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NEW RESULTS AND IMPROVEMENTS RELATED TO THE STUDY OF MULTI-SPECIALIZATION WHIST TOURNAMENT DESIGNS

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ABSTRACT

In this thesis we introduce a new whist construction which we refer to as The Liaw Variant. This construction is shown to be versatile in that, under appropriate conditions, it can produce every known whist design specialization. No other whist construction in the existing literature can do this.

Furthermore, The Liaw Variant is shown to be more powerful than previously published constructions, in that it improves upon the known results related to each of the whist specializations. In particular, under certain applications, we have been able to dramatically reduce previously published asymptotic bounds related to the construction of $\mathbb{Z}$-cyclic directed-triplewhist and ordered-triplewhist designs.

This thesis also introduces a new whist specialization, $\mathbb{Z}$-cyclic whist designs that are balanced, directed and ordered but whose initial round partner pairs do not form a patterned starter. We give a new construction capable of producing such designs, and investigate its ability. In some cases, this construction was found to give significant improvements when compared with prior known results.

Finally, this thesis introduces a new generalized whist specialization, defining the property of balance on $(h, 2h)$ GWhD(v). A number of existing whist design constructions are shown to either automatically possess the property of balance, or to be modifiable such that balanced designs can be easily obtained. We also show how a modification of another construction will produce balanced $(h, 2h)$ GWhD(v) for certain primes v.

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With many thanks to everyone who has ever taught me: naturally this includes parents, grandparents, brothers, sisters and all of my family; my many friends; my colleagues (and students) at Providence Country Day, Portsmouth Abbey, and, of course, the University of Rhode Island. I am so fortunate to have known all of you!

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Last, but certainly not least, I am forever indebted to Norman (Skip) Finizio. He is, quite simply, the greatest advisor in the history of advisors! Dr. Finizio, I never would have made it without your help and guidance. Thank you for everything!

DEDICATION

I first walked into Dr. James T. Lewis' class in 1987. It was a summer course in differential equations, and after the five weeks had passed, neither one of us probably thought we would ever cross paths again.

But 12 years later, when I was searching for a class to take while contemplating a career as a math teacher, I recognized his name in the course listings. I asked his permission to join his number theory class in the spring term, and he remembered me as well.

The following term, it was graph theory. Then, when a spot in the department became available, he offered me a position as a teaching assistant, along with a chance to work on a Masters Degree. Over the next three semesters, I took every graduate course he taught: complex analysis, and block design I and II (the subject of this research).

After I finished the M.S. program, I taught high school, but we rarely went more than a few weeks without talking. Every time I went back to Kingston, whether to watch a soccer game, visit other friends, or play golf, I looked forward to catching up with Dr. Lewis.

Knowing that I had the itch to come back one more time for a chance at a Ph.D. in 2007, he was the one who made it happen.

For being an incredible teacher, for giving me the opportunity and the motivation to succeed, and for being a truly great friend, I dedicate this dissertation to Jim Lewis.

May every graduate student have a teacher, mentor and friend like him!

PREFACE

This thesis consists of three separate papers, each of which was co-authored with Dr. Norman J. Finizio.

The first paper, "The Liaw Variant - A Versatile Multi-Specialization Whist Construction," was submitted for publication in the *Bulletin of the Institute of Combinatorics and Its Applications* in 2014.

The second paper, "Results Related to the Existence of Non $\mathcal{Z}$CPS-BDOWh(p) For Primes of the Form $p = 4u + 1$," was submitted for publication in the journal, *Congressus Numerantium* in 2014.

The third paper, "Balance in Whist and Generalized Whist Designs: Some Classic and Recent Constructions," was submitted for publication in the journal, *Congressus Numerantium* in 2014.

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CHAPTER 1

The Liaw Variant: A Versatile Multi-Specialization Whist Construction

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Abstract

In this paper we introduce, for primes $p \equiv 1 \pmod 4$, a whist construction which we refer to as The Liaw Variant. This construction is shown to be versatile in that, for such p and under appropriate conditions, it can produce every known whist design specialization. Indeed, it can produce a variety of multi-specialization whist designs. In fact, any whist design produced by The Liaw Variant is automatically a $\mathbb{Z}$-cyclic balanced whist design.That is to say, every whist design produced by The Liaw Variant exhibits at least two whist specializations. Additionally, for the production of directed-triplewhist and ordered-triplewhist designs on p players, The Liaw Variant is shown to be more powerful than previously published constructions. It is also demonstrated that applications of the methodology of M. Buratti and A. Posatti [1] to The Liaw Variant enable us to dramatically reduce previously published asymptotic bounds related to the construction of $\mathbb{Z}$-cyclic directed-triplewhist and ordered-triplewhist designs on p players.

keywords: Whist Tournaments; $\mathbb{Z}$-Cyclic Designs; Triplewhist Designs, Directedwhist Designs; Orderedwhist Designs; Three Person Whist Designs; Balanced Whist Designs; Splittable Whist Designs.

1.1 Introduction

A **whist tournament** on v players, denoted Wh(v), is a $(v, 4, 3)$ (near) resolvable BIBD. A whist game (alt. table) is a block, (a, b, c, d), of the BIBD and denotes that the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The design is subject to the (whist) conditions that every player is a partner of every other player exactly once and is an opponent of every other player exactly twice. The (near) resolution classes of the BIBD are called the **rounds** of the Wh(v). It has been known since the 1970s that Wh(v) exist for all $v \equiv 0, 1 \pmod 4$ [2, 3]. For an interesting and informative account of the early history of the whist tournament problem see the recent paper of I. Anderson and T. Crilly [4].

If $v = 4u$ then the Wh(v) consists of $4u - 1$ rounds and if $v = 4u + 1$, the Wh(v) consists of $4u + 1$ rounds. In the former case every player plays in exactly one game of each round whereas in the latter case every player plays in exactly one game in each round with the exception of one round in which the player *sits out*. Throughout this paper we consider, exclusively, that v is an odd prime. Hence all subsequent discussion relates to $v = 4u + 1$. For such whist designs we are primarily interested in those that possess, simultaneously, more than one special whist property. These special properties are now defined. For existence results regarding these specializations see [5, 6].

Definition 1.1.1 *A whist design on $4u + 1$ players is said to be* **$\mathbb{Z}$-cyclic** *if the players are elements in $\mathbb{Z}_{4u+1}$. It is also required that the set of rounds be cyclic. That is to say, the rounds can be labeled, $R_1, R_2, \ldots$, in such a way that R_{j+1} is obtained by adding $+1$ (mod $(4u + 1)$) to every element in R_j.*

Although most of the whist specializations to be introduced here relate to non $\mathbb{Z}$-cyclic whist designs as well as $\mathbb{Z}$-cyclic whist designs the considerations of this study relate, exclusively to $\mathbb{Z}$-cyclic whist designs. Since the collection of rounds of a $\mathbb{Z}$-cyclic Wh(v) form a cyclic set it follows that the entire design can be given by any one of its rounds. This representative round is called the *initial round*. For our purposes, however, it is convenient to define the initial round.

Definition 1.1.2 *For $v = 4u+1$ the initial round is defined to be the unique round for which 0 sits out.*

The method of symmetric differences [2] is the primary methodology to be used for verification of the designs claimed in this study.

In a whist game (a, b, c, d) the opponent pairs $\{a, b\}$, $\{c, d\}$ are called **first kind opponents** and the opponent pairs $\{a, d\}$, $\{b, c\}$ are called **second kind opponents**.

Definition 1.1.3 *[7] A whist tournament on v players is said to be a* **triplewhist tournament***, $TWh(v)$, if every player opposes every other player exactly once as an opponent of the first kind (and, hence, exactly once as an opponent of the second kind).*

In a whist game one can define **left hand opponents** and **right hand opponents**. These relationships are the obvious ones associated with the players seated at a table with a at the North position, b at the East position, c at the South position and d at the West position.

Definition 1.1.4 *[3] A whist tournament on v players is said to be a* **directed-whist tournament***, $DWh(v)$, if every player has every other player exactly once as a left hand opponent (and, hence, exactly once as a right hand opponent).*

Definition 1.1.5 *[8] A whist tournament on v players is said to be an* **ordered-whist tournament***, $OWh(v)$, if each player opposes every other player once at North-South and once at East-West.*

A necessary condition for the existence of OWh(v) is $v \equiv 1 \pmod 4$ [9].

For any given round of a Wh(v) the set of all players sitting in the North and South positions is referred to as the **N-S line**. Similarly the **E-W line** is the set of all players sitting in the East and West positions. Let $\{x, y\}$ be any pair of players in a Wh(v). For any round of the Wh(v) for which x and y both play, but not at the same table, x and y are said to be **relative opponents** if they belong to the same line (either N-S or E-W) and **relative partners** if they belong to opposite lines. For $v = 4u + 1$ there are exactly $4u - 4$ rounds in which x and y play at different tables.

Definition 1.1.6 *[10] A whist tournament on v players is said to be a* **balanced whist tournament***, $BWh(v)$, if every pair of players are relative opponents exactly $2u - 2$ times (and, hence, relative partners exactly $2u - 2$ times).*

Definition 1.1.7 *[11] A whist tournament design on v players is said to be* **splittable***, denoted by $SWh(v)$, if and only if the games in the design can be partitioned into two sets $\mathcal{A}$ and $\mathcal{B}$, called the partition sets, such that in each round half the games are in $\mathcal{A}$ and the other half are in $\mathcal{B}$ in such a way that every player opposes every other player exactly once in each partition set.*

It is established in [11] that SWh(v) exist only if $v \equiv 1 \pmod 8$. Sufficient conditions that ensure that a $\mathbb{Z}$-cyclic Wh(v), produced by The Liaw Variant, is splittable are given below in Lemma 1.3.7.

Definition 1.1.8 *Let G be an abelian group of order $2s+1$. The collection of pairs $\{\{x_i, y_i\} : x_i, y_i \in G \setminus \{e_G\}, i = 1, \ldots, s\}$ is called a* **starter** *in G if and only if $\bigcup_{i=1}^{s}\{x_i, y_i)\} = G \setminus \{e_G\}$ and $\bigcup_{i=1}^{s}\{\pm(y_i - x_i)\} = G \setminus \{e_G\}$. If $y_i = -x_i, i = 1, \ldots, s$ the starter is called the* **patterned starter in** *G.*

Definition 1.1.9 *[12] A $\mathbb{Z}$-cyclic Wh(v), $v = 4n + 1$, is said to be a* **$\mathbb{Z}$-cyclic patterned starter** *whist tournament, denoted $\mathbb{Z}$CPS-Wh(v), if the set of initial round partner pairs form the patterned starter in $\mathbb{Z}_v$.*

Definition 1.1.10 *[13] A whist tournament on v players is said to be a* **three person whist tournament***, 3PWh(v), if the intersection of any two games is at most 2.*

Definition 1.1.11 *[14] Let (a, b, c, d) be a game in a $\mathbb{Z}$-cyclic whist tournament design such that $\infty \notin \{a, b, c, d\}$. The a-***centered difference sets** *corresponding to this game are the three sets $\{b-a, c-a\}$, $\{b-a, d-a\}$, $\{c-a, d-a\}$. Similarly one defines b-***centered**, *c-***centered** *and d-***centered difference sets**. *If (∞, a, b, c) is a game in a $\mathbb{Z}$-cyclic whist tournament design the a-centered difference sets are defined to be $\{\infty, b-a\}$, $\{\infty, c-a\}$, $\{b-a, c-a\}$. Similarly for b-centered and c-centered difference sets. ∞-centered difference sets are not formed.*

Theorem 1.1.1 *[14] A $\mathbb{Z}$-cyclic Wh(v), $v \equiv 0, 1 \pmod 4$, has the $3P$ property if and only if all the $(\cdot)$-centered difference sets formed from the u games in the initial round are different.*

It is possible that a $\mathbb{Z}$-cyclic Wh(v) might satisfy more than one of the specializations listed above. In such cases the name and the notation of the design is a concatenation of the component specializations. For example a $\mathbb{Z}$-cyclic directed-triplewhist design is denoted DTWh(v) (alt. TDWh(v)). It is to be noted, however, that I. Anderson and L. H. M. Ellison [15] have proven that it is impossible for a $\mathbb{Z}$-cyclic Wh$(4u+1)$ to simultaneously possess the properties of triplewhist, directedwhist and orderedwhist. That is to say, there is no $\mathbb{Z}$-cyclic DOTWh$(4u+1)$.

1.2 Preliminaries

Previous studies [16, 15, 11, 17, 18, 19, 20, 21] utilized one or both of the generalized Anderson - Ellison Constructions [22] to obtain solutions for the whist designs of interest to the respective study. In each case an analytic asymptotic bound was either obtained or approximated. Then, for those primes less than the analytic asymptotic bound, the data generation involved finding, via the generalized Anderson - Ellison construction(s) being employed, as many solutions as the construction would produce. In some cases these searches were limited by practical time considerations. In every case this data process resulted in a list of primes for which the constructions did not produce the desired whist design. That is to say, there was a list of *exceptions*. More often than not these studies then utilized the version of Liaw's Construction found in [23] to try to find solutions for the exceptions. Here we have modified Liaw's construction so as to increase its capability to produce whist designs that possess, simultaneously, more multiple specializations than the earlier studies. Another goal of our approach is to demonstrate that our modified construction is more powerful than the previously studied constructions. The term *powerful* is meant in the sense of a smaller list of exceptions.

For the remainder of this study the primes $p \equiv 1 \pmod 4$ are taken in the form $p = 2^k t + 1$ where $k \geq 2$ and t is odd. Set $d = 2^k$, $m = 2^{k-1}$ and $n = 2^{k-2}$. Let r denote an arbitrary, but fixed, primitive root of p. For such p the following facts regarding $\mathrm{GF}(p) = \mathbb{Z}_p$ are well known [2]: (1) $x \in \mathbb{Z}_p^*$ is a square if and only if $-x$ is a square; (2) $-1 = r^{mt}$ and (3) if $\mathcal{S}$ denotes the set of squares and $\mathcal{N}$ denotes the set of non-squares then $\mathcal{S}$, $\mathcal{N}$ form a $(4u+1, 2u, 2u-1)$ difference system where $u = nt$ (see Theorem 2.2.5 in [2]). Let C^d denote the subgroup in the multiplicative group of $\mathrm{GF}(p)$ consisting of the powers of r^d. Note that $|C^d| = t$. The cosets of C^d formed by $r^i C^d, i = 0, 1, \ldots, d-1$ will be denoted by C_i^d and are often referred to as the i-th cyclotomic class of order d. Let $\mathrm{PS}(r)$ denote the set $\{r^s : 0 \leq s \leq p-2\}$. If $x, y \in \mathbb{Z}_p^*$ then there exist unique integers i and j such that $x = r^i$, $y = r^j$ and we shall say that x and y are $|i - j|$ units apart in $\mathrm{PS}(r)$. Thus, for example, if $y = -x = r^{mt}x$ we would say that x and y are an odd multiple of m units apart in $\mathrm{PS}(r)$. Suppose that a quadruple of elements in $\mathrm{GF}(p)$, say (a, b, c, d), can be partitioned into pairs such that one pair consists of squares that are an odd multiple of m units apart in $\mathrm{PS}(r)$ and the other pair consists of non-squares that are an odd multiple of m units apart in $\mathrm{PS}(r)$. Then

$(a, b, c, d) \otimes r^{2i}, i = 0, 1, \ldots, n-1$ is a complete system of representatives of the cyclotomic classes of order d. Furthermore if this system of representatives is expanded by the operation $\otimes\, r^{dj}, j = 0, 1, \ldots, t-1$ then one obtains every element in $\mathbb{Z}_p^*$ exactly once. In the sequel, if a quadruple (a, b, c, d) can be partitioned into pairs such that one pair consists of squares that are an odd multiple of m units apart in PS(r) and the other pair consists of non-squares that are an odd multiple of m units apart in PS(r) we shall say that the quadruple possesses the OMMA property or, for short, OMMA.

1.3 The Liaw Variant

Definition 1.3.1 *Let $p = 2^k t + 1$, t odd, denote a prime with $k \geq 2$. Let r denote an arbitrary, but fixed, primitive root of p and let x be a non-square in $\mathbb{Z}_p^*$. Set $a \equiv m - 1 \pmod{d}$. The construction embodied by the collection of whist games*

$$(1, x, x^{1+a}, -x) \otimes r^{dj+2i}, \quad 0 \leq j \leq t-1, \ 0 \leq i \leq n-1, \tag{3.1}$$

is called **The Liaw Variant**.

Note that 1 and x^{1+a} are squares that are an odd multiple of m units apart in PS(r). Likewise x and $-x$ are non-squares that are an odd multiple of m units apart in PS(r). It follows then (OMMA) that the union of the games in Eq. 3.1 equals $\mathbb{Z}_p^*$. Thus it is possible that The Liaw Variant can serve as the initial round of a $\mathbb{Z}$-cyclic Wh(p). Under suitable conditions it will be demonstrated that, for each specialization mentioned in Section 1, The Liaw Variant can produce a $\mathbb{Z}$-cyclic Wh(p) that satisfies the properties of the specialization. Indeed, it will be further demonstrated that The Liaw Variant can produce $\mathbb{Z}$-cyclic Wh(p) that possess, simultaneously, several of the specialized properties. This versatility is demonstrated in the theorems of this section. For convenience of notation the operation $\otimes r^{dj+2i}, \quad 0 \leq j \leq t-1, \ 0 \leq i \leq n-1$ will be abbreviated to $\otimes\mathcal{R}$.

Occasionally, we will refer to the whist game $(1, x, x^{1+a}, -x)$ as the base game (alt. base table) of The Liaw Variant.

Theorem 1.3.1 *The Liaw Variant produces a $\mathbb{Z}$-cyclic Wh(p) if the following two conditions are satisfied: (1) $2(x^{1+a}-1) = \square$ and (2) the set $\{\pm(x-1), \pm x(x^a+1), \pm(x+1), \pm x(x^a-1)\} \otimes \mathcal{R}$ yields every element in $\mathbb{Z}_p^*$ exactly twice.*

Proof: The partner differences for The Liaw Variant are $\pm 2x \otimes \mathcal{R}$ and $\pm(x^{1+a} - 1) \otimes \mathcal{R}$. These differences yield every element in $\mathbb{Z}_p^*$ exactly once provided that $2x(x^{1+a}-1) \neq \square$ (OMMA), that is if $2(x^{1+a}-1) = \square$. Clearly the differences listed in Condition (2) are the opponent differences for The Liaw Variant. ∎

Corollary 1.3.2 *Any $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is a $BWh(p)$.*

Proof: The N-S line in The Liaw Variant is the set of squares in $\mathbb{Z}_p^*$ and the E-W line is the set of non-squares in $\mathbb{Z}_p^*$. Thus, via Theorem 2.2.5 in [2] these two sets form a $(4u+1, 2u, 2u-1)$ difference family with $u = nt$. Thus each pair of elements in $\mathbb{Z}_p^*$, say $\{x, y\}$, occur together in the same line exactly $2u-1$ times. Removing the occurrence when x and y are partners shows that x and y are relative opponents exactly $2u-2$ times. Since x and y appear in the same round but at different tables exactly $4u-4$ times they are relative partners exactly $2u-2$ times. ∎

As a consequence of Theorem 1.3.1 we note that The Liaw Variant will produce a $\mathbb{Z}$-cyclic Wh(p) if (1) $2(x^{1+a}-1) = \square$ and (2) the quadruple $(x+1, x-1, x(x^a+1), x(x^a-1))$ has the OMMA property. As squares or non-squares there are 16 possibilities for the four expressions in this quadruple but only six of these possibilities allow for the quadruple to possibly have the OMMA property. These six cases are given in the following chart and will be referred to as Case (1), Case(2), ..., Case (6).

	$x+1$	$x-1$	$x(x^a+1)$	$x(x^a-1)$	x^a+1	x^a-1
1	$\not\square$	$\not\square$	$\square$	$\square$	$\not\square$	$\not\square$
2	$\square$	$\not\square$	$\square$	$\not\square$	$\not\square$	$\square$
3	$\not\square$	$\square$	$\not\square$	$\square$	$\square$	$\not\square$
4	$\square$	$\square$	$\not\square$	$\not\square$	$\square$	$\square$
5	$\square$	$\not\square$	$\not\square$	$\square$	$\square$	$\not\square$
6	$\not\square$	$\square$	$\square$	$\not\square$	$\not\square$	$\square$

In the subsequent discussion to say that a $\mathbb{Z}$-cyclic Wh(p) is produced by The Liaw Variant will automatically imply that the two conditions of Theorem 1.3.1 have been satisfied. Furthermore, by dint of Corollary 1.3.2, there is no further need to prove the balance property in our subsequent designs. Note that from here onward the designs associated with our theorems satisfy, simultaneously, several whist specializations.

Theorem 1.3.3 *A $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is a BTWh(p) if* $(x-1)(x^a+1) = \square$.

Proof: The opponent first kind differences in The Liaw Variant are $\pm(x-1) \otimes \mathcal{R}$ and $\pm x(x^a+1) \otimes \mathcal{R}$. The quadruple of elements $\{\pm(x-1), \pm x(x^a+1)\}$ will yield every element in $\mathbb{Z}_p^*$ exactly once if $(x-1)x(x^a+1) \neq \square$ (OMMA), i.e. $(x-1)(x^a+1) = \square$. ∎

Certainly one could have used the opponents second kind differences as sufficient conditions in this latter theorem.

Corollary 1.3.4 *A $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is a BTWh(p) if* $(x+1)(x^a-1) = \square$.

Theorem 1.3.5 *A $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is a BDWh(p) if the set* $\{x-1, x(x^a-1), -x(x^a+1), x+1\} \otimes \mathcal{R}$ *yields every element in* $\mathbb{Z}_p^*$ *exactly once.*

Proof: The set offered as a sufficient condition represents the right hand opponents (alt. the first forward) differences in The Liaw Variant. ∎

Definition 1.3.2 *[24] If* (a, b, c, d) *is a whist game in a* $\mathbb{Z}$*-cyclic whist design, the set of differences* $\{a-b, a-d, c-b, c-d\}$ *is called the set of ordered opponent differences for that game.*

It is proven in [24] that if the union of the sets of ordered opponent differences for the games in the initial round of a $\mathbb{Z}$-cyclic whist design on v players covers the non-zero elements in $\mathbb{Z}_v$ exactly once then the whist design is an ordered whist design.

Theorem 1.3.6 *A* $\mathbb{Z}$*-cyclic Wh(p) produced by The Liaw Variant is a BOWh(P) if the set* $\{1-x, 1+x, x(x^a-1), x(x^a+1)\} \otimes \mathcal{R}$ *yields every element in* $\mathbb{Z}_p^*$ *exactly once.*

Proof: The set offered as a sufficient condition represents the ordered opponents differences in The Liaw Variant. ∎

Throughout the remainder of this paper any reference to splittable whist designs produced by The Liaw Variant will assume that the initial round games in the partition sets $\mathcal{A}$ and $\mathcal{B}$ are given by $(1, x, x^{1+a}, -x) \otimes r^{dj+4i}$, $0 \leq j \leq t-1$, $0 \leq i \leq 2^{k-3}-1$ and $(1, x, x^{1+a}, -x) \otimes r^{dj+4i+2}$, $0 \leq j \leq t-1$, $0 \leq i \leq 2^{k-3}-1$ respectively.

Definition 1.3.3 *[11] Let* (a, b, c, d) *be a game in a* $\mathbb{Z}$*-cyclic whist tournament on* p *players. The differences* $a-b$*,* $c-d$*,* $a-d$*,* $b-c$ *are called the* **split opponent differences** *(alt. the split differences).*

The following lemma is an application of materials found in [11] (see their Lemma 1.4 and Theorem 1.6).

Lemma 1.3.7 *If the Liaw Variant produces a* $\mathbb{Z}$*-cyclic BWh(p) then this design will be splittable if the set of split opponent differences formed from the base table constitute a complete set of representatives of the cyclotomic classes of order* 4*.*

Since $k \geq 3$ for splittable designs [11] it follows that $m \geq 4$. Consequently, $1, -1 \in C_0^4$ and it follows that if $z \in C_i^4$ then $-z \in C_i^4$.

Theorem 1.3.8 *A $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is splittable if the following conditions are satisfied.*

- $x^2 - 1 \neq \square$

- $\frac{x(x^a+1)}{x+1} \in C_2^4$

- $\frac{x(x^a-1)}{x-1} \in C_2^4$

Proof: Note that exactly one of $x+1, x-1$ is a square. Without loss of generality, assume that $x+1 = \square$. Thus either $x+1 \in C_0^4$ or $x+1 \in C_2^4$. Utilizing the second hypothesis it follows that $x^{a+1}+x \in C_2^4$ or $x^{a+1}+x \in C_0^4$ respectively. Furthermore either $x-1 \in C_1^4$ or $x-1 \in C_3^4$. Utilizing the third hypothesis it follows that $x^{a+1} - x \in C_3^4$ or $x^{a+1} - x \in C_1^4$ respectively. Since z and $-z$ are in the same cyclotomic class of order 4 we conclude that the set of split opponent differences $\{x+1, 1-x, x-x^{a+1}, x^{a+1}+x\}$ form a complete system of representatives of the cosets of C^4 in the multiplicative group of the Galois field $\mathbb{Z}_p$. ∎

Theorem 1.3.9 *A $\mathbb{Z}$-cyclic Wh(p) produced by The Liaw Variant is a $\mathbb{Z}$CPS-BWh(P) if $(2w+1)(a+1) \equiv mt \pmod{(p-1)}$ and $(x+1)(x-1) \neq \square$. Here w is defined by $x = r^{2w+1}$.*

Proof: Clearly $x^{a+1} = -1$ and it follows that the set of partner pairs in the initial round of The Liaw Variant form the patterned starter in the additive group of the

Galois field $\mathbb{Z}_p$. Hence the partner condition is satisfied. The opponent differences are $\pm(x+1)$ and $\pm(x-1)$ each occurring twice. Since $(x+1)(x-1) \neq \Box$ the opponent condition is satisfied (OMMA). ∎

Example 1.3.1 The first entry for $k=4$ in Appendix III indicates that for $p=17$, $r=3$, $x=10$ and $a=7$ The Liaw Variant produces a $\mathbb{Z}$CPS-Wh(17) whose base table is $(1,10,16,7)$. The complete initial round of the corresponding whist design is given by the four games:

$$(1,10,16,7), \quad (13,11,4,6), \quad (9,5,8,12), \quad (15,14,2,3).$$

This design has the 3P property.

It is a well known fact that every $\mathbb{Z}$CPS-Wh$(4u+1)$ is a DOWh$(4u+1)$ [9]. Thus any $\mathbb{Z}$CPS-Wh(p) produced by The Liaw Variant is a BDOWh(p).

The next few theorems illustrate that The Liaw Variant can produce additional multi-specialization Wh(p) designs. For convenience we formulate an expanded version of Theorem 1.3.3.

Theorem 1.3.10 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BTWh(p) if the following conditions are satisfied.*

- $x \neq \Box$,
- $2(x^{1+a}-1) = \Box$,
- $(x-1)(x^a+1) = \Box$,
- $(x+1)(x^a-1) = \Box$.

One can easily see that if $\{x, a\}$ are such that the conditions of Theorem 1.3.10 are satisfied then the pair $\{-x, a\}$ also has this property (and vice-versa). Thus the following corollary can be stated.

Corollary 1.3.11 $\{x, a\}$ *is a pair for which The Liaw Variant produces a BTWh(p) if and only if* $\{-x, a\}$ *is a pair for which The Liaw Variant produces a BTWh(p).*

Note that Cases (1), (2), (3) and (4) are consistent with Theorem 1.3.10 but it is impossible to obtain a BTWh(p) from Cases (5) and (6).

Theorem 1.3.12 *The Liaw Variant produces a* $\mathbb{Z}$*-cyclic BDTWh(p) if, in addition to the hypotheses of Theorem 1.3.10, the following conditions are satisfied.*

- $x^2 - 1 \neq \square$,
- $\frac{x^{a+2}(x^a-1)}{x-1} \in C_0^d$,
- $\frac{x(x^a+1)}{x+1} \in C_0^d$,

Proof: Without loss of generality, assume that $x-1 = \square$. It follows, via Conditions 3 and 4 of Theorem 1.3.10, that $x^a + 1 = \square$ and $(x^a - 1) \neq \square$. Thus the first forward differences in the base table of The Liaw Variant can be grouped in the pairs $\{x - 1, x(x^a - 1)\}$, $\{x + 1, -x(x^a + 1)\}$. Both members of the first pair are squares and both members of the second pair are non-squares. If, in each pair, the members are an odd multiple of m units apart in PS(r) it will follow that the set of first forward differences equals $\mathbb{Z}_p^*$ (OMMA) and the design is directed. Thus we impose the conditions: (1) $\frac{x(x^a-1)}{x-1} \in C_m^d$ and (2) $\frac{-x(x^a+1)}{x+1} \in C_m^d$. Since $x^{a+1}, -1 \in C_m^d$ Conditions 2 and 3 guarantee these latter requirements. ∎

Lemma 1.3.13 *Let x and a be as defined in The Liaw Variant. Then the conditions (1)* $x^2 - 1 \neq \square$, *(2)* $\frac{x^{a+2}(x^a-1)}{x-1} \in C_0^d$, *and (3)* $\frac{x(x^a+1)}{x+1} \in C_0^d$, *imply that (1)** $(x-1)(x^a+1) = \square$ *and (2) ** $(x+1)(x^a-1) = \square$.

Proof: Without loss of generality assume that $x - 1 = \square$. It then follows from (3) that $(x^a + 1) = \square$. Hence (1)*. Similarly (2) implies that $x^{a+2}(x^a - 1) = \square$. Since $x^{a+1} = \square$ it follows that $x^a - 1 \neq \square$. Thus (2)*. ∎

As a consequence of Lemma 1.3.13, Theorem 1.3.12 can be replaced by the following theorem.

Theorem 1.3.14 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BDTWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $x^2 - 1 \neq \square$,
- $\frac{x^{a+2}(x^a-1)}{x-1} \in C_0^d$,
- $\frac{x(x^a+1)}{x+1} \in C_0^d$,

Example 1.3.2 For $p = 29$, $r = 2$, $x = 19$ and $a = 21$ one can determine that $x = r^9$, $2(x^{1+a} - 1) = r^6$, $x^2 - 1 = r^7$, $\frac{x^{a+2}(x^a-1)}{x-1} = r^4$ and $\frac{x(x^a+1)}{x+1} = r^{24}$. Thus the sufficient conditions of Theorem 1.3.14 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BDTWh(p). The initial round of this design is given by the following seven games. This design has the 3P property.

$(1, 19, 4, 10)$,	$(16, 14, 6, 15)$,	$(24, 21, 9, 8)$,	$(7, 17, 28, 12)$,
$(25, 11, 13, 18)$,	$(23, 2, 5, 27)$,	$(20, 3, 22, 26)$.	

Similar to Corollary 1.3.11 the following corollary can be established.

Corollary 1.3.15 *The pair $\{x, a\}$ produces, via The Liaw Variant, a $\mathbb{Z}$-cyclic BDTWh(p) if and only if the pair $\{-x, a\}$ does.*

Proof: Since $p \equiv 1 \pmod 4$ it is known that x is a non-square if and only if $-x$ is a non-square. If it is assumed that the conditions of the theorem are satisfied by $\{x, a\}$ then the mapping $x \to -x$ is such that Conditions 2 and 3 are unchanged and Condition 4 maps to Condition 5 and vice-versa. Clearly the mapping $-x \to x$ has the same effect. ∎

We note that Cases (2) and (3) are consistent with the conditions of Theorem 1.3.14. Cases (5) and (6) would be consistent with the existence of a BDWh(p) as follows.

Theorem 1.3.16 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BDWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $x^2 - 1 \neq \square$,
- $\frac{x^{a+2}(x^a-1)}{x+1} \in C_0^d$,
- $\frac{x(x^a+1)}{x-1} \in C_0^d$,

Proof: The quadruple of elements from the first forward differences have the OMMA property. ∎

In a similar fashion Cases (1) and (4) can be associated with a BDWh(p).

Theorem 1.3.17 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BDWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $(x - 1)(x^a + 1) = \square$,
- $\frac{x^{a+1}(x-1)}{x+1} \in C_0^d$,
- $\frac{x^a+1}{x^a-1} \in C_0^d$,

Proof: The quadruple of elements from the first forward differences have the OMMA property. ∎

It easily follows from the conditions of Theorem 1.3.17 that $(x+1)(x^a-1) = \Box$. Consequently, the conditions of Theorem 1.3.17 imply those of Theorem 1.3.10 and the whist design is actually a BDTWh(p).

Example 1.3.3 For $p = 37$, $r = 2$, $x = 2$ and $a = 21$ one can determine that $x = r^1$, $2(x^{1+a} - 1) = r^{26}$, $(x-1)(x^a+1) = r^{14}$, $\frac{x^{a+1}(x-1)}{x+1} = r^{32}$ and $\frac{x^a+1}{x^a-1} = r^{16}$. Thus the sufficient conditions of Theorem 1.3.17 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BDTWh(p). The initial round of this design is given by the following nine games. This design has the 3P property.

$(1, 2, 21, 35)$,	$(16, 32, 3, 5)$,	$(34, 31, 11, 6)$,	$(26, 15, 28, 22)$,
$(9, 18, 4, 19)$,	$(33, 29, 27, 8)$,	$(10, 20, 25, 17)$,	$(12, 24, 30, 13)$,
$(7, 14, 36, 23)$.			

Theorem 1.3.18 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BOTWh(p) if, in addition to the hypotheses of Theorem 1.3.10, the following conditions are satisfied.*

- $x^2 - 1 \neq \Box$,
- $\frac{x(x^a-1)}{x-1} \in C_0^d$,
- $\frac{x^{a+2}(x^a+1)}{x+1} \in C_0^d$,

Proof: Analogous to the proof of Theorem 1.3.12. ∎

The proof of the following Lemma is analogous to that of Lemma 1.3.13.

Lemma 1.3.19 *Let x and a be as defined in The Liaw Variant. Then the conditions (1) $x^2 - 1 \neq \Box$, (2) $\frac{x^{a+2}(x^a+1)}{x+1} \in C_0^d$, and (3) $\frac{x(x^a-1)}{x-1} \in C_0^d$, imply that (1)* $(x-1)(x^a+1) = \Box$ and (2) * $(x+1)(x^a-1) = \Box$.*

As a consequence of Lemma 1.3.19 Theorem 1.3.18 can be replaced by the following theorem.

Theorem 1.3.20 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BOTWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $x^2 - 1 \neq \square$,
- $\frac{x^{a+2}(x^a+1)}{x+1} \in C_0^d$,
- $\frac{x(x^a-1)}{x-1} \in C_0^d$,

Example 1.3.4 For $p = 29$, $r = 2$, $x = 26$ and $a = 21$ one can determine that $x = r^{19}$, $2(x^{1+a} - 1) = r^{18}$, $x^2 - 1 = r^3$, $\frac{x^{a+2}(x^a+1)}{x+1} = r^{20}$ and $\frac{x(x^a-1)}{x-1} = r^0$. Thus the sufficient conditions of Theorem 1.3.20 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BOTWh(29). The initial round of this design is given by the following seven games. This design has the 3P property.

$(1, 26, 22, 3)$, $(16, 10, 4, 19)$, $(24, 15, 6, 14)$, $(7, 8, 9, 21)$,
$(25, 12, 28, 17)$, $(23, 18, 13, 11)$, $(20, 27, 5, 2)$.

We note that Cases (2) and (3) are consistent with the conditions of Theorem 1.3.20. Cases (5) and (6) would be consistent with the existence of a BOWh(p) as follows. It is to be noted that we have replaced the ordered difference $1 - x$ by $-(x - 1)$ and then utilized that $-1 \in C_m^d$.

Theorem 1.3.21 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BOWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $x^2 - 1 \neq \square$,
- $\frac{x^{a+2}(x^a-1)}{x+1} \in C_0^d$,
- $\frac{x(x^a+1)}{x-1} \in C_0^d$,

Proof: The quadruple of elements from the ordered differences have the OMMA property. ∎

Observe that the conditions imposed in Theorem 1.3.16 are identical to those of Theorem 1.3.21. Hence any $\mathbb{Z}$-cyclic Wh(p) produced by either theorem is a BDOWh(p).

Corollary 1.3.22 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BDOWh(p) if the conditions of Theorem 1.3.21 (alt. Theorem 1.3.16) are satisfied*

Example 1.3.5 For $p = 37$, $r = 2$, $x = 32$ and $a = 1$ one can determine that $x = r^5$, $2(x^{1+a} - 1) = r^{30}$, $x^2 - 1 = r^{29}$, $\frac{x^{a+2}(x^a-1)}{x+1} = r^4$ and $\frac{x(x^a+1)}{x-1} = r^{16}$. Thus the sufficient conditions of Theorem 1.3.21 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BDOWh(37). The initial round of this design is given by the following nine games. This design has the 3P property.

$$\begin{array}{llll}
(1,32,25,5), & (16,31,30,6), & (34,15,36,22), & (26,18,21,19),\\
(9,29,3,8), & (33,20,11,17), & (10,24,28,13), & (12,14,4,23),\\
(7,2,27,35). & & &
\end{array}$$

In a similar fashion Cases (1) and (4) can be associated with a BOWh(p).

Theorem 1.3.23 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BOWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $(x - 1)(x^a + 1) = \square$,
- $\frac{x-1}{x+1} \in C_0^d$,
- $\frac{x^{a+1}(x^a+1)}{x^a-1} \in C_0^d$,

Proof: The quadruple of elements from the ordered differences have the OMMA property. ∎

It easily follows from the conditions of Theorem 1.3.23 that $(x+1)(x^a-1) = \square$. Consequently, the conditions of Theorem 1.3.23 imply those of Theorem 1.3.10 and the whist design is actually a BOTWh(p).

Example 1.3.6 For $p = 37$, $r = 2$, $x = 19$ and $a = 21$ one can determine that $x = r^{35}$, $2(x^{1+a} - 1) = r^{22}$, $(x-1)(x^a+1) = r^{10}$, $\frac{x-1}{x+1} = r^{28}$ and $\frac{x^{a+1}(x^a+1)}{x^a-1} = r^{12}$. Thus the sufficient conditions of Theorem 1.3.23 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BOTWh(37). The initial round of this design is given by the following nine games. This design has the 3P property.

$(1, 19, 30, 18)$, $(16, 8, 36, 29)$, $(34, 17, 21, 20)$, $(26, 13, 3, 24)$,
$(9, 23, 11, 14)$, $(33, 35, 28, 2)$, $(10, 5, 4, 32)$, $(12, 6, 27, 31)$,
$(7, 22, 25, 15)$.

Theorem 1.3.24 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(p) if, in addition to the hypotheses of Theorem 1.3.10, the following conditions are satisfied.*

- $x^2 - 1 \neq \square$,
- $\frac{x^3(x^a-1)}{x-1} \in C_0^4$,
- $\frac{x^3(x^a+1)}{x+1} \in C_0^4$,

Proof: Taking into consideration Lemma 1.3.7, the proof is analogous to the proof of Theorem 1.3.12. ∎

Similar to Lemmas 1.3.13 and 1.3.19 we can state the following Lemma whose proof follows in a fashion very similar to the proof of Lemma 1.3.13.

Lemma 1.3.25 *Let x and a be as defined in The Liaw Variant. Then the conditions (1) $x^2 - 1 \neq \square$, (2) $\frac{x^3(x^a-1)}{x-1} \in C_0^4$, and (3) $\frac{x^3(x^a+1)}{x+1} \in C_0^4$, imply that (1)* $(x-1)(x^a+1) = \square$ and (2) * $(x+1)(x^a-1) = \square$.*

As a consequence of Lemma 1.3.25, Theorem 1.3.24 can be replaced by the following theorem.

Theorem 1.3.26 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(p) if the following conditions are satisfied.*

- $x \neq \Box$,
- $2(x^{1+a} - 1) = \Box$,
- $x^2 - 1 \neq \Box$,
- $\frac{x^3(x^a-1)}{x-1} \in C_0^4$,
- $\frac{x^3(x^a+1)}{x+1} \in C_0^4$,

Example 1.3.7 For $p = 73$, $r = 5$, $x = 31$ and $a = 11$ one can determine that $x = r^{11}$, $2(x^{1+a} - 1) = r^{56}$, $x^2 - 1 = r^{55}$, $\frac{x^3(x^a-1)}{x-1} = r^{36}$ and $\frac{x^3(x^a+1)}{x+1} = r^{28}$. Thus the sufficient conditions of Theorem 1.3.26 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(73). The initial round of this design is given by the following 18 games, the first 9 giving the partition set $\mathcal{A}$ and the second 9 give $\mathcal{B}$. This design does not have the 3P property.

$(1, 31, 65, 42)$, $(2, 62, 57, 11)$, $(4, 51, 41, 22)$, $(8, 29, 9, 44)$,
$(16, 58, 18, 15)$, $(32, 43, 36, 30)$, $(64, 13, 72, 60)$, $(55, 26, 71, 47)$,
$(37, 52, 69, 21)$,
$(25, 45, 19, 28)$, $(50, 17, 38, 56)$, $(27, 34, 3, 39)$, $(54, 68, 6, 5)$,
$(35, 63, 12, 10)$, $(70, 53, 24, 20)$, $(67, 33, 48, 40)$, $(61, 66, 23, 7)$,
$(49, 59, 46, 14)$.

Cases (2) and (3) are consistent with the conditions of Theorem 1.3.26. Cases (5) and (6) would be consistent with the existence of a BSWh(p) as follows.

Theorem 1.3.27 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BSWh(p) if the following conditions are satisfied.*

- $x \neq \Box$,
- $2(x^{1+a} - 1) = \Box$,

- $x^2 - 1 \neq \square$,
- $\frac{x^3(x^a-1)}{x+1} \in C_0^4$,
- $\frac{x^3(x^a+1)}{x-1} \in C_0^4$,

Proof: The quadruple of elements from the split differences have a property similar to the OMMA property. ∎

In a similar fashion Cases (1) and (4) can be associated with a BSWh(p).

Theorem 1.3.28 *The Liaw Variant produces a $\mathbb{Z}$-cyclic BSWh(p) if the following conditions are satisfied.*

- $x \neq \square$,
- $2(x^{1+a} - 1) = \square$,
- $(x+1)(x^a - 1) = \square$
- $\frac{x^2(x-1)}{x+1} \in C_0^4$,
- $\frac{x^2(x^a+1)}{x^a-1} \in C_0^4$,

Proof: The quadruple of elements from the split differences have a property similar to the OMMA property. Condition 4 guarantees that the relationships associated with Cases (1) and (4) are satisfied. ∎

It is a fact that the conditions in Theorem 1.3.28 imply that $(x-1)(x^a+1) = \square$. Thus the conditions of Theorem 1.3.28 imply those of Theorem 1.3.10 and the whist design is actually a BSTWh(p).

Example 1.3.8 For $p = 137$, $r = 3$, $x = 3$ and $a = 123$ one can determine that $x = r^1$, $2(x^{1+a} - 1) = r^{104}$, $(x+1)(x^a - 1) = r^{70}$, $\frac{x^2(x-1)}{x+1} = r^{128}$ and $\frac{x^2(x^a+1)}{x^a-1} = r^0$.

Thus the sufficient conditions of Theorem 1.3.28 are satisfied and The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(137). The initial round of this design is given by the following 34 games, the first 17 giving the partition set $\mathcal{A}$ and the second 17 $\mathcal{B}$. This design has the 3P property.

$(1, 3, 99, 134)$,	$(122, 92, 22, 45)$,	$(88, 127, 81, 10)$,
$(50, 13, 18, 124)$,	$(72, 79, 4, 58)$,	$(16, 48, 77, 89)$,
$(34, 102, 78, 35)$,	$(38, 114, 63, 23)$,	$(115, 71, 14, 66)$,
$(56, 31, 64, 106)$,	$(119, 83, 136, 54)$,	$(133, 125, 15, 12)$,
$(60, 43, 49, 94)$,	$(59, 40, 87, 97)$,	$(74, 85, 65, 52)$,
$(123, 95, 121, 42)$,	$(73, 82, 103, 55)$,	
$(9, 27, 69, 110)$,	$(2, 6, 61, 131)$,	$(107, 47, 44, 90)$,
$(39, 117, 25, 20)$,	$(100, 26, 36, 111)$,	$(7, 21, 8, 116)$,
$(32, 96, 17, 41)$,	$(68, 67, 19, 70)$,	$(76, 91, 126, 46)$,
$(93, 5, 28, 132)$,	$(112, 62, 128, 75)$,	$(101, 29, 135, 108)$,
$(129, 113, 30, 24)$,	$(120, 86, 98, 51)$,	$(118, 80, 37, 57)$,
$(11, 33, 130, 104)$,	$(109, 53, 105, 84)$.	

Scrutiny of the proof of Theorem 1.3.12 (alt. Theorem 1.3.18) leads to the fact that one can place the first forward (alt. the ordered) differences in the base table of The Liaw Variant into a pair of squares and a pair of non-squares. It then becomes essential that the construction is achieved if one can guarantee that for each pair the members are an odd multiple of m units apart in PS(r). Conditions 2 and 3 (in each theorem), that is Conditions 4 and 5 in Theorem 1.3.14, provide such a guarantee. The following corollary and lemma demonstrate some obvious alternative versions of these conditions.

Corollary 1.3.29 *Condition 4 of Theorem 1.3.28 can be replaced by any one of the following.*

- $(x-1)(x^a+1) = \square$,
- $(x+1)(x^a-1) = \square$,
- $(x-1)(x^a-1) = \square$

Lemma 1.3.30 *Condition 4 of Theorem 1.3.14 can be replaced by any one of the following: (1)* $\dfrac{1}{x^{1+a}}\dfrac{x(x^a-1)}{x-1}$, *(2)* $x^{1+a}\dfrac{x-1}{x(x^a-1)}$, *(3)* $\dfrac{1}{x^{1+a}}\dfrac{x-1}{x(x^a-1)}$. *Condition 5 of Theorem 1.3.14 can be replaced by* $\dfrac{x+1}{x(x^a+1)}$.

Statements very similar to those found in Lemma 1.3.30 can be formulated for Conditions 4 and 5 of Theorem 1.3.20.

Lemma 1.3.31 *Let* $z \in \mathbb{Z}_p^*$. *(a) If* $z \in C_0^d$ *then* $z^{-1} \in C_0^d$. *(b) If* $z \in C_m^d$ *then* $z^{-1} \in C_m^d$.

Proof: (a) Since $z = r^{sd}$, $z^{-1} = r^{p-1-sd} = r^{(t-s)d}$ and $z^{-1} \in C_0^d$. (b) Since $z = r^{(2s+1)m}$, $z^{-1} = r^{(2t-2s-1)m}$ and $z^{-1} \in C_m^d$. ∎

Lemmas 1.3.30 and 1.3.31 are useful in the proof of the following theorem.

Theorem 1.3.32 *The pair* $\{x, a\}$ *satisfies the conditions for Theorem 1.3.14 if and only if the pair* $\{x^{-1}, a\}$ *satisfies the conditions for Theorem 1.3.20.*

Proof: "only if" For convenience the conditions listed in Theorem 1.3.14, in the order given, will be denoted by 1, 2, 3, 4, 5 and those of Theorem 1.3.20 by 1^*, 2^*, 3^*, 4^*, 5^*. As a further convenience x^{-1} will be denoted by y. Since x is a non-square $x = r^s$ with s odd. Therefore $y = r^{p-1-s} = r^w$, w odd and 1^* is satisfied. From 2 it follows that $-x^{1+a}(2(y^{1+a}-1)) = \square$ and 2^* follows since both -1 and x^{1+a} are squares. Similarly 3^* follows since 3 yields $-x^2(y^2-1)$. Since x satisfies 4 we can begin with Condition (2) of Lemma 1.3.30, i.e. $x^{1+a}\dfrac{x-1}{x(x^a-1)}$. This latter expression can be manipulated into the form $\dfrac{y-1}{y(y^a-1)}$ and an application of Lemma 1.3.31 establishes 4^*. Lastly we begin with the alternate version of 5 as given in Lemma 1.3.30, namely $\dfrac{x+1}{x(x^a+1)}$, and manipulate this expression to obtain $\dfrac{y^{a+1}(y+1)}{y(y^a+1)} \in C_0^d$. That is to say $\dfrac{y+1}{y(y^a+1)} \in C_m^d$. Hence via Lemma 1.3.31 $\dfrac{y(y^a+1)}{y+1} \in C_m^d$ and $\dfrac{y^{a+2}(y^a+1)}{y+1} \in C_0^d$. Thus 5^* holds and the "only if" part of the theorem is established.

Clearly manipulations similar to those presented in the "only if" part of this proof can be employed to establish the "if" part. ∎

In a similar, but simpler, fashion one can prove the following theorem.

Theorem 1.3.33 *The pair $\{x, a\}$ satisfies the conditions for Theorem 1.3.17 if and only if the pair $\{x^{-1}, a\}$ satisfies the conditions for Theorem 1.3.23.*

Theorems 1.3.32 and 1.3.33 enable us to state a powerful property of The Liaw Variant.

Theorem 1.3.34 *The Liaw Variant produces a $\mathbb{Z}$-cyclic $BDTWh(p)$ if and only if The Liaw Variant produces a $\mathbb{Z}$-cyclic $BOTWh(p)$.*

1.4 Asymptotics

In this section we will apply the "Buratti - Pasotti Asymptotic Technique" (see Theorem 2.2 in [1]) to obtain asymptotic bounds for solutions obtained via The Liaw Variant. The pertinent materials to be used are given here as Definition 1.4.1 and Theorem 1.4.1.

Definition 1.4.1 *[1] Let $s \geq 2$, $w \geq 1$ and $z \geq 0$ be arbitrary integers. Denote by $Q(s, w, z)$ the number defined by*

$$Q = \frac{1}{4}(U + \sqrt{U^2 + 4s^{w-1}(w + sz)}\ )^2,$$

where

$$U = \sum_{h=1}^{w} \binom{w}{h} (s-1)^h (h-1).$$

Theorem 1.4.1 *[1] Let $q \equiv 1 \pmod{s}$ be a prime power, let $B = \{b_1, b_2, \ldots, b_w\}$ be an arbitrary w-subset of $GF(q)$ and let*

$(\beta_1, \beta_2, \ldots, \beta_w)$ *be an arbitrary element of* $\mathbb{Z}_s^w$*, where* $\mathbb{Z}_s^w = \mathbb{Z}_s \times \cdots \times \mathbb{Z}_s$*. Set* $X = \{x \in GF(q) : x \in C_{\beta_i}^s, 1 \leq i \leq w\}$*. Then*

$$|X| \geq \frac{q - U\sqrt{q} - s^{w-1}w}{s^w}$$

and hence $|X| > z$ *as soon as* $q > Q(s, w, z)$*. Thus, in particular,* X *is not empty for* $q > Q(s, w, 0) = Q(s, w)$*.*

Theorem 1.4.2 *Assume that* $k \geq 3$*. Let* $\alpha = x$*,* $\beta = x^a$ *and* $\gamma = x^{a+1}$*. If* α, β, γ *are such that*

$$\begin{array}{lll} \alpha \in C_1^d, & \alpha + 1 \in C_1^d, & \alpha - 1 \in C_{m+2}^d, \\ \beta \in C_{m-1}^d, & \beta + 1 \in C_0^d, & \beta - 1 \in C_1^d, \\ \gamma \in C_m^d, & \gamma - 1 \in C_2^d. & \end{array}$$

then The Liaw Variant produces a $\mathbb{Z}$*-cyclic BDTWh(*p*) and a* $\mathbb{Z}$*-cyclic BOTWh(*p*).*

Proof: For convenience the conditions listed above will be denoted by C_{ij}, $1 \leq i, j \leq 3$ with $C_{33} = \emptyset$ and the conditions of Theorem 1.3.14 will be denoted $C^i, 1 \leq i \leq 5$. Thus $C_{11} \Rightarrow C^1$. Since $k \geq 3$, 2 is a square and $C_{32} \Rightarrow C^2$. Furthermore, (1) $C_{12} \wedge C_{13} \Rightarrow C^3$, (2) $C_{31} \wedge C_{11} \wedge C_{23} \wedge C_{13} \Rightarrow C^4$ and (3) $C_{11} \wedge C_{22} \wedge C_{12} \Rightarrow C^5$. Consequently these conditions are sufficient to produce a $\mathbb{Z}$-cyclic BDTWh(p) via The Liaw Variant. The $\mathbb{Z}$-cyclic BOTWh(p) follows from Theorem 1.3.32 (alt. Theorem 1.3.34). ∎

Theorem 1.4.3 *Let* $k = 2$ *and define* u *by the condition* $2 \in C_u^4$*. If* α, β, γ *are such that*

$$\begin{array}{lll} \alpha \in C_1^4, & \alpha + 1 \in C_1^4, & \alpha - 1 \in C_0^4, \\ \beta \in C_1^4, & \beta + 1 \in C_0^4, & \beta - 1 \in C_1^4, \\ \gamma \in C_2^4, & \gamma - 1 \in C_u^4. & \end{array}$$

then The Liaw Variant produces a $\mathbb{Z}$-cyclic BDTWh(p) and a $\mathbb{Z}$-cyclic BOTWh(p).

Proof: With the same notations as in the proof of Theorem 1.4.2 note that $(2 \in C_u^4) \wedge C_{32} \Rightarrow C^2$. The rest of the proof follows as in the proof of Theorem 1.4.2. ∎

Theorem 1.4.4 *Let p be prime such that $p = 2^k t + 1$ with $k \geq 2$ and t odd. The Liaw Variant produces a $\mathbb{Z}$-cyclic BDTWh(p) and a $\mathbb{Z}$-cyclic BOTWh(p) whenever $p > Q(d, 3)$ where $d = 2^k$.*

Proof: Consider that $k \geq 3$. Assume that $p > Q(d, 3)$. Apply Theorem 1.4.1 with $s = d$, $w = 3$, $B = \{0, 1, -1\}$ and $(\beta_1, \beta_2, \beta_3) = (1, 1, m + 2)$. Then the set $X_1 = \{\alpha : \alpha \in C_1^d, \alpha + 1 \in C_1^d, \alpha - 1 \in C_{m+2}^d\}$ is not empty. Arbitrarily fix $\alpha \in X_1$. Apply Theorem 1.4.1 with $s = d$, $w = 3$, $B = \{0, -1, 1\}$ and $(\beta_1, \beta_2, \beta_3) = (m - 1, 0, 1)$. Then the set $X_2 = \{\beta : \beta \in C_{m-1}^d, \beta + 1 \in C_0^d, \beta - 1 \in C_1^d\}$ is not empty. Arbitrarily fix $\beta \in X_2$. Apply Theorem 1.4.1 with $s = d$, $w = 2$, $B = \{0, 1\}$ and $(\beta_1, \beta_2) = (m, 2)$. Then, since $Q(d, 3) > Q(d, 2)$ the set $X_3 = \{\gamma : \gamma \in C_m^d, \gamma - 1 \in C_2^d\}$ is not empty. Arbitrarily fix $\gamma \in X_3$. Thus by Theorem 1.4.2 the existence is established. The case $k = 2$ follows in a similar fashion. ∎

In the statement and proof of the following theorem we will use the same assignments and the same notations as those given in Theorem 1.4.2 and its proof. The exception here is that $C^i, 1 \leq i \leq 5$ are the conditions of Theorem 1.3.26.

Theorem 1.4.5 *If α, β, γ are such that*

$$\alpha \in C_1^4, \quad \alpha + 1 \in C_3^4, \quad \alpha - 1 \in C_0^4,$$
$$\beta \in C_1^4, \quad \beta + 1 \in C_0^4, \quad \beta - 1 \in C_1^4,$$
$$\gamma \in C_2^4, \quad \gamma - 1 \in C_2^4.$$

then The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(p).

Proof: Assume that $k \geq 3$. Note that (1) $C_{11} \Rightarrow C^1$, (2) $C_{32} \Rightarrow C^2$, (3) $C_{12} \wedge C_{13} \Rightarrow C^3$, (4) $C_{11} \wedge C_{23} \wedge C_{13} \Rightarrow C^4$, and (5) $C_{11} \wedge C_{22} \wedge C_{12} \Rightarrow C^5$. For the case $k = 2$ replace C_{32} by $\gamma - 1 \in C_u^4$ where $2 \in C_u^4$. ∎

Theorem 1.4.6 *Let p be prime such that $p = 2^k t + 1$ with $k \geq 2$ and t odd. The Liaw Variant produces a $\mathbb{Z}$-cyclic BSTWh(p) whenever $p > Q(d,3)$ where $d = 2^k$.*

Proof: Analogous to that of Theorem 1.4.4. ∎

It is clear from the theorems of this section that the number $Q(d,3)$ serves as an analytic bound for The Liaw Variant. For $2 \leq k \leq 8$ the following chart compares this analytic bound with previously published analytic bounds for the specializations involved in this study.

k	$Q(d,3)$	Previous
2	6593	16384 [25]
3	694017	2111209 [15]
4	55131137	254370601 [13]
5	3901878273	21184511401 [21]
6	262145032193	1493628623881 [21]
7	17182293426177	102584081424400 [21]
8	1112744488861697	6798134420640000 [18]

1.5 The Data Study and Some Comparisons of the Results

This section is divided into four subsections wherein each subsection contains a summary of the performance of The Liaw Variant as a producer of specific multi-specialization whist designs. In the categories "$\mathbb{Z}$-Cyclic (BDT/BOT)Wh(p)" and in the category "$\mathbb{Z}$-Cyclic BSTWh(p)" The Liaw Variant outperforms previously published constructions. In the category "$\mathbb{Z}$CPS-BDOWh(p)" The Liaw Variant exhibited no exceptions. The one category in which we feel that The Liaw Variant exhibited a larger than expected list of exceptions was the category "non-$\mathbb{Z}$CPS-BDOWh(p)". Data samples for each category are given in the appendices.

1.5.1 Data Related to $\mathbb{Z}$-Cyclic (BDT/BOT)Wh(p)

As mentioned earlier, previous studies [15, 16, 20, 21, 18, 19, 11, 17] utilized one or both of the generalized Anderson - Ellison Constructions [22] to obtain solutions for the whist designs of interest to the respective study. In each case an analytic asymptotic bound was obtained (or approximated). Then, for those primes less than the analytic asymptotic bound, the data generation involved finding, via the generalized Anderson - Ellison Construction being employed, as many solutions as one could. Of course, in some cases these searches were limited by practical time considerations. In every case case this data process resulted in a list of primes for which the constructions did not produce the desired whist design. That is to say, there was a list of *exceptions*. The chart below provides a comparison of The Liaw Variant (denoted as LV) to the Generalized Anderson - Ellison Construction (denoted Gen A-E) in terms of the number of primes less than 1,000,000 for which the respective construction could not produce a solution. It is to be noted that, with the exclusion of the OTWh(p) for the case $k = 2$ [16], every exception to The Liaw Variant is also an exception to the Generalized Anderson - Ellison Construction. A sampling of the results in this case is presented in Appendix I.

k	Gen A-E	LV	LV Exceptions
2	7 [5], 1 [16]	0	$\emptyset$
3	15 [15]	2	$\{41, 73\}$
4	36 [20]	0	$\emptyset$
5	58 [21]	4	$\{97, 353, 673, 929\}$
6	75 [21]	6	$\{193, 449, 577, 1217, 1601, 2753\}$
7	58 [21]	6	$\{641, 1153, 1409, 2689, 3457, 4481\}$
8	184 [18]	5	$\{257, 769, 3329, 7937, 9473\}$
9	54 [19]	7	$\{7681, 10753, 11777, 17921, 23041, 26113, 36353\}$
10	76 [19]	15	$\{13313, 15361, 19457, 25601, 37889\}$ $\{39937, 50177, 58369, 70657, 76801\}$ $\{80897, 87041, 95233, 101377, 138241\}$
11	31 [19]	11	$\{18433, 59393, 83969, 120833, 133121, 301057\}$ $\{329729, 366593, 428033, 694273, 706561\}$
12	25 [19]	19	$\{12289, 61441, 86017, 151553, 176129\}$ $\{184321, 249857, 307201, 331777, 380929\}$ $\{430081, 471041, 495617, 577537, 643073\}$ $\{667649, 675841, 724993, 765953\}$
13	9 [19]	9	$\{40961, 188417, 270337, 286721, 319489\}$ $\{417793, 778241, 925697, 974849\}$
14	4 [19]	3	$\{114689, 147457, 737281\}$
15	2 [19]	2	$\{163841, 557057\}$
16	1 [19]	1	$\{65537\}$
17	$\emptyset$ [19]	$\emptyset$	$\emptyset$
18	1 [19]	1	$\{786433\}$

1.5.2 Data Related to $\mathbb{Z}$-Cyclic BSTWh(p)

In this subsection The Liaw Variant is compared to the construction used by D.R. Berman and N. J. Finizio [17]. This latter study considered the case $k = 3$ only. The Liaw Variant exceptions listed below for $k = 3$ are also exceptions to the construction employed in [17]. Our data study considered all primes ($\equiv 1 \pmod 4$) less than 1,000,000. A sampling of the results in this case is presented in Appendix II.

k	B-F	LV	LV Exceptions
3	6	2	$\{41, 89\}$
4		2	$\{17, 113\}$
5		1	$\{97\}$
6		0	$\emptyset$
7		1	$\{641\}$
8		1	$\{257\}$
9		0	$\emptyset$
10		0	$\emptyset$
11		0	$\emptyset$
12		0	$\emptyset$
13		0	$\emptyset$
14		0	$\emptyset$
15		1	$\{163841\}$
16		1	$\{65537\}$
17	$\emptyset$	$\emptyset$	$\emptyset$
18		0	$\emptyset$

1.5.3 Data Related to $\mathbb{Z}$CPS-BDOWh(p)

In [10] it is noted that the classic construction of R. C. Bose and J. M. Cameron [26] produces a $\mathbb{Z}$CPS-BDOWh(p) for every prime $p \equiv 1 \pmod 4$. Our data study considered all primes ($\equiv 1 \pmod 4$) less than 1,000,000. There were no exceptions, that is to say, The Liaw Variant produced a $\mathbb{Z}$CPS-BDOWh(p) for all such primes. A sampling of the results in this case is presented in Appendix III.

1.5.4 Data Related to non-$\mathbb{Z}$CPS-BDOWh(p)

Since $\mathbb{Z}$CPS-BDOWh(p) are known for all $p \equiv 1 \pmod 4$ it was decided to investigate, using The Liaw Variant, the existence of non $\mathbb{Z}$CPS-BDOWh(p). A construction for non-$\mathbb{Z}$CPS-DOWh(p) was introduced in [27]. In that paper the property of balance was not considered and no data was generated. Consequently there are no comparisons to be made for this multi-specialization category. Our data study considered all $p = 4u + 1$ less than 1,000,000 and related to the conditions associated with Corollary 1.3.22. A sampling of the results in this case is presented in Appendix IV. However, a comparison of the data information in the previous three subsections with the chart below shows that for non-$\mathbb{Z}$CPS-DOWh(p) The Liaw Variant has considerably more exceptional values in this case. The specific exceptional values are presented in Appendix V.

k	Number of LV Exceptions
2	4
3	4
4	6
5	6
6	11
7	15
8	15
9	21
10	27
11	14
12	20
13	9
14	4
15	2
16	1
17	$\emptyset$
18	1

Appendix I

This appendix contains a sampling of the data related to the BDTWh(p) and BOTWh(p) generated by The Liaw Variant. For each value of k the data study was carried out for all primes less than 1,000,000. Consequently the data sets are quite extensive. For efficiency of space we restrict our lists to the data for the first 4 solutions. For $k = 13, 15, 16, 17, 18$ no solutions were obtained. The complete data sets for $p < 1,000,000$ are available from either of the authors. For $k \geq 10$ a further attempt at conserving space is to abbreviate the BDT base table, $(1, x, x^{1+a}, -x)$ and the BOT base table, $(1, x^{-1}, (x^{-1})^{1+a}, -x^{-1}$, respectively, to (x^{1+a}) and $(x^{-1}, (x^{-1})^{1+a})$.

$\{p, r, x, a\}$	BDT base table	BOT base table
$k = 2$		
$\{29, 2, 19, 21\}$	$(1, 19, 4, 10)$	$(1, 26, 22, 3)$
$\{37, 2, 2, 21\}$	$(1, 2, 21, 35)$	$(1, 19, 30, 18)$
$\{53, 2, 34, 29\}$	$(1, 34, 6, 19)$	$(1, 39, 9, 14)$
$\{61, 2, 24, 17\}$	$(1, 24, 52, 37)$	$(1, 28, 27, 33)$
$k = 3$		
$\{89, 3, 54, 51\}$	$(1, 54, 73, 35)$	$(1, 61, 50, 28)$
$\{137, 3, 27, 11\}$	$(1, 27, 78, 110)$	$(1, 66, 65, 71)$
$\{233, 3, 22, 139\}$	$(1, 22, 162, 211)$	$(1, 53, 105, 180)$
$\{281, 3, 194, 35\}$	$(1, 194, 119, 87)$	$(1, 239, 196, 42)$
$k = 4$		
$\{113, 3, 39, 103\}$	$(1, 39, 83, 74)$	$(1, 29, 64, 84)$
$\{241, 7, 199, 135\}$	$(1, 199, 10, 42)$	$(1, 109, 217, 132)$
$\{337, 10, 257, 135\}$	$(1, 257, 324, 80)$	$(1, 139, 311, 198)$
$\{401, 3, 27, 135\}$	$(1, 27, 145, 374)$	$(1, 104, 177, 297)$
$k = 5$		
$\{1249, 7, 636, 1199\}$	$(1, 636, 1103, 613)$	$(1, 163, 1172, 1086)$
$\{1697, 3, 344, 271\}$	$(1, 344, 93, 1353)$	$(1, 74, 73, 1623)$
$\{1889, 3, 1198, 1231\}$	$(1, 1198, 1230, 691)$	$(1, 1427, 923, 462)$
$\{2017, 5, 1897, 1487\}$	$(1, 1897, 175, 120)$	$(1, 1227, 1118, 790)$
$k = 6$		
$\{2113, 5, 1895, 287\}$	$(1, 1895, 1857, 218)$	$(1, 126, 1593, 1987)$
$\{3137, 3, 1844, 1375\}$	$(1, 1844, 1732, 1293)$	$(1, 427, 2005, 2710)$
$\{4289, 3, 113, 543\}$	$(1, 113, 1732, 4176)$	$(1, 2581, 2563, 1708)$
$\{4673, 3, 2707, 159\}$	$(1, 2707, 1119, 1966)$	$(1, 1091, 593, 3582)$
$k = 7$		
$\{4993, 5, 3704, 2623\}$	$(1, 3704, 2631, 1289)$	$(1, 2328, 2116, 2665)$
$\{6529, 7, 217, 5695\}$	$(1, 217, 3290, 6312)$	$(1, 4122, 6273, 2407)$
$\{7297, 5, 6977, 6079\}$	$(1, 6977, 3762, 320)$	$(1, 4401, 3536, 2896)$
$\{9601, 13, 6455, 703\}$	$(1, 6455, 8719, 3146)$	$(1, 9012, 8175, 589)$

$\{p, r, x, a\}$	BDT base table	BOT base table
$k = 8$		
$\{14081, 3, 10989, 11647\}$	$(1, 10989, 8039, 3092)$	$(1, 9823, 8835, 4258)$
$\{14593, 5, 9805, 7551\}$	$(1, 9805, 11409, 4788)$	$(1, 7647, 10931, 6946)$
$\{22273, 5, 2961, 1407\}$	$(1, 2961, 22065, 19312)$	$(1, 850, 5247, 21423)$
$\{23297, 3, 1494, 14207\}$	$(1, 1494, 21082, 21803)$	$(1, 11898, 15335, 11399)$
$k = 9$		
$\{32257, 15, 5992, 4351\}$	$(1, 5992, 14199, 26265)$	$(1, 24112, 14603, 8145)$
$\{45569, 3, 33443, 38655\}$	$(1, 33443, 40272, 12126)$	$(1, 32465, 41259, 13104)$
$\{51713, 3, 22002, 45823\}$	$(1, 22002, 16347, 29711)$	$(1, 48909, 28237, 2804)$
$\{67073, 3, 49449, 19199\}$	$(1, 49449, 4952, 17624)$	$(1, 5271, 25342, 61802)$
$k = 10$		
$\{64513, 5, 63789, 39423\}$	(44846)	$(60325, 44815)$
$\{187393, 7, 46601, 35327\}$	(16832)	$(154045, 72666)$
$\{119809, 11, 102828, 41471\}$	(70695)	$(14372, 92856)$
$\{136193, 3, 75113, 96767\}$	(40405)	$(121247, 57767)$
$k = 11$		
$\{79873, 7, 67715, 19455\}$	(34770)	$(75294, 65936)$
$\{202753, 10, 78209, 136191\}$	(10138)	$(122737, 86897)$
$\{464897, 3, 408845, 5119\}$	(206059)	$(89617, 109858)$
$\{473089, 29, 243549, 308223\}$	(297658)	$(59706, 392910)$
$k = 12$		
$\{520193, 3, 286952, 59391\}$	(350610)	$(356051, 444055)$
$k = 14$		
$\{638977, 7, 236474, 335871\}$	(262137)	$(396212, 276374)$

Appendix II

This appendix contains data related to BSTWh(p) produced by The Liaw Variant. For the partition cells see the comment prior to Theorem 1.3.28. As in Appendix I the data for the first 4 solutions is presented. Complete data sets for primes less than 1,000,000 can be obtained from either of the authors.

k	$\{p, r, x, a\}$	BST Base Table
3	$\{73, 5, 31, 11\}$	$(1, 31, 65, 42)$
	$\{137, 3, 12, 115\}$	$(1, 12, 99, 125)$
	$\{233, 3, 10, 211\}$	$(1, 10, 29, 223)$
	$\{281, 3, 27, 35\}$	$(1, 27, 126, 254)$

k	$\{p, r, x, a\}$	Base Table
4	$\{241, 7, 178, 39\}$	$(1, 178, 226, 63)$
	$\{337, 10, 195, 71\}$	$(1, 195, 42, 142)$
	$\{401, 3, 27, 263\}$	$(1, 27, 177, 374)$
	$\{433, 5, 94, 71\}$	$(1, 94, 235, 339)$
5	$\{353, 3, 206, 111\}$	$(1, 206, 166, 147)$
	$\{673, 5, 306, 111\}$	$(1, 306, 418, 367)$
	$\{929, 3, 174, 239\}$	$(1, 174, 354, 755)$
	$\{1249, 7, 452, 399\}$	$(1, 452, 1233, 797)$
6	$\{193, 5, 37, 31\}$	$(1, 37, 109, 156)$
	$\{449, 3, 241, 31\}$	$(1, 241, 90, 208)$
	$\{577, 5, 265, 415\}$	$(1, 265, 193, 312)$
	$\{1217, 3, 389, 415\}$	$(1, 389, 961, 828)$
7	$\{1153, 5, 623, 191\}$	$(1, 623, 503, 530)$
	$\{1409, 3, 27, 63\}$	$(1, 27, 702, 1382)$
	$\{2689, 19, 2219, 575\}$	$(1, 2219, 308, 470)$
	$\{3457, 7, 343, 703\}$	$(1, 343, 1937, 3114)$
8	$\{769, 11, 691, 639\}$	$(1, 691, 361, 78)$
	$\{3329, 3, 243, 383\}$	$(1, 243, 359, 3086)$
	$\{7937, 3, 2533, 6271\}$	$(1, 2533, 5543, 5404)$
	$\{9473, 3, 4227, 4479\}$	$(1, 4227, 7980, 5246)$
9	$\{7681, 17, 2337, 3327\}$	$(1, 2337, 3119, 5344)$
	$\{10753, 11, 1331, 3327\}$	$(1, 1331, 2507, 9422)$
	$\{11777, 3, 492, 9471\}$	$(1, 492, 10944, 11285)$
	$\{17921, 3, 27, 4863\}$	$(1, 27, 2294, 17894)$
10	$\{13313, 3, 4078, 4607\}$	$(1, 4078, 1411, 9235)$
	$\{15361, 7, 12740, 1535\}$	$(1, 12740, 9905, 2621)$
	$\{19457, 3, 243, 1535\}$	$(1, 243, 11208, 19214)$
	$\{25601, 3, 2187, 18943\}$	$(1, 2187, 2560, 23414)$
11	$\{18433, 5, 1499, 15359\}$	$(1, 1499, 3785, 16934)$
	$\{59393, 5, 3125, 37887\}$	$(1, 3125, 49689, 56268)$
	$\{79873, 7, 24813, 35839\}$	$(1, 24813, 66681, 55060)$
	$\{83969, 3, 243, 80895\}$	$(1, 243, 10318, 83726)$
12	$\{12289, 11, 5812, 2047\}$	$(1, 5812, 6241, 6477)$
	$\{61441, 17, 50953, 14335\}$	$(1, 50953, 54353, 10488)$
	$\{86017, 5, 59330, 38911\}$	$(1, 59330, 55326, 26687)$
	$\{151553, 3, 243, 38911\}$	$(1, 243, 126583, 151310)$
13	$\{40961, 3, 13906, 4095\}$	$(1, 13906, 9110, 27055)$
	$\{188417, 3, 86987, 151551\}$	$(1, 86987, 181966, 101430)$
	$\{270337, 10, 23437, 61439\}$	$(1, 23437, 253665, 246900)$
	$\{286721, 11, 11, 217087\}$	$(1, 11, 251064, 286710)$

k	$\{p, r, x, a\}$	Base Table
14	$\{114689, 3, 31459, 40959\}$	$(1, 31459, 105801, 83230)$
	$\{147457, 10, 54340, 40959\}$	$(1, 54340, 22773, 93117)$
	$\{638977, 7, 128588, 155647\}$	$(1, 128588, 461375, 510389)$
	$\{737281, 11, 317865, 24575\}$	$(1, 317865, 39221, 419416)$
15	$\{557057, 3, 365241, 376831\}$	$(1, 365241, 93279, 191816)$
18	$\{786433, 10, 51897, 131071\}$	$(1, 51897, 392449, 734536)$

Appendix III

This appendix contains a sampling of the $\mathbb{Z}$CPS-Wh(p) generated by The Liaw Variant. As in the previous appendices our data lists are for the first 4 solutions only. The complete data sets are available from either author.

k	$\{p, r, x, a\}$	$\mathbb{Z}$CPS Base Table
2	$\{5, 2, 2, 1\}$	$(1, 2, 4, 3)$
	$\{13, 2, 8, 1\}$	$(1, 8, 12, 5)$
	$\{29, 2, 12, 1\}$	$(1, 12, 28, 17)$
	$\{37, 2, 31, 1\}$	$(1, 31, 36, 6)$
3	$\{41, 6, 6, 19\}$	$(1, 6, 40, 35)$
	$\{73, 5, 10, 3\}$	$(1, 10, 72, 63)$
	$\{89, 3, 37, 3\}$	$(1, 37, 88, 52)$
	$\{137, 3, 27, 67\}$	$(1, 27, 136, 110)$
4	$\{17, 3, 10, 7\}$	$(1, 10, 16, 7)$
	$\{113, 3, 40, 7\}$	$(1, 40, 112, 73)$
	$\{241, 7, 197, 7\}$	$(1, 197, 240, 44)$
	$\{337, 10, 199, 23\}$	$(1, 199, 336, 138)$
5	$\{97, 5, 28, 15\}$	$(1, 28, 96, 69)$
	$\{353, 3, 67, 15\}$	$(1, 67, 352, 286)$
	$\{673, 5, 118, 15\}$	$(1, 118, 672, 555)$
	$\{929, 3, 701, 15\}$	$(1, 701, 928, 228)$
6	$\{193, 5, 158, 31\}$	$(1, 158, 192, 35)$
	$\{449, 3, 412, 31\}$	$(1, 412, 448, 37)$
	$\{577, 5, 557, 31\}$	$(1, 557, 576, 20)$
	$\{1217, 3, 910, 31\}$	$(1, 910, 1216, 307)$
7	$\{641, 3, 243, 63\}$	$(1, 243, 640, 398)$
	$\{1153, 5, 1096, 63\}$	$(1, 1096, 1152, 57)$
	$\{1409, 3, 261, 63\}$	$(1, 261, 1408, 1148)$
	$\{2689, 19, 1410, 63\}$	$(1, 1410, 2688, 1279)$

k	$\{p, r, x, a\}$	$\mathbb{Z}$CPS Base Table
8	$\{257, 3, 27, 127\}$	$(1, 27, 256, 230)$
	$\{769, 11, 214, 127\}$	$(1, 214, 768, 555)$
	$\{3329, 3, 2775, 127\}$	$(1, 2775, 3328, 554)$
	$\{7937, 3, 2805, 127\}$	$(1, 2805, 7936, 5132)$
9	$\{7681, 17, 5722, 255\}$	$(1, 5722, 7680, 1959)$
	$\{10753, 11, 4894, 255\}$	$(1, 4894, 10752, 5859)$
	$\{11777, 3, 7795, 255\}$	$(1, 7795, 11776, 3982)$
	$\{17921, 3, 12162, 255\}$	$(1, 12162, 17920, 5759)$
10	$\{13313, 3, 10076, 511\}$	$(1, 10076, 13312, 3237)$
	$\{15361, 7, 11457, 511\}$	$(1, 11457, 15360, 3904)$
	$\{19457, 3, 15841, 511\}$	$(1, 15841, 19456, 3616)$
	$\{25601, 3, 12725, 511\}$	$(1, 12725, 25600, 12876)$
11	$\{18433, 5, 17660, 1023\}$	$(1, 17660, 18432, 773)$
	$\{59393, 5, 2678, 1023\}$	$(1, 2678, 59392, 56715)$
	$\{79873, 7, 13725, 1023\}$	$(1, 13725, 79872, 66148)$
	$\{83969, 3, 27450, 1023\}$	$(1, 27450, 83968, 56519)$
12	$\{12289, 11, 8105, 2047\}$	$(1, 8105, 12288, 4184)$
	$\{61441, 17, 39003, 2047\}$	$(1, 39003, 61440, 22438)$
	$\{86017, 5, 7923, 2047\}$	$(1, 7923, 86016, 78094)$
	$\{151553, 3, 52786, 2047\}$	$(1, 52786, 151552, 98767)$
13	$\{40961, 3, 243, 4095\}$	$(1, 243, 40960, 40718)$
	$\{188417, 3, 3995, 4095\}$	$(1, 3995, 188416, 184422)$
	$\{270337, 10, 247085, 4095\}$	$(1, 247085, 270336, 23252)$
	$\{286721, 11, 53388, 4095\}$	$(1, 53388, 286720, 233333)$
14	$\{114689, 3, 28269, 8191\}$	$(1, 28269, 114688, 86420)$
	$\{147457, 10, 91750, 8191\}$	$(1, 91750, 147456, 55707)$
	$\{638977, 7, 461405, 8191\}$	$(1, 461405, 638976, 177572)$
	$\{737281, 11, 667463, 8191\}$	$(1, 667463, 737280, 69818)$
15	$\{163841, 3, 94740, 16383\}$	$(1, 94740, 163840, 69101)$
	$\{557057, 3, 459996, 16383\}$	$(1, 459996, 557056, 97061)$
	$\{1146881, 3, 1118503, 16383\}$	$(1, 1118503, 1146880, 28378)$
	$\{2654209, 11, 1985530, 16383\}$	$(1, 1985530, 2654208, 668679)$
16	$\{65537, 3, 27, 32767\}$	$(1, 27, 65536, 65510)$
	$\{1376257, 5, 485685, 32767\}$	$(1, 485685, 1376256, 890572)$
	$\{1769473, 5, 418362, 32767\}$	$(1, 418362, 1769472, 1351111)$
	$\{2424833, 3, 2097965, 32767\}$	$(1, 2097965, 2424832, 326868)$
17	$\{1179649, 19, 736781, 65535\}$	$(1, 736781, 1179648, 442868)$
	$\{2752513, 20, 2722767, 65535\}$	$(1, 2722767, 2752512, 29746)$
	$\{6684673, 5, 5291143, 65535\}$	$(1, 5291143, 6684672, 1393530)$
	$\{6946817, 3, 5033378, 65535\}$	$(1, 5033378, 6946816, 1913439)$

k	$\{p, r, x, a\}$	$\mathbb{Z}$CPS Base Table
18	$\{786433, 10, 1000, 131071\}$	$(1, 1000, 786432, 785433)$
	$\{8650753, 10, 2518187, 131071\}$	$(1, 2518187, 8650752, 6132566)$
	$\{10223617, 5, 4568826, 131071\}$	$(1, 4568826, 10223616, 5654791)$
	$\{11272193, 3, 6233218, 131071\}$	$(1, 6233218, 11272192, 5038975)$

Appendix IV

This appendix contains a sampling of the non-$\mathbb{Z}$CPS-BDOWh(p) generated by The Liaw Variant. As in the previous appendices our data lists are for the first 4 solutions only. The complete data sets are available from either author.

k	$\{p, r, x, a\}$	BDO Base Table
2	$\{37, 2, 32, 1\}$	$(1, 32, 25, 5)$
	$\{61, 2, 24, 1\}$	$(1, 24, 27, 37)$
	$\{101, 2, 8, 1\}$	$(1, 8, 64, 93)$
	$\{109, 6, 39, 65\}$	$(1, 39, 71, 70)$
3	$\{137, 3, 12, 99\}$	$(1, 12, 64, 125)$
	$\{281, 3, 27, 219\}$	$(1, 27, 202, 254)$
	$\{313, 10, 47, 299\}$	$(1, 47, 210, 266)$
	$\{409, 21, 21, 75\}$	$(1, 21, 392, 388)$
4	$\{593, 3, 114, 247\}$	$(1, 114, 194, 479)$
	$\{881, 3, 540, 599\}$	$(1, 540, 710, 341)$
	$\{977, 3, 685, 375\}$	$(1, 685, 396, 292)$
	$\{1009, 11, 228, 231\}$	$(1, 228, 418, 781)$
5	$\{929, 3, 802, 399\}$	$(1, 802, 354, 127)$
	$\{1889, 3, 567, 1455\}$	$(1, 567, 985, 1322)$
	$\{2017, 5, 51, 1743\}$	$(1, 51, 1828, 1966)$
	$\{2273, 3, 325, 1487\}$	$(1, 325, 723, 1948)$
6	$\{4673, 3, 3693, 2719\}$	$(1, 3693, 1674, 980)$
	$\{4801, 7, 2078, 2463\}$	$(1, 2078, 4268, 2723)$
	$\{5441, 3, 4016, 3807\}$	$(1, 4016, 3087, 1425)$
	$\{5569, 13, 2754, 4575\}$	$(1, 2754, 5410, 2815)$
7	$\{7297, 5, 5305, 2623\}$	$(1, 5305, 626, 1992)$
	$\{9601, 13, 4336, 1599\}$	$(1, 4336, 4207, 5265)$
	$\{15233, 3, 4589, 8639\}$	$(1, 4589, 5313, 10644)$
	$\{16001, 3, 5663, 4543\}$	$(1, 5663, 6951, 10338)$

k	$\{p, r, x, a\}$	BDO Base Table
8	$\{36097, 5, 19573, 25727\}$	$(1, 19573, 21003, 16524)$
	$\{37633, 5, 21681, 6015\}$	$(1, 21681, 2619, 15952)$
	$\{41729, 3, 21, 15999\}$	$(1, 21, 11831, 41708)$
	$\{43777, 5, 4524, 17023\}$	$(1, 4524, 33873, 39253)$
9	$\{102913, 5, 47335, 63231\}$	$(1, 47335, 75902, 55578)$
	$\{113153, 3, 103972, 26367\}$	$(1, 103972, 35774, 9181)$
	$\{118273, 5, 90700, 45823\}$	$(1, 90700, 79807, 27573)$
	$\{119297, 3, 112460, 3327\}$	$(1, 112460, 4481, 6837)$
10	$\{240641, 3, 176388, 194047\}$	$(1, 176388, 143180, 64253)$
	$\{285697, 5, 250739, 183807\}$	$(1, 250739, 89592, 34958)$
	$\{295937, 3, 152158, 239103\}$	$(1, 152158, 9190, 143779)$
	$\{320513, 3, 46373, 303615\}$	$(1, 46373, 174204, 274140)$
11	$\{329729, 3, 49091, 222207\}$	$(1, 49091, 256308, 280638)$
	$\{366593, 3, 171020, 111615\}$	$(1, 171020, 263382, 195573)$
	$\{534529, 11, 71332, 166911\}$	$(1, 71332, 71610, 463197)$
	$\{575489, 3, 55881, 373759\}$	$(1, 55881, 160351, 519608)$
12	$\{921601, 11, 180200, 325631\}$	$(1, 180200, 614734, 741401)$

Appendix V

This appendix contains the primes for which The Liaw Variant failed to produce $\mathbb{Z}$-cyclic BDOWh(p) designs. The data study was restricted to primes less than 1,000,000.

k	exceptions set
2	$\{5, 13, 29, 53\}$
3	$\{41, 73, 89, 233\}$
4	$\{17, 113, 241, 337, 401, 433, \}$
5	$\{97, 353, 673, 1249, 1697, 2081\}$
6	$193, 449, 577, 1217, 1601, 2113, 2753, 3137, 4289, 6337, 8513\}$
7	$\{641, 1153, 1409, 2689, 3457, 4481, 4993, 6529, 9857, 10369, 11393, 12161, 13441, 13697, 20353\}$
8	$\{257, 769, 3329, 7937, 9473, 14081, 14593, 22273, 23297, 26881, 30977, 31489, 40193, 49921, 60161\}$
9	$\{7681, 10753, 11777, 17921, 23041, 26113, 32257, 36353, 45569, 51713, 67073, 76289, 81409, 84481, 87553, 96769, 112129, 115201, 125441, 133633, 161281\}$

k	exceptions set
10	$\{13313, 15361, 19457, 25601, 37889, 39937, 50177, 58369, 64513, 70657, 76801, 80897, 87041, 95233, 101377, 119809, 136193, 138241, 187393, 211969, 228353, 242689, 254977, 279553, 310273, 365569, 463873\}$
11	$\{18433, 59393, 79873, 83969, 120833, 133121, 202753, 301057, 428033, 464897, 473089, 514049, 649217, 878593\}$
12	$\{12289, 61441, 86017, 151553, 176129, 184321, 249857, 307201, 331777, 380929, 430081, 471041, 495617, 520193, 577537, 643073, 667649, 675841, 724993, 765953, 790529, 946177, 962561, 995329\}$
13	$\{40961, 188417, 270337, 286721, 319489, 417793, 778241, 925697, 974849\}$
14	$\{114689, 147457, 638977, 737281\}$
15	$\{163841, 557057\}$
16	$\{65537\}$
18	$\{786433\}$

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CHAPTER 2

Results Related to the Existence of Non $\mathcal{Z}$CPS-BDOWh(p) For Primes of the Form $p = 4u + 1$

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Abstract

The concept of balance is a relatively new specialization in the study of whist designs, having been introduced in 2010 [1]. It was noted at that time that, for primes of the form $p = 4u+1$, a classic whist construction of R. C. Bose and J. M. Cameron produces whist designs that not only have the property of balance but also two other whist specializations, that of being directed and ordered. The Bose - Cameron construction is such that the set of initial round partner pairs form the patterned starter in the additive group of the galois field $\mathbb{Z}_p$. For primes of the form $p = 4u + 1 = 2^k t + 1$, where $k \geq 2$ and t is odd, this study addresses the existence of whist designs that are balanced, directed and ordered but whose initial round partner pairs do not form the patterned starter in $\mathbb{Z}_p$. We establish that for $2 \leq k \leq 6$ this new multi-specialization whist design exists for all such primes except for $p = 5, 13, 17$ and possibly for $p = 97, 193$. For $k > 6$ and $p < 1,000,000$ these designs exist except, possibly, for $p = 257, 769, 12289, 40961, 65537, 786433$.

keywords: Balanced whist designs; Directed whist designs; Ordered whist designs; $\mathbb{Z}$-cyclic designs.

2.1 Introduction

For some history and interesting information related to the whist tournament problem see the recent article by I. Anderson and T. Crilly [2]. Although whist tournament designs are known to exist for all $v \equiv 0, 1 \pmod 4$ [3] we restrict our attention, here, to $v \equiv 1 \pmod 4$.

Definition 2.1.1 *A* **whist tournament design** *on $v = 4u + 1$ players, denoted* **Wh**(v)*, is a $(v, 4, 3)$ near resolvable BIBD. A* **whist game** *(alt. whist table) is a block, (a, b, c, d), of the BIBD and denotes that the* **partnership** $\{a, c\}$ **opposes**

the **partnership** $\{b, d\}$. *The design is subject to the* **whist conditions** *that every player is a* **partner** *of every other player* **exactly once** *and is an* **opponent** *of every other player* **exactly twice**. *The near resolution classes of the BIBD are called the* **rounds** *of the* $Wh(v)$.

It follows that a $\text{Wh}(4u + 1)$ consists of $4u + 1$ rounds and every player plays in exactly one game in each round except one round in which the player "sits out". For convenience we visualize the whist game (a, b, c, d) as representing four players seated round a table with a seated at the North position, b at the East position, c at the South position and d at the West position. Thus, for example, one can speak of a as b's right hand opponent and c as b's left hand opponent.

Definition 2.1.2 *A whist design is said to be* $\mathbb{Z}$**-cyclic** *if the players are elements in* $\mathbb{Z}_{4u+1}$. *It is also required that the rounds be cyclic. That is to say, the rounds can be labeled,* $R_1, R_2, \ldots$, *in such a way that* R_{j+1} *is obtained by adding* $+1$ *(mod* $4u + 1$*) to every element in* R_j.

Note that the entire set of $4u+1$ rounds of a $\mathbb{Z}$-cyclic $\text{Wh}(4u+1)$ can be obtained by development of any specific round of the design. The specific round is referred to as **the initial round**. Conventionally, the **initial round** of a $\mathbb{Z}$-cyclic $\text{Wh}(4u+1)$ is the round in which 0 sits out. Consequently the set of partner pairs for the initial round of a $\mathbb{Z}$-cyclic $\text{Wh}(4u + 1)$ forms a **starter** in $\mathbb{Z}_{4u+1}$ [3].

Definition 2.1.3 *Let* G *be an abelian group of order* $2k + 1$ *and let* e_G *denote the identity element. The collection of pairs* $PS = \{\{x, -x\} : x \in G, x \neq e_G\}$ *is called the* **patterned starter** *in* G.

Definition 2.1.4 *[4] A* $\mathbb{Z}$*-cyclic* $Wh(4u + 1)$ *is said to be a* $\mathbb{Z}$**-cylic patterned starter whist tournament**, *denoted by* $\mathbb{Z}$**CPS-Wh**$(4u + 1)$, *if the set of initial round partner pairs forms the patterned starter in* $\mathbb{Z}_{4u+1}$.

Definition 2.1.5 *[5] A whist tournament on v players is said to be a* **directed-whist tournament**, *DWh(v), if every player has every other player exactly once as a left hand opponent (and, hence, exactly once as a right hand opponent).*

Definition 2.1.6 *[6] A whist tournament on v players is said to be an* **ordered-whist tournament**, *OWh(v), if each player opposes every other player once at North-South and once at East-West.*

A necessary condition for the existence of an OWh(v) is $v \equiv 1 \pmod 4$.

For **any given round** of a Wh(v) the set of all players sitting in the North and South positions is referred to as the **N-S line**. Similarly the **E-W line** is the set of all players sitting in the East and West positions. Let $\{x, y\}$ be any pair of players in a Wh(v). For any round of the Wh(v) for which x and y both play, but not at the same table, x and y are said to be **relative opponents** if they belong to the **same line** (either N-S or E-W) and **relative partners** if they belong to **opposite lines**. For $v = 4u + 1$ there are exactly $4u - 4$ rounds in which x and y play at different tables.

Definition 2.1.7 *[1] A whist tournament on v players is said to be a* **balanced whist tournament**, *BWh(v), if every pair of players are relative opponents exactly $2u - 2$ times (and, hence, relative partners exactly $2u - 2$ times).*

In 1965 R.C. Bose and J. M. Cameron [7] introduced a construction that produces $\mathbb{Z}$CPS-Wh(p) for all primes $p \equiv 1 \pmod 4$. It has been known since 2010 that these whist designs of Bose and Cameron are, **simultaneously**, a balanced whist design, a directedwhist design and an orderedwhist design. That is to say, a $\mathbb{Z}$CPS - BDOWh(p). The goal of this paper is to produce $\mathbb{Z}$-cyclic BDOWh(p) that are **NOT** $\mathbb{Z}$CPS designs and to discuss the existence of this new type of multi-specialization whist design. It is a fact that Costa et al. [8] introduced a construction that was designed to produce non $\mathbb{Z}$CPS-DOWh(p). In this latter paper, however, there is no discussion of balance nor is there any information regarding the existence of such designs. In [9] the present authors introduced a versatile whist construction called The Liaw Variant and demonstrated that this construction is capable of producing, under varying sets of sufficient conditions, every known whist specialization. In particular The Liaw Variant is capable of producing non $\mathbb{Z}$CPS-BDOWh(p). It turned out, however, that for this one multi-specialization (the non $\mathbb{Z}$CPS-BDOWh(p)) the results of the data study were disappointing in that there was a large set of primes for which The Liaw Variant was not able to produce the desired design. In the next section a new construction is introduced that produces much more satisfactory results.

2.2 The Main Construction

To facilitate our study the primes $p \equiv 1 \pmod 4$ are taken in the form $p = 2^k t + 1$ where $k \geq 2$ and t is odd. Furthermore, let $d = 2^k$, $m = 2^{k-1}$ and $n = 2^{k-2}$. r will denote an arbitrary, but fixed, primitive root of p.

Let p and r be as above and let x be a non-square, y a square in $\mathbb{Z}_p^*$. Consider the following collection of whist games.

$$(x, y, x^{m+1}, yx^m) \otimes r^{dj+2i},\ 0 \leq j \leq t-1,\ 0 \leq i \leq n-1. \tag{2.2}$$

In the sequel this collection of whist games will be referred to as The Main Construction and the game (x, y, x^{m+1}, yx^m) will be called the "base game". When convenient to do so the operation $\otimes r^{dj+2i}$, $0 \leq j \leq t-1$, $0 \leq i \leq n-1$ will be abbreviated to $\otimes\mathcal{R}$. Theorem 2.2.5 below contains a set of sufficient conditions that, when satisfied, guarantee that The Main Construction produces the initial round of a $\mathbb{Z}$-cyclic BDOWh(p). Note, however, that if $x = r^t$ the corresponding design will be a $\mathbb{Z}$CPS design. Thus $x = r^t$ is not a permissible choice given our goal.

Theorem 2.2.1 *If The Main Construction produces the initial round of a whist design then the design is automatically a $\mathbb{Z}$-cyclic $BWh(p)$.*

Proof: Since x is a non-square and m is even it is clear that the N-S line is the set of non-squares in $\mathbb{Z}_p^*$. Likewise since y is a square it follows that the E-W line is the set of squares in $\mathbb{Z}_p^*$. It is well known that these two sets form a $(4u+1, 2u, 2u-1)$ difference family (see, e.g., Theorem 2.2.5 in [3]). Thus every pair of elements, say $\{z, w\}$, appear in the same line exactly $2u-1$ times. Removal of the time that z and w are partners shows that they are relative opponents exactly $2u-2$ times. Hence they are relative partners exactly $2u-2$ times. ∎

Definition 2.2.1 *Given the games of The Main Construction the collection of differences $\{y-x, x^{m+1}-y, yx^m-x^{m+1}, x-yx^m\} \otimes \mathcal{R}$ is called the first forward differences (alt. the right hand differences).*

Observe that the right hand differences are obtained by computing "counter-clockwise" differences around each initial round whist table. Correspondingly, the left hand differences are obtained by computing "clockwise" differences. Clearly the left hand differences are the additive inverses of the right hand differences. The following theorem is well known [3].

Theorem 2.2.2 *If the first forward differences in the initial round of a $\mathbb{Z}$-cyclic $Wh(4u+1)$ cover $\mathbb{Z}_{4u+1}^*$ exactly once then the $Wh(4u+1)$ is a directed whist design.*

Remark 2.2.3 *This latter theorem is predicated on the fact that one begins with the initial round of a $\mathbb{Z}$-cyclic $Wh(4u+1)$. If, however, we are testing whether or not a collection of games forms an initial round for a directed whist design it is enough to show that the partner whist condition is satisfied and that the right (alt. left) hand differences cover the non-zero elements exactly once.*

Definition 2.2.2 *Given the games of The Main Construction the collection of differences* $\{x-y, x^{m+1}-y, x^{m+1}-yx^m, x-yx^m\} \otimes \mathcal{R}$ *is called the ordered differences.*

The following theorem is well known [10].

Theorem 2.2.4 *If the ordered differences in the initial round of a* $\mathbb{Z}$*-cyclic* $Wh(4u+1)$ *cover* $\mathbb{Z}^*_{4u+1}$ *exactly once then the* $Wh(4u+1)$ *is an ordered whist design.*

Definition 2.2.3 *Let* q *be a power of a prime and let* θ *denote a primitive element for the Galois Field* $GF(q)$*. If* f *divides* $q-1$ *then the set* $C^f = \{\theta^{jf} : j = 0, 1, \ldots, h-1\}$ *with* $q-1 = fh$ *is a subgroup in the multiplicative group of* $GF(q)$*. The cosets of* C^f*,* $C^f_i = \theta^i C^f$*,* $i \in \{0, 1, \ldots, f-1\}$*, are often called the cyclotomic classes of order* f *and index* i*.*

Theorem 2.2.5 *The Main Construction produces the initial round of a* $\mathbb{Z}$*-cyclic BDOWh(p) if the following conditions are satisfied: (1)* x *is a non-square; (2)* y *is a square; (3)* $y-x$ *is a square; (4)* $x^{m+1}-y$ *is a non-square; (5)* $x-yx^m$ *is a non-square and (6)* $\dfrac{x^{m+1}-y}{x-yx^m} \in C^d_m$*.*

Proof: The partner condition is automatically satisfied since the partner differences from the base game are $\pm x(x^m-1)$ and $\pm y(x^m-1)$ with x non-square and y square. Partition the first forward differences into the two sets $\{y-x, x^m(y-x)\} \otimes \mathcal{R}$ and $\{x^{m+1}-y, x-yx^m\} \otimes \mathcal{R}$. Clearly the first set covers the squares in $\mathbb{Z}^*_p$ exactly once and the second set, using Hypotheses (4), (5) and (6) covers the non-squares in $\mathbb{Z}^*_p$ exactly once. At this point we have established that The Main Construction has produced the initial round of a $\mathbb{Z}$-cyclic BDWh(p) (see the above Remark). Next, partition the ordered differences into the two sets $\{x-y, x^m(x-y)\} \otimes \mathcal{R}$ and $\{x^{m+1}-y, x-yx^m\} \otimes \mathcal{R}$. For the primes considered here it is well known [3] that $z \in \mathbb{Z}_p^*$ is a square if and only if $-z$ is a square. Thus, once again, the first set covers the squares in $\mathbb{Z}_p^*$ exactly once and the second set covers the non-squares in $\mathbb{Z}_p^*$ exactly once. Hence the design is ordered. ∎

Corollary 2.2.6 *If $x \neq r^t$ then the $\mathbb{Z}$-cyclic BDOWh(p) of Theorem 2.2.5 is non-$\mathbb{Z}$CPS.*

Example 2.2.1 For $p = 29$ we have $k = 2$ and $t = 7$. Take $r = 2$, $x = 10$, and $y = 6$ then $x = r^{23}$, $y = r^6$, $y - x = r^{16}$, $x^{m+1} - y = r^3$, $x - yx^m = r^9$, and $\dfrac{x^{m+1} - y}{x - yx^m} = r^{22}$. Thus the sufficient conditions of Theorem 2.2.5 are satisfied and The Main Construction produces the initial round of a $\mathbb{Z}$-cyclic BDOWh(29). This initial round is given by the following 7 games.

$(10, 6, 14, 20)$, $(15, 9, 21, 1)$, $(8, 28, 17, 16)$, $(12, 13, 11, 24)$,
$(18, 5, 2, 7)$, $(27, 22, 3, 25)$, $(26, 4, 19, 23)$.

Example 2.2.2 For $p = 61$ we have $k = 2$ and $t = 15$. Take $r = 2$, $x = 2$, and $y = 1$ then $x = r^1$, $y = r^0$, $y - x = r^{30}$, $x^{m+1} - y = r^{49}$, $x - yx^m = r^{31}$, and $\dfrac{x^{m+1} - y}{x - yx^m} = r^{18}$. Thus the sufficient conditions of Theorem 2.2.5 are satisfied and The Main Construction produces the initial round of a $\mathbb{Z}$-cyclic BDOWh(61). This initial round is given by the following 15 games.

$(2, 1, 8, 4)$, $(32, 16, 6, 3)$, $(24, 12, 35, 48)$, $(18, 9, 11, 36)$,
$(44, 22, 54, 27)$, $(33, 47, 10, 5)$, $(40, 20, 38, 19)$, $(30, 15, 59, 60)$,
$(53, 57, 29, 45)$, $(55, 58, 37, 49)$, $(26, 13, 43, 52)$, $(50, 25, 17, 39)$,
$(7, 34, 28, 14)$, $(51, 56, 21, 41)$, $(23, 42, 31, 46)$.

Example 2.2.3 For $p = 73$ we have $k = 3$ and $t = 9$. Take $r = 5$, $x = 47$, and $y = 1$ then $x = r^{31}$, $y = r^0$, $y - x = r^{18}$, $x^{m+1} - y = r^{15}$, $x - yx^m = r^{27}$, and $\dfrac{x^{m+1} - y}{x - yx^m} = r^{60}$. Thus the sufficient conditions of Theorem 2.2.5 are satisfied and The Main Construction produces the initial round of a $\mathbb{Z}$-cyclic BDOWh(73).

This initial round is given by the following 18 games.

$(47, 1, 31, 69)$,	$(21, 2, 62, 65)$,	$(42, 4, 51, 57)$,	$(11, 8, 29, 41)$,
$(22, 16, 58, 9)$,	$(44, 32, 43, 18)$,	$(15, 64, 13, 36)$,	$(30, 55, 26, 72)$,
$(60, 37, 52, 71)$,	$(7, 25, 45, 46)$,	$(14, 50, 17, 19)$,	$(28, 27, 34, 38)$,
$(56, 54, 68, 3)$,	$(39, 35, 63, 6)$,	$(5, 70, 53, 12)$,	$(10, 67, 33, 24)$,
$(20, 61, 66, 48)$,	$(40, 49, 59, 23)$.		

2.3 Existence Results - Asymptotics

2.3.1 The Buratti-Pasotti Technique

In 2009 M. Buratti and A. Pasotti [11] produced a new asymptotic technique that is useful for establishing existence of designs whose element set is a Galois Field. The major ingredients of their technique that will be used here are contained in Definition 2.3.1 and Theorem 2.3.1.

Definition 2.3.1 *Let $s \geq 2$, $w \geq 1$ and $z \geq 0$ be arbitrary integers. Denote by $Q(s, w, z)$ the number defined by*

$$Q(s, w, z) = \frac{1}{4}(U + \sqrt{U^2 + 4s^{w-1}(w + sz)}\ )^2,$$

where

$$U = \sum_{h=1}^{w} \binom{w}{h} (s-1)^h (h-1).$$

Theorem 2.3.1 *Let $q \equiv 1 \pmod{s}$ be a prime power, let $B = \{b_1, b_2, \ldots, b_w\}$ be an arbitrary w-subset of GF(q) and let $(\beta_1, \beta_2, \ldots, \beta_w)$ be an arbitrary element of $\mathbb{Z}_s^w$, where $\mathbb{Z}_s^w = \mathbb{Z}_s \times \cdots \times \mathbb{Z}_s$. Set $X = \{x \in GF(q) : x \in C^s_{\beta_i}, 1 \leq i \leq w\}$. Then*

$$|X| \geq \frac{q - U\sqrt{q} - s^{w-1}w}{s^w}$$

and hence $|X| > z$ as soon as $q > Q(s, w, z)$. Thus, in particular, X is not empty for $q > Q(s, w, 0) \equiv Q(s, w)$.

Theorem 2.3.2 *Assume that $p > Q(d, 2)$ and let $i \in \{0, 1, \ldots, d-1\}$ be arbitrary but fixed. Set $\alpha = x$, $\beta = y$, $\gamma = x^{m+1}$ and $\delta = yx^m$. There exists a $\mathbb{Z}$-cyclic*

BDOWh(p) *if the following conditions are satisfied.*

$$\begin{array}{ll} \alpha \in C_1^d & \\ \beta \in C_0^d, & \beta - \alpha \in C_m^d \\ \gamma \in C_{m+1}^d, & \gamma - \beta \in C_{m+i}^d \\ \delta \in C_m^d, & \delta - \alpha \in C_{mt+i}^d \end{array}$$

Proof: Arbitrarily fix $\alpha \in C_1^d$. Apply Theorem 2.3.1 with $s = d$, $w = 2$ $B = \{0, -\alpha\}$, $(\beta_1, \beta_2) = (0, m)$. Note, then, that the set $X_0 = \{x \in \mathbb{Z}_p : x \in C_0^d, x - \alpha \in C_m^d\}$ is not empty since $p > Q(d, 2)$. Arbitrarily fix an element $\beta \in X_0$. Now, apply Theorem 2.3.1 with $s = d$, $w = 2$ $B = \{0, -\beta\}$, $(\beta_1, \beta_2) = (m + 1, m + i)$. It then follows that the set $X_1 = \{x \in \mathbb{Z}_p : x \in C_{m+1}^d, x - \beta \in C_{m+i}^d\}$ is not empty since $p > Q(d, 2)$. Arbitrarily fix an element $\gamma \in X_1$. Next, apply Theorem 2.3.1 with $s = d$, $w = 2$ $B = \{0, -\alpha\}$, $(\beta_1, \beta_2) = (m, mt + i)$. Now, the set $X_2 = \{x \in \mathbb{Z}_p : x \in C_m^d, x - \alpha \in C_{mt+i}^d\}$ is not empty since $p > Q(d, 2)$. Arbitrarily fix an element $\delta \in X_2$. It is easy to show that the conditions stated in the theorem guarantee that

$$\frac{x^m(x^{m+1} - y)}{x - yx^m} \in C_0^d,$$

which is an equivalent form of Condition (6) in Theorem 2.2.5. Consequently the sufficient conditions for The Main Construction (see Theorem 2.2.5) are satisfied.

∎

2.3.2 The Existence Results

Theorem 2.3.2 provides the basis for our existence results. Indeed, the following Theorem is an immediate consequence of Theorem 2.3.2.

Theorem 2.3.3 *Let* $p = 2^k t + 1$ *with* $k \geq 2$ *and* t *odd. Set* $d = 2^k$. *A non* $\mathbb{Z}$*CPS* $\mathbb{Z}$*-cyclic BDOWh*(p) *exists whenever* $p > Q(d, 2)$.

Thus if one wishes to show, for a specific value of k, that non $\mathbb{Z}$CPS $\mathbb{Z}$-cyclic BDOWh(p) exist for all corresponding primes p it is enough to provide solutions only for those $p \leq Q(d,2)$. Certainly, then, we can study the existence question for separate values of k. For each $k \in \{2,3,4,5,6,7,8\}$, the chart below indicates the asymptotic bound $Q(d,2)$ and the results of our data study. All of the results were computer generated. For $k > 8$ the data study was conducted for all primes less than 1,000,000. A sampling of the data is given in the appendix. Complete data sets are available from the authors. One can observe from this chart that there is a very small set of primes for which The Main Construction was unable to produce a non $\mathbb{Z}$CPS $\mathbb{Z}$-cyclic BDOWh(p). For $k > 8$ the only primes for which The Main Construction was unable to produce a non $\mathbb{Z}$CPS $\mathbb{Z}$-cyclic BDOWh(p) are $p = 12289, 40961, 65537, 786433$.

k	$Q(d,2)$	Existence
2	97	ALL except $5, 13$
3	2433	ALL
4	50689	ALL except 17
5	923649	ALL except 97
6	15573217	ALL except 193
7	260145153	ALL*
8	4228251649	ALL* except $257, 769$

ALL indicates that a non $\mathbb{Z}$CPS $\mathbb{Z}$-cyclic BDOWh(p) exists for all such primes. ALL* indicates that a non $\mathbb{Z}$CPS $\mathbb{Z}$-cyclic BDOWh(p) exists for all such primes less than 1,000,000.

Appendix I

k	$\{p, r, x, y\}$	BDO Base Table
2	$\{29, 2, 8, 1\}$	$(8, 1, 19, 6)$
	$\{37, 2, 13, 1\}$	$(13, 1, 14, 21)$
	$\{53, 2, 12, 1\}$	$(12, 1, 32, 38)$
	$\{61, 2, 2, 1\}$	$(2, 1, 8, 4)$
	$\{101, 2, 2, 1\}$	$(2, 1, 8, 4)$
	$\{109, 6, 10, 1\}$	$(10, 1, 19, 100)$
	$\{149, 2, 126, 1\}$	$(126, 1, 51, 82)$
	$\{157, 5, 83, 1\}$	$(83, 1, 150, 138)$
	$\{173, 2, 32, 1\}$	$(32, 1, 71, 159)$
	$\{181, 2, 28, 1\}$	$(28, 1, 51, 60)$

k	$\{p, r, x, y\}$	BDO Base Table
3	$\{41, 6, 6, 1\}$	$(6, 1, 27, 25)$
	$\{73, 5, 47, 1\}$	$(47, 1, 31, 69)$
	$\{89, 3, 61, 81\}$	$(61, 81, 7, 2)$
	$\{137, 3, 106, 1\}$	$(106, 1, 13, 4)$
	$\{233, 3, 160, 1\}$	$(160, 1, 6, 201)$
	$\{281, 3, 161, 1\}$	$(161, 1, 234, 265)$
	$\{313, 10, 153, 1\}$	$(153, 1, 91, 287)$
	$\{409, 21, 21, 1\}$	$(21, 1, 236, 206)$
	$\{457, 13, 356, 1\}$	$(356, 1, 123, 130)$
	$\{521, 3, 66, 1\}$	$(66, 1, 187, 437)$

k	$\{p, r, x, y\}$	BDO Base Table
4	$\{113, 3, 23, 9\}$	$(23, 9, 84, 82)$
	$\{241, 7, 46, 1\}$	$(46, 1, 220, 141)$
	$\{337, 10, 130, 1\}$	$(130, 1, 77, 335)$
	$\{401, 3, 104, 1\}$	$(104, 1, 67, 205)$
	$\{433, 5, 302, 1\}$	$(302, 1, 195, 177)$
	$\{593, 3, 243, 1\}$	$(243, 1, 360, 287)$
	$\{881, 3, 504, 1\}$	$(504, 1, 92, 161)$
	$\{977, 3, 942, 1\}$	$(942, 1, 845, 255)$
	$\{1009, 11, 998, 1\}$	$(998, 1, 553, 867)$
	$\{1201, 11, 718, 1\}$	$(718, 1, 439, 755)$

k	$\{p, r, x, y\}$	BDO Base Table
5	$\{353, 3, 199, 1\}$	$(199, 1, 250, 168)$
	$\{673, 5, 22, 1\}$	$(22, 1, 476, 144)$
	$\{929, 3, 669, 1\}$	$(669, 1, 13, 418)$
	$\{1249, 7, 223, 1\}$	$(223, 1, 522, 372)$
	$\{1697, 3, 1516, 1\}$	$(1516, 1, 1372, 1080)$
	$\{1889, 3, 419, 1\}$	$(419, 1, 628, 272)$
	$\{2017, 5, 314, 1\}$	$(314, 1, 436, 1967)$
	$\{2081, 3, 1968, 1\}$	$(1968, 1, 312, 1802)$
	$\{2273, 3, 2049, 1\}$	$(2049, 1, 1563, 267)$
	$\{2593, 7, 689, 1\}$	$(689, 1, 2136, 816)$

k	$\{p, r, x, y\}$	BDO Base Table
6	$\{449, 3, 430, 9\}$	$(430, 9, 319, 227)$
	$\{577, 5, 411, 25\}$	$(411, 25, 328, 326)$
	$\{1217, 3, 436, 1\}$	$(436, 1, 557, 1115)$
	$\{1601, 3, 1088, 1\}$	$(1088, 1, 584, 648)$
	$\{2113, 5, 30, 1\}$	$(30, 1, 1123, 2080)$
	$\{2753, 3, 551, 1\}$	$(551, 1, 1617, 2666)$
	$\{3137, 3, 2045, 1\}$	$(2045, 1, 2316, 1101)$
	$\{4289, 3, 1984, 1\}$	$(1984, 1, 2256, 2872)$
	$\{4673, 3, 3322, 1\}$	$(3322, 1, 2613, 593)$
	$\{4801, 7, 3302, 1\}$	$(3302, 1, 1496, 3426)$

k	$\{p, r, x, y\}$	BDO Base Table
7	$\{641, 3, 362, 1\}$	$(362, 1, 248, 284)$
	$\{1153, 5, 205, 625\}$	$(205, 625, 860, 1019)$
	$\{1409, 3, 683, 1\}$	$(683, 1, 1015, 346)$
	$\{2689, 19, 2130, 1\}$	$(2130, 1, 213, 269)$
	$\{3457, 7, 1268, 1\}$	$(1268, 1, 1431, 909)$
	$\{4481, 3, 687, 1\}$	$(687, 1, 4324, 3535)$
	$\{4993, 5, 1353, 1\}$	$(1353, 1, 2201, 736)$
	$\{6529, 7, 4902, 1\}$	$(4902, 1, 2221, 3843)$
	$\{7297, 5, 5599, 1\}$	$(5599, 1, 4464, 281)$
	$\{9601, 13, 5411, 1\}$	$(5411, 1, 2981, 7357)$

k	$\{p,r,x,y\}$	BDO Base Table
8	$\{3329,3,1969,9\}$	$(1969,9,1344,2106)$
	$\{7937,3,1317,1\}$	$(1317,1,5241,6585)$
	$\{9473,3,2717,1\}$	$(2717,1,3373,8369)$
	$\{14081,3,4344,1\}$	$(4344,1,5129,9489)$
	$\{14593,5,5160,1\}$	$(5160,1,6505,6718)$
	$\{22273,5,20067,1\}$	$(20067,1,9067,21653)$
	$\{23297,3,21382,1\}$	$(21382,1,19396,7569)$
	$\{26881,11,11417,1\}$	$(11417,1,16528,8454)$
	$\{30977,3,23234,1\}$	$(23234,1,16597,19473)$
	$\{31489,7,10668,1\}$	$(10668,1,3737,6503)$

k	$\{p,r,x,y\}$	BDO Base Table
9	$\{7681,17,1441,1\}$	$(1441,1,4663,2194)$
	$\{10753,11,7832,1\}$	$(7832,1,6699,3852)$
	$\{11777,3,4816,1\}$	$(4816,1,9873,8490)$
	$\{17921,3,2498,1\}$	$(2498,1,8081,6338)$
	$\{23041,11,12874,1\}$	$(12874,1,16216,19044)$
	$\{26113,7,9043,1\}$	$(9043,1,25474,14314)$
	$\{32257,15,6309,1\}$	$(6309,1,14316,1168)$
	$\{36353,3,32195,1\}$	$(32195,1,18326,1342)$
	$\{45569,3,24216,1\}$	$(24216,1,32486,39432)$
	$\{51713,3,28738,1\}$	$(28738,1,44341,34733)$

k	$\{p,r,x,y\}$	BDO Base Table
10	$\{13313,3,13303,1\}$	$(13303,1,12516,1411)$
	$\{15361,7,3350,1\}$	$(3350,1,12373,8748)$
	$\{19457,3,17578,1\}$	$(17578,1,15997,7095)$
	$\{25601,3,25325,1\}$	$(25325,1,2928,7781)$
	$\{37889,3,33079,1\}$	$(33079,1,23560,10708)$
	$\{39937,5,3714,1\}$	$(3714,1,36314,26828)$
	$\{50177,3,16025,1\}$	$(16025,1,873,40317)$
	$\{58369,7,34031,1\}$	$(34031,1,11392,12907)$
	$\{64513,5,23138,1\}$	$(23138,1,36633,30390)$
	$\{70657,7,29243,1\}$	$(29243,1,54618,66503)$

k	$\{p, r, x, y\}$	BDO Base Table
11	$\{18433, 5, 2454, 1\}$	$(2454, 1, 4296, 14649)$
	$\{59393, 5, 49785, 1\}$	$(49785, 1, 59348, 51054)$
	$\{79873, 7, 73522, 1\}$	$(73522, 1, 15049, 66112)$
	$\{83969, 3, 82104, 1\}$	$(82104, 1, 79265, 63891)$
	$\{120833, 3, 101094, 1\}$	$(101094, 1, 18575, 5184)$
	$\{133121, 3, 105183, 1\}$	$(105183, 1, 34583, 1757)$
	$\{202753, 10, 166814, 1\}$	$(166814, 1, 194072, 41370)$
	$\{301057, 15, 258198, 1\}$	$(258198, 1, 21518, 117222)$
	$\{329729, 3, 246666, 1\}$	$(246666, 1, 59577, 66165)$
	$\{366593, 3, 127301, 1\}$	$(127301, 1, 295959, 132496)$

k	$\{p, r, x, y\}$	BDO Base Table
12	$\{61441, 17, 40795, 1\}$	$(40795, 1, 27779, 18994)$
	$\{86017, 5, 76076, 1\}$	$(76076, 1, 65070, 39450)$
	$\{151553, 3, 79903, 1\}$	$(79903, 1, 77352, 27674)$
	$\{176129, 3, 128074, 1\}$	$(128074, 1, 159610, 124447)$
	$\{184321, 13, 107278, 1\}$	$(107278, 1, 75789, 86371)$
	$\{249857, 3, 96845, 1\}$	$(96845, 1, 32834, 223988)$
	$\{307201, 14, 291862, 1\}$	$(291862, 1, 146224, 155804)$
	$\{331777, 5, 308016, 1\}$	$(308016, 1, 242639, 213276)$
	$\{380929, 7, 62025, 1\}$	$(62025, 1, 120553, 233491)$
	$\{430081, 13, 299398, 1\}$	$(299398, 1, 307483, 253502)$

k	$\{p, r, x, y\}$	BDO Base Table
13	$\{188417, 3, 164318, 1\}$	$(164318, 1, 176089, 145502)$
	$\{270337, 10, 91443, 1\}$	$(91443, 1, 80799, 8932)$
	$\{286721, 11, 26455, 1\}$	$(26455, 1, 231085, 41085)$
	$\{319489, 23, 295699, 1\}$	$(295699, 1, 186800, 31283)$
	$\{417793, 5, 312672, 1\}$	$(312672, 1, 13735, 223703)$
	$\{778241, 6, 638670, 1\}$	$(638670, 1, 541167, 623248)$
	$\{925697, 3, 568795, 1\}$	$(568795, 1, 613397, 329432)$
	$\{974849, 3, 396926, 1\}$	$(396926, 1, 957012, 499115)$
	$\{1073153, 3, 556424, 1\}$	$(556424, 1, 405088, 639884)$

k	$\{p, r, x, y\}$	BDO Base Table
14	$\{114689, 3, 110252, 9\}$	$(110252, 9, 16969, 62777)$
	$\{147457, 10, 15336, 1\}$	$(15336, 1, 78906, 22216)$
	$\{638977, 7, 434751, 1\}$	$(434751, 1, 256450, 418)$
	$\{737281, 11, 546429, 1\}$	$(546429, 1, 473384, 512415)$

k	$\{p, r, x, y\}$	BDO Base Table
15	$\{163841, 3, 91624, 1\}$	$(91624, 1, 49392, 89739)$
	$\{557057, 3, 201403, 1\}$	$(201403, 1, 497198, 440342)$

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CHAPTER 3

Balance in Whist and Generalized Whist Designs: Some Classic and Recent Constructions

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Abstract

The concept of balance is a relatively new specialization in the study of whist designs, having been introduced in 2010 [1]. We demonstrate that a number of previously introduced whist design constructions either automatically possess the property of balance, or can be modified to obtain balanced designs. We also define balance for $(h, 2h)$ generalized whist tournament designs on v players, and show how a modification of a construction due to Hanani [2] will produce balanced $(h, 2h)$ GWhD(v) for certain primes v.

keywords: Balanced whist designs; Generalized whist tournaments; $\mathbb{Z}$-cyclic designs.

3.1 Introduction

For some history and interesting information related to the whist tournament problem see the recent article by I. Anderson and T. Crilly [3].

A **whist tournament** on v players, denoted Wh(v), is a $(v, 4, 3)$ (near) resolvable BIBD. A whist game or 'table' is a block, (a, b, c, d), of the BIBD and denotes that the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. Conventionally, the player listed first in a game, i.e. player a in the game (a, b, c, d), will be in the 'North' position. Player b above will be seated to the left of player a in the 'East' position, player c will be 'South,' and player d is 'West.' The design is subject to the whist conditions that every player is a partner of every other player exactly once and is an opponent of every other player exactly twice. The (near) resolution classes of the BIBD are called the **rounds** of the Wh(v). If $v = 4u$ then the Wh(v) consists of $v - 1$ rounds and if $v = 4u + 1$, the Wh(v) consists of v rounds.

In the former case every player plays in exactly one game of each round, whereas in the latter case every player plays in exactly one game in each round with the exception of one round in which that player *sits out*. It has been known [4, 5] since the 1970s that Wh(v) exist for all $v \equiv 0, 1 \pmod 4$, but throughout this paper, for reasons to be given shortly, we will consider exclusively that v is an odd prime. Hence all subsequent discussion relates to $v = 4u + 1$.

Definition 3.1.1 *A Wh(v) for $v = 4u + 1$ is said to be* $\mathbb{Z}$**-cyclic** *if the players are elements in $\mathbb{Z}_v$. It is also required that the set of rounds be cyclic. That is to say, the rounds can be labeled, $R_1, R_2, \ldots$, in such a way that R_{j+1} is obtained by adding $+1$ (mod v) to every element in R_j.*

Although most of the whist specializations to be introduced here relate to non $\mathbb{Z}$-cyclic whist designs as well as $\mathbb{Z}$-cyclic whist designs the considerations of this study relate primarily to $\mathbb{Z}$-cyclic whist designs.

Since the collection of rounds of a $\mathbb{Z}$-cyclic Wh(v) form a cyclic set it follows that the entire design can be given by any one of its rounds. This representative round is called the *initial round*. For our purposes, however, it is convenient to define the initial round for $v = 4u + 1$ players to be the unique round for which player 0 sits out.

For any given round of a Wh(v) the set of all players sitting in the North and South positions is referred to as the **N-S line**. Similarly the **E-W line** is the set of all players sitting in the East and West positions. Let $\{x, y\}$ be any pair of players in a Wh(v). For any round of the Wh(v) for which x and y both play, but not at the same table, x and y are said to be **relative opponents** if they belong to the same line (either N-S or E-W) and **relative partners** if they belong to opposite lines. For $v = 4u + 1$ there are exactly $4u - 4$ rounds in which x and y play at different tables.

Definition 3.1.2 *[1] A whist tournament on v players is said to be a* **balanced whist tournament**, *$BWh(v)$, if every pair of players are relative opponents exactly $2u - 2$ times (and, hence, relative partners exactly $2u - 2$ times).*

3.2 Recent Constructions

The whist designs generated by the following construction of Anderson and Ellison [6], as generalized by Finizio [7], are automatically balanced.

Theorem 3.2.1 *Consider a prime $p = 2^k t + 1$ where t is odd and $k \geq 2$. Let $d = 2^k$, $m = 2^{k-1}$, and $n = 2^{k-2}$. Let r be an arbitrary, but fixed, primitive root of p and let x be a non-square in $\mathbb{Z}_p^*$. When the following collection of games*

$$(1, x, x^m, -x) \otimes r^{dj+2i} : 0 \leq i \leq n - 1, 0 \leq j \leq t - 1 \tag{2.3}$$

constitutes the initial round of a $\mathbb{Z}$-cyclic Wh(p), the design will possess the property of balance.

Proof: Note that $1\ (= r^{p-1})$ and x^m are both even powers of r, since p is odd and m is even. Because d is even, the N-S line in the initial round, namely $\{1, x^m\} \otimes r^{dj+2i}$: $0 \leq i \leq n - 1, 0 \leq j \leq t - 1$, is exactly the set of all squares in $\mathbb{Z}_p^*$. Also, since x is a non-square and -1 is a square [4], the E-W line in the initial round, namely $\{x, -x\} \otimes r^{dj+2i}$: $0 \leq i \leq n - 1, 0 \leq j \leq t - 1$, is the set of all non-squares in $\mathbb{Z}_p^*$. These 2 sets form a $(4u + 1, 2u, 2u - 1)$ difference system [4]. Therefore, as the rounds of the design play out, every pair of elements, say $\{z, w\}$, will appear in the same line exactly $2u - 1$ times. Of course, precisely one of those times will occur when they are seated at the same table as partners, meaning that z and w will be relative opponents exactly $2u - 2$ times. ∎

Example 3.2.1 *The initial round of a $\mathbb{Z}$-cyclic $BWh(13)$ is given by the following three games. Note that $k = 2, t = 3, d = 4, m = 2, n = 1,$ and $r = 2$. If we let $x = r^3 = 8$, the whist conditions are satisfied.*

$(1, 8, 12, 5)$, $(3, 11, 10, 2)$, $(9, 7, 4, 6)$.

A second recent construction due to Costa, Finizio and Teixeira [8], also produces designs that are automatically balanced, for the same reasons as the generalized Anderson-Ellison construction above.

Theorem 3.2.2 *Consider a prime $p = 2^k t + 1$ where t is odd and $k \geq 2$. Let $d = 2^k$, $m = 2^{k-1}$, and $n = 2^{k-2}$. Let r be a primitive root of p, and let y be a square in $\mathbb{Z}_p^*$. When the following collection of games*

$$(r, y, r^{m+1}, yr^m) \otimes r^{dj+2i} : 0 \leq i \leq n-1, 0 \leq j \leq t-1 \tag{2.4}$$

constitutes the initial round of a $\mathbb{Z}$-cyclic Wh(p), the design will possess the property of balance.

Proof: Note that since m and d are even, r^{dj+2i} will always be a square and $\{r, r^{m+1}\} \otimes r^{dj+2i} : 0 \leq i \leq n-1, 0 \leq j \leq t-1$ gives the set of all non-squares in $\mathbb{Z}_p^*$, while $\{y, yr^m\} \otimes r^{dj+2i} : 0 \leq i \leq n-1, 0 \leq j \leq t-1$ is the set of all squares in $\mathbb{Z}_p^*$. The proof then follows the logic used to verify Theorem 3.2.1. ∎

Example 3.2.2 *The initial round of a $\mathbb{Z}$-cyclic $BWh(13)$ is given by the following three games. Note that $k = 2, t = 3, d = 4, m = 2, n = 1,$ and $r = 2$. If we let $y = 1$, the whist conditions are satisfied, and the design is a different BWh(13) than that given in Example 2.1.*

$(2, 1, 8, 4)$, $(6, 3, 11, 12)$, $(5, 9, 7, 10)$.

Example 3.2.3 The initial round of a $\mathbb{Z}$-cyclic BWh(41) is given by the following ten games. Note that $k = 3, t = 5, d = 8, m = 4, n = 2$, and $r = 6$. If we let $y = 1$, the whist conditions are satisfied.
$(6, 1, 27, 25)$, $(19, 10, 24, 4)$, $(26, 18, 35, 40)$, $(14, 16, 22, 31)$, $(17, 37, 15, 23)$,
$(11, 36, 29, 39)$, $(28, 32, 3, 21)$, $(34, 33, 30, 5)$, $(12, 2, 13, 9)$, $(38, 20, 7, 8)$.

3.3 Modifying A Classic Whist Construction to Achieve Balance

A classic whist construction for primes $p = 4u + 1$ is due to Baker [5]. We will slightly modify it to achieve balance. This construction is of particular interest, because the modification shown here will be used again in the next section of this paper for our primary result.

Theorem 3.3.1 *Let $p = 4u + 1$, u be an odd integer, and r be a primitive root of p. Then there exists a BWh(p).*

Proof: Baker's construction [5] gives the following initial round games:

$$(1, r^u, r^{2u}, r^{3u}) \otimes r^i : 0 \leq i \leq u - 1. \tag{3.5}$$

Note that if u is odd, then 1 and r^{2u} and all even powers of r are squares, while r^u and r^{3u} and all odd powers of r are non-squares. For $i = 1$, the i-th table in the initial round is $(r, r^{u+1}, r^{2u+1}, r^{3u+1})$. The odd and even powers of r have all shifted seats, so the squares and non-squares in this game have shifted lines. In fact, squares and non-squares will shift lines whenever i is odd in our initial round collection of games. Hence for i odd, if one subjects the i-th initial round game to a clockwise rotation of one seating position then the players in North and South positions at every initial round table will consist exclusively of even powers of r and all East-West players will be, exclusively, odd powers of r. The game $(r, r^{u+1}, r^{2u+1}, r^{3u+1})$ has now become $(r^{3u+1}, r, r^{u+1}, r^{2u+1})$, the third table

$(r^3, r^{u+3}, r^{2u+3}, r^{3u+3})$ has now become $(r^{3u+3}, r^3, r^{u+3}, r^{2u+3})$, and so on. As in the previous cases, the squares and non-squares are now completely in separate lines, hence they form the appropriate difference system. Our modified initial round games therefore yield a $\mathbb{Z}$-cyclic BWh(p). ∎

Example 3.3.1 The initial round of a $\mathbb{Z}$-cyclic BWh(29) is given by the following seven games. Note that $u = 7$ and $r = 2$.

$(1, 12, 28, 17)$, $(5, 2, 24, 27)$, $(4, 19, 25, 10)$, $(20, 8, 9, 21)$,
$(16, 18, 13, 11)$, $(22, 3, 7, 26)$, $(6, 14, 23, 15)$.

3.4 Balance in Generalized Whist Tournament Designs

Definition 3.4.1 *[9] Let e, f, h, v be positive integers such that $v \equiv 0, 1 \pmod f$ and $f = eh$. Let a be a positive rational number. A (h, f) generalized whist tournament design (or GWhD) on v players, having parameter a, is a $(v, f, a(f-1))$ (N)RBIBD that satisfies the conditions indicated below. Each block of the BIBD is considered to be a game in which e teams of h players each compete simultaneously. Players on the same team are called partners and players in the same game but not on the same team are called opponents. For each pair of players, say $\{x, y\}$, x is to be a partner of y exactly $a(h-1)$ times and x is to be an opponent of y exactly $a(f-h)$ times. Such a design is denoted by (h, f) GWhD$_a(v)$. The parameter a is usually omitted if it equals 1, as it will be for most GWhDs in this paper. When $v \equiv 1 \pmod f$ consistency with the definition of a NRBIBD requires that a be an integer. When $v \equiv 0 \pmod f$, practical reasons require that each of $a(v-1)$, $a(f-1)$, $a(h-1)$ and $a(f-h)$ be an integer.*

For the sequel we consider $a = 1$, $e = 2$ and $v \equiv 1 \pmod f$. Games for a corresponding $(h, 2h)$ GWhD(v) will be expressed as a $2h$ - tuple $(x_1, x_2, \ldots, x_{2h})$ in such a way that the odd subscripted elements belong to one team and the even subscripted players as the other team. One can visualize this latter game by considering that the $2h$ players are seated at a round table with x_1 sitting at the North position and with x_{i+1} seated to the left of x_i, $i = 1, \ldots, 2h-1$.

Definition 3.4.2 *At each table of a $(h, 2h)$ GWhD(v) the team whose player is seated at the North position is designated as the* **ONE Team** *and the other team is designated as the* **TWO Team**. *For any round of the $(h, 2h)$ GWhD(v) the collection of all the ONE Teams is called the* **ONE - Line** *for that round and the collection of all the TWO Teams is called the* **TWO - Line** *for that round. Players playing in the same round but at different tables are said to be* **relative opponents** *if they belong to the same line (either the ONE - Line or the TWO - Line) and are said to be* **relative partners** *if they belong to opposite lines.*

Since $v \equiv 1 \pmod{2h}$ there exists a positive integer w such that $v = 2hw + 1$. Consider an arbitrary pair of players in the $(h, 2h)$GWhD(v), say x and y. There are exactly $2hw - 1$ rounds in which both x and y play and exactly $2hw - 1 - (2h - 1) = 2h(w - 1)$ rounds in which they both play but at different tables.

Definition 3.4.3 *A $(h, 2h)GWhD(v)$ is said to be* **balanced** *(alt. is said to have the balance property) if for each pair of players, say x and y, x and y are relative partners exactly $h(w-1)$ times and relative opponents exactly $h(w-1)$ times. Such designs will be denoted $(h, 2h)$ BGWhD(v).*

Definition 3.4.4 *Let G be an abelian group of order $2hw + 1$. Let $g_0 = e_G$ denote the identity of G and arbitrarily order the remaining elements of G as $g_1, g_2, \ldots, g_{2hw}$. A $(h, 2h)GWhD(2hw + 1)$ is said to be* G**-cyclic** *if the player set is G and if the rounds, say $R_0, R_1, \ldots, R_{2hw}$, are such that R_0 is the round for which e_G sits out and R_i is obtained from R_0 by adding g_i to every element in R_0.*

If $v \equiv 1 \pmod{f}$ is a prime power it is shown in [9] that a construction due to H. Hanani [2] can be used to construct the initial round of a GF-cyclic (h, f)GWhD(v). Utilizing the approach associated with the Hanani Construction found in [9] one obtains the following Theorem.

Theorem 3.4.1 *Let p be a prime and s a positive integer such that $p^s = 2hw + 1$ where w is a positive integer. Let θ denote a primitive element for the Galois Field GF(p^s). Then the following collection of w games*

$$(1, \theta^w, \theta^{2w}, \ldots, \theta^{(2h-1)w}) \otimes \theta^i : 0 \leq i \leq w-1. \tag{4.6}$$

constitutes the initial round of a $(h, 2h)$GWhD(p^s).

The game $(1, \theta^w, \theta^{2w}, \ldots, \theta^{(2h-1)w})$ is typically referred to as the base game (alt. table) of the construction. As additional nomenclature the initial round game obtained by multiplying the base game by θ^i is said to be the i-th initial round game. If w is an odd integer then the ONE Team in the base game consists of all even powers of θ and the TWO Team in the base game consists of all odd powers of θ. For the next game (the 1-th game) the reverse situation occurs. That is to say, the ONE Team in the 1-th game consists of all odd powers of θ and the TWO Team in the 1-th game consists of all even powers of θ. Thus in an alternating fashion one can describe the ONE Team in terms of even/odd powers of θ. Note, then, that for i odd if one subjects the i-th initial round game to a clockwise rotation of one seating position then the ONE Team at every initial round table will consist of even powers of θ and the TWO Team at every initial round table will consist of odd powers of θ. This rotation procedure together with the requirement that h be an even positive integer leads to the following theorem.

Theorem 3.4.2 *Let p be a prime and s a positive integer such that $p^s = 2hw + 1$ with h an even positive integer, say $h = 2\mu$ and w an odd positive integer. Then there exists a $(h, 2h)$BGWhD(p^s).*

Proof: Apply the rotation procedure to the initial round games of Theorem 3.4.1 and note that the ONE-Line consists of the squares in $\mathrm{GF}(p^s)^*$ and the TWO-Line consists of the non-squares in $\mathrm{GF}(p^s)^*$. The conditions on h and w ensure that $p^s = 4u + 1$ with $u = \mu w$. We again use the fact [4] that if A denotes the squares in $\mathrm{GF}(p^s)^*$ and B denotes the non-squares in $\mathrm{GF}(p^s)^*$, then $\{A, B\}$ is a $(4u + 1, 2u, 2u - 1) = (4\mu w + 1, 2\mu w, 2\mu w - 1)$ difference system. Hence, if x and y are arbitrary elements in $\mathrm{GF}(p^s)^*$ then x and y appear in the same line exactly $2\mu w - 1$ times. Subtracting the times that x and y are partners it follows that x and y are relative opponents exactly $h(w - 1)$ times. Consequently x and y are relative partners exactly $h(w - 1)$ times. ∎

Example 3.4.1 The initial round of a $(6, 12)$BGWhD(37) is given by the following three games. Note that $\theta = 2$.
$(1, 8, 27, 31, 26, 23, 36, 29, 10, 6, 11, 14)$,
$(28, 2, 16, 17, 25, 15, 9, 35, 21, 20, 12, 22)$,
$(4, 32, 34, 13, 30, 18, 33, 5, 3, 24, 7, 19)$.

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