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Vorticity fluctuations in turbulent counterflow of superfluid helium

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A model of vorticity fluctuations in turbulent helium based on Vinen's dynamical equation is developed. Its predictions are compared with measurements of $\langle \delta L^2 \rangle$ recently reported by Mantese, Bischoff, and Moss. The result is interpreted as supporting the validity of Vinen's equation.

I. INTRODUCTION

Recent experimental studies[1-4] of turbulent counterflow of superfluid helium have provided considerable support for a phenomonological model first proposed by Vinen.[5] He presumed that the state could be adequately described by the length of randomly oriented quantized vortex line per unit volume, L, and that L satisfied the dynamical equation

$$\frac{\partial L}{\partial t}=F(L,v)\,, \tag{1a}$$

where v is the relative velocity of the two fluids. He argued further that $F(L,v)$ should have the functional form

$$F(L,v)=\frac{\chi_1 B\rho_n}{2\rho}L^{3/2}\left[1-\left(\frac{\alpha}{d}\right)L^{-1/2}\right]v-\frac{\chi_2\kappa}{2\pi}L^2\,, \tag{1b}$$

where χ_1, χ_2, B, and α are dimensionless constants of order unity, κ is the quantum of circulation, and d is an effective diameter characterizing the counterflow channel.

At equilibrium, $\partial L/\partial t=0$ and the equation has solutions $L_0(v)$ only for values of v greater than a threshold velocity v_c:

$$v_c\equiv\frac{4\kappa\rho}{\pi\rho_n}\left(\frac{\chi_2}{\chi_1 B}\right)\frac{\alpha}{d}\,. \tag{2}$$

We can express them conveniently in terms of the reduced velocity $x\equiv v/v_c$.

$$L_0=4(\alpha/d)^2x^2[1\pm(1-1/x)^{1/2}]^2\,. \tag{3}$$

The measurements cited above have involved measurements only in steady counterflow. Consequently, their results serve only to confirm Eq. (3) and do not bear directly on the validity of the dynamical equation (1). Earlier experiments[5] did attempt to study the dynamical behavior of L by modulating v, but interpretation of the results[6] is complicated by the effect of the propagating turbulent wave front.[7] Mention should also be made of a microscopic calculation of the properties of the vortex array by Schwarz.[8] His model, at present, deals only with steady flow well above threshold, and doesn't attempt to describe the dynamical response of L to changes in v.

In a recent letter, Mantese, Bischoff, and Moss[9] (MBM) reported experimental measurements of both the length of line present in steady-state turbulence and the mean-square fluctuation about that value. The results were obtained from a study of the attenuation of second sound which propagates across a counterflow channel. Measurements of the fluctuations in L are particularly interesting because they provide a potential test of the dynamical predictions of Vinen's model. MBM suggest that the physical origin of the effect is a random component of the counterflow velocity v, which drives the fluctuations in a manner determined by Eq. (1). They then introduce an instability model and use it to explain their experimental results. However, their argument is of questionable validity, for, among other difficulties, it predicts a definite relationship between the mean-square fluctuation in L and that in v, independent of the spectral compositions of the fluctuations. This results is physically unreasonable, since it is easy to show from Eq. (1) that L will respond differently to slow variations in v than to rapid ones.

The purpose of this paper is to show that it is possible to understand the experimental measurements of MBM quantitatively in terms of a model which explicitly accounts both for the spectral composition of the fluctuations and for the dynamical aspects of Vinen's equation. The method is fairly common[10,11]; in fact, similar arguments were first applied to the problem of fluctuations in turbulent helium by Hoch, Busse, and Moss.[12]

II. FLUCTUATIONS IN L NEAR THRESHOLD

Let us presume, as did MBM, that there is a small random component of the counterflow velocity which drives fluctuations in L in a manner determined by Eq. (1). At mean velocities, v_0, near the threshold, the line density is low and as a first approximation it is reasonable to neglect any feedback reaction of the fluctuations in L on those in v. The equilibrium value of L, L_0, is defined by $F(L_0, v_0) \equiv 0$, which leads to Eq. (3) with $x \equiv v_0/v_c$. Next we presume that the fluctuations $\delta L(t) \equiv L(t) - L_0$ and $\delta v(t) \equiv v(t) - v_0$ are small enough so that we may linearize Eq. (1) about the equilibrium point (L_0, v_0). The result is that $\delta L(t)$ obeys a Langevin equation

$$\frac{1}{f}\left(\frac{\partial}{\partial t}\,\delta L + \frac{\delta L}{\tau}\right) = \delta v, \tag{4}$$

where the coefficients are conveniently expressed in terms of the reduced velocity by

$$\begin{aligned} f &= [(16b/v_c)(\alpha/d)^4]\;\; f_1(x)\,, \\ 1/\tau &= [4b(\alpha/d)^2]\;\; f_2(x)\,, \end{aligned} \tag{5}$$

where

$$f_1(x) \equiv x^3[1 + (1 - 1/x)^{1/2}]^4\,,$$
$$f_2(x) \equiv x^2(1 - 1/x)^{1/2}[1 + (1 - 1/x)^{1/2}]\,,$$
$$b \equiv \chi_2\kappa/2\pi\,.$$

Note that the relaxation time τ diverges at threshold ($x = 1$) while the gain f remains well behaved.

For a linear, shift-invariant system such as that described by Eq. (4), the power spectrum of δL, $G_L(\omega)$, is related to the power spectrum of δv, $G_V(\omega)$, by

$$G_L(\omega) = |H_L(\omega)|^2 G_V(\omega)\,, \tag{7}$$

where

$$H_L(\omega) = f\tau/(1 - i\omega\tau) \tag{8}$$

is the transfer function for the system. The mean-square fluctuation in L, $\langle\delta L^2\rangle$, is then related to the integral of the power spectrum by Parseval's theorem:

$$\begin{aligned} \langle\delta L^2\rangle &= \int_0^\infty G_L(\omega)\,d\omega \\ &= \int_0^\infty \left(\frac{f^2\tau^2}{1 + \omega^2\tau^2}\right) G_V(\omega)\,d\omega\,. \end{aligned} \tag{9}$$

This is a precise version of the previous statement that $\langle\delta L^2\rangle$ depends on the explicit form of the power spectrum of δv and not just on $\langle\delta v^2\rangle$. It is clear that in order to make further progress it will be necessary to say something about the power spectrum of δv.

The function $|H_L(\omega)|^2$ is illustrated in Fig. 1 for various x, and we observe that it is a very sensitive function of both x and ω near threshold. In particular, since τ becomes very large near $x = 1$, the function is sharply peaked at $\omega = 0$. If we now assume only that $G_V(\omega)$ is a slowly varying function of ω near zero, a first approximation to $\langle\delta L^2\rangle$ valid near threshold may be had by replacing $G_V(\omega)$ by

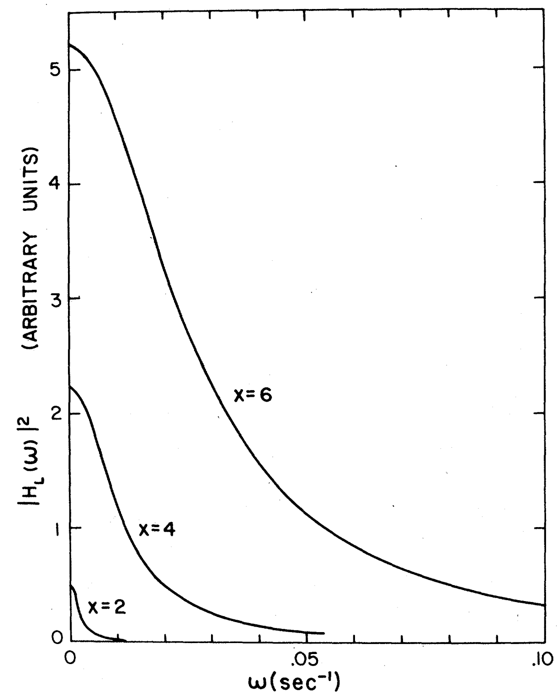

FIG. 1. $|H_L(\omega)|^2$ for various x near threshold.

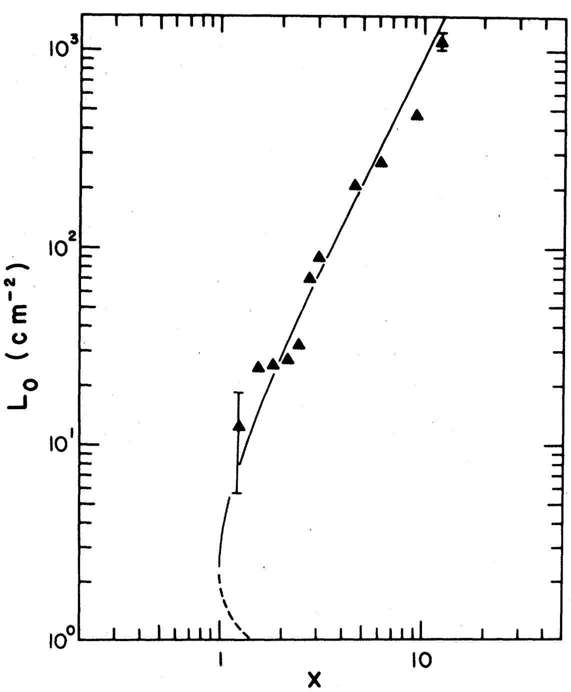

FIG. 2. Line-density measurements of Mantese *et al.* (Ref. 9) vs reduced velocity $x \equiv v_0/v_c$. The solid line is the stable branch (+) of Eq. (3) with $\alpha/d = 0.78\ \mathrm{cm}^{-1}$. The dashed line is the unstable branch (−).

$G_V(0)$ in Eq. (9), which gives

$$\langle \delta L^2 \rangle = (\pi/2) f^2 \tau G_V(0) ,$$
$$= A f_1^2(x)/f_2(x) , \qquad (10)$$

where A is a constant.[13] By examination of Eq. (10), one sees that $\langle \delta L^2 \rangle$ should have an extremely narrow divergence at $x=1$. It should then pass through a sharp minimum at $x \simeq 1.02$, and ultimately rise as x^4 for large x.

In order to compare this prediction with the measurements of L_0 and $\langle \delta L^2 \rangle$ reported by MBM, we proceed as follows. The critical heat current $\dot{q}_c = \rho_s S T v_c$ is chosen by eye as that value at which $\langle \delta L^2 \rangle$ first rises above its background value. The data are then replotted in Fig. 2 and 3 as a function of $x = \dot{q}/\dot{q}_c = v_0/v_c$, for a choice of $\dot{q}_c = 3.2$ mW/cm^2, or alternatively, $v_c = 4.9 \times 10^{-2}$ cm/sec. Next we choose a value of α/d to match Eq. (3) to the L_0 data. The result for $\alpha/d = 0.78$ cm^{-1} is shown as the solid curve in Fig. 2. Finally, we adjust the constant A in Eq. (10) to match the fluctuation data near threshold. The result for $A = 6 \times 10^{-4}$ cm^{-4} is shown as the solid curve labeled 1 in Fig. 3. It can be seen that the agreement is quite good for values of x less than roughly 5, with the exception that there is no evidence in the data of the predicted narrow divergence at $x=1$. There are at least three reasons why the latter outcome is not surprising. First, the divergence arises from increasing power in the lowest-frequency fluctuations, but the lowest frequencies are not detected experimentally since data are taken with a finite record length. Second, the effect of temperature regulation will be to suppress low-frequency fluctuations in δv. Finally, the linearization hypothesis undoubtedly fails in the immediate vicinity of threshold.

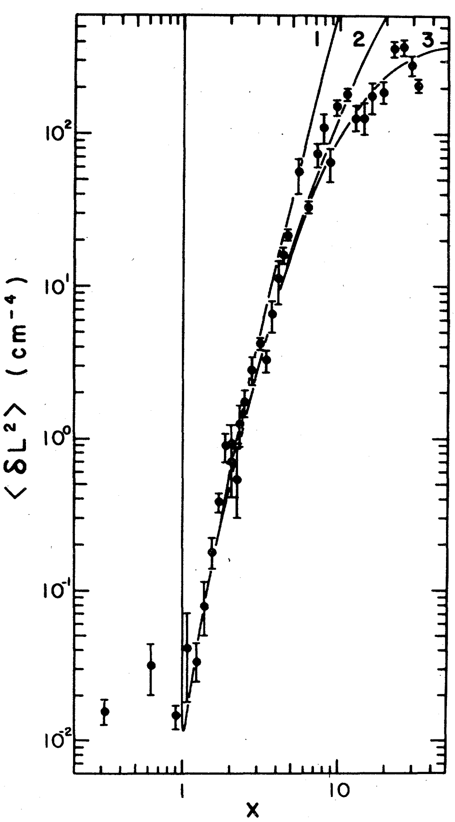

FIG. 3. Mean-square line-density fluctuations measured by Mantese *et al.* (Ref. 9) vs reduced velocity. The solid curves denoted (1–3) are the theoretical predictions of Eqs. (10), (24), and (37) described in the text.

It should be pointed out that taking A, and hence $G_V(0)$, to be independent of x is an additional assumption initially motivated by the agreement with experiment. This, in turn, implies that the mechanism responsible for the fluctuations in δv is most likely not normal fluid turbulence, as suggested by MBM, for this would be expected to lead to fluctuations that increase with increasing v_0. Instead, it appears likely that the fluctuations are an intrinsic feature of the liquid, a point we will return to below.

III. FLUCTUATIONS IN v

In order to make further progress, it will be necessary to introduce a model to describe the power spectrum of the fluctuations in v. We take the view that there are randomly varying temperature and pressure gradients in the fluid whose origin we won't question further. These gradients then drive the fluctuations in v in a manner described by the usual two fluid equations of motion.[14] Let us study the dynamical response of v in a slit of width d coupling two reservoirs. The equations of motion are

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{\vec{\nabla} p}{\rho} + S\vec{\nabla} T , \qquad (11)$$

$$\frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_n \cdot \vec{\nabla})\vec{v}_n = -\frac{\vec{\nabla} p}{\rho} - \frac{\rho_s}{\rho_n} S\vec{\nabla} T + \frac{\eta_n}{\rho_n}\nabla^2 \vec{v}_n , \qquad (12)$$

and we presume no slip boundary conditions of $\vec{v}_n$ at $z = \pm d/2$. We make the following simplifying assumptions. First, we note that Eq. (11) acts to strongly couple pressure and temperature fluctuations so that at low frequencies it is reasonable to expect that they are not independent but, instead, are coupled by the thermomechanical relation $\vec{\nabla} p = \rho S \vec{\nabla} T$. In other words, we assume that the fluctuations of importance at low frequencies take place at constant chemical potential. In this case we have $\vec{v}_s = \text{const}$ and Eq. (12) becomes the ordinary Navier-Stokes equation:

$$\frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_n \cdot \vec{\nabla})\vec{v}_n = -\frac{\vec{\nabla}_p}{\rho_n} + \frac{\eta_n}{\rho_n}\nabla^2\vec{v}_n . \tag{13}$$

Our second assumption is that at low frequencies we may treat the flow as unidirectional[15]; i.e., we presume the solutions are of the form $\vec{v}_n = v_n(z,t)\hat{x}$. Equation (13) then reduces to

$$\frac{\partial v_n}{\partial t} - \frac{\eta_n}{\rho_n}\frac{\partial^2 v_n}{\partial z^2} = g(t) , \tag{14}$$

where $g(t) \equiv -|(\vec{\nabla}p)/p_n|$ is a random function of time only. Since the equation is linear, it also holds for fluctuations for v_n and g about their average in counterflow.

Given these assumptions, one can show[16] that the power spectrum $G_V(\omega)$ of the fluctuations in the spatial average relative velocity v is related to the power spectrum of the gradient term $G_g(\omega)$ by

$$G_V(\omega) = |H(\omega)|^2 G_g(\omega) , \tag{15}$$

where

$$H(\omega) = \sum_{j=0}^{\infty} \frac{8\tau_0}{\pi^2(2j+1)^4}\frac{1}{1-i\omega\tau_j} , \tag{16}$$

$$\tau_j = \tau_0/(2j+1)^2; \quad \tau_0 = \rho_n d^2/\eta_n \pi^2 . \tag{17}$$

We can simplify things considerably by noting that $H(\omega)$ is strongly dominated by its first term

$$H^0(\omega) = (8\tau_0/\pi^2)[1/(1-i\omega\tau_0)] . \tag{18}$$

In particular, it may be shown that

$$\frac{\pi^4}{96} = \sum_{j=0}^{\infty}\frac{1}{(2j+1)^4} \le \left|\frac{H(\omega)}{H^0(\omega)}\right| \le \sum_{j=0}^{\infty}\frac{1}{(2j+1)} = \frac{\pi^2}{8} \tag{19}$$

and consequently that the fractional error in $G_V(\omega)$, made by replacing $H(\omega)$ by $H^0(\omega)$, ranges from 0.971 at $\omega = 0$ to 0.657 at $\omega = \infty$. Since we are primarily interested in the fluctuation spectrum at low frequencies it should be a very good approximation to take

$$G_V(\omega) \simeq (8\tau_0/\pi^2)^2 G_g(\omega)/(1+\omega^2\tau_0^2) . \tag{20}$$

As a final step we will also assume that the fluctuations in the pressure gradient are essentially uncorrelated even for times much shorter than τ_0. It is therefore possible to replace $G_g(\omega)$ by a constant and obtain for an approximate power spectrum of δv

$$G_V(\omega) = G_V(0)/(1+\omega^2\tau_0^2); \quad \tau_0 = \rho_n d^2/\eta_n \pi^2 . \tag{21}$$

For values appropriate to the experiments of MBM ($T = 1.65$ K, $d = 0.5$ cm), the relaxation time τ_0 is 55.9 sec.

We may combine this result with Eqs. (7) and (9) to get

$$G_L(\omega) = G_V(0) f^2\tau^2/(1+\omega^2\tau^2)(1+\omega^2\tau_0^2) \tag{22}$$

and

$$\langle \delta L^2 \rangle = \frac{\frac{1}{2}\pi G_V(0) f^2 \tau}{1+\tau_0/\tau} . \tag{23}$$

This expression may be rewritten explicitly as a function of x to facilitate comparison with experiment:

$$\langle \delta L^2 \rangle = \frac{A f_1^2(x)/f_2(x)}{1 + C f_2(x)} , \tag{24}$$

where the constants A and C are given by

$$A = \tfrac{1}{2}\pi G_V(0)[(64b/v_c^2)(\alpha/d)^6] ,$$
$$C = \tau_0 4b(\alpha/d)^2 . \tag{25}$$

Recalling that b is related to the parameter χ_2 of the Vinen expression by Eq. (6), one might hope to treat C as an adjustable constant and obtain, in this way, an experimental value of χ_2. The agreement with experiment, however, is quite insensitive to the value of χ_2 as long as $0 \le \chi_2 < 1.4$. It seems preferable, then, to use the accepted value[1] $\chi_2 = 1.08$ to evaluate C, obtaining $C = 2.3\times10^{-2}$. One then can adjust the single parameter A to fit the experimental curve. The result is shown as the solid curve labeled 2 in Fig. 3, again with $A = 6\times10^{-4}$ cm^{-4}. Agreement with experiment at larger values of x is somewhat improved, but the appparent leveling off of the experimental results starting at values of x near 10 is clearly not reproduced by this model either.

One of the advantages of using an integrable spectrum $G_V(\omega)$ is that one can now obtain an experimental estimate of the mean-square velocity fluctuation $\langle \delta v^2 \rangle$. By integrating Eq. (21), we find

$$\langle \delta v^2 \rangle = \tfrac{1}{2}\pi G_V(0)/\tau_0 , \tag{26}$$

which can be related to the constants A and C by using Eq. (25):

$$\frac{\langle \delta v^2 \rangle}{v_c^2} = \frac{A/C}{16(\alpha/d)^4} . \tag{27}$$

The resulting experimental rms velocity fluctuation is $(\langle \delta v^2 \rangle)^{1/2} \simeq 0.067 v_c$. This corresponds to a fluctuating velocity on the order of 3×10^{-3} cm/sec, or to a fluctuating heat current on the order of 200 μW/cm^2. These do not seem so large as to be easily excluded by other experiments. It is interesting to note that were we to accept both A and C as experimentally determined, this result would be independent of any assumption about the value of either b or τ_0.

It does not appear likely that the apparent leveling off of the experimental curve for large x can be explained by simply adopting some more realistic choice of the velocity power spectrum. It could perhaps be accounted for if for some reason the magnitude of the fluctuations in v were to become a decreasing function of v_0. The most obvious way in which fluctuations in v might depend on v_0 at all would be if the normal fluid flow were turbulent. However, this would almost certainly lead to fluctuations which increase with increasing v_0, which is the wrong direction to explain the results. A possible resolution of the problem may be had if we recall that one of our earliest approximations was the neglect of the feedback reaction of the fluctuations in L on those of v. For large enough fluctuations this will certainly no longer be justified. An increase in v will produce an increase in L; the extra line will then lead to an additional force on the normal fluid acting to oppose the increase in v. This mechanism will lead to fluctuations whose magnitude will decrease with increasing v_0, which is the correct direction to account for the results. In order to properly describe this situation it will be necessary to consider the fluctuations in v and L simultaneously as part of a single coupled system.

IV. FLUCTUATIONS IN L INCLUDING FEEDBACK

A relatively simple but clearly approximate model of the system which includes the effect of feedback may be developed by using the following arguments. The principal result of Sec. III is that the power spectrum of δv is very similar to that which would be obtained if δv obeyed a Langevin equation of the form

$$\frac{\partial}{\partial t}\,\delta v+\frac{\delta v}{\tau_0}=a_0 g(t)\,. \tag{28}$$

Let us presume, then, that in the absence of fluctuations in L, δv actually does satisfy this dynamical equation. In the presence of a fluctuation δL, however, there will be an additional force acting on the normal fluid which tends to reduce δv. This force should be proportional to δL and also to the mean relative velocity between the vortex array and the normal fluid. This in turn may be shown to be proportional to v or to x.[17] Consequently, it is reasonable to modify Eq. (28) to account for feedback as follows:

$$\frac{\partial}{\partial t}\,\delta v+\frac{\delta v}{\tau_0}=a_0\, g(t)-\beta x\delta L\,, \tag{29}$$

where β is considered to be an adjustable constant. Since we have already seen that δL satisfies

$$\frac{1}{f}\left(\frac{\partial}{\partial t}\,\delta L+\frac{\delta L}{\tau}\right)=\delta v\,, \tag{4}$$

we can insert this into Eq. (29) and eliminate δv.

$$\frac{1}{a_0 f}\left(\frac{\partial^2}{\partial t^2}\,\delta L+\gamma\frac{\partial}{\partial t}\,\delta L+c\,\delta L\right)=g(t)\,, \tag{30}$$

where

$$\gamma\equiv\frac{1}{\tau}+\frac{1}{\tau_0}\,;\quad c\equiv\frac{1}{\tau\tau_0}+\beta x f\,. \tag{31}$$

Consequently, δL satisfies the dynamical equation of a damped harmonic oscillator driven by a random forcing function $g(t)$. The transfer function for the system is

$$H_f(\omega)=a_0 f/[(c-\omega^2)-i\omega\gamma]\,, \tag{32}$$

and the power spectrum of δL is related to that of $g(t)$ by

$$G_L(\omega)=|H_f(\omega)|^2 G_g(\omega)\,. \tag{33}$$

If we make our usual assumption that $G_g(\omega)$ can be replaced by $G_g(0)$ we obtain for the power spectrum of δL

$$G_L(\omega)=\frac{G_V(0)}{\tau_0^2}\,\frac{f^2}{(c-\omega^2)^2+\gamma^2\omega^2}\,, \tag{34}$$

where $G_V(0)\equiv(a_0\tau_0)^2 G_g(0)$. Using

$$\int_0^\infty\frac{d\omega}{(c-\omega^2)^2+\gamma^2\omega^2}=\frac{\pi}{2\gamma c}\,, \tag{35}$$

we can integrate Eq. (34) to obtain $\langle\delta L^2\rangle$.

$$\langle\delta L^2\rangle=\frac{\frac{1}{2}\pi G_V(0)f^2\tau}{(1+\tau_0/\tau)(1+\beta\tau_0 x f\tau)}\,, \tag{36}$$

where we have used Eq. (33) to replace γ and c. We note that this reduces to Eq. (23) when $\beta=0$, and to Eq. (10) when $\tau_0=0$.

In order to compare Eq. (36) with experiment we can rewrite it to make the dependence on x more explicit:

$$\langle\delta L^2\rangle=\frac{Af_1^2(x)/f_2(x)}{[1+Cf_2(x)]\{1+[B_0 x f_1(x)/f_2(x)]\}}\,, \tag{37}$$

where A and C are still defined by Eq. (25), and B_0 is an adjustable constant given by

$$B_0=\beta(4\tau_0/v_c)(\alpha/d)^2\,. \tag{38}$$

We note that for large x, $\langle\delta L\rangle^2$ now reaches a limiting value $\lim_{x\to\infty}\langle\delta L^2\rangle=A/B_0C$. In addition, $\langle\delta L^2\rangle$ no longer diverges at $x=1$ but instead reaches the finite value A/B_0. Equation (37), evaluated with $B_0=5\times10^{-4}$ and with the same values of A and C used earlier, is shown as the curve labeled 3 in Fig. 3. The result essentially reproduces curve 2 for values of $x\lesssim7$, but begins to flatten off for large x. The good qualitative agreement with the data certainly provides support for the hypothesis that large-x behavior is strongly influenced by the effects of feedback. However, the experimental results are clearly too imprecise to offer a convincing test of the details of the model.

V. SUMMARY

The physical model that we have used to explain the fluctuations in L begins with the suggestion of MBM that they are driven by fluctuations in v in a manner described by the Vinen equation. However, in contrast to MBM who assume that the fluctuations arise from turbulent flow of the normal fluid, we presume that small fluctuations in v are an intrinsic feature of the liquid even at rest, and not a product of the applied heat flow. By linearizing the Vinen equation if proves possible to directly relate $\langle \delta L^2 \rangle$ to the power spectrum of δv. This relation is used in two ways. First, we note that behavior near threshold is very sensitive to the details of the Vinen equation and very insensitive to the details of the power spectrum δv. The resulting prediction of threshold behavior agrees well with experiment and thus constitutes perhaps the best confirmation yet of the validity of Vinen's complete dynamical equation. Second, we attempt a model calculation of the low-frequency behavior of the fluctuations in v. The fluctuations in v are presumed driven by fluctuating temperature gradients in the fluid and damped by ordinary viscous effects. No attempt is made to identify the source of the fluctuating gradients. Calculation of $\langle \delta L^2 \rangle$ with this more realistic $G_V(\omega)$ does not alter the previous good agreement with experiment near threshold, nor significantly improve the poor agreement well above it. One benefit of the introduction of an integrable G_V is that it allows us to extract a value of $\langle \delta v^2 \rangle$ from the data and show that it is reasonably small.

In order to understand the large velocity data we introduce a simplified model which assumes that fluctuations in L and v are strongly coupled in a feedback loop which is driven by small fluctuating temperature gradients. The predictions of the model agree well enough with the measurements of $\langle \delta L^2 \rangle$ to suggest that the physical principles are probably correct. The data are clearly inadequate to provide a test of the details of the model, however. We note that the strength of this model lies not in its ability to predict $\langle \delta L^2 \rangle$, but in its quite explicit predictions about the dependence of the power spectrum $G_L(\omega)$ on both ω and x. Experimental measurements of $G_L(\omega)$ should soon be available[18] and will then provide a much more rigorous test of the arguments presented here.

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