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Rank-one inverse scattering problem: Reformulation and analytic solutions

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Using the $K$-matrix formalism, we give a simplified reformulation of the $S$-wave rank-one inverse scattering problem. The resulting Cauchy integral equation, obtained differently by Gourdin and Martin in their first paper, is tailored to rational representations of $F(k)=k\cot(\delta_0)$. Use of such $F(k)$ permits a simple but general solution without integration, giving analytic form factors having a pole structure like the $S$ matrix that are reducible to rational expressions using Padé approximants. Finally, we show a bound state pole condition is necessary, and makes the form factor unique.

Finite rank NN potentials are used primarily because they simplify calculations of structure and reactions. Such potentials are being developed to reproduce NN phase shifts with increasing accuracy. In particular, rank-two and rank-three interactions possessing form factors that are rational functions of momentum have recently been introduced. It would be desirable for there to be a convergent procedure for constructing analytic low-rank potentials that (a) are interchangeable with meson-theoretic potentials in an appropriate energy range and (b) have a simple and direct connection with two-body data. Such a procedure would simplify realistic nuclear calculations, in some cases decisively, while helping to justify the use of finite rank representations, known not to be fully equivalent, at higher energies, to local potentials. In an approach which emphasizes the first goal, recent work based upon new Padé approximant methods for bound states can generate rational form factors for a variety of local potentials such as sums of Yukawas. Here we focus upon the second goal. We present a new formulation of rank-one inverse scattering theory that yields simple analytic form factors, and is designed to take advantage of the discovery that rational representations of the scattering function $F(k)=k\cot(\delta_0)$ can be accurate. Relating to the first goal, we employ our formulation to construct a rank-one potential phase-shift equivalent to a Yukawa potential.

Despite an interesting attempt with a rank-one interaction, it is generally accepted that at least a rank of two is required to reproduce one salient feature of the NN interactions: a long range attraction together with a strong short range repulsion. A rank-two inverse scattering formalism, that of Fiedeldey, already exists. It allows for the initial introduction of a somewhat arbitrary long range attraction such as could be constructed using the formulation we present here. We will address the rank-two inverse scattering problem in another paper.

We present a $K$-matrix formulation of rank-one inverse scattering theory that accomplishes several things: simpler physical and mathematical analysis than previous formulations; direct use of the known solutions of the inhomogeneous Riemann boundary value problem, and presence of data in the form of $k\cot(\delta_0)$, which provides a context for introducing Padé approximants, facilitates analytical continuation, and enables a pole analysis of form factors not previously carried out. When the scattering function $F(k)$ is rational, we obtain the form factor without integrations. For a rational $S$ matrix $S(k)$ we find that the poles of $S(k)$ and of the squared form factor $h(k)$ coincide and are of the same order in the upper $k$ plane, except at the bound state pole, $ik_B$, and we obtain a general bound state pole condition: The order of the pole in $h(ik_B)$ must be one less than the order in $S(ik_B)$. This condition makes $h(k)$ unique, and as seen in our example, is easily enforced.

Our work is aligned most closely with the initial approach of Gourdin and Martin (GMI). Using the Schrödinger equation, GMI arrived at our Eq. (5), which was analyzed extensively and partially solved for rational $F(k)$. The authors did not make use of Riemann boundary value theory, and their lengthy discussion of rational $F(k)$ did not utilize a bound state pole condition. Subsequently, Gourdin and Martin (GMI) (Ref. 12) transformed Eq. (5) into a form which enabled them to find a complete solution, with $h(k)$ proportional to

$$
\sin(\delta(k)) \exp \left[ -\frac{2}{\pi} \int_0^\infty dp \frac{\delta(p)}{p^2-k^2} \right].
$$

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Although this expression can presumably be evaluated when \( F(k) \) is given in rational form, it would appear immensely more practical to understand the simpler approach we present and to avoid integrations altogether. Subsequent work has led to the same expression for \( h(k) \).

Bolsterli and MacKenzie obtained this result starting with the \( T \) matrix,\(^{13} \) while Chadan and Sabatier explicitly referred to the Riemann problem while supplying extensions and more rigor to GMII.\(^{10} \) Our approach uses the same analyticity properties of scattering states but differs significantly in detail by working with a Cauchy integral equation. Coupled with the high accuracy of rational scattering functions, our formulation provides a powerful tool for inverse scattering problems.

We specialize to \( S \)-wave scattering states and an attractive interaction. The repulsive case is easily traced through upon changing the sign in Eq. (1). Given a scattering function \( F(k) = k \cot(\delta_0) \), we find the rank-one interaction producing \( F(k) \), if it exists, by (a) solving the \( S \)-wave \( K \)-matrix equation for a rank-one interaction, and (b) treating the formal solution as an integral equation in the squared form factor, \( h(k) = g^2(k) \). The \( S \)-wave interaction is

\[
V_0(k',k) = -\frac{4\pi}{M} g(k') g(k),
\]

where \( \hbar = c = 1 \) and \( M/2 \) is the reduced mass. We solve

\[
K_0(k',k) = V_0(k',k) + \int_0^\infty dy y^2 V_0(k',y) K_0(y,k,E) \frac{1}{E - y^2/M},
\]

where Cauchy principal values are understood, obtaining

\[
K_0(k',k,E) = -\frac{4\pi}{M} g(k') g(k) + 2\pi J(ME),
\]

where we have defined \( h(k) \) to be an even function and

\[
J(\omega) = \int_{-\infty}^{\infty} dy y^2 h(y) \frac{1}{\omega - y^2}.
\]

The fully on-shell \( K \) matrix is related to the scattering function by

\[
K_0(k,k^2/M) = -2/[\pi M F(k)],
\]

giving

\[
1 + 2\pi \int_{-\infty}^{\infty} dy \frac{h(y)^2}{y^2} = 2\pi^2 F(k) h(k).
\]

Equation (5) can be put in the standard form of the dominant integral equation of the Cauchy type for which a general solution is known.\(^{11} \) We write

\[
a(k)h(k) + b(k) \int_{-\infty}^{\infty} dy \frac{h(y)}{y - k} = f(k),
\]

with \( a(k) = k \cot(\delta_0), b(k) = ik \), and \( f(k) \) becoming the constant

\[
f = f(k) = -\frac{1}{2\pi} - \frac{1}{\pi} \int_{-\infty}^{\infty} dy h(y).
\]

In turn, Eq. (6) is reducible to a special case of the inhomogeneous Riemann boundary value problem, which is to find functions \( \phi^+(z) \) and \( \phi^-(z) \) that are analytic, respectively, in the upper \( (D^+) \) and lower \( (D^-) \) half complex plane with limiting values on the real \( k \) axis, \( \phi^2(k) \), that satisfy

\[
\phi^+(k) = G(k) \phi^-(k) + t(k).
\]

The coefficient of the Riemann problem is

\[
G(k) = \frac{a(k) - b(k)}{a(k) + b(k)},
\]

which is simply related to the \( S \) matrix by \( G(k) = S^*_0(k) \).

The free term \( t(k) \) of the Riemann problem is proportional to the scattering amplitude:

\[
t(k) = \frac{f(k)}{a(k) + b(k)}.
\]

Finally, the form factor is given upon application of the well-known Plemlj formulas,\(^{11} \) also used to establish the connection between the Riemann problem and our integral equation:

\[
h(k) = \phi^+(k) - \phi^-(k),
\]

where the total change of the argument of \( G(k) \) over the real line is denoted by \( \Delta[\arg(G)] \). The number \( \chi \) is easily calculated in terms of phase shifts, becoming

\[
\chi = 2[\delta(0) - \delta(\infty)]/\pi.
\]

For nucleons, it is realistic to assume there is no zero energy resonance and \( |\delta(0) - \delta(\infty)| \leq \pi \), and we shall make these assumptions here. Then \( \chi \) is twice the number of bound states.\(^{9,10} \) If \( \chi = 0 \), the requirement that \( h(k) \to 0 \) as \( k \to \infty \) makes the solution unique. If \( \chi = 2 \), our method will be shown to give the unique form factor consistent with the binding energy.

The general algorithm for solving the Riemann problem is somewhat complicated, especially in the case \( \chi \neq 0 \), requiring the introduction of ancillary functions.\(^{11} \) When \( F(k) \) is rational, the algorithms become simple. As the basic application of our formulation, we specialize in what follows to a rational \( F(k) \). Then

\[
G(k) = \frac{c(k)}{d(k)} = \frac{c_+(k)c_-(k)}{d_+(k)d_-(k)},
\]

where polynomials \( c_+, d_+ \) (\( c_-, d_- \)) have roots in the upper (lower) half plane. Then \( \chi = m_+ - n_+ \), where \( m_+ \) (\( n_+ \)) is the number of zeros of \( c_+ \) (\( d_+ \)) and the number of bound states is half the difference between the number of poles and zeros of the \( S \) matrix in the upper half plane. When poles and zeros are counted here, it is their multiplicities that are added.

The decomposition of \( G \) in Eq. (12) permits the use of
the generalized Liouville theorem. One writes the
Riemann equation as
\[
\frac{d_{-}(k)}{c_{-}(k)} \phi^{+}(k) - \frac{c_{+}(k)}{d_{+}(k)} \phi^{-}(k) = \frac{d_{-}(k)}{c_{-}(k)} g(k).
\]  

(13)

The terms on the left are boundary values of functions analytic in \(D^{+}\) and \(D^{-}\). It is convenient to specify the analytic structure of Eq. (13) by requiring \(F(k)\) to have the form \(P_{L}(k^{2})/Q_{M}(k^{2})\), just \([L/M]\) in Padé approximant notation, where \(P_{L}(k^{2})\) and \(Q_{M}(k^{2})\) are polynomials in \(k^{2}\) of degrees \(L\) and \(M\). We require \(M < L\) to ensure the vanishing of \(\delta(k)\) at infinity. Any such \(F(k)\) leads uniquely to a rational \(S\) matrix with the structure discussed following Eq. (16), and conversely. Then the right
side of Eq. (13) is the rational function
\[
f_{Q_{M}(k^{2})} \big/ d_{+}(k)c_{-}(k),
\]
expressible as \(\psi^{+}(k) - \psi^{-}(k)\), where
\[
\psi^{+}(k) = x(k)/c_{-}(k),
\]
degree\([x(k)] \leq \text{degree}[c_{-}(k)],\)
and
\[
\psi^{-}(k) = -[\psi^{+}(k)^{*}].
\]
Equation (13) expresses the equality on the real axis of functions analytic in \(D^{+}\) and \(D^{-}\), which must therefore together represent an entire function \(R(z)\). On the real axis,
\[
R(k) = \frac{d_{-}(k)}{c_{-}(k)} \phi^{+}(k) - \psi^{-}(k)
= \frac{c_{+}(k)}{d_{+}(k)} \phi^{-}(k) - \psi^{-}(k).
\]  

(14)

We require \(\phi^{\pm}(\infty) = 0\) to restrict to solutions \(h(k)\) that also vanish at infinity. Consequently, by the generalized Liouville theorem, \(R(z)\) can be an arbitrary polynomial of degree \(\chi - 1\), written \(R_{\chi-1}(z)\), but with \(R_{-1}(z) = 0\). The \(\phi^{\pm}(k)\) are found by taking the real limit of \(z, z \to k^{\pm}\), of those functions analytic in \(D^{+}\) and \(D^{-}\).

For a detailed discussion of the solution both for \(\chi = 0\) and \(\chi = 2\), we make use of the function
\[
h(k) = [x(k)c_{+}(k) + x(k)c_{-}(k)d_{-}(k)] [c_{+}(k)c_{+}(k) - d_{-}(k)d_{+}(k)] R_{\chi-1}(k) / d_{-}(k)c_{+}(k).
\]  

(17)

When \(\chi = 0\), then \(R_{-1}(k) = 0\), and \(h(k)\) is an even function. When \(\chi = 2\), then \(h(k)\) is real and even only if \(R_{1}(k)\) is imaginary and odd. Hence, \(R_{1}(k) = ikC_{1}\), where \(C_{1}\) is real. The bound state condition, which we establish now, is used to determine \(C_{1}\). A negative energy bound state pole would occur at \(z = ik_{B}\), with \(k_{B} > 0\). It follows directly from the Schrödinger equation for the separable interaction that \(A(ik_{B}) = 0\). Since \(D^{+}\) represents the domain of continuation of the function \(A_{-}(k)\), then \(ik_{B}\) is a pole of the analytically-continued \(S\) matrix. Clearly, \(ik_{B}\) is a simple zero of \(A(z)\) because \(dA(z)/dz \neq 0\) if \(z \neq 0\). The numerator of \(S(k)\) at \(ik_{B}\) becomes \(-4\pi^{2}h(ik_{B})k_{B}\), and therefore if \(N\) is the order of the pole of \(S(ik_{B})\), then \(N - 1\) is the order of the pole of \(h(ik_{B})\). From Eqs. (12) and (17) it is evident that \(h(k)\) has the same poles as those of \(S(k)\) in \(D^{+}\), and that these poles have the same order unless the remaining constant \(C_{1}\) in \(R_{1}(z)\) is assigned the value that creates a common factor \(k^{2} + k_{B}^{2}\) in the numerator. This bound state condition in practice is simple to apply.

Our first example is for a shape-independent \(F(k)\) consisting of the first two terms in the effective range expansion. The choice of scattering length \(a > 0\) corresponds to a low energy approximation of the triplet np interaction with one bound state. Then
\[
c_{+} = (r_{0}/2)[k - i(\alpha + \beta)][k - i(\alpha - \beta)]
\]
and \(c_{-} = 1\), where \(\alpha = 1/r_{0}\) and
\[ \beta = \left[ 1 - 2r_0/a \right]^{1/2}/r_0. \]
If we assume \( k_B = \alpha - \beta \), then
\[ h(k) = \frac{\alpha/\pi^2}{k^2 + (\alpha + \beta)^2}. \]  
(18)

Although the form factor \( g(k) = \sqrt{h(k)} \) is hardly rational, Padé approximants of \( g(k) \) converge rapidly, as seen in Fig. 1, where the values \( a = 5.414 \text{ fm} \) and \( r_0 = 1.750 \text{ fm} \) are used. Here, Padé approximants express \( g(k) \) as a combination of functions of the type \( (k^2 + \gamma^2)^{-1} \).

The shape-independent approximation illustrates two other cases. When \( a < 0 \), as characteristic of the \( ^1S_0 \) np state, we find \( \beta > \alpha \), and the poles shift so that \( \chi = 0 \). The analysis, simpler because the bound state pole condition is not needed, results in the same expression for \( h(k) \) given in Eq. (18). Finally, when \( \beta = 0 \), \( S(k) \) has a double pole at \( ik_B \). The bound state pole condition, easily applied, lowers the order of the pole of \( h(k) \) at \( ik_B \) to unity. The solution is still given by Eq. (18) with \( \beta = 0 \). These simple results may be compared with those given in Ref. 10.

As a final example we consider a Yukawa potential that fits the \( ^1S_0 \) np effective range parameters. Figure 2 shows our rational least squares fits of \( F(k) \) that we computed for this potential. The \([4/3]\) approximant curve is indistinguishable from the data in the energy range shown, while the \([3/2]\) approximant gives a reasonable fit for \( E_{\text{cm}} < 120 \text{ MeV} \). The \([3/2]\) approximant to \( F(k) \) is exactly reproduced by
\[ h(k) = \frac{\alpha_1}{k^2 + k_1^2} + \frac{\alpha_2 + \alpha_3 k}{(k + k_{21})^2 + k_2^2} + \frac{\alpha_2 - \alpha_3 k}{(k - k_{21})^2 + k_2^2}, \]  
(19)

with \( \alpha_1 = 0.0201873 \text{ fm}^{-1} \), \( \alpha_2 = 0.000811496 \text{ fm}^{-1} \), \( \alpha_3 = -0.0175540 \), \( k_1 = 0.0667608 \text{ fm}^{-1} \), \( k_{21} = 0.585295 \text{ fm}^{-1} \), \( k_{22} = 1.20083 \text{ fm}^{-1} \). Here, rational approximants to \( g(k) \) would be most easily obtained by least squares methods.6

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12M. Gourdin and A. Martin, Nuovo Cimento 6, 757 (1957), referred to as GM1; 6, 699 (1958), referred to as GMII.