + \phi_{n-1}}, \quad n = 1, 2, \ldots, \qquad \phi_0 \geq 0,\ \phi_1 \geq 0 \tag{77}$$

Here $\rho = \sqrt{\frac{7}{8}}$. Proposition 1 implies that solutions $\{\phi_n\}$ to (77) satisfy

$$\text{(a)} \limsup \left| \phi_n - \frac{7}{4} \right|^{1/n} = \sqrt{\frac{7}{8}} \quad \text{and} \quad \text{(b)}\, \phi_n = \frac{7}{4} + \psi_n + O\left(\left(\frac{7}{8}\right)^n\right), \tag{78}$$

where ψ_n is a precisely the solution to the linearized equation (74) for which

$$\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} \frac{4}{7} & -\frac{8}{7} \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \phi_{n-1} \\ \phi_n \end{pmatrix}. \tag{79}$$

See Fig. 1.

2.6 Numerical Examples

In this section, examples are provided to showcase the results of Theorems 1 and 2 and their corollaries. The goal of Examples 2 and 3 below is to compare the effect of changes in the smaller eigenvalues of J on the asymptotic behavior of x_n. The matrices J in Examples 2 and 3 below have the same norm and similar Jordan block structure, but different smallest eigenvalue. In each case a solution is calculated for which $\rho = \lim \|x_n\|^{1/n} = 0.25$. The residual $x_n - J^n\, y$ behaves

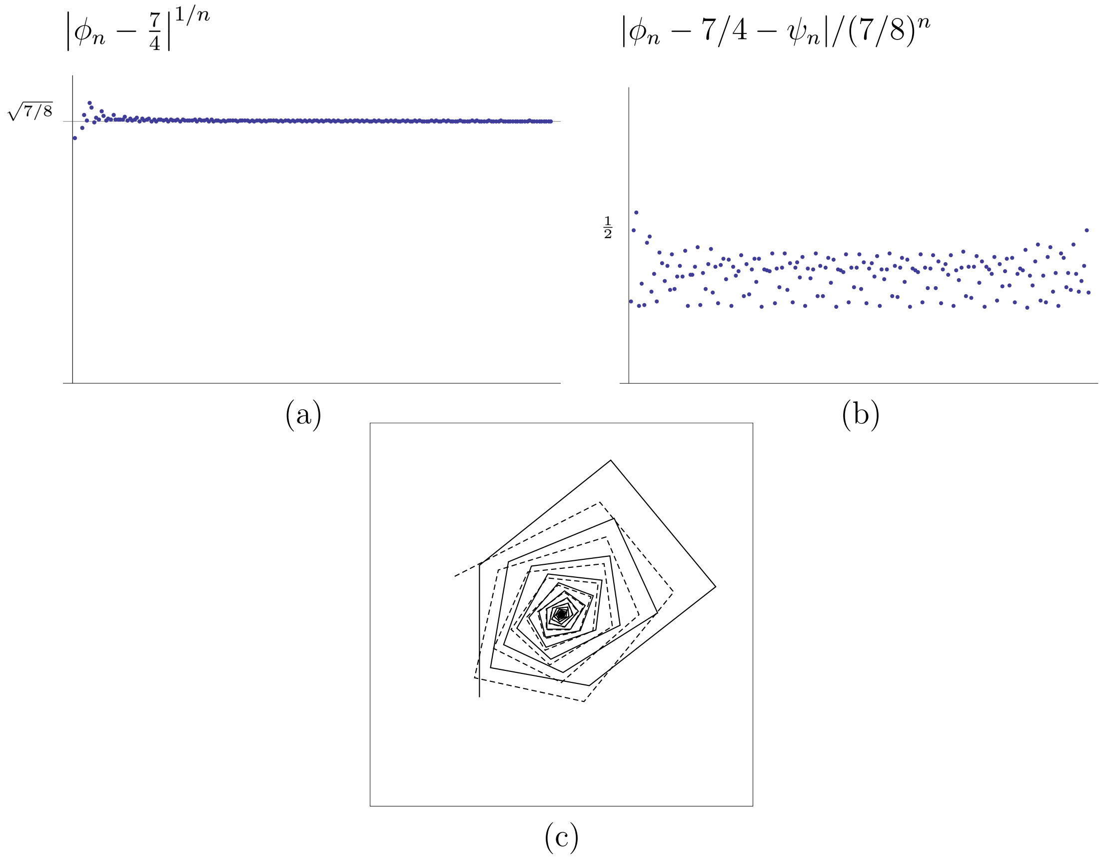


Figure 1. (a) The figure shows $|\phi_n - 7/4|^{1/n}$ as a function of n, where $\phi_0 = 1$ and $\phi_1 = 1$. Note that the values oscillate, with least upper bound $\rho = \sqrt{7/8} \approx 0.9354$, as guaranteed by part (a) of (78). (b) With formula (79), one gets the approximation $(\psi_0, \psi_1) \approx (0.793593488474, 0.778353526370)$. The graph shows $|\phi_n - 7/4 - \psi_n|/(7/8)^n$ as a function of n. (c) Phase plane plot, showing points (ϕ_n, ϕ_{n+1}) of a solution (solid) and points (ψ_n, ψ_{n+1}) of the associated solution to the linearized system, $\{\psi_n\}$ given by Proposition 1 (dash). The characteristic values of the linearized equation are complex numbers, which results in rotation of phase plane points about the equilibrium.

differently in both examples, as predicted by Theorem 1. Example 4 is a Poincaré-type difference system where the perturbation matrices B_n have norm that goes to zero geometrically with rate $\eta = 0.1$.

In the examples presented in this section, all calculations were performed with extended precision arithmetic using 200 decimal places of precision.

Example 2: Consider the difference equation $x_{n+1} = J\,x_n + f(x_n)$ with

$$J = \begin{pmatrix} 0.25 & 1 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.05 \end{pmatrix} \quad \text{and} \quad f(x_n) = \begin{pmatrix} x_n^T\,B_1\,x_n \\ x_n^T\,B_2\,x_n \\ x_n^T\,B_3\,x_n \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} -0.23 & -0.08 & -0.07 \\ -0.23 & -0.09 & -0.13 \\ 0.44 & -0.13 & -0.27 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.46 & 0.09 & -0.19 \\ 0.38 & 0.36 & -0.20 \\ 0.15 & -0.18 & 0.44 \end{pmatrix}, \quad \text{and} \quad B_3 = \begin{pmatrix} -0.43 & -0.38 & -0.30 \\ 0.44 & 0.32 & 0.31 \\ -0.42 & -0.36 & -0.12 \end{pmatrix}.$$

For the initial point $x_0 = (-0.40, 0.19, 0.38)^T$, we have $\lim_{n\to\infty} x_n = 0$. Figure 2 (a) shows $\|x_n\|^{1/n}$ as a function of n. It is seen that for the given initial point $\|x_n\|^{1/n} \to \frac{1}{4}$. A numerical calculation gives the approximation $\tilde{y} \approx (-1.2137598816, 0.1851109665, 0.3800000000)^T$ to the vector $\tilde{y}$ in Theorem 1. Figure 2 (b) is a log-plot of the relative error $\|J^n\tilde{y} - x_n\|/\|x_n\|$ between the linear and non-linear iterations as a function of n. From this plot it can be seen that as n increases, the error is diminishing quickly relative to the norm of x_n.

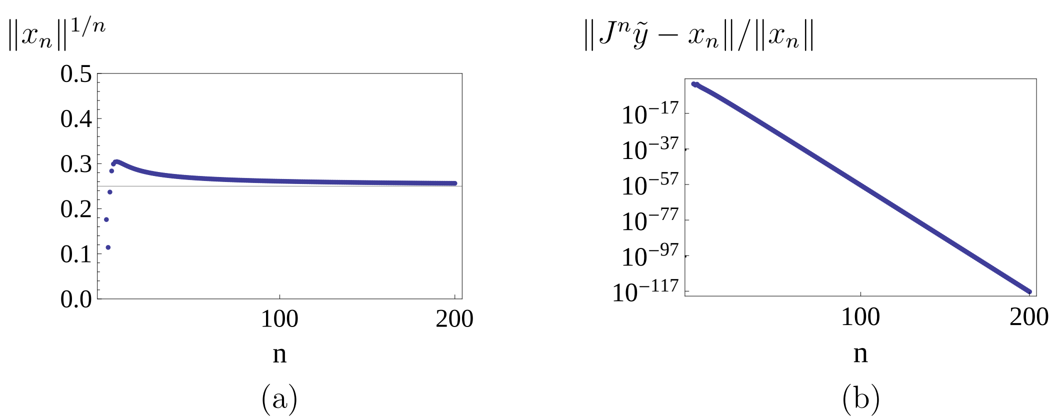
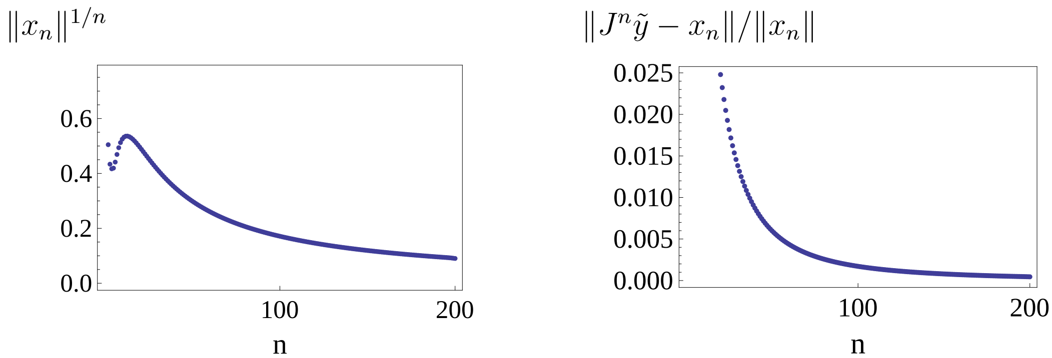


Figure 2. (a) Convergence of $\|x_n\|^{1/n}$ to 1/4. (b) Log plot of the relative error between the x_n and $J^n\,\tilde{y}$.

J has a 2×2 elementary Jordan block associated with $\rho = 0.25$, so $k_* = 1$ in Theorem 1. The elementary Jordan block associated with $\rho = 0.25$ is also the largest elementary Jordan block of J, so $k = 1$. Further, $f(x)$ is of class C^2 in a neighborhood of the origin, so $\alpha = 2$. Theorem 1 indicates that $\beta = \alpha(k_*) + 2k + 1 = 5$. However, since $\varrho_2 \neq \rho^\alpha$ we may take $\beta = 4$. In Figure 3 the plots show $\|J\,\tilde{y} - x_n\|/(n^i\,(0.25)^{2n})$ as a function of n for various values of i.



(a) (b)

Figure 3. Plots of $\|J^n\tilde{y} - x_n\|/(n^i\,\rho^{2n})$ with (a) $i = 3$, (b) $i = 4$. Theorem 1 asserts that $\|J^n\tilde{y} - x_n\| = O(n^\beta\,\rho^{2n})$ with $\beta = 4$.

Example 3: Consider the difference equation $x_{n+1} = J\,x_n + f(x_n)$ with

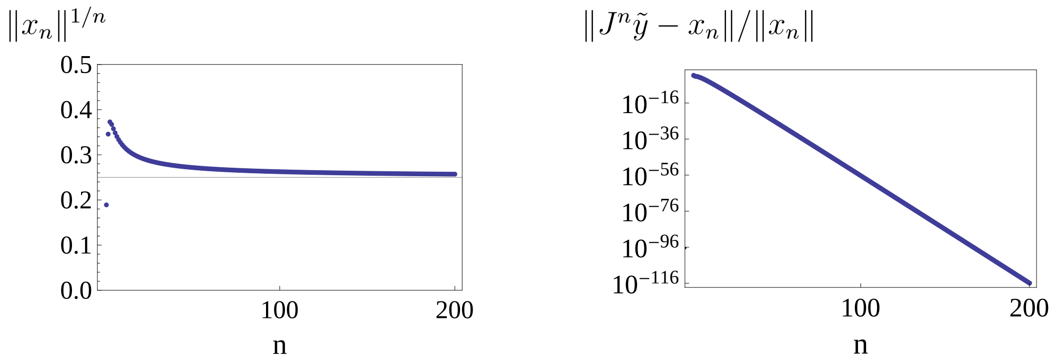
$$J = \begin{pmatrix} 0.25 & 1 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.0625 \end{pmatrix} \quad \text{and} \quad f(x_n) = \begin{pmatrix} x_n^T\,B_1\,x_n \\ x_n^T\,B_2\,x_n \\ x_n^T\,B_3\,x_n \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} -0.24 & -0.10 & 0.11 \\ -0.05 & -0.37 & 0.35 \\ -0.46 & 0.01 & -0.12 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.25 & 0.33 & 0.31 \\ 0.37 & 0.15 & -0.06 \\ -0.05 & 0.35 & 0.04 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} -0.17 & -0.43 & 0.28 \\ -0.36 & -0.12 & 0.50 \\ -0.33 & 0.07 & 0.06 \end{pmatrix}.$$

If the initial vector is $x_0 = (0.36, 0.060, 0.40)^T$, then $\tilde{y} \approx (-0.6887409072, 0.3340680336, 0.4000000000)^T$ and $\lim_{n\to\infty} x_n = 0$. Figure 4 (a) shows that $\|x_n\|^{1/n} \to 0.25 = \rho$. Figure 4 (b) shows that the relative error between the linear and non-linear iterates becomes small quickly.

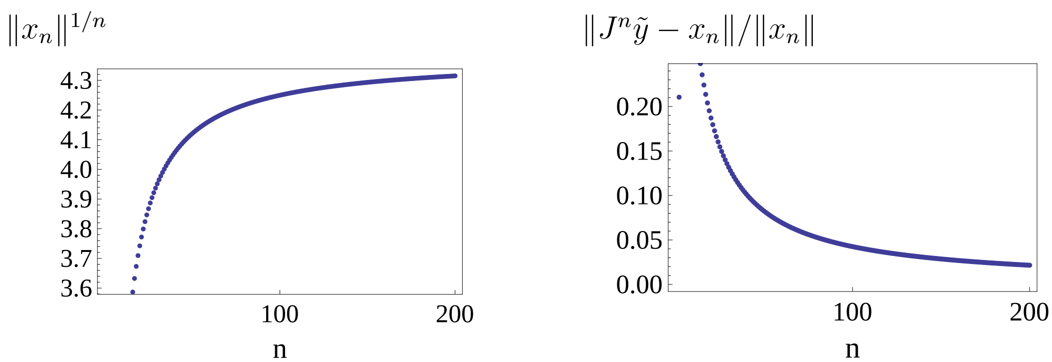
As in Example 2, the structure of J implies that $k_* = 1$ and $k = 1$, and $f(x)$ is again such that $\alpha = 2$. In contrast to Example 2, here $\varrho_2 = \rho^\alpha = (0.25)^2 = 0.0625$.



(a) (b)

Figure 4. (a) Convergence of $\|x_n\|^{1/n}$ to 1/4. (b) Log plot of the relative error between the x_n and $J^n\,\tilde{y}$.

Thus Theorem 1 gives $\beta = 5$. Figure 4 shows the quantity $\|J^n\tilde{y} - x_n\|/(n^i\,(0.25)^{2n})$ as a function of n for various values of i. From a comparison of the plots in Figure 5 to those in Figure 3, one can see that the polynomial term in the error does appear to be have higher degree in Example 3 than in Example 2.



(a) (b)

Figure 5. Plots of $\|J^n\tilde{y} - x_n\|/(n^i\,\rho^{2n})$ with (a) $i = 3$, (b) $i = 4$. Theorem 1 asserts that $\|J^n\tilde{y} - x_n\| = O(n^\beta\,\rho^{2n})$ with $\beta = 5$.

Example 4: Consider the difference equation $x_{n+1} = J\,x_n + B_n\,x_n$ where for some constant $c > 0$, $\|B_n\| \leq c\,(0.1)^n$ for all $n \geq 0$, and

$$J = \begin{pmatrix} 0.25 & 1 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.16 \end{pmatrix}.$$

As a numerical example, we set $B_n = (0.1)^n\, U_n$, where the matrices U_n were chosen with entries of the form $\frac{i}{100}$ with i a randomly generated integer chosen uniformly between -100 and 100. If the initial vector is $x_0 = (-0.62, 0.14, -0.84)$, then $\lim_{n\to\infty} x_n = 0$, and $\tilde{y} \approx (0.3307208470, -0.5738212397, -0.6766558855)^T$. Figure 6 (a) illustrates the relation $\lim \|x_n\|^{1/n} = \rho = 0.25$. One can see from the structure of J that $k = 1$ in Theorem 2. Theorem 2 predicts that $\|J^n\tilde{y} - x_n\| = O(n^{\beta}\,(\eta\rho)^n)$, where $\beta = 3k+1 = 4$. Figure 7 shows plots of $\|J^n\tilde{y} - x_n\|/(n^i\,(.025)^n)$ for various values of i as a function of the number of iterations n. As predicted, the plot corresponding to $i = 4$ appears to be bounded.

Here $\eta = 0.1$, $\rho = 0.25$, $\alpha = 1$, and since $\eta\,\rho^{\alpha} = 0.1\cdot 0.25 = 0.025$, by Corollary 2 a vector $\tilde{y}$ exists and is unique so that

$$x_n \;=\; J^n\,\tilde{y} \;+\; O(\,n^4\,(0.025)^n\,) \quad \text{as} \quad n \to \infty\,. \tag{80}$$

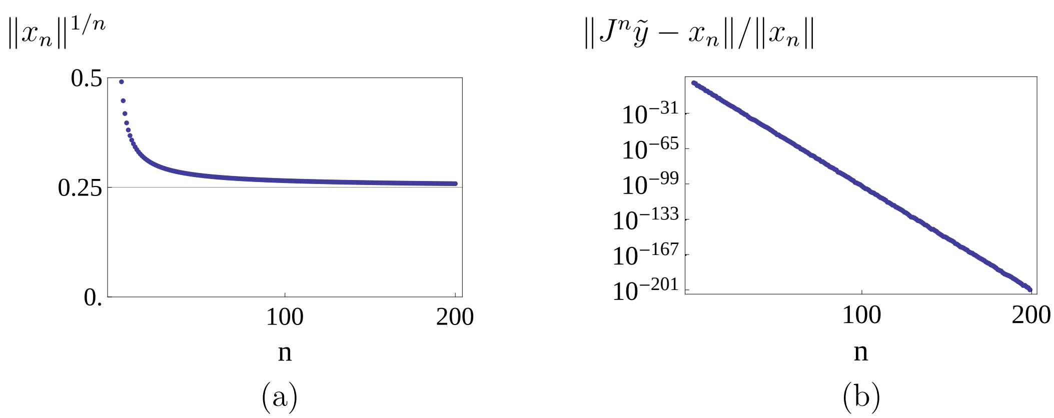


Figure 6. (a) Convergence of $\|x_n\|^{1/n}$ to $1/4$. (b) Log plot of the relative error between the x_n and $J^n\,\tilde{y}$.

We conclude this section with two remarks about the numerical examples. Although solutions $\{x_n\}$ for which $\|x_n\|^{1/n} \to \rho = |\lambda|$ for some λ that is not the eigenvalue of J with the largest modulus do exist, they are difficult to produce

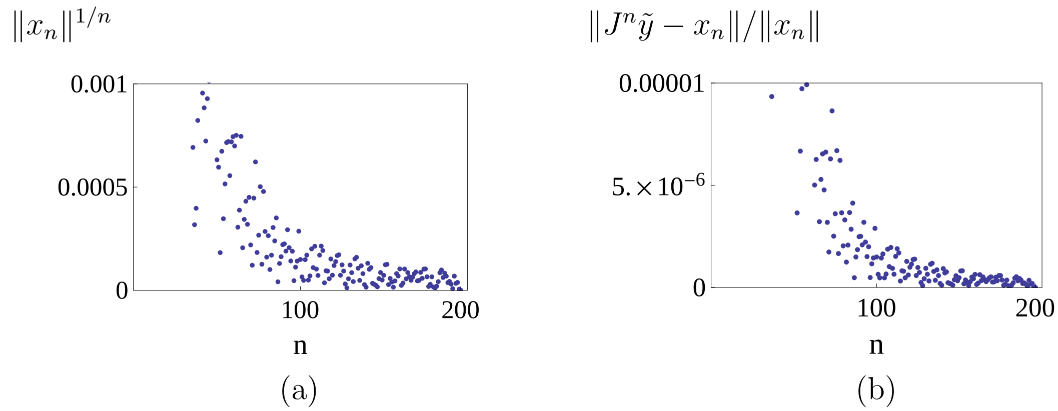


Figure 7. Plots of $\|J^n\tilde{y} - x_n\|/(n^i\,\rho^{2n})$ with (a) $i = 3$, (b) $i = 4$. Theorem 1 asserts that $\|J^n\tilde{y} - x_n\| = O(n^\beta\,\rho^{2n})$ with $\beta = 4$.

numerically, and are not generic. Thus we do not present any such numerical examples. Finally, we note that the values for β given in Theorems 1 and 2 may not be the smallest possible for hypothesis (H). By looking at Figs. 3, 5, and 7, it appears that the values for β given in Theorems 1 and 2 could be lowered.

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MANUSCRIPT 3

Local Dynamics of Planar Maps with a Non-isolated Fixed Point Exhibiting 1-1 Resonance

William T. Jamieson and Orlando Merino

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Abstract. Planar maps with a continuum of fixed points may have fixed points exhibiting 1-1 resonance. We give a complete classification of all possible dynamical behavior scenarios valid in a neighborhood of such points for maps that are real analytic. Examples from the literature are discussed.

3.1 Introduction

Planar maps with a curve of fixed points may have fixed points exhibiting 1-1 resonance. The main purpose of this paper is to give a complete classification of all possible dynamical behavior scenarios valid in a neighborhood of such points for maps that are real analytic.

Since the early 1990s there has been a large amount of activity in the study of difference equations in general, especially in rational difference equations and monotone difference equations. A considerable amount of work has been done in these areas by many authors, see Elaydi [12], Smith [8], Agarwal [13], Ladas [15], Kulenović [16] and references cited therein.

The study of the dynamical behavior near an equilibrium point of a planar difference equation or system (i.e. a fixed point of the associated map) is separated into cases depending on whether the fixed point is hyperbolic or non-hyperbolic. It is assumed from now on that the map in question is at least continuously differentiable on a planar domain. In the hyperbolic case, the fixed point is either a sink, a source, or a saddle, and local dynamics are well understood [14]. In the non-hyperbolic case, i.e. when at least one of the characteristic values have modulus 1, the dynamics are also understood with some notable exceptions. If the characteristic values λ_1, λ_2 associated with a fixed point are complex conjugate of each other and not a first, second, third, or fourth root of unity, then techniques from Kolmogorov-Arnold-Moser (KAM) theory can be used to establish the local dynamical behavior; see [2] for an overview or [3] for an application of the theory.

These cases where KAM theory cannot be applied are called strong resonances (see [5] p.396.) In this paper, we are interested in a special case of a strong resonance, called 1-1 resonance. A fixed point of a planar map is said to be *1-1 resonant* if the jacobian of the map at the fixed point is similar to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The local dynamical behavior near a 1-1 resonant fixed point is not yet resolved.

In the case when one characteristic value is not on the unit circle and the second one is ± 1, the local dynamical behavior has not been completely established. However, the existence of invariant center-stable and center-unstable manifolds has been established. If one is able to find center manifolds associated with a fixed point, a center manifold may provide important information about the local dynamical properties near the fixed point. In some cases, for example part (a) of Theorem 5 in Section 3.4, the center manifold forms a boundary between the basins of attraction of different fixed points.

In 1977 Hirsch, Pugh, and Shub [8] proved that in the neighborhood of an isolated fixed point whose jacobian matrix has at least one eigenvalue outside the unit circle, a C^k-system possesses a C^k-unstable manifold, establishing the existence of a center-unstable manifold associated with the fixed point. In the case when the map is invertible, their theorem also establishes a center-stable manifold associated with the fixed point. The local behavior in this case has been established for some subclasses; for example Kulenović and Merino [4] described the local and global dynamical behavior in the case when $|\lambda_1| < \lambda_2 = 1$ for competitive and cooperative systems of difference equations. Unfortunately, any general procedure to identify a center manifold, for example the Lyapunov-Perron method or the Hadamard Graph Transform method (see [9] for an overview), require that the fixed point has *at least one hyperbolic eigenvalue.* There is not a general theory that can be applied to the remaining configurations of λ_1, λ_2 or for more general classes of functions. A 1-1 resonant fixed point has only non-hyperbolic eigenvalues, so finding center manifolds in this case is problematic.

We shall be concerned with the case of non-isolated fixed points, which has not been considered so far in the literature. A fixed point of a planar system of difference equations is called *isolated* if there exists a neighborhood of the fixed point that does not contain any other fixed points. In all other cases the fixed point is called *non-isolated*. If a map has a locally defined manifold consisting entirely of fixed points, one of the associated characteristic values at the fixed points must be equal to 1. To see this, translate the fixed point to the origin and perform a local change of coordinates to map the curve of fixed points to either the x- or y-axis. Then it is clear that $(1, 0)^T$ or $(0, 1)^T$ is an eigenvector with eigenvalue 1.

To illustrate the situation when a system has a non-isolated 1-1 resonant fixed point, consider the following example:

Example 1. Clark-Kulenovic [10] and Clark-Kulenovic-Selgrade [11] studied a class of maps that included the following:

$$(\tilde{x}, \tilde{y}) = S(x, y) = \left(\frac{x}{a + y}, \frac{y}{1 + x} \right), \quad (x, y) \in [0, \infty) \times [0, \infty), \, 0 < a < 1. \tag{81}$$

In a modeling setting, x and y represent the population levels of two competing species, and $\tilde{x}$ and $\tilde{y}$ represent the population levels one generation later. This map has a continuum of equilibrium points. In fact, every point $(0, y)$ is a fixed point of system (81), and only those points are fixed points [10]. The characteristic values at a point $(0, y)$ are $1/(a + y)$ and 1. Thus all fixed points are non-hyperbolic of stable type if $y > 1 - a$, and non-hyperbolic of unstable type if $y < 1 - a$. Global behavior of solutions was not completely determined in [10], [11]. With tools developed later [4], it can be shown that each of these fixed points other than $(0, 1 - a)$ has, respectively, an associated local stable, respectively unstable, invariant manifold, which can be extended to a monotonic curve. A complete dynamical description requires understanding the dynamical characteristics at the 1-1 resonant fixed point $(0, 1-a)$, to determine which orbits converge to $(0, 1-a)$ under forward iteration of the map.

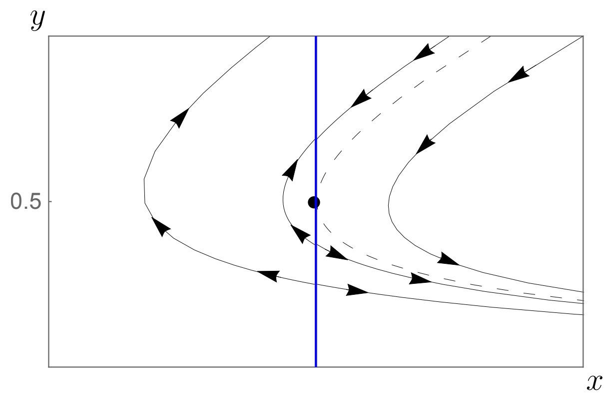

Figure 8. Clark-Kulenovic system (81) with $a = 0.5$: simulations suggest the existence of a curve (dashed) that passes through $(0, 0.5)$, which appears to be a boundary of regions with different dynamical behavior. The point in the center of the plot is $(0, .5)$ and the solid vertical line consists entirely of fixed points. The curves depict paths followed by orbits under iteration of S.

In 2009, Brett et al. [15] studied a monotone rational difference equation possessing a non-isolated fixed point of the 1-1 resonant type; the local behavior near this fixed point was not established, nor has the local dynamical behavior near a 1-1 resonant fixed point been established in the literature. Fixed points that are 1-1 resonant have appeared in other recent papers [3]. Theorem 5 of this paper describes the local qualitative behavior for the system studied in [10], [15], and any other system containing a non-isolated 1-1 resonant fixed point.

A continuous analog to 1-1 resonance for ordinary differential equations has been studied in the 1950s and 1960s by N. A. Gubar [1] (see also [16] for an English translation). In [1], a classification of the local dynamical behavior of a system of planar differential equations near an isolated fixed point whose jacobian was similar to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ was made under the assumption that the planar system is real analytic and that the fixed point is isolated (this case corresponds to 1-1 resonance in the difference equations context). This suggests that a classification would be possible for a 1-1 resonant fixed point of an analytic system of planar difference equations.

Computer simulations suggest that the criteria for the classification made in the differential equations setting are similar to the those needed for the classification in the difference equations setting. In the continuous case, the classification for isolated 1-1 resonant fixed points made by Gubar [1] was obtained by putting the system into a normal form that is particularly helpful. We shall use a normal form analogous to Gubar's to produce a classification of the possible dynamical scenarios near a 1-1 resonant non-isolated fixed point of a map. We shall show in Section 3.2 that every 1-1 resonant fixed point of a real analytic map is conjugate to, and hence can be reduced to, the adaptation of the normal form used by Gubar. Thus results for the normalized form are valid for any real analytic map with a 1-1 resonant non-isolated fixed point. In order to describe the dynamical behavior, we will introduce in Section 3.3 the notion of a *sector*, which is also used in [16] in a different form.

Gubar [1] defines systems of differential equations with a 1-1 resonance at the origin to be in normal form to be those of the form

$$x' = y\,, \quad y' = g(x, y) \tag{82}$$

for all (x, y) in a neighborhood of the origin, and $g(x, y)$ is a real valued analytic function that satisfies $\frac{\partial}{\partial y} g(0, 0) = 0$ and $g(x, 0) = 0$ for $(x, 0)$ in the neighborhood of the origin. Any such real analytic function $g(x, y)$ can be written as

$$g(x, y) = P(x)x^k + Q(x)x^\ell y + R(x, y)y^2,$$

where P, Q, and R are real analytic functions such that $P(0), Q(0) \neq 0$. The main theorem in [3] makes the classification in terms of the sign of $P(x)$ and parity of k.

Our main result, Theorem 5 given in Section 3.4, is a discrete version of Gubar's classification for differential equations systems. However, the method of proof is completely different for the one used by Gubar.

We now give an overview of the manuscript. In Section 3.2 we introduce normal forms. Conditions are given for the normal form to exist and it is shown how to obtain information relevant to the normal form without obtaining the normal form explicitly. Section 3.3 introduces *sectors* and *sector boundary curves*, which we will use to describe dynamical behavior in the statement of the main theorem, Theorem 5. The proof of the main theorem, which relies on Theorems 6 and 7, is given in Section 3.4. The proofs of Theorems 6 and 7 are presented separately from the proof of the main theorem in Sections 3.5 and 3.6.

3.2 Curves of Fixed Points and Normal Forms

Near any 1-1 resonant non-isolated fixed point, the set of zeros of a real analytic planar map is a real analytic curve. This is stated formally in the next result.

Theorem 3. *Let W be an open set in $\mathbb{R}^2$ containing a point $(\bar{x}, \bar{y})$. Let $S : W \to \mathbb{R}^2$ be a real analytic map for which $(\bar{x}, \bar{y})$ is a non-isolated, type 1-1 resonant fixed point. Then there exists a neighborhood $U \subset W$ of $(\bar{x}, \bar{y})$ on which the set $\mathcal{C}$ of fixed points of S is a real analytic curve $\mathcal{C}_0$ through $(\bar{x}, \bar{y})$. The curve $\mathcal{C}_0$ is tangential at $(\bar{x}, \bar{y})$ to the one dimensional eigenspace of the jacobian of S at $(\bar{x}, \bar{y})$.*

Proof. Throughout this proof and the rest of this paper, for any function $f : \mathbb{R}^2 \to \mathbb{R}$, let the result of taking the n^{th} partial derivative with respect to the first variable and the m^{th} partial derivative of the second component be $f_{(n,m)}$, and

$$S_i(x, y) = \text{ the } i^{\text{th}} \text{ component of } S,\ i = 1, 2$$

for any function $S : \mathbb{R}^2 \to \mathbb{R}^2$.

Without loss of generality it may be assumed that $(\bar{x}, \bar{y}) = (0,0)$ and that the jacobian matrix of S at $(0,0)$ is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore there exist real analytic functions f and g such that

$$S(x,y) = (\ x + y + f(x,y)\ ,\ y + g(x,y)\ )\,, \qquad (x,y) \in U\,, \tag{83}$$

with

$$f_{(1,0)}(0,0) = f_{(0,1)}(0,0) = g_{(1,0)}(0,0) = g_{(0,1)}(0,0) = 0 \tag{84}$$

In particular, $1 + f_{(0,1)}(x,y) \neq 0$ for (x,y) close enough to $(0,0)$. Set $F(x,y) = y + f(x,y)$. If (x,y) is a fixed point of S, then necessarily $F(x,y) = 0$. Then by these considerations and the Real Analytic Implicit Function Theorem (Theorem 1.8.3 in [6]), there exists a neighborhood $I \subset \mathbb{R}$ of 0 and a real analytic function $y = \xi(x)$ on I such that for $x \in I$, $1 + f_{(0,1)}(x, \xi(x)) \neq 0$, $F(x, \xi(x)) = 0$, and

$$\xi'(x) = \frac{-f_{(1,0)}(x, \xi(x))}{1 + f_{(0,1)}(x, \xi(x))}\,. \tag{85}$$

Thus every fixed point of S in W must be of the form $(x, \xi(x))$ for some x near 0. We now show that points of the form $(x, \xi(x))$ satisfy $g(x, \xi(x)) = 0$, and consequently they are fixed points of S. Since $(0,0)$ is a non-isolated fixed point, there exists a sequence (x_n, y_n) of fixed points that converges to $(0,0)$. By the previous discussion, $y_n = \xi(x_n)$ for n larger than some $N \in \mathbb{N}$, and thus we have $g(x_n, \xi(x_n)) = 0$, $n \geq N$. Then $g(\cdot, \xi(\cdot))$ is a real analytic function on I that has a sequence of zeros that accumulates in the interior of I. By Corollary 1.2.6 in [6], $g(x, \xi(x)) = 0$ for every $x \in I$. Finally, relations (84) and (85) give $\xi'(0) = \frac{-f_{(1,0)}(0,0)}{1+f_{(0,1)}(0,0)} = 0$, which is the last statement in the theorem. $\square$

The definition below is an adaptation of the normal form used by Gubar in [1].

Definition 1. *A real analytic map T defined on a neighborhood $V \subset \mathbb{R}^2$ of the origin is said to be* in normal form *if there exist real analytic functions Q and R where either $Q(\cdot)$ is the zero function, or $Q(0) \neq 0$ and there exists an integer $\ell \geq 1$ such that*

$$T(x,y) = (\, x + y \,,\, y + Q(x)\, x^{\ell}\, y + R(x,y)\, y^2 \,) \quad \textit{for} \quad (x,y) \in V. \tag{86}$$

For $(x,y) \in V$, $T(x,y) = (x,y)$ if and only if $y = 0$.

Definition 2. *Let W be an open set in $\mathbb{R}^2$ containing a point $(\bar{x}, \bar{y})$. Let $S : W \to \mathbb{R}^2$ be a real analytic map for which $(\bar{x}, \bar{y})$ is a non-isolated, type 1-1 resonant fixed point. A planar map T is a* normal form *of S relative to $(\bar{x}, \bar{y})$ and W if T is in normal form and there exists a real analytic change of coordinates under which $(\bar{x}, \bar{y})$ is mapped to the origin and the map S conjugates to T on W.*

It should be noted that the normal form T is not unique- for example, if T is a normal form, then T conjugated with a map reflecting over the origin also will be a normal form for a possibly different choice of W. However, the local dynamical behavior of a map S and *any* of its normal forms will be the same. Theorem 4 shows that every real analytic planar mapping with a 1-1 resonant fixed point has a normal form.

Theorem 4. *Let W be an open set in $\mathbb{R}^2$ containing a point $(\bar{x}, \bar{y})$. Let $S : W \to \mathbb{R}^2$ be a real analytic map for which $(\bar{x}, \bar{y})$ is a non-isolated, type 1-1 resonant fixed point. Then there exists a neighborhood $U \subset W$ of $(\bar{x}, \bar{y})$ and a normal form of S relative to $(\bar{x}, \bar{y})$ and U.*

Proof. Without loss of generality we assume $(\bar{x}, \bar{y}) = (0,0)$ and that the jacobian matrix of S at $(0,0)$ is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. By Theorem 3 the set of fixed points $\mathcal{C}_0$ near the origin has a real analytic parametrization of the form $y = \xi(x)$ such that $\xi(0) = 0$ and $\xi'(0) = 0$. The map $\Theta(x,y) := (x, y - \xi(x))$, defined on a neighborhood U of the origin chosen so that $\Theta(U) \subset W$, is real analytic, 1-1, and satisfies $\Theta(0,0) = (0,0)$ and $\Theta^{-1}(x,y) = (x, y + \xi(x))$ for (x,y) near the origin. The conjugate map of S through Θ given by $T = \Theta^{-1} S \Theta$ satisfies

$$T(x,0) = (x,0) \qquad (x,0) \in U\,. \tag{87}$$

Since $(0,0)$ is a 1-1 resonant fixed point, there exist real analytic functions f and g such that

$$T(x,y) = (\ x + y + f(x,y)\ ,\ y + g(x,y)\ )\,, \qquad (x,y) \in U\,, \tag{88}$$

with

$$f_{(1,0)}(0,0) = f_{(0,1)}(0,0) = g_{(1,0)}(0,0) = g_{(0,1)}(0,0) = 0. \tag{89}$$

Write $(\tilde{x}, \tilde{y}) = T(x,y)$. By the real analytic character of T and by (87), there exist real analytic functions ϕ and ψ such that

$$\left\{ \begin{array}{lcl} \tilde{x} & = & x + y + y\,\phi(x,y) \\ \tilde{y} & = & y + y\,\psi(x,y) \end{array} \right. , \qquad (x,y) \in U\,. \tag{90}$$

Set

$$z = y + \phi(x,y)\,y \qquad \text{and} \qquad \tilde{z} = \tilde{y} + \phi(x,\tilde{y})\,\tilde{y}. \tag{91}$$

Combine (90) and (91) to obtain

$$\left\{ \begin{array}{lcl} \tilde{x} & = & x + z \\ \tilde{z} & = & (y + y\,\psi(x,y))\,\phi(x, y + y\,\psi(x,y)) + y + y\,\psi(x,y) \end{array} \right. , \qquad (x,y) \in U\,. \tag{92}$$

The function ϕ satisfies $\phi(0,0) = 0$, so there exists a neighborhood $U' \subset U$ of the origin such that

$$\phi(x,0) \neq -1 \quad \text{for} \quad (x,0) \in U'\,. \tag{93}$$

The latter relation, the first equation of (91), and the Real Analytic Implicit Function Theorem (Theorem 1.8.3 in [6]) imply that there exist a neighborhood

$V \subset U'$ of the origin and a real analytic function h on V such that $y = h(x, z)$ for $(x, z) \in V$. From substituting $y = h(x, z)$ in (92), the latter may be rewritten as

$$\left\{ \begin{array}{rcl} \tilde{x} & = & x + z \\ \tilde{z} & = & H(x, z) \end{array} \right. , \qquad (x, y) \in V \tag{94}$$

where $H(x, z)$ is a real analytic function on V. Now $H(x, 0) = 0$ for $(x, 0) \in V$, by the choice of $U' \supset V$ and by the first equation of (91). Thus there exists H_1 real analytic on V such that

$$H(x, z) = z\, H_1(x, z) \quad \text{for} \quad (x, z) \in V\,. \tag{95}$$

Next we verify that $H_1(0, 0) = 1$. By (91) and (93), $z = 0$ if and only if $y = 0$, so (91), (94) and (95) imply that for $(x, z) \in V$ with $z \neq 0$,

$$H_1(x, z) = \frac{\tilde{z}}{z} = \frac{\tilde{y}(1 + \phi(\tilde{x}, \tilde{y}))}{y(1 + \phi(x, y))} = \frac{(1 + \psi(x, y))(1 + \phi(\tilde{x}, \tilde{y}))}{(1 + \phi(x, y))}. \tag{96}$$

Since the last term in (96) is a continuous function of (x, z) near the origin and it has the value 1 there, it follows that $H_1(0, 0) = 1$. Now $H_1(x, 0) - 1$ is a real analytic function of x near zero. Write $H_1(x, 0) - 1 = Q(x)\, x^\ell$, where either Q is the zero function, or $\ell \geq 1$ and $Q(0) \neq 0$. Then $H_1(x, z) - 1 - Q(x)\, x^\ell$ is real analytic and we may write $H_1(x, z) = 1 + Q(x)\, x^\ell + R(x, z)\, z$, where R is real analytic. Then equations (94) and (95) give a normal form (86). $\square$

As a consequence of the proof we have that if an analytic expression for the curve $\mathcal{C}_0$ of fixed points of S is known, if can be used to define the map $T = \Theta^{-1} S \Theta$ which has a specific structure. The following corollary shows that T allows a direct calculation of the terms $Q(x)$ and ℓ of a normal form (86) of S, which will be useful in applications given in Section 3.4.1

Corollary 10. *Let T be a real analytic map on an open neighborhood U of the origin be of the form*

$$T(x, y) = (\ x + y + y\, \phi(x, y)\ ,\ y + y\, \psi(x, y)\ )\,, \qquad (x, y) \in U\,, \tag{97}$$

where ϕ and ψ are real analytic on U. Then T has a normal form (86) *where* $Q(x)\,x^\ell = \psi(x,0)$ *for all x close enough to* 0.

Proof. By proceeding as in the proof of Theorem 4, a normal form,

$$\begin{cases} \tilde{x} &= x+z \\ \tilde{z} &= z+Q(x)\,x^\ell\,z+R(x,z)\,z^2 \end{cases}, \qquad (x,z)\in V\,, \tag{98}$$

is obtained by putting $z = y+\phi(x,y)\,y$ and $\tilde{z} = \tilde{y}+\phi(x,\tilde{y})\,\tilde{y}$. The latter relations and (98) give for $(x,0)\in V$,

$$\begin{aligned} Q(x)\,x^\ell &= \lim_{z\to 0}\frac{\tilde{z}-z}{z} = \lim_{y\to 0}\frac{\tilde{y}+\phi(x,\tilde{y})\,\tilde{y}-y-\phi(x,y)\,y}{y+\phi(x,y)\,y} \\ &= \lim_{y\to 0}\frac{y+y\,\psi(x,y)+\phi(x,y+y\,\psi(x,y))\,(y+y\,\psi(x,y))-y-\phi(x,y)\,y}{y+\phi(x,y)\,y} \\ &= \lim_{y\to 0}\frac{\psi(x,y)+\phi(x,y+y\,\psi(x,y))\,(1+\psi(x,y))-\phi(x,y)}{1+\phi(x,y)} \\ &= \frac{\psi(x,0)+\phi(x,0)\,(1+\psi(x,0))-\phi(x,0)}{1+\phi(x,0)} = \psi(x,0)\,. \end{aligned} \tag{99}$$

□

3.3 Sector Boundary Curves and Sectors

We now introduce the concepts of *sector boundary curves* and *sectors*. Sector boundary curves are curves that serve as boundaries of regions (sectors) with different dynamical behavior.

In this section S is a planar real analytic map defined on an open set W containing a fixed point $(\bar{x},\bar{y})$, and V is an open connected neighborhood of $(\bar{x},\bar{y})$ such that if $\mathrm{cl}(V)$ denotes the closure of V, then $\mathrm{cl}(V)\subset W$ and S^{-1} exists on $\mathrm{cl}(V)$.

Definition 3. *A C^1 simple curve $\mathcal{C}$ is a* sector boundary curve of S relative to $(\bar{x},\bar{y})$ and V *if it has endpoints $(\bar{x},\bar{y})$ and $(\tilde{x},\tilde{y})\in\partial V$, so that $\mathcal{C}\setminus\{(\tilde{x},\tilde{y})\}\subset V$ and such that one of the following condition holds:*

a. *For every* $(x, y) \in \mathcal{C}$, $S(x, y) \in \mathcal{C}$, $S^n(x, y) \to (\bar{x}, \bar{y})$, *and either for every* $\epsilon > 0$ *there exists* $(z, w) \in B((x, y), \epsilon) \cap V$ *such that* $S^n(z, w) \notin V$ *for some* $n > 0$, *or* $S^n(z, w) \not\to (\bar{x}, \bar{y})$.

b. *For every* $(x, y) \in \mathcal{C}$, $S^{-1}(x, y) \in \mathcal{C}$, $S^{-n}(x, y) \to (\bar{x}, \bar{y})$, *and either for every* $\epsilon > 0$ *there exists* $(z, w) \in B((x, y), \epsilon) \cap V$ *such that* $S^{-n}(z, w) \notin V$ *for some* $n > 0$, *or* $S^{-n}(z, w) \not\to (\bar{x}, \bar{y})$.

c. *For every* $(x, y) \in \mathcal{C}$, $S(x, y) = (x, y)$, *and for every* $\epsilon > 0$ *there exists* $(z, w) \in B((x, y), \epsilon) \cap V$ *such that* $S(z, w) \neq (z, w)$.

Thus under iteration of the map, all points on a sector boundary curve march towards the fixed point, or away from it, or are fixed points. In addition, under iteration of the map or its inverse, some points near a point on a sector boundary curve will behave in a different manner than points on the curve.

Definition 4. *Let* $\mathcal{C}_1, \ldots, \mathcal{C}_r$ *be sector boundary curves of* S *relative to* $(\bar{x}, \bar{y})$ *and* V *such that any two sector curves have* $(\bar{x}, \bar{y})$ *as their only common point. A connected component* $\mathcal{R}$ *of* $V \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r)$ *is called one of the following:*

a. a hyperbolic sector of S relative to $(\bar{x}, \bar{y})$ *and* V *if for every* $(x, y) \in \mathcal{R}$ *there exist* m, n *in* $\mathbb{N}$ *such that* $S^\ell(x, y) \in \mathcal{R}$ *for* $-m \leq \ell \leq n$, *and* $S^\ell(x, y) \notin V$ *for* $\ell = -m - 1$ *or* $\ell = n + 1$.

b. an attracting parabolic sector of S relative to $(\bar{x}, \bar{y})$ *and* V *if for every* $(x, y) \in \mathcal{R}$, *there exists* $m \in \mathbb{N}$ *such that* $S^\ell(x, y) \in \mathcal{R}$ *for* $-m \leq \ell < \infty$, $S^{-m-1}(x, y) \notin V$, *and* $\{S^\ell(x, y)\}_{\ell=0}^{\infty}$ *converges to a point in one of the sector boundary curves* $\mathcal{C}_k$.

c. a repelling parabolic sector of S relative to $(\bar{x}, \bar{y})$ *and* V *if for every* $(x, y) \in \mathcal{R}$, *there exists* $n \in \mathbb{N}$ *such that* $S^\ell(x, y) \in \mathcal{R}$ *for* $-\infty < \ell < n$, $S^{n+1}(x, y) \notin$

V, and $\{S^{-\ell}(x,y)\}_{\ell=0}^{\infty}$ converges to a point in one of the sector boundary curves $\mathcal{C}_k$.

d. an elliptic sector of S relative to $(\bar{x},\bar{y})$ *and V if there exists an open disk $B((\bar{x},\bar{y});\delta) \subset V$ such that $S^{\ell}(x,y) \in \mathcal{R}$ for every $(x,y) \in \mathcal{R} \cap B((\bar{x},\bar{y});\delta)$ and every $\ell \in \mathbb{Z}$, and both $\{S^{\ell}(x,y)\}_{\ell=0}^{\infty}$ and $\{S^{-\ell}(x,y)\}_{\ell=0}^{\infty}$ converge to points in sector boundary curves $\mathcal{C}_k$ and $\mathcal{C}_i$.*

See Figure 9 for an illustration.

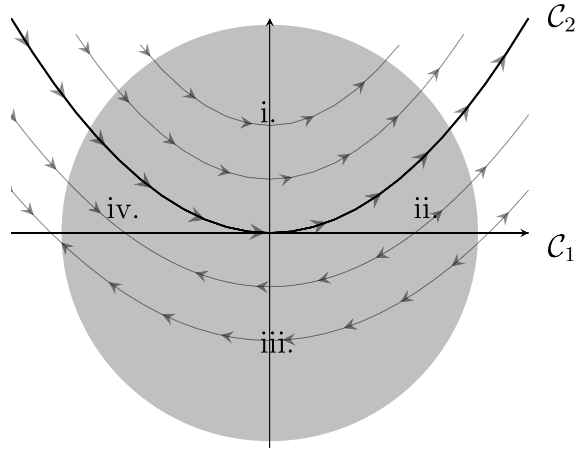

Figure 9. A neighborhood (shaded region) with two sector boundary curves $\mathcal{C}_1$ and $\mathcal{C}_2$. The resulting sectors are i. hyperbolic, ii. repelling parabolic, iii. elliptic, and iv. attracting parabolic.

3.4 The Classification Theorem

The next theorem is the main result of this paper. It gives a classification of possible dynamic scenarios near $(\bar{x},\bar{y})$ in terms of the curve of fixed points and of the normal form.

Theorem 5. *Let S be a real analytic planar map on an open set $W \subset \mathbb{R}^2$ containing a type 1-1 resonant, non-isolated fixed point $(\bar{x},\bar{y})$. Suppose W is so small that the set of fixed points of S in W is a real analytic curve $\mathcal{C}_0$, and that there* *exists a normal form T of S relative to $(\bar{x}, \bar{y})$ and W. Let Q and ℓ be associated to T and W as in equation* (86)*.*

If $Q = 0$, then there exists a closed neighborhood $V \subset W$ of $(\bar{x}, \bar{y})$ such that $V \setminus \mathcal{C}_0$ consists of two hyperbolic sectors relative to $(\bar{x}, \bar{y})$ and V. If $Q \neq 0$, there exists a closed neighborhood V of $(\bar{x}, \bar{y})$ and a smooth curve $\mathcal{C}_1$ in V with endpoints in ∂V and tangential to $\mathcal{C}_0$ at $(\bar{x}, \bar{y})$, such that $V \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)$ consists of four sectors relative to $(\bar{x}, \bar{y})$ and V. Furthermore,

(a) *If ℓ is odd, the four sectors, in either clockwise or counterclockwise orientation, are of elliptic, attracting parabolic, hyperbolic, and repelling parabolic type, respectively. Also, $\mathcal{C}_1 \setminus \{(\bar{x}, \bar{y})\}$ has two connected components $\mathcal{C}_1^-$ and $\mathcal{C}_1^+$ that are sector boundary curves such that $S^{-n}(x, y) \to (\bar{x}, \bar{y})$ for every $(x, y) \in \mathcal{C}_1^-$ and $S^n(x, y) \to (\bar{x}, \bar{y})$ for every $(x, y) \in \mathcal{C}_1^+$.*

(b) *If $Q(0) > 0$ and ℓ is even, all four sectors are of repelling parabolic type, and $S^{-n}(x, y) \to (\bar{x}, \bar{y})$ for every $(x, y) \in \mathcal{C}_1$.*

(c) *If $Q(0) < 0$ and ℓ is even, all four sectors are of attracting parabolic type, and $S^n(x, y) \to (\bar{x}, \bar{y})$ for every $(x, y) \in \mathcal{C}_1$.*

The four cases in Theorem 5 are illustrated in Figure 10. Theorems 6 and 7, given in Sections 3.5 and 3.6 respectively, give specific details about the dynamics in in the right and left half planes individually. The proof of Theorem 5 will be given in Section 3.4.3. Before discussing the proof, we present some examples.

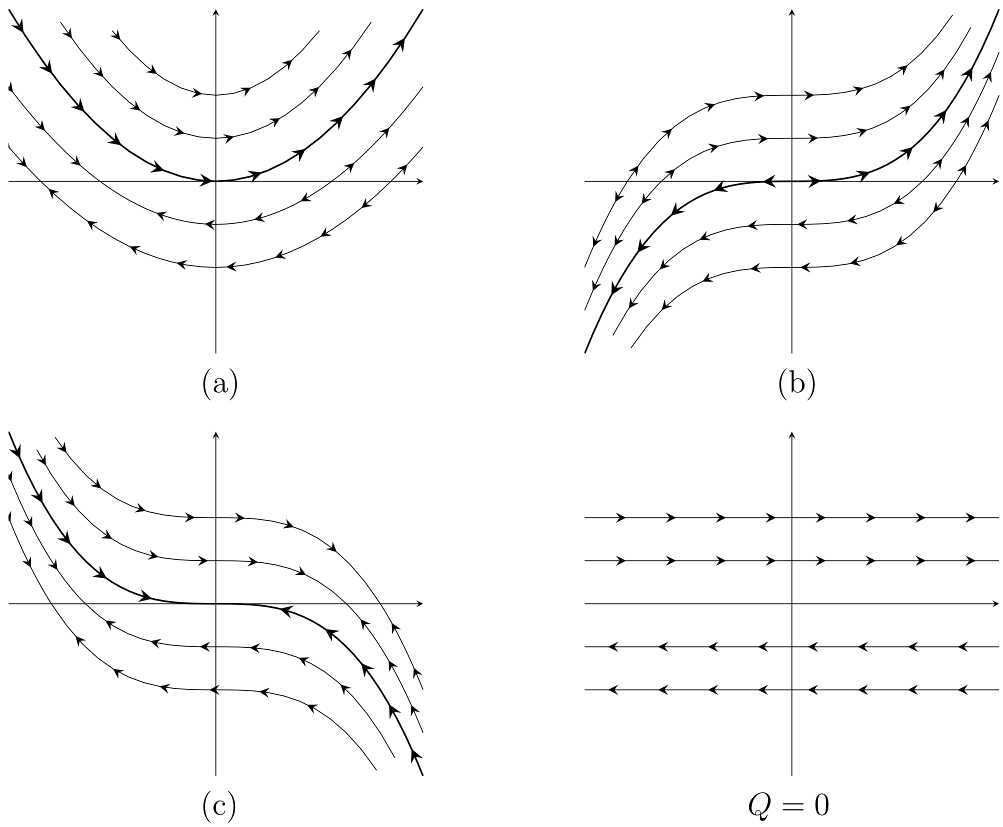

Figure 10. The four cases, corresponding to scenarios (a) - (c) in Theorem 5. The curves represent the paths and direction taken by orbits under forward iteration of T.

3.4.1 Examples

Example 1 Continued. (Kulenović et al. [10])

The map, for a such that $0 < a < 1$, is

$$(\tilde{x}, \tilde{y}) = S(x, y) = \left(\frac{x}{a+y}, \frac{y}{1+x} \right), \quad (x, y) \in [0, \infty) \times [0, \infty). \tag{100}$$

In this example, the y-axis is a curve of fixed points, and the jacobian matrix of S at the fixed point $(0, 1-a)$ is

$$\begin{pmatrix} 1 & 0 \\ (a-1) & 1 \end{pmatrix} = J^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} J, \quad \text{where } J = \begin{pmatrix} 0 & \frac{1}{a-1} \\ 1 & 0 \end{pmatrix},$$

so $(0, 1-a)$ is a 1-1 resonant fixed point of S. Translating $(0, 1-a)$ to the origin and conjugating by the similarly matrix yields the map in the form needed for

Corollary 10,

$$T(x,y) = \left(x + y - \frac{xy + y^2}{(a-1)+y}, y - \frac{xy}{1+x}\right). \tag{101}$$

Comparing equation (101) with equation (97), we have $\dfrac{xy}{1+x} = y\,\psi(x,y)$ so that

$$Q(x)x^\ell = \psi(x,0) = \frac{-x}{1+x} = -x + x^2 - x^3 + \ldots. \tag{102}$$

From equation (102), we see that $Q(0) = -1$ and $\ell = 1$. By Theorem 5, the dynamics of equation (100) are conjugate to the dynamics pictured in Figure 10 (a), which agrees with the phase portrait in Figure 8.

Example 2. Consider the planar map

$$(\tilde{x},\tilde{y}) = S(x,y) = \left(\frac{2(x+y)}{2+y}, \frac{4xy}{1+2x+x^2+y}\right). \tag{103}$$

For the map S in (103), the x-axis is a curve of fixed points, and the jacobian at the fixed point $(1,0)$ is

$$\begin{pmatrix} 1 & .5 \\ 0 & 1 \end{pmatrix} = J^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} J\,, \quad \text{where } J = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

so $(1,0)$ is a 1-1 resonant fixed point of S. Translating $(1,0)$ to the origin and conjugating by J gives the map

$$T(x,y) = \left(x + y + \frac{1 + y - xy - y^2}{1+y}, y - \frac{x^2y + 2y^2}{(x+2)^2 + 2y}\right). \tag{104}$$

Comparing equation (104) with equation (97), we have $-\frac{x^2y+2y^2}{(x+2)^2+2y} = y\,\psi(x,y)$ so that

$$Q(x)x^\ell = \psi(x,0) = -\frac{x^2}{(x+2)^2} = -\frac{x^2}{4} + \frac{x^3}{4} - \frac{3x^4}{16} + \ldots \tag{105}$$

Thus $Q(0) < 0$ and $\ell = 2$, so by Theorem 5 the local dynamical behavior of (103) is conjugate to the dynamics pictures in Figure 10 (c), see Figure 11.

Example 3.

Consider the following map, studied by Brett et al. [15]:

$$(\tilde{x},\tilde{y}) = S(x,y) = \left(\frac{ax}{a+y}, \frac{by}{Bx+y}\right) \quad (x,y) \in [0,\infty) \times [0,\infty)\,, \tag{106}$$

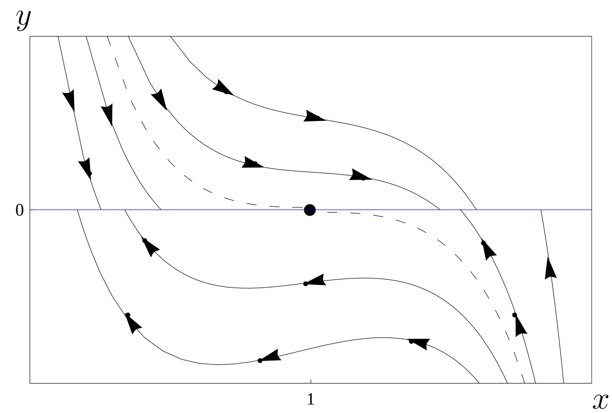

Figure 11. The phase portrait of system (103). The dynamical behavior of (103) is described by Theorem 5 part (c). Every point on the x-axis is a fixed point (blue line). The curves depict the path and direction followed by orbits under forward iteration of (103). The dashed line is the sector boundary curve $\mathcal{C}$.

where $a, b, B > 0$. In equation (106), the x-axis is a curve of fixed points, and the jacobian at the fixed point $(b/B, 0)$ is

$$\begin{pmatrix} 1 & -\frac{b}{aB} \\ 0 & 1 \end{pmatrix} = J^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} J\,, \quad \text{where } J = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{aB}{b} \end{pmatrix},$$

so $(b/B, 0)$ is a 1-1 resonant fixed point of S. Translating $(b/B, 0)$ to the origin and conjugating by J gives the map

$$T(x, y) = \left(x + y - \frac{Bxy + By^2}{-b + By}, y - \frac{By(bx - ay)}{-b^2 - bBx + aBy} \right). \tag{107}$$

Comparing equation (107) with equation (97), we have $-\frac{By(bx-ay)}{-b^2-bBx+aBy} = y\,\psi(x, y)$ so that

$$Q(x)x^\ell = \psi(x, 0) = -\frac{bBx}{-b(b + Bx)} = -\frac{B}{b}x + \frac{B^2}{b^2}x^2 - \frac{B^3}{b^3}x^3 + \dots. \tag{108}$$

Thus $Q(0) < 0$ and $\ell = 1$, so by Theorem 5 the local dynamical behavior of (106) is conjugate to the dynamics pictures in Figure 10 (a), see Figure 12.

3.4.2 Overview of the Proof of the Main Theorem

By Theorem 3 and Theorem 4, a map S with a 1-1 resonant non-isolated fixed point $(\bar{x}, \bar{y})$ has a normal form and a real analytic curve of fixed points near $(\bar{x}, \bar{y})$.

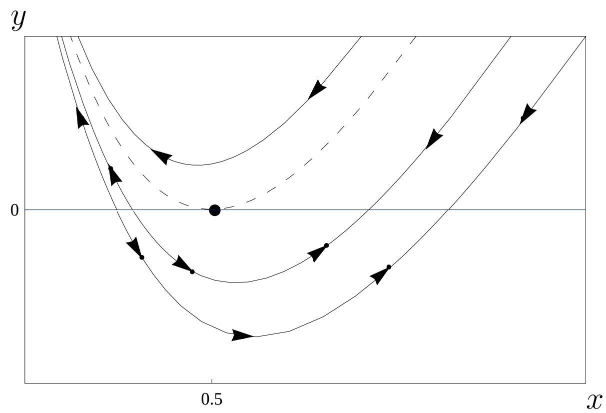

Figure 12. The phase portrait of system (106) with $a = 1$, $b = 1$, and $B = 2$. The dynamical behavior of (106) is described by Theorem 5 part (a). Every point on the x-axis is a fixed point (blue line). The fixed point $(b/B, 0) = (1/2, 0)$ is 1-1 resonant. The curves depict the path and direction followed by orbits under forward iteration of (106). The dashed line represents the sector boundary curve $\mathcal{C}$.

Thus to prove the Classification Theorem 5, it suffices to do so for maps that are in normal form.

From now on, T denotes a map in normal form defined in a neighborhood of the origin. The proof proceeds by first establishing local dynamical behavior in the right half plane (R.H.P.), and then in the left half plane (L.H.P.).

The case when $Q(0) > 0$ is considered first. Theorem 6 states that there is a T^{-1}-invariant curve in the first quadrant, one of whose endpoints is the origin. This curve is C^1 and tangential to x-axis at the origin. The curve separates first quadrant regions where T has different dynamical behavior. In addition, points in the fourth quadrant belong to unstable manifolds of fixed points on the axis.

The case $Q(0) < 0$ and in general behavior on the L.H.P. when $Q(0)$ is not zero can be obtained from the study of the relationship between normal forms of a map and both its inverse and conjugation with a reflection over the y-axis. This is the contents of Propositions 10 and 11.

Theorem 5 uses Propositions 10, 11, and Theorem 6 to give dynamical behavior in the left half plane. All the pieces are put together in Section 3.4.3, where the proof of the Main Classification Theorem 5 is completed, including details of the case when $Q = 0$.

The proofs of Theorems 6 and 7 are deferred until Sections 3.5 and 3.6 respectively.

Theorem 6. *Let T be a map in normal form (86). If $Q(0) > 0$, then there exists $\delta > 0$ and a C^1 curve $\mathcal{C}_1^- \subset \mathcal{B}_\delta \cap \mathcal{Q}_1$ with one endpoint at $(0,0)$ and the other one on the line $x = \delta$ such that*

i. *$\mathcal{C}_1^-$ is T^{-1}-invariant.*

ii. $\mathcal{C}_1^- = \left\{ (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_1 : \begin{array}{c} T^{-n}(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_1 \quad \forall n \geq 0, \text{ and} \\ \lim T^{-n}(x,y) = (0,0) \end{array} \right\}$

iii. *The set $\mathcal{B}_\delta \cap \mathcal{Q}_1 \setminus \mathcal{C}_1^-$ has two connected components, henceforth denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$, such that where $\mathcal{S}_1$ is a repelling parabolic sector of T relative to $(0,0)$ and $\mathcal{B}_\delta$, and for (x,y) in $\mathcal{S}_2$, both $T^n(x,y)$ and $T^{-n}(x,y)$ eventually leave $B_\delta \cap \mathcal{Q}_1$.*

iv. *Every nonzero point (x,y) in $\mathcal{B}_\delta \cap \mathcal{Q}_4$ belongs to the unstable manifold of a fixed point of T.*

Theorem 7. *Let T be a map in normal form (86). If $Q(0) > 0$ and ℓ is odd, or if $Q(0) < 0$ and ℓ is even, then there exists $\delta > 0$, a set $\mathcal{B}$ such that $(\mathcal{B}_\delta \cap \mathcal{Q}_2) \subset \mathcal{B} \subset \mathcal{Q}_2$, and a C^1 curve $\mathcal{C}_1^+ \subset \mathcal{B}$ with one endpoint at $(0,0)$ and the other one on the line $x = \delta$ such that*

i. *$\mathcal{C}_1^+$ is T-invariant.*

ii. $\mathcal{C}_1^+ = \left\{ (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 : \begin{array}{c} T^{n}(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 \quad \forall n \geq 0, \text{ and} \\ \lim T^{n}(x,y) = (0,0) \end{array} \right\}.$

iii. *The set* $\mathcal{B}_\delta \setminus \mathcal{C}_1^+$ *has two connected components, henceforth denoted by* $\mathcal{S}_1'$ *and* $\mathcal{S}_2'$*, where* $\mathcal{S}_1'$ *is a repelling parabolic sector of* T *relative to* $(0,0)$ *and* $\mathcal{B}_\delta$*, and for* (x,y) *in* $\mathcal{S}_2'$*, both* $T^n(x,y)$ *and* $T^{-n}(x,y)$ *eventually leave* $\mathcal{B}_\delta \cap \mathcal{Q}_2$*.*

iv. *Every nonzero point* (x,y) *in* $\mathcal{B}_\delta \cap \mathcal{Q}_3$ *belongs to the stable manifold of a fixed point of* T *on the negative* x *semi-axis.*

3.4.3 Proof of the Classification Theorem

For $\delta > 0$, denote with $\mathcal{B}_\delta$ the set $\{\ (x,y) : -\delta \leq x \leq \delta\ ,\ -\delta \leq x \leq \delta\ \}$. For $\ell = 1, \ldots, 4$ denote with $\mathcal{Q}_\ell$ the usual closed quadrants with respect to the origin. Also, let

$$\begin{aligned} \mathbb{X}_+ &:= \{(x,0) : x \geq 0\}, & \mathbb{X}_- &:= \{(x,0) : x \leq 0\}, \\ \mathbb{Y}_+ &:= \{(0,y) : y \geq 0\}, & \mathbb{Y}_- &:= \{(0,y) : y \leq 0\}. \end{aligned} \tag{109}$$

Given a map S, a set A is S-invariant if $S(A) \subset A$.

Let T be a map in normal form, defined on a neighborhood V of the origin. There are some properties of T that follow immediately from (86). In particular, every point in V of the form $(x,0)$ is a fixed point of T, and any sufficiently small neighborhood V of the origin has the property

$$(x,y) \in V \ \text{ and } \ (x',y') = T(x,y) \implies \begin{cases} x < x' \text{ and } y' > 0 \text{ whenever } y > 0, \text{ and} \\ x > x' \text{ and } y' < 0 \text{ whenever } y < 0. \end{cases} \tag{110}$$

For small enough $\delta > 0$, T is injective on $\mathcal{B}_\delta$.

Proposition 2. *If* $\delta > 0$ *is sufficiently small, for all* $(x,y),(z,w) \in \mathcal{B}_\delta$*,*

$$T(x,y) = T(z,w) \quad \textit{implies that} \quad (x,y) = (z,w).$$

Proof. For $\delta > 0$, if $(x,y),(z,w) \in \mathcal{B}_\delta$ are such that $T(x,y) = T(z,w)$, then $(x+y, y+g(x,y)) = (z+w, w+g(z,w))$. Let $t := x+y$. Thus $y = t-x$, $w = t-z$, and $t-x+g(x,t-x)=t-z+g(z,t-z)$. Therefore, $y+g(x,y)=w+g(z,w)$ is equivalent to

$$z-x+g(x,t-x)-g(z,t-z)=0\,. \tag{111}$$

Substituting $h(x):=g(x,t-x)$ by its expansion about z in (111),

$$z-x+g_{(1,0)}(z,t-z)(x-z)+g_{(0,1)}(z,t-z)(z-x)+(x-z)^2\,O(1)=0\,, \tag{112}$$

or,

$$(z-x)\,\big(1-g_{(1,0)}(z,t-z)+g_{(0,1)}(z,t-z)+|x-z|\,O(1)\,\big)=0\,. \tag{113}$$

Provided $\delta>0$ is sufficiently small, $1-g_{(1,0)}(z,t-z)+g_{(0,1)}(z,t-z)+|x-z|\,O(1)>0$, so it follows from this and from (113) that $z=x$ and therefore $w=y$. $\square$

Establishing additional dynamical behavior characteristics of T near the origin takes considerably more work. Next, we will combine Theorems 6 and 7 to see the complete dynamical picture. There are four non-conjugate dynamic scenarios; we will consider each one separately.

Case 1: $Q(0)>0$ and ℓ odd

If $Q(0)>0$ and ℓ is odd, then the behavior in the right- and left-half planes are described by Theorems 6 and 7 respectively. Let $\mathcal{C}_1:=\mathcal{C}_1^-\cup\mathcal{C}_1^+$. It is easy to see that $\mathcal{C}_1$ is a manifold since $\mathcal{C}_1^-$ and $\mathcal{C}_1^+$ both intersect the origin tangentially to the x-axis. We have immediately that $\mathcal{S}_1$ and $\mathcal{S}_1'$ are respectively repelling and attracting parabolic sectors with respect to the origin and $\mathcal{B}_\delta$.

We now show that a neighborhood of the origin lying in the lower-half plane is an elliptic sector. By Theorems 6 and 7, for $\delta'>0$ sufficiently small, $(0,y)\in\mathcal{B}_{\delta'}$ lies both in the stable manifold of a fixed point on the negative x semi-axis and the unstable manifold of a fixed point on the positive x semi-axis. By Proposition 2, we have that T is injective in $\mathcal{B}_{\delta'}$ for $\delta'>0$ sufficiently small. Thus it must be that the stable and unstable manifolds passing through the point $(0, y) \in \mathcal{B}_{\delta'}$ are equal.

Let $\mathcal{B}_e$ be the set of all stable and unstable manifolds intersecting the line segment from the origin to $(0, -\delta')$. Choose $\delta > 0$ such that $(\mathcal{B}_\delta \cap \mathbb{Y}_-) \subset \mathcal{B}_e$. Then for any $(x, y) \in \mathcal{B}_\delta$, we have that $\{T^n(x, y)\}$ and $\{T^{-n}(x, y)\}$ both converge to a point on the x-axis, and thus $\mathcal{B}_\delta \cap \mathbb{Y}_-$ is an elliptic sector with respect to the origin and $\mathcal{B}_\delta$.

Finally, we claim that if $\mathcal{S}_2$ and $\mathcal{S}_2'$ are defined as in Theorems 6 and 7, then $\mathcal{S}_2 \cup \mathcal{S}_2'$ is a hyperbolic sector. By Theorem 7, if $(x, y) \in \mathcal{S}_2$, then there exist indices $m, k \in \mathbb{N}$ such that $T^k(x, y) \notin \mathcal{B}_\delta \cap \mathcal{Q}_1$ and $T^{-m}(x, y) \notin \mathcal{B}_\delta \cap \mathcal{Q}_1$. By (110), $T^k(x, y) \notin \mathcal{Q}_2$. If $T^{-m}(x, y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2$, then Theorem 6 implies that there exists an index $m' \in \mathbb{N}$ such that $T^{-m'}(T^{-m}(x, y)) \notin \mathcal{B}_\delta \cap \mathcal{Q}_2$. By (110), $T^{-(m+m')}(x, y) \notin \mathcal{Q}_1$, thus $\mathcal{S}_1 \cup \mathcal{S}_1'$ is a hyperbolic sector with respect to the origin and $\mathcal{B}_\delta$, completing Case 1.

Case 2: $Q(0) < 0$ and ℓ odd

If $Q(0) < 0$ and ℓ is odd, then reflecting about the origin gives a system in which $Q(0) > 0$ and ℓ is odd, so this situation is conjugate to the first case.

Case 3: $Q(0) > 0$ and ℓ even

If $Q(0) > 0$ and ℓ is even, then the behavior in the right-half plane is described by Theorem 6. Further, reflecting about the origin gives a system in which $Q(0) > 0$ and ℓ is even, so the dynamical behavior in the left-half plane is equivalent to the dynamics of the right-half plane after a reflection about the origin. Let $\hat{\mathcal{C}}_1^-$ be the invariant manifold (lying in $\mathcal{Q}_3$ after reflection about the origin) guaranteed by Theorem 6 and let $\mathcal{C}_1 := \mathcal{C}_1^- \cup \hat{\mathcal{C}}_1^-$. Since the one-sided derivatives of both $\mathcal{C}_1^-$ and $\hat{\mathcal{C}}_1^-$ are zero at the origin, $\mathcal{C}_1$ is a manifold. We have immediately that $\mathcal{S}_1$ and its reflection about the origin two repelling parabolic sectors.

We wish to show the existence of two more repelling parabolic sectors. By Theorem 7, if $(x, y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2$, then (x, y) lies in the unstable manifold of a fixed point $(\bar{x}, 0)$ lying on the negative x semi-axis. We claim that these unstable manifolds can be extended to intersect the set $\mathcal{B}_\delta \cap \mathcal{Q}_1$. Consider an unstable manifold through the point $(0, \delta')$ for $\delta' > 0$. Since $(0, \delta') \in \mathcal{B}_{\delta'}$, then by Theorem 6, for $\delta' > 0$ sufficiently small, then $\{T^n(0, \delta')\}$ eventually leaves $\mathcal{B}_{\delta'} \cap \mathcal{Q}_1$. This implies that the unstable manifold through $(0, \delta')$ leaves $\mathcal{B}_{\delta'}$. Let $\mathcal{D}$ be the open region bounded by the unstable manifold, $\mathcal{B}_{\delta'}$, $\mathcal{C}_1^-$, and the negative x semi-axis. We can ensure with an appropriately small choice of $\delta' > 0$ that T is injective in $\mathcal{D}$. We claim that for any $(x, y) \in \mathcal{D}$, $T^{-n}(x, y) \in \mathcal{D}$. To prove the claim, consider any path from the origin to (x, y) that is contained completely within $\mathcal{D}$. Then if $T^{-n}(x, y) \notin \mathcal{D}$, there exists a point lying on the curve whose preimage under T lies on either $\mathcal{C}_1^-$, the invariant curve, or the x-axis. In any case, this contradicts the invariance of these sets under T^{-1}. Thus $\{T^{-n}(x, y)\}$ lies in $\mathcal{D}$ for all $n \geq 0$. By (110), $\{x_n\}$ is a monotone decreasing sequence that is bounded below, and thus converges. This implies that $\{T^{-n}(x, y)\}$ converges to a point lying in the negative x semi-axis. Then choosing $\delta > 0$ such that $\mathcal{B}_\delta \cap \mathcal{Q}_2 \cap \mathcal{S}_2 \subset \mathcal{D}$ shows that $\mathcal{B}_\delta \cap \mathcal{Q}_2 \cap \mathcal{S}_2$ is a repelling parabolic sector with respect to the origin and $\mathcal{B}_\delta$. A similar argument holds for the remaining sector.

Case 4: $Q(0) < 0$ **and** ℓ **odd**

The proof is similar to the proof of the case $Q(0) > 0$ and ℓ even, so we skip it.

Case 5: $Q = 0$

We wish to show that for any $\delta > 0$ sufficiently small, if $(x, y) \in \mathcal{B}_\delta \cap \mathbb{Y}_+$ or $\mathcal{B}_\delta \cap \mathbb{Y}_-$, then both $\{T^n(x, y)\}$ and $\{T^{-n}(x, y)\}$ eventually leave $\mathcal{B}_\delta \cap \mathbb{Y}_+$ or $\mathcal{B}_\delta \cap \mathbb{Y}_-$. Let $T^n(x, y) := (x_n, y_n)$. We will reach a contradiction by showing that an orbit remaining in $\mathcal{B}_\delta$ must converge to a point on the x-axis, and then show that no orbit can converge to the x-axis. Choose $\delta > 0$ such that

$$|R(x,y)y| < 1 \quad \text{for all } (x,y) \in \mathcal{B}_\delta. \tag{114}$$

Suppose that $(x_n, y_n) \in \mathcal{B}_\delta$ for all $n \geq 1$. Then by the definition of T,

$$|x_{n+1} - x_n| = |y_n| \quad \text{for all } n \geq 1. \tag{115}$$

By (110), $\{x_n\}$ is a monotone sequence that is bounded by $\pm\delta$, so $\{x_n\}$ converges, say to $\bar{x}$. Then equation (115) implies that $\{y_n\}$ converges to 0. Thus $\{(x_n, y_n)\} \in \mathcal{B}_\delta$ for all $n \geq 1$ implies that $\{(x_n, y_n)\}$ converges to a point $(\bar{x}, 0) \in \mathcal{B}_\delta$.

We now show that $\{(x_n, y_n)\}$ cannot converge to $(\bar{x}, 0)$ for $\{(x_n, y_n)\} \subset \mathcal{B}_\delta \cap \mathbb{Y}_+$ and $\bar{x} > 0$; the proof for the remaining combinations of $\bar{x} > 0$, $\bar{x} < 0$, $\mathbb{Y}_+$, and $\mathbb{Y}_-$ are similar. If $\{x_n\}$ converges to $\bar{x}$, then by (110), it must be that $x - \bar{x} < 0$. This along with equation (114) yields

$$0 > (x - \bar{x})y(1 + R(x,y)y) - y^2 = (x - \bar{x})g(x,y) - y^2, \tag{116}$$

and since $(x_n, y_n) \to (\bar{x}, 0)$ then by (110) we have additionally that
$0 > T(x,y) - \bar{x} = x - \bar{x} + y$, so

$$(x - \bar{x})g(x,y) - y^2 < 0.$$

That is,

$$\frac{T_2(x,y)}{T_1(x,y) - \bar{x}} = \frac{y + g(x,y)}{x - \bar{x} + y} < \frac{y}{x - \bar{x}}. \tag{117}$$

In light of inequalities (116) and (117), the slopes of the line segment between (x_n, y_n) and $(\bar{x}, 0)$ form a decreasing sequence of negative terms, and thus converge to a fixed negative constant m in the extended real line. Further we have

$$\begin{aligned}
\frac{y_{n+1}}{x_{n+1}-\bar{x}} &= \frac{y_n + g(x_n, y_n)}{x_n - \bar{x} + y_n} \\
&= \frac{\dfrac{y_n}{x_n-\bar{x}} + \dfrac{g(x_n,y_n)}{x_n-\bar{x}}}{1 + \dfrac{y_n}{x_n-\bar{x}}} \\
&= \frac{\dfrac{y_n}{x_n-\bar{x}} + \dfrac{y_n}{x_n-\bar{x}}R(x_n,y_n)y_n}{1 + \dfrac{y_n}{x_n-\bar{x}}}.
\end{aligned} \tag{118}$$

Set $m := \lim y_n/(x_n - \bar{x})$. Taking the limit of both sides of equation (118) as $n \to \infty$, since $R(x_n, y_n)y_n \to 0$ as $(x_n, y_n) \to (\bar{x}, 0)$ we have

$$m = \frac{m}{1+m}, \tag{119}$$

whose only solution on the extended real line is $m = 0$, contradicting the fact that $m < 0$. Thus $\{(x_n, y_n)\}$ cannot converge to $(\bar{x}, 0)$. A similar argument can be used for the remaining cases of $\{(x_n, y_n)\}$ in the upper- and lower-half planes and $(\bar{x}, 0)$ in the positive or negative x semi-axes. Thus $\{T^n(x, y)\}$ must eventually leave $\mathcal{B}_\delta$.

Applying Proposition 11 together with Corollary 12 and repeating the argument above with $\tilde{T}$ gives that $\{T^{-n}(x, y)\}$ must eventually leave $\mathcal{B}_\delta$, completing the case. □

3.5 Proof of Theorem 6

The proof of the Theorem 6 will be broken into two parts. In the first part, we will show the existence of an invariant curve using the *Hadamard graph transform*; see [8] for an introduction. Informally, the Hadamard graph transform $T_\#$ of a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be pictured as follows- consider the image of the graph of a function under T. If the image of the function can also be parametrized as a function of x, then let $T_\#(x)$ be the resulting function. If there exists a function that is a fixed point for $T_\#$, then the function must be an invariant function of T.

The normal course of action is to find a Banach space of functions over which $T_{\#}$ is a contraction and use the Banach contraction principle to conclude that the space has a unique fixed point under iteration of $T_{\#}$; see [8] Section 5 for an example.

Proofs like those in [8] rely on one eigenvalue of the jacobian at the fixed point being off the unit circle in order to find a metric for which $T_{\#}$ is a contraction; we were not able to use the same method. We shall find a space $\mathcal{R}_\delta$ that has the following monotonicity property : *If graph*(ψ), *graph*$(\phi) \subset \mathcal{R}_\delta$, *then* $\psi(x) < \phi(x)$, *implies that* $T_{\#}(\psi(x)) < T_{\#}(\phi(x))$. This monotonicity property can be used to find a sequence $\{\tau_n(x)\}$ generated by iteration of $T_{\#}$ which is monotone and bounded, and thus converges pointwise to a function $\tau(x)$ whose graph we later show is invariant under $T_{\#}$.

In order to show that $\tau(x)$ is continuously differentiable, we find bounds on $\{\tau_n'(x)\}$ and $\{\tau_n''(x)\}$ and apply the Arzèla-Ascoli Theorem (see section 7.5 in [17]) to find a uniformly convergent subsequence $\{\tau_{n_k}'(x)\}$ of $\{\tau_n'(x)\}$. We then apply Theorem B which implies that $\tau(x)$ is continously differentiable. Finally, setting $\mathcal{C}_1^- := \cup_{x\in[0,\delta]}(x, \tau(x))$ completes the proof of the existence of an invariant curve. To begin the proof of Theorem 4, assume $Q(0) > 0$, and choose arbitrary real numbers α, β that satisfy

$$0 \ < \ \frac{Q(0)}{\ell+2} \ < \ \alpha \ < \ \frac{Q(0)}{\ell+1} \ < \ \beta \tag{120}$$

We will need the following definitions:

Definition 5. *For a function* $\phi : A \to \mathbb{R}$ *where* A *is a set of real numbers, the* graph of ϕ *is the set graph*$(\phi) = \{(x, \phi(x)) : x \in A\}$, *and the* graph function of ϕ *is the function* $\Gamma(\phi)$ *given by* $\Gamma(\phi)(x) = (x, \phi(x))$ *for* $x \in A$.

For $\delta > 0$ fixed, define the sets

$$\mathcal{R}_\delta \quad := \quad \{ \ (x,y): \ 0 \le x \le \delta, \quad \alpha\, x^{\ell+1} \le y \le \beta\, x^{\ell+1} \ \} \tag{121}$$

$$\mathcal{W}_\delta \quad := \quad \{ \ \phi \in \mathcal{C}([0,\delta]) \ : \ \phi \text{ is nondecreasing, and } graph(\phi) \subset \mathcal{R}_\delta \ \} \tag{122}$$

$$\mathcal{W}'_\delta \quad := \quad \left\{ \phi \in \mathcal{C}^2([0,\delta]) \ : \ \begin{array}{c} (\ell+1)\ \alpha\ x^\ell \le \phi'(x) \le (\ell+1)\ \beta\ x^\ell, \\ \text{and } 0 \le \phi''(x) \text{ for all } x \in [0,\delta] \end{array} \right\}. \tag{123}$$

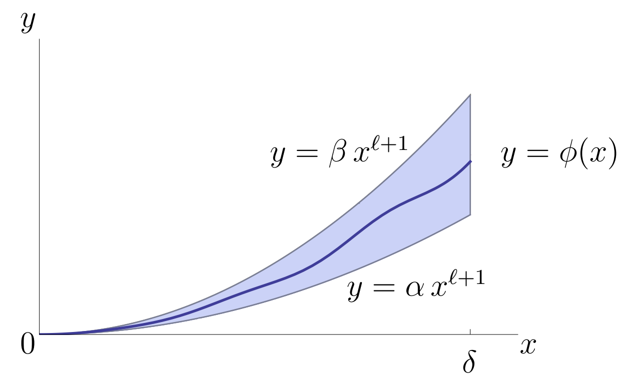

Figure 13. The graph of any $\phi \in \mathcal{W}_\delta$ must be nondecreasing and lie entirely in $\mathcal{R}_\delta$ (shaded region) for $x \in [0,\delta]$.

Definition 6. *The* graph transform *of T is the function $T_\# : \mathbb{R} \to \mathbb{R}$ given by*

$$T_\#(\phi)(x) = T_2 \circ \Gamma(\phi) \circ (T_1 \circ \Gamma(\phi))^{-1}(x), \quad \phi \in \mathcal{W}_\delta. \tag{124}$$

Proposition 4 part 1) will show that $(T_1 \circ \Gamma(\phi))^{-1}(x)$ exists and is well defined for $\phi(x) \in \mathcal{W}_\delta$.

3.5.1 Properties of $\mathcal{R}_\delta$

In this subsection, we will show that $\mathcal{R}_\delta$ possesses useful properties with respect to $T_\#$ and the parametrically defined derivative of $T_\#$.

Define the *north-east* partial ordering $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{ne} (z,w)$ if and only if $x \le z$ and $w \le y$, and the *south-east* partial ordering $\preceq_{se}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{se} (z,w)$ if and only if $x \le z$ and $y \le w$.

Proposition 3 (Properties of the restriction of T to $\mathcal{R}_\delta$)**.** *For all $\delta > 0$ sufficiently small, the following statements are true.*

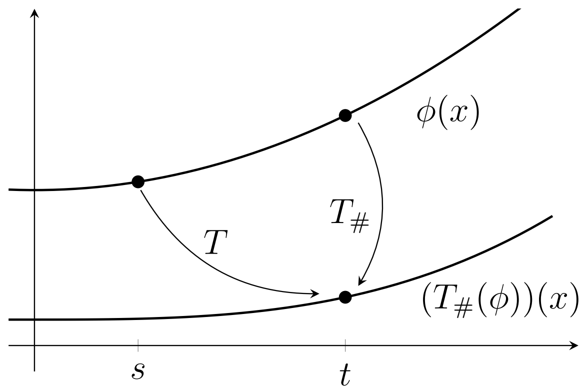

Figure 14. The graph transform $T_\#$ of T. The lower curve is the image of $\phi(x)$ under T, and $T_\#$ is the parametrization of the lower curve as a function of x. Here, $s = (T_1 \circ \Gamma(\phi))^{-1}(t)$.

1. $\forall_{(x,y),(z,w)\in\mathcal{R}_\delta}\ (\ (x,y) \preceq_{ne} (z,w) \implies T(x,y) \preceq_{ne} T(z,w)\ ).$

2. $\forall_{(x,y),(z,w)\in\mathcal{R}_\delta}\ (\ T(x,y) \preceq_{se} T(z,w) \implies (x,y) \preceq_{se} (z,w)\ ).$

Proof (1): Suppose (x,y) is in $\mathcal{R}_\delta \setminus \{(0,0)\}$. Since $\alpha\, x^{\ell+1} \le y \le \beta\, x^{\ell+1}$, then for some functions O_1 and O_2 bounded on $\mathcal{R}_\delta$,

$$\begin{aligned} g_{(1,0)}(x,y) &= (Q'(x)\, x^\ell + Q(x)\, \ell\, x^{\ell-1})\, y + R_1(x,y)\, y^2 \\ &\ge\ Q(0)\, \alpha\, x^{2\ell} + x^{2\ell+1}\, O_1(1), \text{ and} \\ g_{(0,1)}(x,y) &= Q(x)\, x^\ell + R_2(x,y)\, y^2 + R(x,y)\, 2\, y \ge Q(0)\, x^\ell + x^{\ell+1}\, O_2(1). \end{aligned} \tag{125}$$

Then $g_{(1,0)}(x,y) > 0$ and $1 + g_{(0,1)}(x,y) > 0$, provided $\delta > 0$ is sufficiently small. Also, we have for $\delta > 0$ sufficiently small that

$$-\, g_{(1,0)}(x,y) + g_{(0,1)}(x,y) > 0 \quad \text{for all } (x,y) \in \mathcal{R}_\delta. \tag{126}$$

For such δ, the jacobian matrix of T at (x,y),

$$J_T(x,y) = \begin{pmatrix} 1 & 1 \\ g_{(1,0)}(x,y) & 1 + g_{(0,1)}(x,y) \end{pmatrix},$$

has positive entries for all (x,y) in $\mathcal{R}_\delta \setminus \{(0,0)\}$. Then the components of T are increasing in each variable.

Proof (2): For $(x, y) \in \mathcal{R}_\delta \setminus \{(0,0)\}$, $\det(J_T(x,y)) = 1 + g_2(x,y) - g_1(x,y)$ is positive, given that δ is sufficiently small. By [18], T is orientation preserving. $\square$

Proposition 4 (Properties of $T_\#$ in $\mathcal{R}_\delta$)**.** *For all $\delta > 0$ sufficiently small, the following statements are true.*

1. *$T_\#$ exists and is well-defined for any $\phi \in \mathcal{W}_\delta$.*

2. *For all ϕ, $\psi \in \mathcal{W}_\delta$, $\phi(t) < \psi(t)$ for $0 < t \leq \delta$*
$$\Longrightarrow \; T_\#(\phi)(t) < T_\#(\psi)(t) \text{ for } 0 < t \leq \delta.$$

3. *For all ϕ, $\psi \in \mathcal{W}_\delta$, $\phi(t) \leq \psi(t)$ for $0 < t \leq \delta$*
$$\Longrightarrow \; T_\#(\phi)(t) \leq T_\#(\psi)(t) \text{ for } 0 < t \leq \delta.$$

4. *If $\phi \in \mathcal{W}_\delta$, then $T_\#(\phi) \in \mathcal{W}_\delta$.*

5. *For every ψ, ϕ in $\mathcal{W}_\delta$ with $\phi(x) \leq \psi(x)$ whenever $x \in (0, \delta]$, the relations*
$$s, t \in (0, \delta]\,, \;\; s + \psi(s) = t\,, \quad \text{and } \; \phi(s) < \psi(s) \tag{127}$$
imply
$$0 \; < \; \frac{T_\#(\psi)(t) - T_\#(\phi)(t)}{t^{\ell+2}} \; < \; \frac{\psi(s) - \phi(s)}{s^{\ell+2}}\,, \tag{128}$$
while if $\phi(s) = \psi(s)$ is assumed in addition to (127), then $T_\#(\psi)(t) = T_\#(\phi)(t)$.

Proof (1): $T_\#$ exists and is well defined if and only if $(T_1 \circ \Gamma(\phi))^{-1}(t)$ exists and is well defined. Notice that
$$(T_1 \circ \Gamma(\phi))^{-1}(t) = s \qquad \Longleftrightarrow \qquad s + \phi(s) = t. \tag{129}$$

The function $I(s) := s + \phi(s)$ is continuous and increasing on $[0, \delta]$,
$$I(0) = 0 + \phi(0) = 0, \qquad \text{and} \qquad I(t) = t + \phi(t) > t. \tag{130}$$

By the Intermediate Value Theorem and equation (130), there must exist $s \in [0, t]$ such that $I(s) = s + \phi(s) = t$. Since $I(s)$ is increasing on $[0, \delta]$, the point s is unique in this interval, concluding the proof.

Proof (2): Suppose $\phi, \psi \in \mathcal{W}_\delta$ are such that $\phi(t) < \psi(t)$ for $0 < t \leq \delta$. Our first observation is that $T_\#(\phi)(t) \neq T_\#(\psi)(t)$ for $t \in (0, \delta]$. Indeed, if $t > 0$ and $(x, y) \in graph(\phi)$ and and $(z, w) \in graph(\psi)$ are such that $T(x, y) = (t, T_\#(\phi)(t))$ and $T(z, w) = (t, T_\#(\psi)(t))$, then $T(x, y) \neq T(z, w)$ by (3) of Proposition 1. Also, note that if $(x, y) \in graph(\phi)$ and $(z, w) \in graph(\psi)$ are comparable in the south-east ordering $\preceq_{se}$, then either

$$(x, y) \preceq_{se} (z, w) \quad \text{or} \quad (z, w) \preceq_{se} (x, y). \tag{131}$$

The former case implies that $x \leq z$ and $w \leq y$. By the monotonicity of ϕ, we have

$$y = \phi(x) \leq \phi(z) < \psi(z) = w, \tag{132}$$

contradicting $w \leq y$. Thus it must be the case that

$$(z, w) \preceq_{se} (x, y)\,. \tag{133}$$

We now prove that $T_\#(\phi)(t) > T_\#(\psi)(t)$ is not possible for $t \in (0, \delta]$. Arguing by contradiction, assume $t \in (0, \delta]$ is such that $T_\#(\phi)(t) > T_\#(\psi)(t)$, so in particular $(t, T_\#(\phi)(t)) \preceq_{se} (t, T_\#(\psi)(t))$. Set $(x, y) := T^{-1}(t, T_\#(\phi)(t))$ and $(z, w) := T^{-1}(t, T_\#(\psi)(t))$. Then (x, y) and (z, w) are elements of $graph(\phi)$ and $graph(\psi)$ respectively such that $T(x, y) \preceq_{se} T(z, w)$, and by (2) of Proposition 1, $(x, y) \preceq_{se} (z, w)$. The latter relation and (133) imply $(x, y) = (z, w)$, which is a contradiction. This proves (1).

The proof of (3) is similar to the proof of (2) so we skip it.

Proof (4): Let $\phi \in \mathcal{W}_\delta$. We show first that $T_\#(\phi)(\cdot)$ is nondecreasing on $[0, \delta]$. Let $t_1, t_2 \in [0, \delta]$ with $t_1 < t_2$. Find $s_1, s_2 \in [0, \delta]$ such that $s_1 + \phi(s_1) = t_1$,

$s_2 + \phi(s_2) = t_2$. Then necessarily $s_1 < s_2$ and $\phi(s_1) \leq \phi(s_2)$. The partial derivatives of $T_2(x,y) = y + g(x,y)$ are positive on $\mathcal{R}_\delta \setminus \{(0,0)\}$. Thus $T_2(s_1, \phi(s_1)) < T_2(s_2, \phi(s_2))$, that is, $T_\#(\phi)(t_1) < T_\#(\phi)(t_2)$. We conclude that $T_\#(\phi)(\cdot)$ is nondecreasing on $[0,\delta]$.

To prove $T_\#(\phi)(\cdot)$ is continuous on $[0,\delta]$, let $\epsilon > 0$ and $t, \tilde{t} \in [0,\delta]$ be given. By continuity, there exists $\eta > 0$ such that if $(\tilde{s}, \phi(\tilde{s})) \in B((s,\phi(s)),\eta)$, then $T(\tilde{s},\phi(\tilde{s})) \in B(T(s,\phi(s)),\epsilon)$. By continuity and the injectivity of the function $(x,y) \to x+y$, the image of the set $\{(\tilde{s},\phi(\tilde{s})) : s \in [0,\delta] \cap B((s,\phi(s)),\eta)\}$ is a neighborhood V of $t = s + \phi(s)$ in $\mathbb{R}$. Choose $\rho > 0$ such that $(s-\rho, s+\rho) \subset V$. Thus for every $\tilde{t} \in (s-\rho, s+\rho)$, the element $\tilde{s}$ defined by the equation $\tilde{s} + \phi(\tilde{s}) = \tilde{t}$ satisfies $T(\tilde{s},\phi(\tilde{s})) \in B(T(s,\phi(s)),\epsilon)$. Since $T(\tilde{s},\phi(\tilde{s})) = (\tilde{s}, T_\#(\phi)(\tilde{t}))$, it follows that $|T_\#(\phi)(\tilde{t}) - T_\#(\phi)(t)| < \epsilon$. This completes the proof of continuity.

Let $\sigma_0, \tau_0 \in \mathcal{W}_\delta$ be given by $\sigma_0(t) = \alpha\, t^{\ell+1}$ and $\tau_0(t) = \beta\, t^{\ell+1}$ for $0 \leq t \leq \delta$. We claim that $T_\#(\sigma_0)$ and $T_\#(\tau_0)$ are both in $\mathcal{W}_\delta$. Indeed, $\alpha\, t^{\ell+1} \leq T_\#(\sigma_0)(t) \leq \beta\, t^{\ell+1}$ if and only if, for s chosen so that $s + \sigma_0(s) = t$,

$$\alpha\,(s + \sigma_0(s))^{\ell+1} \leq \sigma_0(s) + g(s,\sigma_0(s)) \leq \beta\,(s+\sigma_0(s))^{\ell+1}. \tag{134}$$

That is,

$$\alpha\,(s + a\, s^{\ell+1})^{\ell+1} \leq \alpha\, s^{\ell+1} + g(s, \alpha\, s^{\ell+1}) \leq \beta\,(s + \alpha\, s^{\ell+1})^{\ell+1}. \tag{135}$$

Since $g(s,\alpha\, s^{\ell+1}) = \alpha\, s^{\ell+1} + Q(0)\, s^{2\,\ell+1} + s^{2\ell+2}\, O(1)$, we have from this and (135) that

$$\begin{aligned}\alpha\, s^{\ell+1} &+ (\ell+1)\,\alpha^2\, s^{2\ell+1} + s^{2\ell+2}\, O_1(1)\\ &\leq \alpha\, s^{\ell+1} + \alpha\, Q(0)\, s^{2\ell+1} + s^{2\ell+2}\, O(1)\\ &\qquad \leq \beta\, s^{\ell+1} + \alpha\,\beta\,(\ell+1)\, s^{2\ell+1} + s^{2\ell+2}\, O_2(1).\end{aligned} \tag{136}$$

Inequalities in (136) give

$$\begin{aligned}\alpha &+ (\ell+1)\,\alpha^2\, s^{\ell} + s^{\ell+1}\, O_1(1)\\ &\leq \alpha + \alpha\, Q(0)\, s^{\ell} + s^{\ell+1}\, O(1)\\ &\qquad \leq \beta + \alpha\,\beta\,(\ell+1)\, s^{\ell} + s^{\ell+1}\, O_2(1).\end{aligned} \tag{137}$$

Simplification of (137) yields

$$\begin{aligned}(\ell+1)\,\alpha\, s^{\ell} + s^{\ell+1}\, O_1(1)&\\ \le Q(0)\, s^{\ell} + s^{\ell+1}\, O(1)&\\ \le \tfrac{b-\alpha}{\alpha} + \beta\,(\ell+1)\, s^{\ell} + s^{\ell+1}\, O_2(1).&\end{aligned} \tag{138}$$

Since $(\ell+1)\,\alpha < Q(0)$, (138) holds for all sufficiently small δ and all $s \in [0, \delta]$. Thus $T_{\#}(\sigma_0) \in \mathcal{W}_\delta$ for such δ. An analogous argument gives that $T_{\#}(\tau_0) \in \mathcal{W}_\delta$ for all sufficiently small δ.

If now $\phi \in \mathcal{W}_\delta$, then $\sigma_0(t) \le \phi(t) \le \tau_0(t)$. By the previous claim and by statement (2),

$$\sigma_0(t) \le T_{\#}(\sigma_0)(t) \le T_{\#}(\phi)(t) \le T_{\#}(\tau_0)(t) \le \tau_0(t). \tag{139}$$

Thus $T_{\#}(\phi) \in \mathcal{W}_\delta$. This completes the proof of (4).

Proof (5): From the definition of $T_{\#}$, $T_{\#}(\phi)(t) = \phi(s) + g(s, \phi(s))$. Let $\tilde{s} \in (0, \delta]$ be such that $T_{\#}(\psi)(t) = \psi(\tilde{s}) + g(\tilde{s}, \psi(\tilde{s}))$. Then since $\psi(\tilde{s}) \ge \phi(s)$ and $s - \tilde{s} = \psi(\tilde{s}) - \phi(s)$, we have that $s - \tilde{s} \ge 0$. Expanding g about $(s, \phi(s))$,

$$\begin{aligned}T_{\#}(\psi)(t) - T_{\#}(\phi(t) &= \psi(\tilde{s}) + g(\tilde{s}, \psi(\tilde{s}) - \phi(s) - g(s, \phi(s))\\ &= (s-\tilde{s}) + g_{(0,1)}(s, \phi(s))(s-\tilde{s}) - g_{(1,0)}(s, \phi(s))(s-\tilde{s}) + (s-\tilde{s})^2\, O(1)\\ &= (s-\tilde{s})\left(1 + g_{(0,1)}(s, \phi(s)) - g_{(1,0)}(s, \phi(s)) + (s-\tilde{s})\, O(1)\right).\end{aligned} \tag{140}$$

By (126) and (140), we have for $\delta > 0$ sufficiently small that

$$T_{\#}(\psi)(t) - T_{\#}(\phi(t) = (s-\tilde{s})\left(1 + g_{(0,1)}(s, \phi(s)) - g_{(1,0)}(s, \phi(s)) + (s-\tilde{s})\, O(1)\right) > 0 \tag{141}$$

which completes the proof of the first inequality in (127).

Now proving the second inequality, and again expanding g about $(s, \phi(s))$, the relations $s + \phi(s) = t$ and $\psi(\tilde{s}) \ge \psi(s)$ imply that the second inequality in (128) is equivalent to

$$
\begin{aligned}
&s^{\ell+2}\,(T_{\#}(\psi)(t) - T_{\#}(\phi)(t)) - t^{\ell+2}\,(\psi(s) - \phi(s)) \\
&= s^{\ell+2}\,(\psi(\tilde{s}) + g(\tilde{s}, \psi(\tilde{s})) - \phi(s) - g(s, \phi(s))) - (s + \phi(s))^{\ell+2}\,(\psi(s) - \phi(s)) \\
&\leq s^{\ell+2}\,\big(\psi(\tilde{s}) - \phi(s) + g_{(1,0)}(s, \phi(s))(\tilde{s} - s) + g_{(0,1)}(s, \phi(s))(\psi(\tilde{s}) - \phi(s))\big) \\
&\qquad + O(1)\,s^{\ell+2}\,(|\tilde{s} - s|^2 + |\psi(\tilde{s}) - \phi(s)|^2) - (s + \phi(s))^{\ell+2}\,(\psi(\tilde{s}) - \phi(s)).
\end{aligned}
\tag{142}
$$

Substituting $\psi(\tilde{s}) - \phi(s) = s - \tilde{s}$ into (142), using $\alpha\, s^{\ell+1} \leq \phi(s)$, and rearranging,

$$
\begin{aligned}
&s^{\ell+2}\,(T_{\#}(\psi)(t) - T_{\#}(\phi)(t)) - t^{\ell+2}\,(\psi(s) - \phi(s)) \\
&\quad\leq s^{\ell+2}\,(s - \tilde{s})\left(1 - g_{(1,0)}(s, \phi(s)) + g_{(0,1)}(s, \phi(s)) + 2O(1)\,(s - \tilde{s}) - (1 + \tfrac{\phi(s)}{s})^{\ell+2}\right) \\
&\quad\leq s^{\ell+2}\,(s - \tilde{s})\left(-g_{(1,0)}(s, \phi(s)) + g_{(0,1)}(s, \phi(s)) + 2O(1)\,(s - \tilde{s}) - (\ell + 2)\,\tfrac{\phi(s)}{s}\right) \\
&\quad\leq s^{\ell+2}\,(s - \tilde{s})\left(-g_{(1,0)}(s, \phi(s)) + g_{(0,1)}(s, \phi(s)) + 2O(1)\,(s - \tilde{s}) - (\ell + 2)\,\alpha\, s^{\ell}\right).
\end{aligned}
\tag{143}
$$

From (143) follows that to have (128), it is sufficient to have $h(s, \tilde{s}) < 0$, where

$$
h(s, \tilde{s}) := -g_{(1,0)}(s, \phi(s)) + g_{(0,1)}(s, \phi(s)) + 2O(1)\,(s - \tilde{s}) - (\ell + 2)\,\alpha\, s^{\ell}.
$$

Note that for $s, \tilde{s}$ such that $0 < \tilde{s} < s$ with s small enough, for some functions O_1 and O_2 bounded in $\mathcal{R}_\delta$ that

$$
-\,g_{(1,0)}(s, \phi(s)) + g_{(0,1)}(s, \phi(s)) = Q(0)\,s^{\ell} + s^{\ell+1}\,O_1(1) \tag{144}
$$

$$
c\,(s - \tilde{s}) \;=\; c\,(\psi(\tilde{s}) - \phi(s)) = s^{\ell+1}\,O_2(1). \tag{145}
$$

Relations (144), (145) and the definition of h imply for some function O_3 bounded in $\mathcal{R}_\delta$,

$$
h(s, \tilde{s}) = (\,Q(0) - (\ell + 2)\,\alpha\,)\,s^{\ell} + s^{\ell+1}\,O_3(1). \tag{146}
$$

Since $Q(0) < (\ell+2)\,\alpha$, it follows that $h(s, \tilde{s}) < 0$ for all sufficiently small $\delta > 0$ and all $s, \tilde{s} \in (0, \delta]$. In this case the right-hand side term of (143) is negative. That is, the second inequality in (128) holds. $\square$

In order to establish the differentiability of $\mathcal{C}_1^-$, it will be useful to have an expression for the derivative of an image of a function under the graph transformation of T. Define $T'_{\#}(\phi)(t)$ and $T''_{\#}(\phi)(t)$ to be respectively the first and second parametric derivatives of T evaluated at $(t, \phi(t))$. That is, for $s + \phi(s) = t$,

$$\begin{aligned} T'_{\#}(\phi)(t) &:= \left(\frac{d}{dt} T(t, \phi(t)) \right)\Big|_{t=s} \\ &= \frac{\phi'(s) + g_{(1,0)}(s, \phi(s)) + \phi'(s) g_{(0,1)}(s, \phi(s))}{1 + \phi'(s)} \end{aligned} \tag{147}$$

and

$$\begin{aligned} &T''_{\#}(\phi)(t) := \left(\frac{d^2}{dt^2} T(t, \phi(t)) \right)\Big|_{t=s} \\ &= \frac{\phi''(s) + g_{(2,0)}(s, \phi(s)) + 2\phi'(s) g_{(1,1)}(s, \phi(s)) + \phi''(s) g_{(0,1)}(s, \phi(s)) + (\phi'(s))^2 g_{(0,2)}(s, \phi(s))}{(1 + \phi'(s))^3} \\ &\qquad - \frac{\phi''(s)(\phi'(s) + g_{(1,0)}(s, \phi(s)) + \phi'(s) g_{(0,1)}(s, \phi(s))}{(1 + \phi'(s))^3}. \end{aligned} \tag{148}$$

We will later use $T'_{\#}$ and $T''_{\#}$ to show that a subsequence of the sequence $\{\tau'_n(t)\}$ of derivatives of $\{\tau_n(t)\}$ converges uniformly as $n \to \infty$. Let us first establish some monotonicity properties for $T'_{\#}$:

Proposition 5. (Properties of $T'_{\#}$) *For all $\delta > 0$ sufficiently small, the following statements are true.*

1. *For all $\sigma', \tau' \in \mathcal{W}'_\delta$ and $\sigma, \tau \in \mathcal{W}_\delta$,*
$$\left\{ \begin{array}{l} \sigma(t) \leq \tau(t), \\ \sigma'(t) < \tau'(t) \end{array} \right. \quad \textit{for} \quad 0 < t \leq \delta \implies T'_{\#}(\sigma)(t) < T'_{\#}(\tau)(t).$$

2. *For all $\sigma', \tau' \in \mathcal{W}'_\delta$ and $\sigma, \tau \in \mathcal{W}_\delta$,*
$$\left\{ \begin{array}{l} \sigma(t) \leq \tau(t), \\ \sigma'(t) \leq \tau'(t) \end{array} \right. \quad \textit{for} \quad 0 < t \leq \delta \implies T'_{\#}(\sigma)(t) \leq T'_{\#}(\tau)(t).$$

3. *If $\sigma \in \mathcal{W}_\delta$ and $\sigma' \in \mathcal{W}'_\delta$, then $T'_{\#}(\sigma)(t) \in \mathcal{W}'_\delta$.*

Proof (1): Fix $t \in [0, \delta]$, and choose s_1, s_2 such that $s_1 + \sigma(s_1) = s_2 + \tau(s_2) = t$. Since $\sigma(x) \leq \tau(x)$, we have that

$$s_1 + \sigma(s_1) = s_2 + \tau(s_2) \geq s_2 + \sigma(s_2).$$

The function $t + \sigma(t)$ is nondecreasing, so it must be that $s_2 \leq s_1$. Since $\phi''(t) > 0$ for $0 \leq t \leq \delta$, we also have

$$\sigma'(s_1) < \tau'(s_2).$$

We then have

$$\begin{aligned} T'_{\#}(\tau)(t) - T'_{\#}(\sigma)(t) = & \frac{\tau'(s_2) + g_{(1,0)}(s_2, \tau(s_2)) + \tau'(s_2) \cdot g_{(0,1)}(s_2, \tau(s_2))}{1 + \tau'(s_2)} \\ & - \frac{\sigma'(s_1) + g_{(1,0)}(s_1, \sigma(s_1)) + \sigma'(s_1) \cdot g_{(0,1)}(s_1, \sigma(s_1))}{1 + \sigma'(s_1)}. \end{aligned} \tag{149}$$

After combining the fractions in equation (149), the resulting numerator is of the form $\tilde{h}(s_1, s_2)$, where

$$\begin{aligned} \tilde{h}(s_1, s_2) := & [\tau'(s_2) - \sigma'(s_1)] + [g_{(1,0)}(s_2, \tau(s_2))] - g_{(1,0)}(s_1, \sigma(s_1))] + [\sigma'(s_1) g_{(1,0)}(s_2, \tau(s_2)) \\ & - \tau'(s_2) g_{(1,0)}(s_1, \sigma(s_1))] + [\tau'(s_2) g_{(0,1)}(s_2, \tau(s_2)) - \sigma'(s_1) g_{(0,1)}(s_1, \sigma(s_1))] \\ & + [\sigma'(s_1)\tau'(s_2)(g_{(0,1)}(s_2, \tau(s_2)) - g_{(0,1)}(s_1, \sigma(s_1)))] \\ & := h_1(s_1, s_2) + h_2(s_1, s_2) + h_3(s_1, s_2) + h_4(s_1, s_2) + h_5(s_1, s_2). \end{aligned} \tag{150}$$

It is unclear how large $\tau'(s_2) - \sigma'(s_1)$ is in comparison to the other terms of $\tilde{h}(s_1, s_2)$, so we wish to consider this term separately. Define

$$\epsilon := \tau'(s_2) - \sigma'(s_1) > 0. \tag{151}$$

Now, let us investigate the asymptotic behavior of the terms h_i of $\tilde{h}(s_1, s_2)$, where O is a different bounded function on $\mathcal{R}_\delta$ in each equality:

$$h_1(s_1, s_2) = \tau'(s_2) - \sigma'(s_1) = \epsilon \tag{152}$$

$$\begin{aligned}
h_2(s_1,s_2) &= g_{(1,0)}(s_2,\tau(s_2)) - g_{(1,0)}(s_1,\sigma(s_1)) \\
&= (s_2-s_1)(g_{(2,0)}(s_1,\sigma(s_1)) - g_{(1,1)}(s_1,\sigma(s_1)) + O(|s_1-s_2|^2)) \\
&= s_1^{\ell-1}(-\ell Q(0) + s_1 O(1))
\end{aligned} \tag{153}$$

$$\begin{aligned}
h_3(s_1,s_2) &= \sigma'(s_1) g_{(1,0)}(s_2,\tau(s_2)) - \tau'(s_2) g_{(1,0)}(s_1,\sigma(s_1)) \\
&= \sigma'(s_1)(g_{(1,0)}(s_2,\tau(s_2)) - g_{(1,0)}(s_1,\sigma(s_1))) - \epsilon g_{(1,0)}(s_1,\sigma(s_1)) \\
&= \sigma'(s_1)(s_2-s_1)(g_{(2,0)}(s_1,\phi_1(s_1)) - g_{(1,1)}(s_1,\sigma(s_1)) \\
&\qquad\qquad + O(|s_1-s_2|^2)) - \epsilon g_{(1,0)}(s_1,\sigma(s_1)) \\
&= (s_2-s_1)s_1^{2\ell-1}(-\ell Q(0) + s_1 O(1)) - \epsilon g_{(1,0)}(s_1,\sigma(s_1))
\end{aligned} \tag{154}$$

$$\begin{aligned}
h_4(s_1,s_2) &= \tau'(s_2) g_{(0,1)}(s_2,\tau(s_2)) - \sigma'(s_1) g_{(0,1)}(s_1,\sigma(s_1)) \\
&= \sigma'(s_1)(g_{(0,1)}(s_2,\tau(s_2)) - g_{(0,1)}(s_1,\sigma(s_1))) + \epsilon g_{(0,1)}(s_1,\sigma(s_1)) \\
&= \sigma'(s_1)(s_2-s_1)(g_{(1,1)}(s_1,\sigma(s_1)) - g_{(0,2)}(s_1,\sigma(s_1)) \\
&\qquad\qquad + O(|s_1-s_2|^2)) + \epsilon g_{(0,1)}(s_1,\sigma(s_1)) \\
&= (s_2-s_1)s_1^{\ell} O(1) + \epsilon g_{(0,1)}(s_1,\sigma(s_1))
\end{aligned} \tag{155}$$

$$\begin{aligned}
h_5(s_1,s_2) &= \sigma'(s_1)\tau'(s_2)(g_{(0,1)}(s_2,\tau(s_2)) - g_{(0,1)}(s_1,\sigma(s_1))) \\
&= \sigma'(s_1)\tau'(s_2)(s_2-s_1)(g_{(1,1)}(s_1,\sigma(s_1)) - g_{(0,2)}(s_1,\sigma(s_1)) + O(|s_1-s_2|^2)) \\
&= (s_2-s_1)s_1^{2\ell} O(1)
\end{aligned} \tag{156}$$

Combining equations (152) - (156) yields, for some O bounded on $\mathcal{R}_\delta$,

$$\tilde{h}(s_1,s_2) = \epsilon(1+g_{(0,1)}(s_1,\sigma(s_1)) - g_{(1,0)}(s_1,\sigma(s_1))) + (s_2-s_1)s_1^{\ell-1}(-\ell Q(0) + s_1 O(1)). \tag{157}$$

By (151) and the fact that $s_1 \geq s_2$, we have for δ sufficiently small

$$T'_{\#}(\tau)(t) - T'_{\#}(\sigma)(t) > 0. \tag{158}$$

The proof of (2) is similar to the proof of (1) and we skip it.

Proof of (3): We begin by showing that $T''_{\#}(\phi)(t) \geq 0$ for all $t \in [0, \delta]$. By assumption, we have that $\phi \in \mathcal{W}_\delta$ and $\phi' \in \mathcal{W}'_\delta$. By (125), the definitions of $\mathcal{W}_\delta$ and $\mathcal{W}'_\delta$, we have that $\phi'(s), \phi''(s), g_{(1,0)}(s, \phi(s))$, and $g_{(0,1)}(s, \phi(s))$ are positive. The second derivative of $T_{\#}(\phi)(t)$ at $t \in [0, \delta]$ is given by

$$\begin{aligned}
&T''_{\#}(\phi)(t) \\
&= \frac{\phi''(s) + g_{(2,0)}(s, \phi(s)) + 2\phi'(s) g_{(1,1)}(s, \phi(s)) + \phi''(s) g_{(0,1)}(s, \phi(s)) + (\phi'(s))^2 g_{(0,2)}(s, \phi(s))}{(1 + \phi'(s))^3} \\
&\qquad\qquad\qquad - \frac{\phi''(s)(\phi'(s) + g_{(1,0)}(s, \phi(s)) + \phi'(s) g_{(0,1)}(s, \phi_n(s))}{(1 + \phi'(s))^3} \\
&\leq \frac{\phi''(s) + g_{(2,0)}(s, \phi(s)) + 2\phi'(s) g_{(1,1)}(s, \phi(s)) + \phi''(s) g_{(0,1)}(s, \phi(s)) + (\phi'(s))^2 g_{(0,2)}(s, \phi(s))}{(1 + \phi'(s))^3}
\end{aligned} \tag{159}$$

where $s + \phi(s) = t$. We have that for δ sufficiently small and functions O_1, O_2 bounded in $\mathcal{R}_\delta$ that

$$\begin{aligned}
\phi''(s) + g_{(2,0)}(s, \phi(s)) + {}&2\phi'(s) g_{(1,1)}(s, \phi(s)) + \phi''(s) g_{(0,1)}(s, \phi(s)) + (\phi'(s))^2 g_{(0,2)}(s, \phi(s)) \\
&\geq \phi''(s)(1 + g_{(0,1)}(s, \phi(s))) + 3\ell^2 \, Q(0) \, \alpha \, s^{2\ell - 1} + s^{2\ell} O_3(1) \\
&\geq \phi''(s).
\end{aligned} \tag{160}$$

Choosing $\delta > 0$ sufficiently small, for all $t \in [0, \delta]$ we have

$$\begin{aligned}
T''_{\#}(\phi)(t) &\geq \phi''(s) - \phi''(s)(\phi'(s) + g_{(1,0)}(s, \phi(s)) + \phi'(s) g_{(0,1)}(s, \phi(s)) \\
&= \phi''(s) \left(1 - g_{(1,0)}(s, \phi(s)) - \phi'(s) g_{(0,1)}(s, \phi(s))\right) \\
&\geq 0.
\end{aligned} \tag{161}$$

In order to show that $T'_\#(\sigma)(t) \in \mathcal{W}'_\delta$, we have to understand what the derivative of the image of σ_0 and τ_0 is. We will first show that $\sigma'_1(t) := T'_\#(\sigma_0)(t) > \sigma'_0(t)$ and $\tau'_1(t) := T'_\#(\tau_0)(t) < \tau'_0(t)$. Indeed, for some functions O bounded on $\mathcal{R}_\delta$,

$$\begin{aligned}
\sigma'_1(t)\,(1+\sigma'_0(s)) &= \sigma'_0(s) + g_{(1,0)}(s,\sigma_0(s)) + \sigma'_0(s) g_{(0,1)}(s,\sigma_0(s)) \\
&= (\ell+1)\alpha s^\ell + (2\ell+1)\alpha Q(0) s^{2\ell} + s^{2\ell+1} O(1) \\
&> (\ell+1)\alpha s^\ell + (2\ell+1)(\ell+1)\alpha^2 s^{2\ell} + s^{2\ell+1} O(1) \qquad (162)\\
&= (\ell+1)\alpha (s+\sigma_0(s))^\ell\,(1+\sigma'_0(s)) \\
&= \sigma'_0(t)\,(1+\sigma'_0(s))
\end{aligned}$$

and

$$\begin{aligned}
\tau'_1(t)\,(1+\tau'_0(s)) &= \tau'_0(s) + g_{(1,0)}(s,\tau_0(s)) + \tau'_0(s) g_{(0,1)}(s,\tau_0(s)) \\
&= (\ell+1)\beta s^\ell + (2\ell+1)\beta Q(0) s^{2\ell} + s^{2\ell+1} O(1) \\
&< (\ell+1)\beta s^\ell + (2\ell+1)(\ell+1)\beta^2 s^{2\ell} + s^{2\ell+1} O(1) \qquad (163)\\
&= (\ell+1)\beta (s+\tau_0(s))^\ell\,(1+\tau'_0(s)) \\
&= \tau'_0(t)\,(1+\tau'_0(s)).
\end{aligned}$$

If $\sigma \in \mathcal{W}_\delta$ and $\sigma' \in \mathcal{W}'_\delta$, then

$$\sigma_0(t) \le \sigma(t) \le \tau_0(t) \quad \text{and} \quad \sigma'_0(t) \le \sigma'(t) \le \tau'_0(t) \quad \text{for all } t \in [0,\delta]. \qquad (164)$$

By part 2) of this proposition, we have

$$\sigma'_0(t) < \sigma'_1(t) \le T'_\#(\sigma)(t) \le \tau'_1(t) < \tau'_0(t). \qquad (165)$$

A proof analogous to that of the continuity of $T_\#(\sigma)(t)$ shows that $T'_\#(\sigma)(t)$ is continuous, thus

$$T'_\#(\sigma)(t) \in \mathcal{W}'_\delta, \qquad (166)$$

completing the proposition. $\square$

The remainder of this subsection will be devoted to showing that there exist sequences of functions which converge to the invariant curve $\mathcal{C}_1^-$, and that they converge in such a way that their limit $\mathcal{C}_1^-$ is continuously differentiable. These sequences will be the iterates of the boundaries of $\mathcal{W}_\delta$ under $T_\#$, which are defined at each $t \in [0, \delta]$ as follows:

$$\sigma_0(t) := \alpha\, t^{\ell+1} \quad \text{and} \quad \sigma_n(t) := T_\#(\sigma_{n-1})(t),\ n = 1, 2, \dots \tag{167}$$

$$\tau_0(t) := \beta\, t^{\ell+1} \quad \text{and} \quad \tau_n(t) := T_\#(\tau_{n-1})(t),\ n = 1, 2, \dots. \tag{168}$$

By (147) and (148), we can define the sequences $\{\tau_n'\}$ and $\{\tau_n''\}$:

$$\tau_0'(t) := (\ell+1)\ \beta\ t^{\ell} \quad \text{and} \quad \tau_n'(t) := T_\#'(\tau_{n-1})(t),\ n = 1, 2, \dots \tag{169}$$

$$\tau_0''(t) := (\ell+1)\ \ell\ \beta\ t^{\ell-1} \quad \text{and} \quad \tau_n''(t) := T_\#''(\tau_{n-1})(t),\ n = 1, 2, \dots. \tag{170}$$

In Proposition 7 we will use the Arzèla-Ascoli Theorem to find a uniformly convergent subsequence of $\{\tau_n'\}$. The following proposition shows that $\{\tau_n'\}$ satisfies the hypotheses of the Arzèla-Ascoli Theorem:

Proposition 6. (Properties of $\{\tau_n\}$)

There exists $\delta > 0$ such that

1. *$\{\tau_n'\}$ is uniformly bounded by $(\ell+1)\,\beta\,\delta^{\ell}$ on $[0, \delta]$.*

2. *$\{\tau_n''\}$ is uniformly bounded by $(\ell+1)\,\ell\,\beta\,\delta^{\ell-1}$ on $[0, \delta]$.*

Proof (1): Since $\tau_0 \in \mathcal{W}_\delta$, statement (3) of Proposition 4 implies that $\tau_n \in \mathcal{W}_\delta$ for all $n \in \mathbb{N}$. By statement (3) of Proposition 5 and the fact that $\tau_0' \in \mathcal{W}_\delta'$, we have that $\tau_n' \in \mathcal{W}_\delta'$ for all $n \in \mathbb{N}$ so that

$$\tau_n'(t) \le (\ell+1)\,\beta\,t^{\ell} \le (\ell+1)\,\beta\,\delta^{\ell} \quad \text{for all} \quad t \in [0, \delta]. \tag{171}$$

Proof (2): We will proceed inductively. Notice that

$$0 \le \tau_0''(t) = (\ell+1)\,\ell\,\beta\,t^{\ell-1}$$

for $t \in [0,\delta]$. Now suppose for $n > 1$ that $0 \le \sigma_n''(t) \le (\ell+1)\,\ell\,\beta\,t^{\ell-1}$. By statement (3) of Propositions 4 and 5, we have that $\tau_n \in \mathcal{W}_\delta$ and $\tau_n' \in \mathcal{W}_\delta'$ for all $t \in [0,\delta]$. By (125), the definitions of $\mathcal{W}_\delta$ and $\mathcal{W}_\delta'$, and the induction hypothesis, we have that $\tau_n'(s), \tau_n''(s), g_{(1,0)}(s,\tau(s))$, and $g_{(0,1)}(s,\tau(s))$ are positive. The second derivative of τ_{n+1} at $t \in [0,\delta]$ is given by

$$\begin{aligned}
&\tau_{n+1}''(t) = \\
&\frac{\tau_n''(s) + g_{(2,0)}(s,\tau_n(s)) + 2\tau_n'(s)g_{(1,1)}(s,\tau_n(s)) + \tau_n''(s)g_{(0,1)}(s,\tau_n(s)) + (\tau_n'(s))^2 g_{(0,2)}(s,\tau_n(s))}{(1+\tau_n'(s))^3} \\
&\qquad\qquad - \frac{\tau_n''(s)(\tau_n'(s) + g_{(1,0)}(s,\tau_n(s)) + \tau_n'(s)g_{(0,1)}(s,\tau_n(s))}{(1+\tau_n'(s))^3} \\
&\le \\
&\frac{\tau_n''(s) + g_{(2,0)}(s,\tau_n(s)) + 2\tau_n'(s)g_{(1,1)}(s,\tau_n(s)) + \tau_n''(s)g_{(0,1)}(s,\tau_n(s)) + (\tau_n'(s))^2 g_{(0,2)}(s,\tau_n(s))}{(1+\tau_n'(s))^3}
\end{aligned} \tag{172}$$

where $s + \tau_n(s) = t$. We have that for δ sufficiently small and functions O_1, O_2 bounded in $\mathcal{R}_\delta$

$$\begin{aligned}
&\tau_n''(s) + g_{(2,0)}(s,\tau_n(s)) + 2\tau_n'(s)g_{(1,1)}(s,\tau_n(s)) + \tau_n''(s)g_{(0,1)}(s,\tau_n(s)) + (\tau_n'(s))^2 g_{(0,2)}(s,\tau_n(s)) \\
&\qquad \le (\ell+1)\,\ell\,\beta\,s^{\ell-1} + \ell(3\ell+1)Q(0)\,\beta\,s^{2\ell-1} + s^{2\ell}O_1(1) \\
&\qquad < (\ell+1)\,\ell\,\beta\,s^{\ell-1} + \ell(3\ell+1)(\ell+2)\alpha\,\beta\,s^{2\ell-1} + s^{2\ell}O_2(1) \\
&\qquad \le (\ell+1)\,\ell\,\beta\,s^{\ell-1} + \ell(4\ell^2+6\ell+2)\alpha\,\beta\,s^{2\ell-1} + s^{2\ell}O_2(1) \\
&\qquad = (\ell+1)\,\ell\,\beta\,t^{\ell-1}(1+\tau_n'(s))^3.
\end{aligned} \tag{173}$$

It follows that $\tau_{n+1}''(t) < (\ell+1)\,\ell\,\beta\,\delta^{\ell-1}$ for all $t \in [0,\delta]$. By Proposition 5, $\tau_n \in \mathcal{W}_\delta'$ for all n, thus $\tau_n''(t) \ge 0$ for all $t \in [0,\delta]$ and $n \in \mathbb{N}$, completing the proposition.

□

3.5.2 Existence of an Invariant Curve

We are now ready to prove the existence of the invariant center-unstable manifold $\mathcal{C}_1^-$.

Proposition 7. *There exists $\delta > 0$ and a unique $\tau \in \mathcal{W}_\delta$ such that $T_\#(\tau) = \tau$ and $\tau \in C^1[0, \delta]$*

Proof: Consider the sequences $\{\sigma_n\}$ and $\{\tau_n\}$. An induction argument and statement (2) of Proposition 4 imply that

$$\sigma_n(t) < \sigma_{n+1}(t) \quad \text{and} \quad \tau_n(t) > \tau_{n+1}(t) \quad \text{for} \quad t \in (0, \delta], \quad n \in \mathbb{N}. \tag{174}$$

We claim that *there exist σ, τ such that (a) $\sigma_n \to \sigma$ and $\tau_n \to \tau$ pointwise, and (b) σ and τ are invariant under $T_\#$.* To prove the claim note that for each t, the sequences $\{\sigma_n(t)\}$ and $\{\tau_n(t)\}$ are monotonic and bounded, thus convergent. Set $\sigma(t) := \lim \sigma_n(t)$ and $\tau(t) = \lim \tau_n(t)$. If $0 \leq s < t \leq \delta$, then $\sigma(s) = \lim \sigma_n(s) \leq \lim \sigma_n(t) = \sigma(t)$, so σ is non-decreasing. An analogous argument gives that τ is nondecreasing. To prove that σ and τ are invariant under $T_\#$, consider an arbitrary $t \in (0, \delta]$. Choose $s \in (0, \delta]$ such that $s + \sigma(s) = t$. Statement (5) of Proposition 4 applied to σ_n and σ gives

$$\left| \frac{\sigma_{n+1}(t) - T_\#(\sigma)(t)}{t^{\ell+1}} \right| = \left| \frac{T_\#(\sigma_n)(t) - T_\#(\sigma)(t)}{t^{\ell+1}} \right| < \left| \frac{\sigma_n(s) - \sigma(s)}{s^{\ell+1}} \right| . \tag{175}$$

By taking the limit as $n \to \infty$ in (175) one gets $T_\#(\sigma)(t) = \sigma(t)$. Since t was arbitrary, it follows that σ is invariant under $T_\#$. One can prove in similar fashion that τ is invariant as well. This concludes the claim.

We will now show that $\tau \in C^1([0, \delta])$. By Proposition 6, $\{\tau_n'(t)\}$ and $\{\tau_n''(t)\}$ are uniformly bounded. Applying the Arzèla-Ascoli Theorem (see section 7.5 in [17]), $\{\tau_n'(t)\}$ has a subsequence $\{\tau_{n_k}'(t)\}$ that converges uniformly, say to $\zeta(t)$. Since $\tau_n'(t)$ is continuous for all $n \in \mathbb{N}$, it follows that $\zeta(t)$ is continuous as well.

Applying Theorem B to $\{\tau'_{n_k}\}$, we have that $\{\tau_{n_k}\}$ converges uniformly to $\tau(t)$, and thus $\tau(t)$ is continuous and $\tau \in \mathcal{W}_\delta$, and also $\zeta(t) = \tau'(t)$, implying that $\tau \in C^1([0, \delta])$. A similar proof shows that $\sigma \in \mathcal{W}'_\delta$ as well.

We now prove uniqueness of the invariant element of $T_\#$ in $\mathcal{W}_\delta$. If $\phi \in \mathcal{W}_\delta$, then

$$\sigma_0(t) \leq \phi(t) \leq \tau_0(t) \quad \text{for} \quad t \in [0, \delta]. \tag{176}$$

If, in addition, $\phi(t)$ is invariant under $T_\#$, then induction, (176), and statement (4) of Proposition 4 imply $\sigma_n(t) \leq \phi(t) \leq \tau_n(t)$ for $t \in [0, \delta]$ and $n \geq 1$. Taking limit as $n \to \infty$, we have $\sigma(t) \leq \phi(t) \leq \tau(t)$ for $t \in [0, \delta]$. Thus for uniqueness of the invariant curve in $\mathcal{W}_\delta$ it is sufficient to prove that $\sigma = \tau$. If $\sigma \neq \tau$, there exist $x_1, y_1 \in (0, \delta)$, $x_1 \neq y_1$, such that $(x_1, y_1) \preceq_{se} (z_1, w_1)$. Since T is orientation preserving in $\mathcal{W}_\delta$, then the sequences $\{(x_n, y_n)\} := \{T^{-n}(x_1, y_1)\}$ and $\{(z_n, w_n)\} := \{T^{-n}(z_1, w_1)\}$ satisfy $(x_n, y_n) \preceq (z_n, w_n)$, i.e.,

$$x_n \leq z_n \quad \text{and} \quad y_n \geq w_n\,, \quad n \geq 1\,. \tag{177}$$

We also must have

$$(x_n, y_n) \to (0,0) \quad \text{and} \quad (z_n, w_n) \to (0,0) \quad \text{as} \quad n \to \infty\,. \tag{178}$$

From $(x_n, y_n) = T(x_{n+1}, y_{n+1})$ and $(z_n, w_n) = T(z_{n+1}, w_{n+1})$ it follows that

$$z_n - x_n = z_{n+1} + w_{n+1} - x_{n+1} - y_{n+1} = z_{n+1} - x_{n+1} + w_{n+1} - y_{n+1} \leq z_{n+1} - x_{n+1}. \tag{179}$$

From (177) and (179), $0 < x_1 - z_1 \leq x_n - z_n$ for all $n \geq 1$. This contradicts (178). Therefore $\sigma = \tau$, concluding the proof of Theorem 6 parts i) and ii). □

From this point forward, set $\mathcal{C}_1^-(x) := \tau(x) = \sigma(x)$ for $x \in [0, \delta]$.

3.5.3 Behavior off the Invariant Curve

In this subsection, we will establish the behavior of solutions lying in the region $\{(x, y) \in \mathcal{Q}_1 : \max\{x, y\} < \delta\}$ off of $\mathcal{C}_1^-$. To facilitate the proof, we must break this region into the following three pieces, defined for each $\delta > 0$ fixed:

$$\mathcal{A}_1 := \{(x,y) : 0 \le x \le \delta, \quad x^\ell < y < \delta\} \tag{180}$$

$$\mathcal{A}_2 := \{(x,y) : 0 \le x \le \delta, \quad \mathcal{C}_1^-(x) < y \le x^\ell\} \tag{181}$$

$$\mathcal{A}_3 := \{(x,y) : 0 \le x \le \delta, \quad 0 \le y < \mathcal{C}_1^-(x)\}. \tag{182}$$

It is clear that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ depend upon $\delta > 0$, but to ease the notation we will suppress this dependence. Proposition 8 below will show that $\mathcal{A}_3$ is a repelling hyperbolic sector with respect to the origin and Proposition 9 will show that $\mathcal{A}_1 \cup \mathcal{A}_2$ is a subset of a hyperbolic sector with respect to the origin.

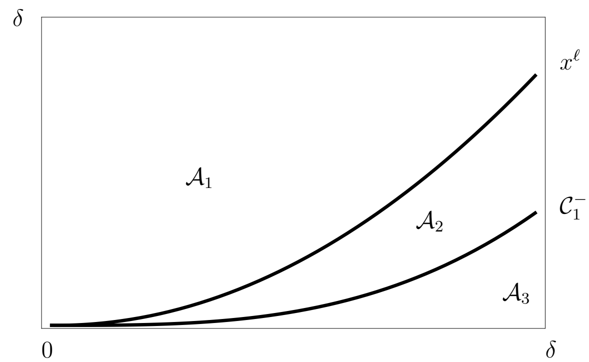

Figure 15. The region $[0,\delta] \times [0,\delta]$ is partitioned into three regions $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$.

Proposition 8. *If $(x,y) \in \mathcal{A}_3$, then $T^{-n}(x,y) \to (\bar{x}, 0)$ for some $\bar{x} \in (0,\delta)$.*

We begin the proof by showing that the preimage of a vertical line segment lying in $\mathcal{A}_3$ is of the form $\{(u,v) \in \mathcal{A}_3 : v = -u + x\}$. Suppose that $\{(u,v) \in \mathcal{A}_3 : v = -u + x\}$ intersects $\mathcal{C}_1^-$ at (u_0, v_0), so that the endpoints of this line segment are (u_0, v_0) and $(x, 0)$. If $(u,v) \in \{(u,v) \in \mathcal{A}_3 : v = -u + x\}$, then

$$T_1(u,v) = u + v = u + (-u + x) = x. \tag{183}$$

For points $(u,v) \in \{(u,v) \in \mathcal{A}_3 : v = -u + x\}$,

$$\frac{d}{du}\left(g(u, -u + x)\right) = g_u(u, -u + x) - g_v(u, -u + x) = -Q(0)u^\ell + u^{\ell+1}O(1) < 0. \tag{184}$$

Equation (184) implies that $T_2(u, v) = v + g(u, v)$ is monotone decreasing as u increases along the line $v = -u + x$. By the invariance of $\mathcal{C}_1^-$ and the x-axis under T, we have that $T(u_0, v_0) \in \mathcal{C}_1^-$ and $T(x, 0) = (x, 0)$ respectively. Thus

$$0 \leq T_2(u, v) \leq T_2(u_0, v_0), \quad \text{for all } (u, v) \in \{(u, v) \in \mathcal{A}_3 : v = -u + x\}. \tag{185}$$

By the continuity of T_2 and the Intermediate Value Theorem, for all $(x, y) \in \mathcal{A}_3$ there exists $(u, v) \in \{(u, v) \in \mathcal{A}_3 : v = -u + x\}$ such that $T_2(u, v) = y$. By (183) we have that $T(u, v) = (x, y)$.

Now that we have established the preimage of the set $\{(u, v) \in \mathcal{A}_3 : u = x\}$, we can use its structure to show that the x-coordinates of a backwards orbit of a point $(x, y) \in \mathcal{A}_3$ must converge. Let $(x_{-n}, y_{-n}) := T^{-n}(x, y)$. Proposition 2 tells us that $(x_{-n}, y_{-n}) \in \mathcal{B}_\delta$ has a unique preimage under T in $\mathcal{B}_\delta$, thus

$$(x_{-(n+1)}, y_{-(n+1)}) = T^{-1}(x_{-n}, y_{-n}) \in \{(u, v) \in \mathcal{A}_3 : v = -u + x_{-n}\}. \tag{186}$$

Equation (186) shows that x_{-n} is a decreasing sequence which is bounded below by 0, so $\{x_{-n}\}$ converges to $\bar{x}$ for some $\bar{x} > 0$. Turning our attention to the y-coordinate, $y_{-n} = y_{-(n+1)} + g(x_{-(n+1)}, y_{-(n+1)})$, and $g(x, y) = Q(0)x^\ell + x^{\ell+1}O(1) > 0$ for $(x, y) \in \mathcal{A}_3$, so $\{y_n\}$ is also a decreasing sequence bounded below by 0, and thus converges to $\bar{y}$. The only fixed points in the closure of $\mathcal{A}_3$ lie on the x-axis, so $T^{-n}(x, y) \to (\bar{x}, 0)$, completing the proposition.

Proposition 9. *For every $(x, y) \in \mathcal{A}_1 \cup \mathcal{A}_2$, there exists indices m, k such that $T^n(x, y) \in \mathcal{A}_1 \cup \mathcal{A}_2$ for $-m \leq n \leq k$, and $T^n(x, y) \notin \mathcal{A}_1 \cup \mathcal{A}_2$ for $n = -m - 1$ or $n = k + 1$.*

Define $(x_n, y_n) := T^n(x, y)$ for all $n \in \mathbb{Z}$. Suppose that $(x_n, y_n) \in \mathcal{A}_1 \cup \mathcal{A}_2$. First, we narrow down the locations where (x_{n+1}, y_{n+1}) may lie. Notice that $x_{n+1} = x_n + y_n$ and $x_n, y_n > 0$, so (x_{n+1}, y_{n+1}) must lie in either $\mathcal{Q}_1$ or $\mathcal{Q}_4$. By

(186), the unique preimages of points in $\mathcal{A}_3$ that lie in $\mathcal{B}_\delta$ must also lie in $\mathcal{A}_3$, so $(x_{n+1}, y_{n+1})) \notin \mathcal{A}_3$. To see that $y_{n+1} > 0$, choose δ sufficiently small so that for $x \in (0, \delta)$,

$$T_2(x,y) = y + g(x,y) = y(1 + Q(x)x^\ell + R(x,y)y) = y(1 + Q(0)x^\ell + O(1)x^{\ell+1}) > 0. \tag{187}$$

Equation (187) shows that $(x_{n+1}, y_{n+1}) \notin \mathcal{Q}_4$. Now consider the sequence $\{(x_n, y_n)\}$, $n \geq 0$. The sequence $\{x_n\}$ is increasing, and since $\mathcal{C}(x)$ is increasing on the interval $[0, \delta]$, we have that

$$x_{n+1} - x_n = y_n \geq C(x_n) \geq C(x_0),$$

so there must exist an index k such that $(x_k, y_k) \in \mathcal{A}_1 \cup \mathcal{A}_2$ and $x_{k+1} > \delta$, that is that $(x_{k+1}, y_{k+1}) \notin \mathcal{A}_1 \cup \mathcal{A}_2$, concluding the first part of the proof.

We will now show that there exists an index m such that $T^{-m}(x,y) \in \mathcal{Q}_2$. Suppose that there exists $(x,y) \in \mathcal{A}_1 \cup \mathcal{A}_2$ whose backwards orbit remains in $\mathcal{A}_1 \cup \mathcal{A}_2$. In order to remain in $\mathcal{A}_1$, (and thus $\mathcal{Q}_1$), the sequence $\{x_{-n}\}$ must converge to some $\bar{x}$ since $\{x_{-n}\}$ is decreasing. This can happen only if $\{y_{-n}\} \to 0$. The only fixed point in the closure of $\mathcal{A}_1 \cup \mathcal{A}_2$ is the origin, so our problem can be reduced to showing that the backwards orbit of any point in $\mathcal{A}_1 \cup \mathcal{A}_2$ does not converge to the origin.

If $(x,y) \in \mathcal{A}_1 \cup \mathcal{A}_2$, then $(x_{-1}, y_{-1}) \in \mathcal{A}_1 \cup \mathcal{A}_2$ because otherwise if $(x_{-1}, y_{-1}) \in \mathcal{A}_3$, then by Proposition 8 we have $(x,y) = T(x_{-1}, y_{-1}) \in \mathcal{A}_3$.

We will consider two cases. First, suppose that $\{(x_{-n}, y_{-n})\} \subset \mathcal{A}_2$ for all $n \in \mathbb{N}$. If $(x,y) \in \mathcal{A}_2$, then for $\delta > 0$ sufficiently small we have

$$g_{(1,0)}(x,y) = y(Q'(x)x^\ell + Q(x)\ell x^{\ell-1} + R_{(1,0)}(x,y)y) \geq y(Q(0)\ell x^{\ell-1} + x^\ell O(1)) > 0, \tag{188}$$

so T is orientation preserving in this region. Choose $(z,w) \in \mathcal{C}$ such that $(z,w) \preceq_{se} (x,y)$. Repeating the same argument as for the uniqueness of τ in Theorem 1, we have $0 < z - x < z_{-n} - x_{-n}$, and since $z_{-n} \to 0$, it must be that $x_{-n_0} < 0$ for some $n_0 \in \mathbb{N}$, completing the first case.

For the second case, suppose there exists an index n_0 such that $(x_{-n_0}, y_{-n_0}) \in \mathcal{A}_1$. If $(x, y) \in \mathcal{A}_1$ and δ sufficiently small,

$$\begin{aligned}
x\, g(x,y) - y^2 &= Q(x)x^{\ell+1}y + (xR(x,y) - 1)y^2 \\
&= y(Q(x)x^{\ell+1} + (xR(x,y) - 1)y) \\
&\leq y(-x^{\ell} + x^{\ell+1}O(1)) \\
&< 0.
\end{aligned} \tag{189}$$

Equation (189) implies that

$$\frac{y_{-(n+1)}}{x_{-(n+1)}} = \frac{y_{-n} + g(x_{-n}, y_{-n})}{x_{-n} + y_{-n}} < \frac{y_{-n}}{x_{-n}}. \tag{190}$$

It can be easily seen geometrically by considering the concavity of $y = x^{\ell}$ and equation (190) that $(x_{-(n+1)}, y_{-(n+1)}) \in \mathcal{A}_1$. Thus the position vector angle y_{-n}/x_{-n} is increasing with each $n \geq n_0$. Suppose that $\{(x_{-n}, y_{-n})\}$ remains in $\mathcal{A}_1$. Then it must be that $\{y_{-n}/x_{-n}\}$ converges, since $\{y_{-n}/x_{-n}\}$ is a monotone and bounded sequence, necessarily to an m that is strictly greater than 0. Further we have

$$\begin{aligned}
\frac{y_{-(n+1)}}{x_{-(n+1)}} &= \frac{y_{-n} + g(x_{-n}, y_{-n})}{x_{-n} + y_{-n}} \\
&= \frac{\dfrac{y_{-n}}{x_{-n}} + \dfrac{g(x_{-n}, y_{-n})}{x_{-n}}}{1 + \dfrac{y_{-n}}{x_{-n}}} \\
&= \frac{\dfrac{y_{-n}}{x_{-n}} + \dfrac{y_{-n}}{x_{-n}}\left(Q(x_{-n})x_{-n}^{\ell} + R(x_{-n}, y_{-n})y_{-n}\right)}{1 + \dfrac{y_{-n}}{x_{-n}}}.
\end{aligned} \tag{191}$$

Set $m := \lim y_{-n}/x_{-n}$. Assuming that $(x_{-n}, y_{-n}) \to (0,0)$ and taking the limit as $n \to \infty$ to both sides, we have

$$m = \frac{m}{1+m}, \tag{192}$$

to which only $m = 0$ is a solution, leading to a contradiction. Setting $S_1 := \mathcal{A}_1 \cap \mathcal{Q}_\delta$ and $S_2 := (\mathcal{A}_2 \cup \mathcal{A}_3) \cap \mathcal{Q}_\delta$ completes the proof of part iii) of Theorem 6. $\square$

3.5.4 Behavior in the Fourth Quadrant

In this section, we will establish the dynamical behavior of solutions lying in $\mathcal{B}_\delta \cap \mathcal{Q}_4$. To begin, for each $\delta > 0$ define

$$\nabla_\delta := \{(x, y) \in \mathcal{Q}_4 : x - y < \delta\}. \tag{193}$$

We will construct a region satisfying part iv) of Theorem 6 that lies inside ∇_δ. We wish to first show that orbits lying in ∇_δ will eventually enter $\mathcal{Q}_3$. Choose $\delta > 0$ so that for all $(x, y) \in \nabla_\delta$,

$$\frac{|g(x, y)|}{|y|} = |Q(x)x^\ell + R(x, y)y| < 1, \tag{194}$$

$$Q(x)x^\ell > 0 \quad \text{and} \quad |R(x, y)| < 1, \tag{195}$$

and so that the angle between the two eigenvectors of the jacobian of each of the fixed points $(x, 0) \in \nabla_\delta$ is less than $\frac{\pi}{4}$. This is possible because the eigenvectors of the jacobian of T at $(x, 0)$ are given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{g_{(0,1)}(x,0)} \\ 1 \end{pmatrix}, \tag{196}$$

and thus the angle between the eigenvectors in (196)

$$\theta(x) := \arccos\left(\frac{1}{\sqrt{1 + g_{(0,1)}(x, 0)}}\right) \tag{197}$$

satisfies $\theta \to 0$ as $x \to 0$. By the continuity $\theta(x)$ for x close to 0, we can choose δ so that $|\theta(x)| < \frac{\pi}{4}$. Notice that if $(x, y) \in \nabla_\delta$, by (194) and (195) respectively, we have

$$T_2(x, y) = y + g(x, y) = y\left(1 + \frac{g(x, y)}{y}\right) < 0, \tag{198}$$

and

$$
\begin{aligned}
0 &> y(Q(x)x^{\ell+1} + (xR(x,y) - 1)y) \\
&= Q(x)x^{\ell+1}y + R(x,y)xy^2 - y^2 \\
&= x\,g(x,y) - y^2.
\end{aligned} \tag{199}
$$

Next, we must find the regions in which the image of a point $(x,y) \in \nabla_\delta$ may lie. By (198), $T(x,y)$ cannot lie in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. If $T(x,y) \in \mathcal{Q}_4 \setminus \nabla_\delta$, then since $(x,y) \in \nabla_\delta$,

$$
x < \delta + y \tag{200}
$$

so that equation (200), $(x,y) \in \nabla_\delta$ and $T(x,y) \in \mathcal{Q}_4$ imply that

$$
\begin{aligned}
\delta < (x+y) - (y + g(x,y)) &= x - g(x,y) \\
&\le x + |g(x,y)| \\
&< \delta + y + |g(x,y)| \\
&= \delta - |y| + |g(x,y)|.
\end{aligned} \tag{201}
$$

Equation (201) implies that

$$
|y| < |g(x,y)|. \tag{202}
$$

However, equation (194) implies that $|y| > |g(x,y)|$, contradicting equation (202). Thus it must be that $T(x,y) \in \nabla_\delta \cup \mathcal{Q}_3$. Now, we must rule out the possibility that an orbit remains in ∇_δ. Since $T_1(x,y) > 0$ and $(x,y) \in \nabla_\delta$, then

$$
x\,g(x,y) - y^2 < 0 \iff \frac{T_2(x,y)}{T_1(x,y)} = \frac{y + g(x,y)}{x+y} < \frac{y}{x}, \tag{203}
$$

so if $(x,y) \in \nabla_\delta$, then the angle formed between the origin and (x_{n+1}, y_{n+1}) is less than the angle formed between the origin and (x,y). Suppose that $\{x_n, y_n\} \subset \nabla_\delta$ for all $n \in \mathbb{N}$, where $\{x_n, y_n\} := T^n(x_0, y_0)$ for an arbitrary $(x_0, y_0) \in \nabla_\delta$. Now by (199) and (203), we have that $\{y_n/x_n\}$ is a decreasing sequence for all $n \in \mathbb{N}$, and thus is either less than -1 for some index or is bounded below by a constant greater than -1. If $y_k/x_k \le -1$ for some index k, then $T_1(x_k, y_k) = x_k + y_k \le 0$, which together with equation (198) implies that $(x_{k+1}, y_{k+1}) \in \mathcal{Q}_3$.

If $\{y_n/x_n\}$ is bounded below, then $\{y_n/x_n\} \to m$ for some $-1 < m < 0$. Since $(x_n, y_n) \in \nabla_\delta \subset \mathcal{Q}_4^\circ$, we have

$$x_{n+1} = x_n + y_n < x_n \quad \text{for all } n \in \mathbb{N}. \tag{204}$$

If $(x_n, y_n) \in \nabla_\delta$ for all n, clearly $\{x_n\}$ is bounded below by 0, and thus converges, say to $\bar{x}$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \ge N$, we have $x_n - \bar{x} < \epsilon/2$. Then

$$y_n = x_{n+1} - x_n = (x_{n+1} - \bar{x}) - (x_n - \bar{x}) < \epsilon. \tag{205}$$

Since $\epsilon > 0$ was arbitrary, it must be that $(x_n, y_n) \to (\bar{x}, 0)$. However, if $\bar{x} \ne 0$, the sequence $\{y_n/x_n\} \to 0$, which contradicts equation (203). Thus $\{(x_n, y_n)\} \to (0, 0)$. Further we have

$$\begin{aligned} \frac{y_{n+1}}{x_{n+1}} &= \frac{y_n + g(x_n, y_n)}{x_n + y_n} \\ &= \frac{\dfrac{y_n}{x_n} + \dfrac{g(x_n, y_n)}{x_n}}{1 + \dfrac{y_n}{x_n}} \\ &= \frac{\dfrac{y_n}{x_n} + \dfrac{y_n}{x_n}\left(Q(x_n)x_n^\ell + R(x_n, y_n)y_n\right)}{1 + \dfrac{y_n}{x_n}}. \end{aligned} \tag{206}$$

Set $m := \lim y_n/x_n$. Taking the limit of both sides of equation (206) as $n \to \infty$, since $Q(x_n)x_n^\ell + R(x_n, y_n)y_n \to 0$ as $(x_n, y_n) \to (0, 0)$ we have

$$m = \frac{m}{1 + m}, \tag{207}$$

which has no solutions in $(-1, 0)$, leading to a contradiction. Thus there exists an index k such that $(x_k, y_k) \in \mathcal{Q}_3$.

By Theorem A in the appendix, the fixed point $(\delta, 0)$ has a local unstable manifold; define $\mathcal{U}_\delta$ to be the subset of the manifold that lies below the x-axis.

The manifold $\mathcal{U}_\delta$ must be tangent to the eigenvector of the jacobian of T at $(\delta, 0)$ which has vector angle less than $\pi/4$, which implies that $\mathcal{U}_\delta \subset \nabla_\delta$. By the claim, $T(\mathcal{U}_\delta) \subset \nabla_\delta \cup \mathcal{Q}_3$ and $T^n(\mathcal{U}_\delta) \cap \mathcal{Q}_3 \neq \emptyset$ for n sufficiently large. Thus $\mathcal{U}_\delta$ can be extended until it intersects the negative y-axis. Define $\mathcal{R}_{\mathcal{U}_\delta}$ to be the region bounded by the x-axis, the y-axis, and $\mathcal{U}_\delta$.

We claim that *if* $(x, y) \in \mathcal{R}_{\mathcal{U}_\delta}$, *then* $T^{-1}(x, y) \in \mathcal{R}_{\mathcal{U}_\delta}$. Define $(x_{-n}, y_{-n}) := T^{-n}(x, y)$ for $n \in \mathbb{N}$. Then we have

$$y_{-(n+1)} = -y_{-n}\left(1 + \frac{g(x_{-n}, y_{-n})}{y_{-n}}\right) \tag{208}$$

and

$$x_{-(n+1)} = x_{-n} - y_{-(n+1)}. \tag{209}$$

If $(x_{-n}, y_{-n}) \in \mathcal{R}_{\mathcal{U}_\delta} \subset \nabla_\delta$, then by (194) and (208), we have $y_{-(n+1)} < 0$. In turn, this implies with (209) that $x_{-(n+1)} > x_{-n}$, so that $(x_{-(n+1)}, y_{-(n+1)}) \in \mathcal{Q}_4$.

Consider any path from the origin to (x_{-n}, y_{-n}) that is a subset of $\mathcal{R}_{\mathcal{U}_\delta}$. Suppose that $T^{-1}(x_{-n}, y_{-n}) \notin \mathcal{R}_{\mathcal{U}_\delta}$. By the discussion in the previous paragraph, the preimage of the path under T must lie in $\mathcal{Q}_4$, so it must intersect $\mathcal{U}_\delta$ at some point. This leads to a contradiction, since $\mathcal{U}_\delta$ is invariant under T.

Finally, we have that $\{(x_{-n}, y_{-n})\}_{n\in\mathbb{N}} \subset \mathcal{R}_{\mathcal{U}_\delta}$, so by (209), $\{x_{-n}\}$ is an increasing sequence which is bounded above by δ. Using arguments analogous to those used in $\mathcal{A}_3$ in the proof of part iii) of Theorem 6 shows that $T^{-n}(x, y) \to (\bar{x}, 0)$ for some $0 < \bar{x} < \delta$. Choosing $\delta' > 0$ such that $\mathcal{B}_{\delta'} \subset \mathcal{R}_{\mathcal{U}_\delta}$ completes the proof of part iv) of Theorem 6.

□

The following corollary describes the behavior of the invariant curve near the origin.

Corollary 11. *If $\mathcal{C}_-$ is the graph of $\phi : [0, \delta] \to \mathbb{R}$, then* $\lim\limits_{x\to 0^+} \frac{\phi(x)}{x^{\ell+1}} = \frac{Q(0)}{\ell+1}$.

Proof. Let $\delta > 0$ be as in Theorem 6. Since $\phi \in \mathcal{W}_\delta$, then $\alpha\, x^{\ell+1} \le \phi(x) \le \beta\, x^{\ell+1}$ for $0 \le x \le \delta$, where α and β satisfy (120) and are otherwise arbitrary. Thus $\alpha \le \liminf_{x\to 0^+} \frac{\phi(x)}{x^{\ell+1}} \le \limsup_{x\to 0^+} \frac{\phi(x)}{x^{\ell+1}} \le \beta$. The statement follows from the fact that both α and β can be taken as close to $\frac{Q(0)}{\ell+1}$ as desired. $\square$

3.6 Proof of Theorem 7

To study the left half plane, we will focus on the scenario when $Q(0) > 0$ and ℓ odd or when $Q(0) < 0$ and ℓ even. The other combinations of signs for $Q(0)$ and parity of ℓ can be reduced to these cases through reflections about the origin. We will show in subsections 3.6.1 and 3.6.2 that the behavior of T^{-1} in the left half plane for $Q(0) > 0$ and ℓ odd or $Q(0) < 0$ and ℓ even is conjugate to the behavior of T in the right half plane for $Q(0) > 0$, which satisfies the hypothesis of Theorem 6.

3.6.1 Normal Form of the Check Map

Let J be the map $J(x, y) = (-x, y)$. For a map T defined on a set V, let $\check{T}$, the check map of T, be the map given by

$$\check{T}(x, y) := J\, T\, J\,(x, y)\,, \qquad (x, y) \in \check{V} := J(V)\,. \tag{210}$$

If T is a normal form, so is $\check{T}$ since

$$\check{T}(x, y) = (x + y, \check{Q}(x)x^\ell + \check{R}(x, y)y^2)\,, \quad (x, y) \in \check{V}\,, \tag{211}$$

where

$$\check{Q}(x)x^\ell + \check{R}(x, y)y^2 := -Q(-x)(-x)^\ell + R(-x, y)y^2 = (-1)^{\ell+1}Q(-x)x^\ell + R(-x, y)y^2$$

satisfies (86). It follows that if $(x, y) \in \check{V}$, then $\check{Q}(\cdot) = 0$ if $Q(\cdot) = 0$, and otherwise

$$\check{Q}(x) = (-1)^{\ell+1}\, Q(-x) \quad \text{and} \quad \check{\ell} = \ell\,. \tag{212}$$

These considerations give the following result.

Proposition 10. *Let T be a map in normal form with Q and ℓ as in* (86)*, and let $\check{T}$ be the map* (210) *with $\check{Q}$ and $\check{\ell}$ as in* (212)*. Then, $Q(0) \neq 0$ if and only if $\check{Q}(0) \neq 0$. In either case, $\check{Q}(0) = (-1)^{\ell+1} Q(0)$.*

3.6.2 Normal Form of the Inverse Map

Proposition 11. *Let T be a map in normal form with Q and ℓ as in* (86)*. If $(-1)^{\ell+1}Q(0) < 0$, then T^{-1} has a normal form $\tilde{T}$ such that $\tilde{Q}(0) > 0$.*

Proof. Given (x, y), set $(\tilde{x}, \tilde{y}) = T(x, y) = (x + y, y + g(x, y))$. With the involution $(x, y) \to (-x, y)$, the map T conjugates to $\bar{T}(x, y) = (\tilde{x}, \tilde{y}) = (x - y, y + g(-x, y))$. We have

$$\begin{aligned} \tilde{x} + \tilde{y} &= x + g(-x, y) \\ \tilde{y} &= y + g(-x, y). \end{aligned} \tag{213}$$

Since $\bar{T}$ is invertible near the origin, both x and y are functions of $\tilde{x}$ and $\tilde{y}$, so for a suitable function Φ we have

$$\Phi(\tilde{x}, \tilde{y}) = -g(-x, y). \tag{214}$$

By invertibility of $\bar{T}$ and the fact that fixed points of $\bar{T}$ are precisely the points on the x-axis, we have

$$y = 0 \text{ if and only if } \tilde{y} = 0, \text{ and in either case } x = \tilde{x}. \tag{215}$$

Now relations (213) and (214) imply

$$(x, y) = (\tilde{x} + \tilde{y} + \Phi(\tilde{x}, \tilde{y}), \tilde{y} + \Phi(\tilde{x}, \tilde{y})). \tag{216}$$

Set

$$w := y + \Phi(x, y) \quad \text{and} \quad \tilde{w} := \tilde{y} + \Phi(\tilde{x}, \tilde{y}). \tag{217}$$

By (216) and (217) we have

$$\tilde{w} = y \tag{218}$$

and

$$(x, w) = (\tilde{x} + \tilde{w}, \tilde{w} + \Phi(x, y)) . \tag{219}$$

Since $\Phi(x, y) = \Phi(\tilde{x} + \tilde{w}, \tilde{w})$, we have that in a neighborhood of the origin T conjugates to the map

$$\tilde{T}(\tilde{x}, \tilde{w}) := (\tilde{x} + \tilde{w}, \tilde{w} + \Phi(\tilde{x} + \tilde{w}, \tilde{w})). \tag{220}$$

Hence $\tilde{T}$ has the form (86). For (220) to be a normal form, it is required that (86) holds, i.e., that

$$\frac{\partial}{\partial \tilde{x}} \Phi(\tilde{x} + \tilde{w}, \tilde{w}) \Big|_{(\tilde{x}, \tilde{w}) = (0,0)} = 0 , \quad \text{and} \quad \Phi(\tilde{x} + \tilde{w}, \tilde{w})|_{\tilde{w}=0} = (\tilde{x}, 0) . \tag{221}$$

The second equality in (221) follows from (215) and (218). We now proceed to prove the first equality in (221).

Since $\Phi(\tilde{x}, \tilde{y}) = -g(-x, y)$ where $x = x(\tilde{x}, \tilde{y})$ and $y = y(\tilde{x}, \tilde{y})$, we have

$$\begin{aligned} \Phi_{(1,0)}(\tilde{x}, \tilde{y}) &= -g_{(1,0)}(-x, y) \left(-\tfrac{\partial x}{\partial \tilde{x}}\right) - g_{(0,1)}(-x, y) \left(\tfrac{\partial y}{\partial \tilde{x}}\right) \\ \Phi_{(0,1)}(\tilde{x}, \tilde{y}) &= -g_{(1,0)}(-x, y) \left(-\frac{\partial x}{\partial \tilde{y}}\right) - g_{(0,1)}(-x, y) \left(\frac{\partial y}{\partial \tilde{y}}\right) . \end{aligned} \tag{222}$$

Now $(\tilde{x}, \tilde{y}) = (0, 0)$ if and only if $(x, y) = (0, 0)$, hence the latter relation, (222) and (86) imply

$$\Phi_{(1,0)}(0, 0) = 0 \quad \text{and} \quad \Phi_{(0,1)}(0, 0) = 0 . \tag{223}$$

Finally, by (223),

$$\begin{aligned} \tfrac{\partial}{\partial \tilde{x}} \Phi(\tilde{x} + \tilde{w}, \tilde{w})\big|_{(\tilde{x}, \tilde{w})=(0,0)} &= \left(\Phi_{(1,0)}(\tilde{x} + \tilde{w}, \tilde{w}) + \Phi_{(0,1)}(\tilde{x} + \tilde{w}, \tilde{w})\right)\big|_{(\tilde{x}, \tilde{w})=(0,0)} \\ &= \Phi_{(1,0)}(0, 0) + \Phi_{(0,1)}(0, 0) = 0 \end{aligned} \tag{224}$$

thus completing the proof of (221).

We now claim that there exist $\tilde{\ell}$ an integer greater than or equal to one and a function $\tilde{Q}$ that is real analytic near $0 \in \mathbb{R}$ with $\tilde{Q}(0) > 0$, such that

$$\Phi(\tilde{x} + \tilde{w}, \tilde{w}) = \tilde{Q}(\tilde{x})\, \tilde{x}^{\tilde{\ell}}\, \tilde{w} + \tilde{R}(\tilde{x}, \tilde{w})\, \tilde{w}^2 . \tag{225}$$

By real analyticity of Φ, to prove the claim it is sufficient to verify that

$$\frac{\partial}{\partial \tilde{w}}\Phi(\tilde{x}+\tilde{w},\tilde{w})\Big|_{\tilde{w}=0} > 0\,, \qquad \text{for all } \tilde{x} \text{ positive and close enough to zero.} \tag{226}$$

From (222) and (215) we have

$$\begin{aligned}\Phi_{(1,0)}(\tilde{x},0) &= g_{(1,0)}(-x,0)\tfrac{\partial x}{\partial \tilde{x}}(\tilde{x},0) - g_{(0,1)}(-x,0)\tfrac{\partial y}{\partial \tilde{x}}(\tilde{x},0) = -g_{(0,1)}(-x,0)\tfrac{\partial y}{\partial \tilde{x}}(\tilde{x},0)\\ \Phi_{(0,1)}(\tilde{x},0) &= g_{(1,0)}(-x,0)\tfrac{\partial x}{\partial \tilde{y}}(\tilde{x},0) - g_{(0,1)}(-x,0)\tfrac{\partial y}{\partial \tilde{y}}(\tilde{x},0) = -g_{(0,1)}(-x,0)\tfrac{\partial y}{\partial \tilde{y}}(\tilde{x},0).\end{aligned} \tag{227}$$

Hence,

$$\frac{\partial}{\partial \tilde{w}}\Phi(\tilde{x}+\tilde{w},\tilde{w})\Big|_{\tilde{w}=0} = \Phi_{(1,0)}(\tilde{x},0)+\Phi_{(0,1)}(\tilde{x},0) = -g_{(0,1)}(-x,0)\left(\frac{\partial y}{\partial \tilde{x}}(\tilde{x},0)+\frac{\partial y}{\partial \tilde{y}}(\tilde{x},0)\right). \tag{228}$$

By taking partial derivatives w.r.t. $\tilde{x}$ and $\tilde{y}$ of the terms in (213), setting $y=\tilde{y}=0$, and manipulating the resulting expressions we get the relations

$$\begin{aligned}(\,1-g_{(0,1)}(-x,y)\,)\,\tfrac{\partial y}{\partial \tilde{x}}(\tilde{x},0) &= 0\\ (\,1-g_{(0,1)}(-x,y)\,)\,\tfrac{\partial y}{\partial \tilde{y}}(\tilde{x},0) &= 1.\end{aligned} \tag{229}$$

For $\tilde{x}$ near zero and $\tilde{y}=0$, it must be the case that x is also near zero, and $g_{(0,1)}(-x,0)\approx 0$ follows. Hence from the latter relation and (229) we have

$$\frac{\partial y}{\partial \tilde{x}}(\tilde{x},0)=0 \quad \text{and} \quad \frac{\partial y}{\partial \tilde{y}}(\tilde{x},0)>0 \quad \text{whenever } \tilde{x} \text{ is near } 0. \tag{230}$$

Finally, since by assumption $(-1)^{\ell}Q(0)<0$, we have $-g_{(0,1)}(-x,0)>0$ for x positive and close enough to zero. The latter relation, (230) and (228) give (226), thus completing the proof of the claim and of the proposition. □

An implication of the proof of Proposition 11 is that the change of coordinates from T^{-1} to the normal form $\tilde{T}$ flips the horizontal axis about the origin, and maps (locally) each vertical semi-axis into itself. (see Corollary 12 below).

Corollary 12. *Let T and $\tilde{T}$ be as in Proposition 11. Then a conjugacy map Θ for which $T^{-1} = \Theta^{-1}\tilde{T}\Theta$ on a neighborhood V of the origin can be chosen to satisfy*

$$\Theta(\mathbb{X}'_+) \subset \mathbb{X}_- \,, \quad \Theta(\mathbb{X}'_-) \subset \mathbb{X}_+ \,, \quad \Theta(\mathbb{Y}'_+) \subset \mathbb{Y}_+ \,, \quad \textit{and} \quad \Theta(\mathbb{Y}'_-) \subset \mathbb{Y}_- \,.$$

Proof. The conjugacy map θ is the composition of two mappings- the first mapping is a reflection over the y-axis. Clearly the upper- and lower-half planes and the x-axis are invariant under this reflection. The second mapping is the change of coordinates $w = y + \Phi(x, y)$, see equation (217). Since $\Phi(x, 0) = 0$ for all $(x, 0) \in V$ and Φ is real analytic, $\Phi(x, y) = y\,\tilde{\Phi}(x, y)$ for some real analytic $\tilde{\Phi}$ in V, and V can be chosen small enough so that $|\Phi(x, y)/y| < 1$. Thus, if $y > 0$,

$$w = y + \Phi(x, y) = y\left(1 + \frac{\Phi(x, y)}{y}\right) > 0$$

and similarly for $y < 0$, completing the proof of the corollary.

Proof of Theorem 7

By Proposition 11, we have that T^{-1} is conjugate to a map $\tilde{T} = \theta T^{-1}\theta^{-1}$, where $\tilde{T}$ is in normal form with $\tilde{Q}(0) > 0$. Applying Theorem 6 to the map $\tilde{T}$, we have that there exists a $\delta' > 0$ and $\mathcal{C}_1^- \subset \mathcal{B}_{\delta'} \cap \mathcal{Q}_1$ such that

a. $\mathcal{C}_1^-$ is $\tilde{T}^{-1}$-invariant and strongly $\preceq_{ne}$-ordered.

b. $\mathcal{C}_1^- = \left\{ (x, y) \in \mathcal{B}_{\delta'} \cap \mathcal{Q}_1 : \begin{array}{c} \tilde{T}^{-n}(x, y) \in \mathcal{B}_{\delta'} \cap \mathcal{Q}_1 \quad \forall n \geq 0, \text{ and} \\ \lim \tilde{T}^{-n}(x, y) = (0, 0) \end{array} \right\}$

c. The set $\mathcal{B}_{\delta'} \cap \mathcal{Q}_1 \setminus \mathcal{C}_1^-$ has two connected components, henceforth denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$, such that where $\mathcal{S}_1$ is a repelling parabolic sector of $\tilde{T}$ relative to $(0, 0)$ and $\mathcal{B}_{\delta'}$, and for (x, y) in $\mathcal{S}_2$, both $\tilde{T}^n(x, y)$ and $\tilde{T}^{-n}(x, y)$ eventually leave $\mathcal{B}_{\delta'} \cap \mathcal{Q}_1$.

d. Every nonzero point (x, y) in $\mathcal{B}_{\delta'} \cap \mathcal{Q}_4$ belongs to the unstable manifold of a fixed point of $\tilde{T}$.

By Corollary 12, it is possible to choose $\delta > 0$ such that

$$\theta(\mathcal{B}_\delta \cap \mathcal{Q}_2) \subset \mathcal{B}_{\delta'} \cap \mathcal{Q}_1. \tag{231}$$

Since $\mathcal{C}_1^- \subset \mathcal{Q}_1$, Corollary 12 also implies that $\theta^{-1}(\mathcal{C}_1^-) \subset \mathcal{Q}_2$. Set $\mathcal{C}_1^+ := \theta^{-1}(\mathcal{C}_1^-) \cap \mathcal{B}_\delta$.

From part a above, $\mathcal{C}_1^-$ is $\tilde{T}^{-1}$-invariant, thus

$$\tilde{T}^{-1}(\mathcal{C}_1^-) \subset \mathcal{C}_1^- \implies (\theta T \theta^{-1})(\mathcal{C}_1^-) \subset \mathcal{C}_1^- \implies T(\mathcal{C}_1^+) \subset \mathcal{C}_1^+. \tag{232}$$

By Corollary 12, $\mathcal{C}_1^+$ is C^1, and $\mathcal{C}_1^+ \in \mathcal{B}_\delta \cap \mathcal{Q}_2$. This completes the proof of (i).

To prove (ii), note that by (231), and the definition of $\tilde{T}$,

$$\begin{aligned}
&(x,y) \in \left\{ (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 : \begin{array}{c} T^n(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 \quad \forall n \geq 0, \text{ and} \\ \lim T^n(x,y) = (0,0) \end{array} \right\} \\
\iff &(x,y) \in \left\{ (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 : \begin{array}{c} \theta^{-1}\tilde{T}^{-n}\theta(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 \quad \forall n \geq 0, \text{ and} \\ \lim \theta^{-1}\tilde{T}^{-n}\theta(x,y) = (0,0) \end{array} \right\} \\
\iff &(x,y) \in \left\{ (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 : \begin{array}{c} \tilde{T}^{-n}\theta(x,y) \in \theta(\mathcal{B}_\delta \cap \mathcal{Q}_2) \quad \forall n \geq 0, \text{ and} \\ \lim \tilde{T}^{-n}\theta(x,y) = (0,0) \end{array} \right\} \\
\iff &(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 \text{ and } \theta(x,y) \in \mathcal{C}_1^- \\
\iff &(x,y) \in \theta^{-1}(\mathcal{C}_1^-) \cap \mathcal{B}_\delta \\
\iff &(x,y) \in \mathcal{C}_1^+,
\end{aligned}$$

which completes the proof of (ii).

To prove (iii), set $\mathcal{S}_1' = \theta^{-1}(\mathcal{S}_1) \cap \mathcal{B}_\delta$ and $\mathcal{S}_2' = \theta^{-1}(\mathcal{S}_2) \cap \mathcal{B}_\delta$. Then by Corollary 12,

$$\begin{aligned}
(x,y) \in \mathcal{S}_1' &\implies (x,y) \in \theta^{-1}(\mathcal{S}_1) \cap \mathcal{B}_\delta \\
&\implies \theta(x,y) \in \mathcal{S}_1 \text{ and } (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2.
\end{aligned}$$

By the definition of $\tilde{T}$ and $\mathcal{S}_1$, for some $\bar{x} \in (0, \delta)$,

$$\begin{aligned}
\theta(x,y) \in \mathcal{S}_1 \text{ and } (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 &\implies \tilde{T}^{-n}\theta(x,y) \to (\bar{x}, 0) \\
&\implies \theta^{-1}\tilde{T}^{-n}\theta(x,y) \to \theta^{-1}(\bar{x}, 0) \qquad (233) \\
&\implies T^n(x,y) \to (-\bar{x}, 0)
\end{aligned}$$

and

$$\begin{aligned}
\theta(x,y) \in \mathcal{S}_1 \text{ and } (x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_2 &\implies \exists k \in \mathbb{N} \text{ such that } \tilde{T}^k\theta(x,y) \notin \mathcal{B}_{\delta'} \cap \mathcal{Q}_1 \\
&\implies \theta^{-1}\tilde{T}^k\theta(x,y) \notin \theta^{-1}(\mathcal{B}_{\delta'} \cap \mathcal{Q}_1) \\
&\implies T^{-k}(x,y) \notin \mathcal{B}_\delta \cap \mathcal{Q}_2.
\end{aligned}$$

The last implication follows from $\theta(\mathcal{B}_{\delta'} \cap \mathcal{Q}_2) \subset \mathcal{B}_\delta \cap \mathcal{Q}_1$. The proof that for (x,y) in $\mathcal{S}_2'$, both $T^n(x,y)$ and $T^{-n}(x,y)$ eventually leave $\mathcal{B}_\delta \cap \mathcal{Q}_2$ is similar and will be omitted, completing the proof of (iii).

Finally, to prove (iv), choose $\delta > 0$ such that $\theta(\mathcal{B}_\delta \cap \mathcal{Q}_3) \subset \mathcal{B}_{\delta'} \cap \mathcal{Q}_4$. If $(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_3$, then $\theta(x,y) \in \mathcal{B}_{\delta'} \cap \mathcal{Q}_4$, so $\theta(x,y)$ belongs to the unstable manifold of a fixed point of $\tilde{T}$ on the positive x semi-axis. By an argument nearly identical to equation (233) we have that every nonzero point $(x,y) \in \mathcal{B}_\delta \cap \mathcal{Q}_3$ belongs to the stable manifold of a fixed point of T on the negative x semi-axis. $\square$

3.7 Appendix

The following theorem is an adaptation of Theorem 5.1 in [8].

Theorem A. *Let S be a real analytic map on a set $\mathcal{R} \subset \mathbb{R}^2$ with nonempty interior, and let $(\bar{x}, \bar{y})$ be an interior fixed point of S. Let the characteristic values of S at $(\bar{x}, \bar{y})$ be α and β, and let E_α be the eigenspace associated to α. If α and β are real numbers and satisfy $|\alpha| < 1 \leq \beta$, then there exists a neighborhood $\mathcal{N}_\delta$ of $(\bar{x}, \bar{y})$ and a C^1 invariant curve $\mathcal{C} \subset \mathcal{N}_\delta$ that is tangential to E_α at $(\bar{x}, \bar{y})$ such that $S^n(x,y) \in \mathcal{N}_\delta$ and $S^n(x,y) \to (\bar{x}, \bar{y})$ whenever $(x,y) \in \mathcal{C}$. If α and β*

are real numbers and satisfy $|\alpha| > 1 \geq \beta$, *then there exists a neighborhood* $\mathcal{N}_\delta$ *of* $(\bar{x}, \bar{y})$ *and a* C^1 *invariant curve* $\mathcal{C} \subset \mathcal{N}_\delta$ *that is tangential to* E_α *at* $(\bar{x}, \bar{y})$ *such that* $S^{-n}(x, y) \in \mathcal{N}_\delta$ *and* $S^{-n}(x, y) \to (\bar{x}, \bar{y})$ *whenever* $(x, y) \in \mathcal{C}$.

The theorem below (with some changes in notation) is Theorem 8.6.4 in [17].

Theorem B. Let A be an open interval in $\mathbb{R}$, $\{f_n\}$ be a sequence of differentiable fns. $f_n : A \to \mathbb{R}$. Suppose that

(a) there exists $x_0 \in A$ such that $\{f_n(x_0)\}$ converges in $\mathbb{R}$,

(b) for each $a \in A$ there exists a neighborhood $(a - \delta_a, a + \delta_a) \subset A$ such that f_n' converges uniformly on $(a - \delta_a, a + \delta_a)$.

Then for each $a \in A$, the sequence $\{f_n\}$ converges uniformly on $(a - \delta_a, a + \delta_a)$. Furthermore, if for each $x \in A$, $f(x) = \lim f_n(x)$ and $g(x) = \lim f_n'(x)$, then $g(x) = f'(x)$ for each $x \in A$.

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APPENDIX

Future Work

There are still many open problems related to the local qualitative behavior of maps near non-hyperbolic fixed points. Some of the techniques used to prove the main theorem in Manuscript 3 can be applied or adapted to be applied other scenarios. Specifically, I would like to prove the following conjecture, which I believe to be tractable with the methods that were developed in Manuscript 3:

Conjecture 1. *Let V be an open neighborhood of the origin in $\mathbb{R}^2$, and let T be a real analytic map on V with an isolated fixed point whose jacobian is similar to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ where $0 < \alpha < 1$. Then if T is in a normalized form, near the origin there exist exactly three non-conjugate dynamic scenarios depending upon the structure of the normal form:*

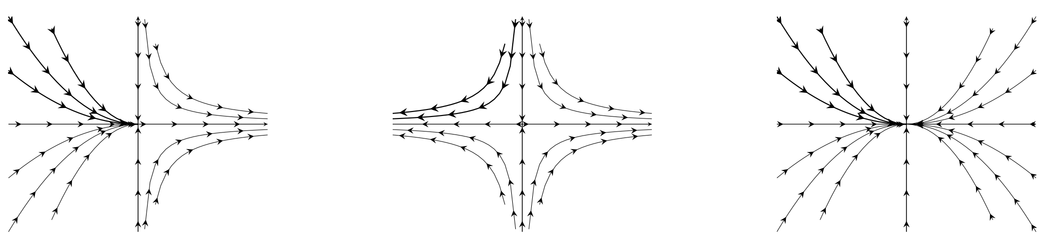

Figure A.1. The curves in the figures above represent the path and direction of orbits under forward iteration of the normalized function T.

A similar conjecture can be made for the case when $\alpha > 1$.

Knowing whether Conjecture 1 is true is an important problem because it describes precisely the (local) basin of attraction of the fixed point, which is an open problem partially addressed for competitive systems in [1], and understanding the basin of attraction of the non-hyperbolic fixed point simplifies the analysis of the global behavior of a difference equation. A more ambitious project that I am interested in is a classification of the possible local dynamical behaviors near an *isolated* 1-1 resonant fixed point. In this situation the local dynamics are considerably more complicated; computer simulations suggest that there will be eight different dynamic scenarios. Some of the dynamic scenarios can be proven using the methods developed in Manuscript 3, but other scenarios will require new techniques.

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