

12-7-2021

The principal Erdős–Gallai differences of a degree sequence

Michael D. Barrus

University of Rhode Island, barrus@uri.edu

Follow this and additional works at: https://digitalcommons.uri.edu/math_facpubs

The University of Rhode Island Faculty have made this article openly available.
Please let us know how Open Access to this research benefits you.

This is a pre-publication author manuscript of the final, published article.

Terms of Use

This article is made available under the terms and conditions applicable towards Open Access Policy Articles, as set forth in our [Terms of Use](#).

Citation/Publisher Attribution

Barrus, Michael D. The principal Erdos-Gallai differences of a degree sequence. *Discrete Mathematics*, vol. 345 (April 2022), no. 4, Article 112755. <https://doi.org/10.1016/j.disc.2021.112755>

This Article is brought to you for free and open access by the Mathematics at DigitalCommons@URI. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of DigitalCommons@URI. For more information, please contact digitalcommons@etal.uri.edu.

The principal Erdős–Gallai differences of a degree sequence

Michael D. Barrus

Department of Mathematics and Applied Mathematical Sciences
University of Rhode Island, USA

December 7, 2021

Abstract

The Erdős–Gallai criteria for recognizing degree sequences of simple graphs involve a system of inequalities. Given a fixed degree sequence, we consider the list of differences of the two sides of these inequalities. These differences have appeared in varying contexts, including characterizations of the split and threshold graphs, and we survey their uses here. Then, enlarging upon properties of these graph families, we show that both the last term and the maximum term of the principal Erdős–Gallai differences of a degree sequence are preserved under graph complementation and are monotonic under the majorization order and Rao’s order on degree sequences.

1 Erdős–Gallai differences

Let $d = (d_1, \dots, d_n)$ be the degree sequence of an arbitrary finite, simple graph, and suppose that the terms of d are listed in nonincreasing order. Let $m(d) = \max\{i : d_i \geq i - 1\}$; this parameter is called the *modified Durfee number* of d . We define the k th Erdős–Gallai difference $\Delta_k(d)$ by

$$\Delta_k(d) = k(k - 1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i\leq k} d_i,$$

for nonnegative integers k .

Though we allow $k = 0$ (so $\Delta_0(d) = 0$) for convenience, we will primarily be interested in the terms of the list

$$\Delta(d) = (\Delta_1(d), \dots, \Delta_{m(d)}(d)).$$

We call $\Delta(d)$ the list of *principal* Erdős–Gallai differences.

The Erdős–Gallai differences have appeared implicitly or explicitly in the work of several authors (see, for example, the work of Koren [21], or that of Li [22], in which the differences appear with the opposite sign). Functioning

almost like a “spectrum” of a degree sequence, they provide some intriguing pieces of information about graphs, but as yet they seem to have attracted limited notice. In this paper we begin by surveying known properties of $\Delta(d)$. In later sections we prove that $\Delta(d)$ behaves “nicely” under the operation of graph complementation (as applied to degree sequences) and under the majorization relation and another degree-sequence-based relation introduced by Rao [27].

We organize our survey by the contexts in which the terms of $\Delta(d)$ have previously appeared, briefly summarizing each.

Origins and nonnegativity. The Erdős–Gallai differences take their name from a classic result of Erdős and Gallai that characterizes the degree sequences of simple graphs.

Theorem 1.1 ([13]). *A list $d = (d_1, \dots, d_n)$ of nonnegative integers, in nonincreasing order, is the degree sequence of a simple graph if and only if its entries have even sum and*

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i > k} \min\{k, d_i\} \quad (1)$$

holds for all $k \in \{1, \dots, n\}$.

Our definition of $\Delta_k(d)$ is simply the difference between the two sides in the inequality (1) with parameter value k . Consequently, a sorted list d of n nonnegative integers with even sum is a degree sequence if and only if $\Delta_k(d) \geq 0$ for all n .

Relations to each other. Authors have improved upon Theorem 1.1 in various ways. One fruitful way weakens the conditions by checking only certain of the inequalities.

Theorem 1.2 (Li [22]; Hammer–Ibaraki–Simeone [17]). *A sequence (d_1, \dots, d_n) of nonnegative numbers with even sum, arranged in nonincreasing order, is a degree sequence if and only if the k th Erdős–Gallai inequality holds for all values of k such that $1 \leq k \leq m(d)$, where $m(d) = \max\{i : d_i \geq i - 1\}$.*

Note here the appearance of the modified Durfee number $m(d)$. It is an upper bound on various graph parameters, such as the clique number. Theorem 1.2’s strengthening of Theorem 1.1 suggests why the terms $\Delta_k(d)$ for $1 \leq k \leq m(d)$ should be the “principal” Erdős–Gallai differences. When we consider complementary sequences in Section 3, we will see another reason.

The way that Theorem 1.2 and similar criteria (as seen in [6, 12, 29, 33], for example) are proved involves showing that certain values of $\Delta(d)$ are bounded by others, so the nonnegativity of $\Delta_k(d)$ need only be checked for certain k . Though we do not review all such statements here, we illustrate this idea with the following results of Li.

Lemma 1.3 (Li [22]). *Let d be a degree sequence $d = (d_1, \dots, d_n)$ in nonincreasing order, and let $m = m(d)$.*

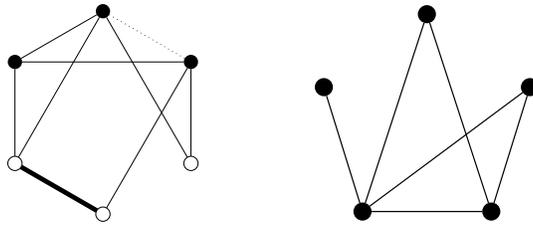


Figure 1: A graph with splittance 2 (left) and a threshold graph (right).

- (i) The differences $\Delta_m(d), \Delta_{m+1}(d), \dots, \Delta_n(d)$ form a strictly increasing sequence.
- (ii) If p denotes $\max\{i : d_i \geq m\}$, then the differences $\Delta_p(d), \Delta_{p+1}(d), \dots, \Delta_m(d)$ are all equal.
- (iii) If $\Delta_q(d)$ is nonnegative, where q is the last index in which the maximum value from d appears, then $\Delta_i(d) \geq 0$ for all $i \in \{1, \dots, \min\{q, m\}\}$.

Splittance and the last term of $\Delta(d)$. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set. In [19], Hammer and Simeone defined the *splittance* of a graph G as the minimum number of edges that can be deleted or added to G in order to change it into a split graph. For example, the graph on the left in Figure 1 has splittance 2; if an edge is supplied in place of the dotted nonadjacency, and if the thick edge is removed from the graph, then the resulting graph is split with the black vertices forming a clique and the white vertices forming an independent set.

Hammer and Simeone showed that splittance is invariant among all realizations of a degree sequence, and if the degree sequence is $d = (d_1, \dots, d_n)$, with terms written in nonincreasing order, then the splittance is exactly

$$\frac{1}{2} \left(m(m-1) + \sum_{i=m+1}^n d_i - \sum_{i=1}^m d_i \right)$$

where $m = m(d)$. Split graphs are precisely the graphs with splittance 0. We restate this result in terms of $\Delta(d)$.

Corollary 1.4. *If d is an arbitrary degree sequence and G is an arbitrary realization of d , then $\Delta_{m(d)}(d) = 2s(G)$, where $s(G)$ denotes the splittance of G .*

Consequently, a graph G having degree sequence d is split if and only if the last term of $\Delta(d)$ equals 0.

Proof. Recall that $\Delta(d)$ has length $m(d)$. Note that though the expression defining splittance varies slightly in form from our definition of $\Delta_{m(d)}(d)$, it is true that $\min\{m, d_i\} = d_i$ for all $i > m$, so $\Delta_m(d)$ is twice the splittance of G . \square

Graph families with degree sequence characterizations. The result of Hammer and Simeone just described shows that whether a graph is split depends only on its degree sequence. Another family of graphs where membership is determined by the degree sequence is that of the threshold graphs. As defined by Chvátal and Hammer in [10], a threshold graph is a graph whose characteristic vectors for independent sets can be separated by a hyperplane from those of non-independent sets. Equivalently, a graph G is threshold if and only if there exists a weighting of its vertices and a real threshold such that two vertices in G are adjacent exactly when the sums of their weights meets or exceeds the threshold. For example, the graph on the right in Figure 1 is a threshold graph, since its adjacencies satisfy the definition if each vertex is given weight equal to its degree and the threshold 4.5 is used.

Threshold graphs have many additional characterizations, in such settings as nested neighborhood conditions, forbidden subgraphs, and the majorization order on degree sequences; see [24] for a book-length survey including all of these, and see [18] for a characterization in terms of Laplacian eigenvalues. The degree sequence characterization was found by Hammer, Ibaraki, and Simeone and is implicit in earlier work by Li; it may be restated in terms of $\Delta(d)$ as the following.

Theorem 1.5 ([16, 17]; see also [22, Theorem 18]). *A graph G having degree sequence d is threshold if and only if $\Delta_k(d) = 0$ for all $k \in \{1, \dots, m(d)\}$. In other words, G is threshold if and only if every term of $\Delta(d)$ is 0.*

Forced graph structure corresponding to small values in $\Delta(d)$. Besides the split and threshold graphs, other classes with degree sequence characterizations include the pseudosplit graphs [23], the matroidal and matrogenic graphs [25, 32], the hereditary unigraphs [4], and the unigraphs [20, 21, 22]. For each of these cases, the characterization relies in part on a (perhaps iterated) decomposition of the vertex set $V(G)$ of a graph G into sets Q, R, S such that the subgraph induced on $Q \cup S$ is a split graph with clique Q and independent set S , and each vertex in R is adjacent to every vertex in Q and to no vertex in S . This structure is illustrated on the left in Figure 2 by the solid and dotted lines and curves; note that there is no restriction on edges between Q and S . This decomposition, repeatedly carried out until all induced subgraphs involved are indecomposable, yields the *canonical decomposition* of graphs, as defined and extensively studied by Tyshkevich and others (see [30] and the references therein).

This decomposition yields insight into how the principal Erdős–Gallai differences are tied to the structure of a graph. One striking connection (and another motivation for focusing attention on the “principal” differences) is the following. Here we use $G[X]$ to denote the induced subgraph of G with a given subset X of the vertex set $V(G)$.

Theorem 1.6 ([5]). *If d is the degree sequence of G , and if $V(G)$ is partitioned into Q, R, S as previously described, then $\Delta(d)$ equals the list formed by appending $\Delta(G[R])$ to $\Delta(G[Q \cup S])$.*

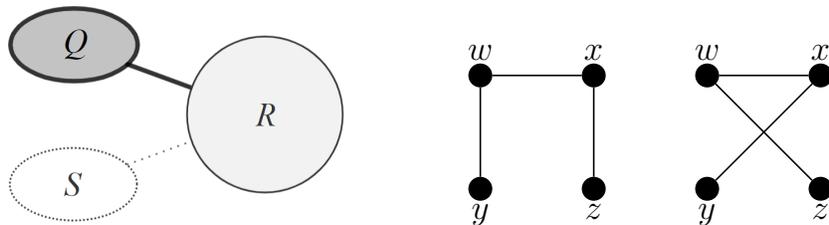


Figure 2: Decomposition corresponding to $\Delta_k(d) = 0$ (left) and the labeled realizations of $(2, 2, 1, 1)$ (right).

There are further connections between $\Delta(d)$ and the canonical decomposition.

Theorem 1.7 ([4]). *A graph G with at least two vertices admits a partition of $V(G)$ into sets Q, R, S as previously described, with $Q \cup S$ and R both nonempty, if and only if one of the following holds.*

- (i) *G has one or more isolated vertices; here we may let S consist of one isolated vertex, with $Q = \emptyset$ and $R = V(G) \setminus (Q \cup S)$.*
- (ii) *$\Delta_k(d) = 0$ for some positive integer k ; here the vertices in Q may be k vertices of highest degree in G , the vertices in S are all those having a degree in G strictly less than k , and $R = V(G) \setminus (Q \cup S)$.*

Note that if the partition Q, R, S exists for one realization of a degree sequence d , then these sets' same prescribed adjacencies and non-adjacencies (i.e., all possible edges within Q , none within S , and the edges/non-edges required between R and the other two sets) continue to hold in every other realization of d having the same vertex set and vertex degrees.

These ideas lead to a new problem. Note that when $\Delta_k(d) = 0$ for some k , the canonical decomposition leads to certain vertices being guaranteed to be adjacent, or guaranteed to be nonadjacent, in every realization of d . We see this on the left in Figure 2 and more concretely on the right. Here we have the two realizations of $(2, 2, 1, 1)$. These are the only two realizations when the vertices are labeled and w and x are required to have degree 2 while y and z are required to have degree 1. Note that $\Delta((2, 2, 1, 1)) = (1, 0)$. Here w and x are adjacent in each realization, and y and z are nonadjacent in each realization. We say that w and x are “forcibly” adjacent, and y and z are “forcibly” nonadjacent.

In [2, 11] the author and Cloteaux independently studied forcible adjacency relationships such as these. Such a phenomenon is not restricted to split graphs like P_4 or to graphs with a nontrivial canonical decomposition; for example, in each of the nine labeled realizations of $d = (4, 4, 3, 3, 3, 1)$, the vertices of degree 4 are adjacent, though the Erdős–Gallai difference list $\Delta(d) = (1, 1, 2, 2)$ contains no 0 term. The paper [2] shows that the occurrence of a forcible adjacency relationship requires that $0 \leq \Delta_k(d) \leq 1$ for some k ; when some minor technicalities are satisfied, the converse is true as well.

Building off these results, and returning to the theme of special graph families like those mentioned earlier, the paper [5] defines the *weakly threshold graphs* as those graphs having degree sequences d for which $\Delta_k(d) \leq 1$ for all k . In addition to their degree sequence characterization, the weakly threshold graphs, like the threshold graphs, split graphs, matroidal/matrogenic graphs, and/or hereditary unigraphs, have characterizations in terms of forbidden induced subgraphs, iterative constructions, and more. These enlarge upon several corresponding results for threshold graphs in pleasing ways.

In light of these several contexts in which $\Delta(d)$ appears, we ask whether more useful information about a degree sequence or its realizations may be obtained through study of $\Delta(d)$. In particular, given the results related to split graphs, threshold graphs, and weakly threshold graphs, the final term $\Delta_{m(d)}(d)$ and the maximum term of $\Delta(d)$, which we denote by $\Delta^*(d)$, may be of interest. In this paper, we present results along these lines.

We begin in Section 2 by presenting matrices that aid in computing and visually representing Erdős–Gallai differences.

In Section 3 we tackle a problem suggested by the classes of split graphs, threshold graphs, and weakly threshold graphs. Each of these families is closed under complementation, and we prove here that degree sequences of a graph and of its complement always agree on their maximum Erdős–Gallai differences and on their final differences.

Finally, in Section 4 we show that certain Erdős–Gallai differences behave monotonically as one moves upward through one of two posets. We first show that both the last term of $\Delta(d)$ (equivalently, the splittance) and the maximum term are nondecreasing while moving “upward” through degree sequences in an induced-subgraph-related poset introduced by Rao [27]. We then improve upon a result from [5], in which it was observed that the k th Erdős–Gallai difference decreases or stays the same as one moves upward through the majorization/dominance poset; here we exactly quantify the changes in Δ_k during such moves. We see that with this poset as well, both the last and the maximum terms of $\Delta(d)$ exhibit monotonic behavior as one moves up the poset.

2 The matrix $M(d)$

We preface our results on complements and posets in later sections by presenting two matrices helpful in studying $\Delta(d)$. The first is a version of the *corrected Ferrers diagram*, which has appeared multiple times in degree sequence literature (our presentation follows [1]; see also [8], for example). Given a degree sequence $d = (d_1, \dots, d_n)$ with terms in nonincreasing order, define $F(d)$ to be the n -by- n matrix having stars on the main diagonal in which for each $i \in \{1, \dots, n\}$ the first d_i non-diagonal entries in the i th row are 1, with all other entries set equal to 0.

When performing matrix computations on a corrected Ferrers diagram $F(d)$ in what follows, we note that the transpose $F(d)^T$ preserves the stars on the

main diagonal. We adopt the convention that the numerical value of a star is 0, though for convenience we may continue to write stars instead of 0's on the main diagonal, even after the result of matrix arithmetic.

We now relate the corrected Ferrers diagram to Erdős–Gallai differences. As observed in [5], for each value $k \in \{1, \dots, m(d)\}$, Erdős–Gallai difference $\Delta_k(d)$ is the difference of (i) the number of nonzero terms below the main diagonal of F within the first k columns of $F(d)$ and (ii) the number of nonzero terms to the right of the main diagonal within the first k rows of $F(d)$. We adapt this result with the definition of a second helpful matrix.

Definition 2.1. Given a degree sequence d , define its (*Erdős–Gallai*) *difference matrix* as the matrix $M(d) = F(d)^T - F(d)$, where \star entries (still having a value of 0) continue to occupy the main diagonal.

Observation 2.2. For any degree sequence d and value $k \in \{0, \dots, m(d)\}$, the k th Erdős–Gallai difference $\Delta_k(d)$ is equal to the sum of the entries in the first k rows of $M(d)$.

We note that the matrix $M(d)$ has the form

$$\begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix},$$

where the 0 block in the top left is a square block of size $m(d)$; more generally, 0 entries in $M(d)$ occur where both the matrices $F(d)$ and $F(d)^T$ agree. Nonzero entries in the blocks B and B^T are ± 1 and occur in locations where one of $F(d), F(d)^T$ is 0 and the other is 1; the definition of $M(d)$ as $F(d)^T - F(d)$ ensures that $M(d)$ is skew-symmetric.

Example. Suppose that $d = (6, 5, 3, 3, 3, 1, 1, 1, 1)$. The matrix $F = F(d)$ is shown here:

$$F = \begin{bmatrix} \star & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \star & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & \star & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \star & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \star & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \end{bmatrix}.$$

Below, we show the difference matrix $M(d) = F^T - F$.

$$M(d) = \begin{bmatrix} \star & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \star & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \star & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \end{bmatrix}.$$

Note that $m(d) = 4$. Summing the entries in the first i rows for $1 \leq i \leq 4$ shows that $\Delta(d) = (2, 1, 2, 2)$. \blacksquare

In light of Observation 2.2, the k th Erdős–Gallai difference is a signed count of asymmetries in a portion of $F(d)$. We will see in Theorem 3.3 that summing the entries in the first i rows for larger i than $m(d)$ also gives useful information.

We conclude this section with an observation illustrated by the matrix $M(d)$ in the example above.

Lemma 2.3. *For any degree sequence d with $m = m(d)$, if $d_m = m - 1$, then $\Delta_m(d) = \Delta_{m-1}(d)$.*

Proof. Under these conditions, the m th row of $M(d)$ contains only zeroes, so Observation 2.2 yields the result. \square

3 Complementation

In this section we show that some of the values appearing in $\Delta(d)$ appear in $\Delta(\bar{d})$, where \bar{d} is the degree sequence of the complement of a realization of d . In particular, the maximum and final principal Erdős–Gallai differences agree, i.e., $\Delta^*(\bar{d}) = \Delta^*(d)$ and $\Delta_{m(d)}(d) = \Delta_{m(\bar{d})}(\bar{d})$.

These result generalize properties observed in those graph classes mentioned in Section 1 whose membership can be recognized through Erdős–Gallai differences. Each of the classes of threshold graphs, weakly threshold graphs, and split graphs is closed under complementation. Proving this is typically done by considering structural characterizations of those graphs or by noting that the set of minimal forbidden induced subgraphs for each class is likewise closed under complementation; see [5, 10, 14]. As a consequence of our work here, we obtain a proof based entirely on vertex degrees.

First, we develop a useful matrix operation that has appeared in various places but seems not to have developed a standard notation. Our presentation is adapted from [15]. Given a matrix M , let M^T denote its usual transpose about the main diagonal. We use the notation M_{\perp} to indicate the transpose of M “about its antidiagonal”; if the entries of M are denoted by $M_{i,j}$, where

$1 \leq i \leq m$ and $1 \leq j \leq n$, then the (i', j') -entry of M_\perp is defined by

$$(M_\perp)_{i', j'} = M_{(m+1-j'), (n+1-i')}$$

for $1 \leq i' \leq n$ and $1 \leq j' \leq m$. So, for example, we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_\perp = \begin{bmatrix} 6 & 3 \\ 5 & 2 \\ 4 & 1 \end{bmatrix}.$$

It is straightforward to verify that the actions of transposing about the main diagonal and the antidiagonal commute with each other, and their composition corresponds to a 180° “rotation” of the matrix. As with the standard transpose, we also find that \perp distributes over sums of matrices and that $(M_\perp)_\perp = M$ for each matrix M . As suggested by [15], if M is an m -by- n matrix and H_t denotes the t -by- t matrix with $H_{i,j} = 1$ if $i + j = t + 1$ and $H_{i,j} = 0$ otherwise (assuming that $1 \leq i, j \leq t$), then

$$M_\perp = H_n M^T H_m.$$

This allows us to conclude, for instance, that for any matrices A, B that are m -by- p and p -by- n , respectively,

$$(AB)_\perp = H_n (AB)^T H_m = H_n B^T A^T H_m = H_n B^T H_p H_p A^T H_m = B_\perp A_\perp.$$

Turning now to Ferrers diagrams, note that, like $F(d)^T$, the matrix $F(d)_\perp$ preserves the appearance of stars on the main diagonal.

For any degree sequence d , observe that the complementary sequence \bar{d} is equal to $(n-1-d_n, \dots, n-1-d_1)$. The matrix $M(\bar{d})$ also has a nice relationship with $M(d)$.

Lemma 3.1. *If d is a degree sequence and \bar{d} is its complementary sequence, then $M(\bar{d}) = M(d)_\perp$.*

Proof. The matrix $F(\bar{d})$ is obtained by rotating $F(d)$ by 180° and switching non-diagonal entries from 0 to 1, and vice versa. Using matrix operations, we write

$$F(\bar{d}) = (J_n - I_n - F(d))_\perp^T,$$

where I_n and J_n denote the n -by- n identity and all-ones matrices, respectively. We then compute:

$$\begin{aligned} M(\bar{d}) &= F(\bar{d})^T - F(\bar{d}) \\ &= [(J_n - I_n - F(d))_\perp^T]^T - (J_n - I_n - F(d))_\perp^T \\ &= (F(d)^T - F(d))_\perp \\ &= M(d)_\perp. \end{aligned}$$

□

Example. Suppose again that $d = (6, 5, 3, 3, 3, 1, 1, 1, 1)$. We compute $\bar{d} = (7, 7, 7, 7, 5, 5, 3, 2)$. The matrices $F(\bar{d})$ and $M(\bar{d})$ are as follows:

$$F(\bar{d}) = \begin{bmatrix} \star & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \star & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & \star & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \star & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \star & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \star & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & \star & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \star \end{bmatrix} = (J_9 - I_9 - F(d))_{\perp}^T$$

(notice how the 0's "trace out" the shape $F(d)$'s 1's, but in "rotated" manner);

$$M(\bar{d}) = \begin{bmatrix} \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \star & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \star \end{bmatrix} = M(d)_{\perp}.$$

Here $m(\bar{d}) = 6$, and summing the entries in the first \bar{d} rows of $M(\bar{d})$ shows that $\Delta(\bar{d}) = (1, 2, 2, 1, 2, 2)$. \blacksquare

As we will see, these connections between matrices related to d and to \bar{d} imply that the Erdős–Gallai differences agree on certain values. We begin with a brief lemma that is not original (see [22, Proposition 16] for a similar result) but is included here for completeness.

Lemma 3.2. *Each degree sequence (d_1, \dots, d_n) and its complementary sequence $(\bar{d}_1, \dots, \bar{d}_n) = (n - 1 - d_n, \dots, n - 1 - d_1)$ satisfy*

$$n \leq m(d) + m(\bar{d}) \leq n + 1.$$

Equality holds in the latter inequality if and only if $d_m = m - 1$.

Proof. Abusing notation, let $m = m(d)$ and $\bar{m} = m(\bar{d})$. Observe that

$$\bar{d}_{n-m} = n - 1 - d_{n+1-(n-m)} = n - 1 - d_{m+1} > n - 1 - m;$$

hence $\bar{m} \geq n - m$.

On the other hand,

$$\begin{aligned} \bar{d}_{n-m+2} &\leq \bar{d}_{n-m+1} = n - 1 - d_{n+1-(n-m+1)} \\ &= n - 1 - d_m \leq n - 1 - (m - 1) < (n - m + 2) - 1; \end{aligned}$$

thus $\bar{m} \leq n - m + 1$.

Now if $m + \bar{m} = n$, then $\bar{d}_{n-m+1} < n - m$, so $n - 1 - d_m < n - m$ and thus $d_m > m - 1$. If $m + \bar{m} = n + 1$, then $\bar{d}_{n+1-m} \geq n - m$ and $n - 1 - d_m \geq n - m$ and so $d_m \leq m - 1$; since $d_m \geq m - 1$, this implies that $d_m = m - 1$. \square

Theorem 3.3. *Let d be a degree sequence. For $i \in \{0, \dots, n\}$, let $\sigma(d, i)$ denote the sum of all entries in the first i rows of $M(d)$. Then*

$$\sigma(d, i) = \begin{cases} \Delta_i(d) & \text{if } 0 \leq i \leq m(d); \\ \Delta_{n-i}(\bar{d}) & \text{if } n - m(\bar{d}) \leq i \leq n. \end{cases}$$

Proof. The first case on the right-hand side follows from Observation 2.2.

For $0 \leq k \leq n$, let h_k denote the 1-by- n vector in which the first k entries are 1 and all other entries are 0. Let $\mathbf{1}$ denote the n -by-1 all-ones vector. For convenience let $I = I_n$ and $J = J_n$. Now observe that for $1 \leq k \leq m(\bar{d})$, since $\Delta_k(\bar{d})$ is a scalar and $M(d)$ is a skew symmetric matrix,

$$\begin{aligned} \Delta_k(\bar{d}) &= h_k M(\bar{d}) \mathbf{1} \\ &= h_k M(d)_{\perp} \mathbf{1} \\ &= (\mathbf{1}^T - h_{n-k})_{\perp}^T M(d)_{\perp} \mathbf{1}_{\perp}^T \\ &= [(\mathbf{1}^T - h_{n-k}) M(d)^T \mathbf{1}]_{\perp}^T \\ &= (\mathbf{1}^T - h_{n-k}) M(d)^T \mathbf{1} \\ &= \mathbf{1}^T M(d)^T \mathbf{1} - h_{n-k} M(d)^T \mathbf{1} \\ &= 0 - h_{n-k} M(d)^T \mathbf{1} \\ &= h_{n-k} M(d) \mathbf{1}. \end{aligned}$$

Hence the k th Erdős–Gallai difference of \bar{d} is the sum of the entries in all but the last k rows of the matrix $M(d)$; this yields the second case on the right-hand side. \square

Theorem 3.3 shows that the sequence of terms $\sigma(d, k)$ lists the Erdős–Gallai differences of d , in order, transitioning in later terms to list the Erdős–Gallai differences of \bar{d} in reverse order.

Example. To conclude the example from Section 2, the terms $\sigma(d, k)$ for $d = (6, 5, 3, 3, 3, 1, 1, 1, 1)$ are, beginning with $\sigma(d, 0)$,

$$0, 2, 1, 2, 2, 1, 2, 2, 1, 0;$$

note that the initial terms of the sequence are $\Delta_0(d)$, followed by the principal Erdős–Gallai differences for d (i.e., 2, 1, 2, 2), while the differences for \bar{d} , which were 1, 2, 2, 1, 2, 2, show up in reverse order before the final term $\Delta_0(\bar{d})$. \blacksquare

That the final principal Erdős–Gallai differences for d and \bar{d} overlap in the example above is no coincidence, and it illustrates our first result on common differences for d and \bar{d} .

Corollary 3.4. *If d is a degree sequence and we let $m = m(d)$ and $\bar{m} = m(\bar{d})$, then $\Delta_{\bar{m}}(\bar{d}) = \Delta_m(d)$.*

Proof. If $m + \bar{m} = n$, then $n - \bar{m} = m$, and Theorem 3.3 yields $\Delta_{\bar{m}}(\bar{d}) = \sigma(d, n - \bar{m}) = \sigma(d, m) = \Delta_m(d)$.

If $m + \bar{m} = n + 1$, then $d_m = m - 1$, so Lemma 2.3 and Theorem 3.3 yield $\Delta_{\bar{m}}(\bar{d}) = \sigma(d, m - 1) = \Delta_{m-1}(d) = \Delta_m(d)$. \square

As we now show, there are often additional values that appear among the principal Erdős–Gallai differences of both d and \bar{d} . These arise from “islands” of nonzero entries in $M(d)$. In the following, use *line* to refer to any row or column of a matrix, and recall that star entries are treated as zero entries.

Lemma 3.5. *The following statements hold for nonzero entries in $M(d)$.*

- (i) *If two entries in a single line of $M(d)$ are both nonzero, then they and all entries between them in that line are equal.*
- (ii) *$M(d)$ contains no submatrix of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where a and c are nonzero.*

Proof. In the difference matrix $M(d) = F(d)^T - F(d)$, a nonzero entry occurs in a position where one of $F(d)$ or $F(d)^T$ has a zero entry and the other does not. Both statements (i) and (ii) follow from the fact that in any line of $F(d)$ or $F(d)^T$, each 1 entry precedes any 0 entries. \square

Lemma 3.6. *For $k \in \{1, \dots, m(d)\}$, suppose that no nonzero entry in the $(k + 1)$ th row of $M(d)$ lies in the same column as a nonzero entry in the k th row. If the leftmost nonzero entry among the first k rows appears in the j th column of $M(d)$, then $\Delta_{n+1-j}(\bar{d}) = \Delta_k(d)$.*

Proof. Note that under the given hypotheses we have $m(d) < j \leq n$, which together with Lemma 3.2 yields $1 \leq n + 1 - j \leq n - m(d) \leq m(\bar{d})$.

Now by hypothesis and Lemma 3.5, the nonzero entries in the first k rows of $M(d)$ lie in columns that contain only 0 entries following the k th entry. Hence the sum of the first k rows of $M(d)$, which equals $\Delta_k(d)$, is also equal to the sum of the columns containing these nonzero entries, which also equals the sum of the entries in each of the columns j, \dots, n in $M(d)$, by Lemma 3.5(ii). This is precisely the sum of the entries in the first $n + 1 - j$ rows of $M(d)_\perp$, and since $M(d)_\perp = M(\bar{d})$, we conclude that $\Delta_k(d) = \Delta_{n+1-j}(\bar{d})$. \square

Theorem 3.7. *Let d be a degree sequence, and let \bar{d} be its complementary sequence. If one of the following is true, then $\Delta_k(d)$ is one of the principal Erdős–Gallai differences of \bar{d} :*

- (i) $k = m(d)$;
- (ii) $k = 1$ and $\Delta_1(d) > \Delta_2(d)$;
- (iii) $k \in \{2, \dots, m(d) - 1\}$ and it is not true that $\Delta_{k-1}(d) < \Delta_k(d) < \Delta_{k+1}(d)$ or that $\Delta_{k-1}(d) > \Delta_k(d) > \Delta_{k+1}(d)$.

In particular, the maximum principal Erdős–Gallai differences coincide, i.e., $\Delta^*(\bar{d}) = \Delta^*(d)$.

Proof. The case $k = m(d)$ was shown in Corollary 3.4.

In each of the other cases, if $\Delta_k(d) = v$ for some v , it suffices to assume that k is the lowest index making $\Delta_k(d) = v$ and satisfying the conditions of the theorem. This guarantees that the k th row of the difference matrix $M(d)$ contains a nonzero entry.

Suppose first that $\Delta_k(d) = \Delta_{k+1}(d)$. Then the $(k+1)$ th row of $M(d)$ must contain only 0 entries, so the hypotheses of Lemma 3.6 are met, and $\Delta_k(d)$ appears as a principal Erdős–Gallai difference of \bar{d} .

If $\Delta_k(d) > \Delta_{k+1}(d)$, then by our assumptions either $k = 1$ or $\Delta_{k-1}(d) < \Delta_k(d)$, so the k th row of $M(d)$ contains positive entries. We also see that the $(k+1)$ th row of $M(d)$ contains no positive entries. By Lemma 3.5, this implies that no nonzero entry in the $(k+1)$ th row of $M(d)$ lies in the same column as a nonzero entry in the k th row, so the hypotheses of Lemma 3.6 are met, and we again have the desired conclusion.

If $\Delta_k(d) < \Delta_{k+1}(d)$, then $\Delta_{k-1}(d) > \Delta_k(d)$, so the k th row of $M(d)$ contains negative entries while the $(k+1)$ th row does not. Again Lemma 3.5 implies that hypotheses of Lemma 3.6 are met, and the claim holds.

Finally, observe that if $\Delta_k(d) = \Delta^*(d)$, then one of the conditions (i)–(iii) holds, so $\Delta^*(d)$ appears among the principal Erdős–Gallai differences of \bar{d} , and $\Delta^*(\bar{d}) \geq \Delta^*(d)$. Interchanging the roles of d and \bar{d} shows that equality holds. \square

Corollary 3.8 (Compare with [5, 10, 14]). *The classes of split, threshold, and weakly threshold graphs are closed under complementation. More generally, for any set I of nonnegative integers, the following classes of graphs are closed under complementation:*

$$\mathcal{A}_I = \{G : G \text{ is a simple graph whose degree sequence } d \text{ satisfies } \Delta_{m(d)}(d) \in I\};$$

$$\mathcal{B}_I = \{G : G \text{ is a simple graph whose degree sequence } d \text{ satisfies } \Delta^*(d) \in I\};$$

Proof. The general statement follows from Corollary 3.4 and Theorem 3.7. By Corollary 1.4 and Theorem 1.5, the families of split and threshold graphs are respectively the families \mathcal{A}_I and \mathcal{B}_I when $I = \{0\}$. The family of weakly threshold graphs is the family \mathcal{B}_I when $I = \{0, 1\}$. \square

4 Two posets

Here we show how values of principal Erdős–Gallai differences compare for degree sequences related under either of two partial orders important to the study of degree sequences. In particular, we will show that both the parameters of last and largest principal Erdős–Gallai difference are monotone under these orders.

4.1 The Rao order

In [27], S.B. Rao introduced a partial order on degree sequences in the following way: for degree sequences e and d , define $e \preceq d$ if there exist realizations H of e , and G of d , such that H is an induced subgraph of G . Rao showed that \preceq does indeed yield a partial order (we call it \mathcal{R}) and posed some questions about this order. In the decades since the poset's introduction, the most celebrated result about it has been the proof by Chudnovsky and Seymour that \mathcal{R} is a well quasi-order [9]. As such, if a family of degree sequences forms an ideal (i.e., a downward-closed subposet) in \mathcal{R} , then it can be characterized by finitely many minimal obstructions. One example of such a characterization has appeared in [3]; another will in [7].

How do the Erdős–Gallai differences behave with respect to the order relation \preceq ? Consider the example where $e = (2, 2, 2, 1, 1)$ and $d = (4, 3, 3, 2, 2, 2)$. Here $e \preceq d$ in \mathcal{R} , and $\Delta(e) = (2, 2, 2)$ while $\Delta(d) = (1, 3, 2)$. Note that entrywise, a term in $\Delta(d)$ may be equal to, less than, or greater than the corresponding term in $\Delta(e)$. For other pairs e, d it may also be the case that $\Delta(d)$ may have more terms than $\Delta(e)$. However, a few relationships between the terms of $\Delta(e)$ and $\Delta(d)$ always hold, as we now show.

Consider first the last terms of $\Delta(e)$ and $\Delta(d)$. Recall from Section 1 that the splittance of a degree sequence d (or of a graph G realizing d) is the minimum number of edges to be deleted from or added to G in order to transform it into a split graph. This number is the same for any realization of d , and it is equal to one-half the value of $\Delta_{m(d)}(d)$.

Theorem 4.1. *If degree sequences e, d satisfy $e \preceq d$ in \mathcal{R} , then $\Delta_{m(e)}(e) \leq \Delta_{m(d)}(d)$.*

Proof. For any degree sequence π , let $s(\pi)$ denote the splittance of any realization of π ; for a graph P , let $s(P)$ denote the splittance of P . Given that $e \preceq d$ in \mathcal{R} , let H and G respectively be realizations of e and d such that H is an induced subgraph of G . Observe that

$$\Delta_{m(e)}(e) = 2s(e) = 2s(H) \leq 2s(G) = 2s(d) = \Delta_{m(d)}(d),$$

since any adjacency modifications that transform G into a split graph necessarily transform its induced subgraph H as well. \square

Now consider the maximum Erdős–Gallai differences of e and d .

Theorem 4.2. *If degree sequences e, d satisfy $e \preceq d$ in \mathcal{R} , then $\Delta^*(e) \leq \Delta^*(d)$.*

Proof. It suffices by induction to assume that d has one more term than e (since a realization H of d that contains an induced subgraph G that is a realization of e can be built up by restoring one vertex of $V(H) \setminus V(G)$ at a time to G ; the degree sequences of the intermediate graphs lie between e and d in \mathcal{R}). Suppose that d is the degree sequence resulting from adding a new vertex v to a realization G of e . If a is the degree of v , then d is obtained by increasing a

terms of e each by 1, inserting a term equal to a in the list, and reordering the list terms as necessary so that d is in nonincreasing order.

Equivalently, $F(d)$ is obtained from $F(e)$ by

- (i) adding one more row of zeroes at the bottom of the matrix and one more column of zeroes at the right side of $F(e)$, with a star in the entry on the main diagonal,
- (ii) changing the first non-diagonal 0 term of a row into a 1, in each of a distinct rows, and
- (iii) changing the first non-diagonal 0 term of a column into a 1, in each of the *first* a columns.

Now let k be an index such that $\Delta_k(e) = \Delta^*(e)$. Let n be the length of e . Since

$$\sum_{\substack{i \leq k \\ 1 \leq j \leq n+1}} (F(d)^T)_{ij} = \sum_{\substack{i \leq k \\ 1 \leq j \leq n}} (F(e)^T)_{ij} + \min\{k, a\}$$

and

$$\sum_{\substack{i \leq k \\ 1 \leq j \leq n+1}} (F(d))_{ij} \leq \sum_{\substack{i \leq k \\ 1 \leq j \leq n}} (F(e))_{ij} + \min\{k, a\},$$

The value $\Delta_k(d)$, which by Observation 2.2 equals the difference of the two left-hand sides in the expressions above, is necessarily at least as large as the difference of the two right-hand sides above. Hence $\Delta^*(d) \geq \Delta_k(d) \geq \Delta_k(e) = \Delta^*(e)$. \square

4.2 The dominance order

In this section we consider the *dominance order* \mathcal{D}_s . The ground set for this partially ordered set consists of all degree sequences having a fixed sum s (which equals twice the number of edges in realizations of the sequences). Two degree sequences d, e in this set satisfy $d \succeq e$ if d *majorizes* e , that is, if

$$\sum_{i \leq k} d_i \geq \sum_{i \leq k} e_i$$

for all integers k , assuming that the terms of d and e are respectively in nonincreasing order.

Classic results on \mathcal{D}_s and the relation \succeq include the fact that \mathcal{D}_s is an ideal, or “downward-closed” subposet when the majorization order is applied to all partitions (not just degree sequences) of a fixed even positive number; this was proved by Ruch and Gutman [28]. In other words, for any partitions d, e of even positive integer s , if $d \succeq e$ and d is the degree sequence of a simple graph, then e is as well. On the other hand, as shown in [26, 5, 28], if degree sequences d and e with the same sum satisfy $d \succeq e$ and e is respectively split, weakly threshold, or threshold, then d must be as well.

A key notion in studying \mathcal{D}_s is the *unit transformation*, which is an operation performed on degree sequences d in \mathcal{D}_s . If the terms of d are d_i , indexed so that $d_1 \geq d_2 \geq \dots$, a unit transformation on d subtracts 1 from some term d_r and adds 1 to d_t for some t such that $d_r \geq d_t + 2$ (if $d_r \geq 2$, we also permit ourselves to imagine that a 0 immediately following nonzero terms of d can be augmented to become 1, even if d was not previously assumed to have any terms equal to 0). A well known result (sometimes called *Muirhead's Lemma*, as in [26]) states that $d \succeq e$ if and only if e can be obtained by a finite sequence of unit transformations on d .

We examine the effect of a single unit transformation on $\Delta(d)$, making use of the matrices $F(d)$ and $M(d)$ introduced in Section 2. As above, suppose that e is obtained from a unit transformation on d , and r, t are the indices where the modifications of the terms of d happen, so

$$e_i = \begin{cases} d_i - 1 & \text{if } i = r; \\ d_i + 1 & \text{if } i = t; \\ d_i & \text{otherwise;} \end{cases}$$

and $r < t$. We further suppose that t is the first index following r such that $d_t \leq d_r - 2$ while r is the last index preceding t such that $d_r \geq d_t + 2$; this ensures that $d_r > d_{r+1}$, that e 's terms remain in nonincreasing order, and that e cannot be produced from d through a composition of two or more unit transformations (as may happen if $d_t < d_u \leq d_r - 2$ for some u , for instance). For convenience, assume that d and e both have n terms, where a trailing 0 is added to d if necessary.

We may write $F(e) = F(d) + C_{rt}(d)$ for some n -by- n matrix $C_{rt}(d)$ having a single -1 in Row r and a 1 in Row t ; let j_r and j_t respectively denote the indices of the columns of $C_{rt}(d)$ where these nonzero entries occur. Every other entry of $C_{rt}(d)$ is equal to 0. Note that the star entries in $F(d)$ cause

$$j_r = \begin{cases} d_r + 1 & \text{if } d_r \geq r; \\ d_r & \text{if } d_r < r; \end{cases} \quad \text{and} \quad j_t = \begin{cases} d_t + 2 & \text{if } d_t \geq t - 1; \\ d_t + 1 & \text{if } d_t < t - 1. \end{cases} \quad (2)$$

We claim that $j_t < j_r$. Indeed, since $d_r \geq d_t + 2$, this is immediate unless $d_t + 2 = j_t = j_r = d_r$; however, this exceptional possibility cannot happen, since it requires both that $d_t \geq t - 1$ and $d_r < r$, causing

$$d_r < r < t \leq 1 + d_t \leq d_r - 1.$$

Observe that $M(e) = M(d) + C'_{rt}(d)$, where $C'_{rt}(d) = C_{rt}(d)^T - C_{rt}(d)$. The matrix $C'_{rt}(d)$ is skew-symmetric and has either two or four nonzero entries: there are entries of $+1$ in positions (r, j_r) and (j_t, t) , or an entry of $+2$ if these positions coincide, and entries equal to -1 in positions (t, j_t) and (j_r, r) , with an entry of -2 instead if these positions coincide. Observe that no location of a positive entry in $C'_{rt}(d)$ is in danger of coinciding with the location of a negative entry, since $r < t$ and $j_t < j_r$ imply that $(r, j_r) \neq (t, j_t)$ and $(j_t, t) \neq (j_r, r)$, and (2) shows that $j_r \neq r$ and $j_t \neq t$, so $(r, j_r) \neq (j_r, r)$ and $(t, j_t) \neq (j_t, t)$.

Using Observation 2.2 and the fact that $M(e) = M(d) + C'_{rt}(d)$, the description of $C'_{rt}(d)$ just given lets us trace the effect on the Erdős–Gallai differences due to unit transformations.

Lemma 4.3. *If d and e satisfy the assumptions given previously, then*

$$\Delta_k(e) = \begin{cases} \Delta_k(d) + c_1(d, r, t, k) - c_{-1}(d, r, t, k) & \text{if } k \leq \min\{m(d), m(e)\}; \\ \Delta_{m(d)}(d) + 2 & \text{if } k = m(e) > m(d), \end{cases}$$

where $c_1(d, r, t, k)$ is the number of elements (counting multiplicities) from $\{r, j_t\}$ that are less than or equal to k , and $c_{-1}(d, r, t, k)$ is the number of elements (counting multiplicities) from $\{t, j_r\}$ that are less than or equal to k .

Proof. The first case was proved above; the second case can only happen when

$$d_{m(d)+1} < m(d) \leq e_{m(d)+1} \leq d_{m(d)+1} + 1;$$

in this situation $m(e) = m(d) + 1$. Here $e_{m(d)+1} = m(d)$, so by Lemma 2.3, $\Delta_{m(e)}(e) = \Delta_{m(d)}(e)$. Since we further know that $(t, j_t) = (m(d)+1, d_{m(d)+1}+1)$ and $r \leq m(d)$ while $j_r > m(d) + 1$, we see that $c_1(d, r, m(d) + 1, m(d)) = 2$ and $c_{-1}(d, r, m(d) + 1, m(d)) = 0$, and the first case in our formula yields the result. \square

Lemma 4.3 provides exact values for differences $\Delta_k(e)$ as unit transformations occur; as consequences, we have the following previously known results.

Corollary 4.4.

- (1) (See [5]) *If e is obtained by a unit transformation on d and $m' = \min\{m(d), m(e)\}$, then for any $k \in \{1, \dots, m'\}$ we have $\Delta_k(d) \leq \Delta_k(e)$.*
- (2) (Li [22]; see also [5] and Corollary 1.4 herein) *For any degree sequence d , $\Delta_{m(d)}(d)$ is an even number.*

Proof. (1): Since $r < t$ and $j_t < j_r$, it follows that $c_1(d, r, t, k) \geq c_{-1}(d, r, t, k)$.

(2): Because of the definition of $m(d)$, for $u \in \{r, t\}$ it follows that $u \leq m(d)$ if and only if $j_u > m(d)$. Thus $c_1(d, r, t, m(d)) - c_{-1}(d, r, t, m(d))$ has either the form $2 - 0$ or $1 - 1$ for any d . Theorem 1.5 implies that any threshold graph's degree sequence π satisfies $\Delta_{m(\pi)}(\pi) = 0$. Since every degree sequence is majorized by a threshold sequence [28], Lemma 4.3 and an inductive argument show that $\Delta_{m(d)}(d)$ is even. \square

Our next lemma shows that the last term of $\Delta(d)$ is monotone with respect to the dominance order.

Lemma 4.5. *If d and e are degree sequences with a common sum such that $d \succeq e$, then $\Delta_{m(d)}(d) \leq \Delta_{m(e)}(e)$.*

Proof. By Muirhead’s Lemma, it suffices to prove the result in the case that some unit transformation changes d into e . Suppose that this is so.

If $m(e) = m(d)$ or $m(e) > m(d)$, then Corollary 4.4 or Lemma 4.3 gives the result. If instead $m(e) < m(d)$, then the unit transformation changing d to e must change a 1 to a 0 in the $(m(d) - 1)$ th column of the $m(d)$ th row in $F(d)$. This implies that $d_{m(d)} = m(d) - 1$, which by Lemma 2.3 implies that $\Delta_{m(d)}(d) = \Delta_{m(e)}(e)$. \square

This gives context to the following result of Merris that degree sequences of split graphs (those d for which the last term of $\Delta(d)$ is 0) appear at the “top” of \mathcal{D}_s .

Corollary 4.6 ([26, Lemma 3.3]). *If d and e are degree sequences such that $d \succeq e$ and e is the degree sequence of a split graph, then d is the degree sequence of a split graph as well.*

Proof. This follows from Lemma 4.5 and Corollary 1.4. \square

We now extend the monotonicity to general indices and beyond individual unit transformations.

Theorem 4.7. *If d and e are degree sequences with a common sum and $d \succeq e$, then $\Delta_k(d) \leq \Delta_k(e)$ for all $k \in \{1, \dots, m'\}$, where $m' = \min\{m(d), m(e)\}$.*

Proof. Let $\pi^0, \pi^1, \dots, \pi^p$ denote a sequence of degree sequences satisfying $\pi^0 = d$, $\pi^p = e$, and for each $i \in \{1, \dots, p\}$ the degree sequence π^i is obtained by performing a unit transformation on π^{i-1} . Suppose that $k \in \{1, \dots, m'\}$. If $k \leq m(\pi^i)$ for all i in $\{0, \dots, p\}$, then $\Delta_k(d) \leq \Delta_k(e)$ follows inductively by Corollary 4.4. It thus suffices to prove the conclusion in the case that $m(\pi^0) = m(\pi^p) = k$ and $m(\pi^i) < k$ for all i such that $0 < i < p$.

Under this assumption, let m_i denote $m(\pi^i)$ for $i \in \{0, \dots, p\}$. Lemma 4.5 implies that

$$\Delta_k(\pi^0) = \Delta_{m_0}(\pi^0) \leq \Delta_{m_1}(\pi^1) \leq \dots \leq \Delta_{m_p}(\pi^p) = \Delta_k(\pi^p),$$

and our claim is proved. \square

To conclude the paper, we return to the maximum term among Erdős–Gallai difference lists.

Theorem 4.8. *If d and e are degree sequences satisfying $d \succeq e$, then $\Delta^*(d) \leq \Delta^*(e)$.*

Proof. This is immediate from the previous theorem if for some index k we have $\Delta_k(d) = \Delta^*(d)$ and $k \leq m(e)$. If instead $\Delta(d)$ achieves the value $\Delta^*(d)$ at some index k that is greater than $m(e)$, then consider a sequence $\pi^0, \pi^1, \dots, \pi^p$ of degree sequences satisfying $\pi^0 = d$, $\pi^p = e$, and for each $i \in \{1, \dots, p\}$ the degree sequence π^i is obtained by performing a unit transformation on π^{i-1} . Since $m(d) > m(e)$ and the length of the Erdős–Gallai difference list of a degree

sequence can change by at most 1 during a unit transformation, at some term π^ℓ in the sequence, the length of $\Delta(\pi^\ell)$ is equal to k , and each of $\Delta(\pi^0), \dots, \Delta(\pi^\ell)$ has length at least k . By Corollary 4.4 we see that $\Delta_{m(\pi^\ell)}(\pi^\ell) \geq \Delta^*(d)$. Then by Lemma 4.5, applied iteratively to the sequence $\pi^\ell, \dots, \pi^p = e$, we see that $\Delta^*(e) \geq \Delta^*(d)$, as claimed. \square

Corollary 4.9 ([28, 5]). *If d and e are degree sequences satisfying $d \succeq e$, where e is the degree sequence of a threshold (respectively, weakly threshold) graph, then d is also the degree sequence of a threshold (resp., weakly threshold) graph.*

Proof. This follows from Theorem 4.8 and Theorem 1.5 and the definition of weakly threshold graphs. \square

References

- [1] S.R. Arikati and U.N. Peled, Degree sequences and majorization, *Linear Algebra and Its Applications* 199 (1994), 179–211.
- [2] M.D. Barrus, Adjacency relationships forced by a degree sequence, *Graphs and Combinatorics* 34 (2018), no. 6, 1411–1427.
- [3] M.D. Barrus, On fractional realizations of graph degree sequences, *Electron. J. Combin.* 21 (2014), no. 2, Paper 2.18.
- [4] M.D. Barrus, Hereditary unigraphs and Erdős–Gallai equalities, *Discrete Mathematics* 313 (2013), no. 21, pages 2469–2481.
- [5] M.D. Barrus, Weakly threshold graphs, *Discrete Mathematics & Theoretical Computer Science* 20 (2018), no. 1, Paper 15.
- [6] M.D. Barrus, S.G. Hartke, K.F. Jao and D.B. West, Length thresholds for graphic lists given fixed largest and smallest entries and bounded gaps, *Discrete Mathematics* 312 (2012), no. 9, 1494–1501.
- [7] M.D. Barrus, A. Trenk and R. Whitman, The hereditary closure of the unigraphs, in preparation.
- [8] C. Berge, *Graphs and hypergraphs*. Translated from the French by Edward Minieka. North-Holland Mathematical Library, Vol. 6. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
- [9] M. Chudnovsky and P. Seymour, Rao’s degree sequence conjecture, *J. Combin. Theory Ser. B* 105 (2014), 44–92.
- [10] V. Chvátal and P.L. Hammer, Aggregation of inequalities in integer programming, In: *Studies in Integer Programming*, P.L. Hammer, E.L. Johnson, B.H. Korte, and G.L. Nemhauser (Editors), North-Holland, New York, 1977, pp. 145–162.

- [11] B.D. Cloteaux, Forced edges and graph structure, *Journal of Research of the National Institute of Standards and Technology*, Gaithersburg, MD, 124 (2019), Article No. 124022. <https://doi.org/10.6028/jres.124.022>.
- [12] R.B. Eggleton, Graphic sequences and graphic polynomials: a report, in: *Infinite and Finite Sets*, Vol. 1, ed. A. Hajnal et al, Colloq. Math. Soc. J. Bolyai 10, North Holland, Amsterdam, 1975, pp. 385–392.
- [13] P. Erdős and T. Gallai, Graphen mit Punkten vorgeschriebenen Grades, *Math. Lapok* 11 (1960), 264–272.
- [14] S. Földes and P.L. Hammer, Split graphs. Proceedings of the 8th Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977), 311–315. *Congressus Numerantium*, No. XIX, Utilitas Math., Winnipeg, Man., 1977.
- [15] V. Golyshev and J. Stienstra, Fuchsian equations of type DN, *Communications in number theory and physics*, Volume 1, Number 2, 323–346, 2007.
- [16] P.L. Hammer, T. Ibaraki, and B. Simeone, Degree sequences of threshold graphs, *Congres. Numer.* 21 (1978), 329–355.
- [17] P.L. Hammer, T. Ibaraki, and B. Simeone, Threshold sequences, *SIAM J. Algebraic Discrete Methods* 2 (1981), 39–49.
- [18] P.L. Hammer and A.K. Kelmans, Laplacian spectra and spanning trees of threshold graphs, *Discrete Applied Mathematics* 65 (1996), no. 1–3, 255–273. *Discrete Applied Mathematics*
- [19] P.L. Hammer and B. Simeone, The splittance of a graph, *Combinatorica* 1 (1981), 275–284.
- [20] D.J. Kleitman and S.-Y. Li, A note on unigraphic sequences, *Studies Appl. Math.* 54 (1975), no. 4, 283–287.
- [21] M. Koren, Sequences with a unique realization by simple graphs, *J. Combin. Theory B* 21 (1976), no. 3, 235–244.
- [22] S.-Y.R. Li, Graphic sequences with unique realization, *J. Combin. Theory B* 19 (1975), no. 1, 42–68.
- [23] F. Maffray and M. Preissmann, Linear recognition of pseudo-split graphs, *Discrete Appl. Math.* 52 (1994), 307–312.
- [24] N.V.R. Mahadev and U.N. Peled, Threshold graphs and related topics. *Annals of Discrete Mathematics*, 56. North-Holland Publishing Co., Amsterdam, 1995.
- [25] P. Marchioro, A. Morgana, R. Petreschi and B. Simeone, Degree sequences of matrogenic graphs, *Discrete Math.* 51 (1984) 46–61.

- [26] R. Merris, Split graphs, *European Journal of Combinatorics* 24 (2003), 413–430.
- [27] S.B. Rao, Towards a theory of forcibly hereditary P-graphic sequences, *Combinatorics and Graph Theory: Proceedings of the Symposium Held at the Indian Statistical Institute, Calcutta, February 25–29*. Springer-Verlag, Lecture Notes in Mathematics, Vol. 885, 1980, pp. 441–458.
- [28] E. Ruch and I. Gutman, The branching extent of graphs, *J. Combin. Inform. System Sci.* 4 (1979), 285–295.
- [29] A. Tripathi and S. Vijay, A note on the theorem of Erdős and Gallai, *Discrete Mathematics* 265 (2003), 417–420.
- [30] R. Tyshkevich, Decomposition of graphical sequences and unigraphs, *Discrete Mathematics* 220 (2000), 201–238.
- [31] R.I. Tyshkevich, Canonical decomposition of a graph, *Dokl. Akad. Nauk BSSR*, 24 (1980), No. 8, 677–679.
- [32] R.I. Tyshkevich, Once more on matrogenic graphs, *Discrete Math.* 51 (1984), 91–100.
- [33] I.E. Zverovich and V.E. Zverovich, Contributions to the theory of graphic sequences, *Discrete Mathematics* 105 (1992), 293–303.