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Research Article

Bifurcation and Global Dynamics of a Leslie-Gower Type Competitive System of Rational Difference Equations with Quadratic Terms

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Received 1 April 2017; Accepted 4 June 2017; Published 2 August 2017

1. Introduction

In this paper we study the global dynamics of the following rational system of difference equations:

\[
\begin{align*}
    x_{n+1} &= \frac{x_n}{A_1 + B_1 x_n + C_1 y_n}, \\
    y_{n+1} &= \frac{y_n^2}{A_2 + B_2 x_n + C_2 y_n^2},
\end{align*}
\]

where the parameters \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) are positive numbers and initial conditions \( x_0 \) and \( y_0 \) are arbitrary nonnegative numbers.

System (1) is a competitive system, and our results are based on recent results about competitive systems in the plane; see [1]. System (1) can be used as a mathematical model for competition in population dynamics. System (1) is related to Leslie-Gower competition model

\[
\begin{align*}
    x_{n+1} &= \frac{x_n}{A_1 + B_1 x_n + C_1 y_n}, \\
    y_{n+1} &= \frac{y_n^2}{A_2 + B_2 x_n + C_2 y_n^2},
\end{align*}
\]

where the parameters \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) are positive numbers and initial conditions \( x_0 \) and \( y_0 \) are arbitrary nonnegative numbers, considered in [2]. System (2) globally exhibits three dynamic scenarios in five parametric regions which are competitive exclusion, competitive coexistence, and existence of an infinite number of equilibrium solutions; see [1–3]. System (2) does not exhibit the Allee effect, which is desirable from modeling point of view. The simplest variation of system (2) which exhibits the Allee effect is probably system

\[
\begin{align*}
    x_{n+1} &= \frac{x_n^2}{A_1 + B_1 x_n^2 + C_1 y_n}, \\
    y_{n+1} &= \frac{y_n^2}{A_2 + B_2 x_n + C_2 y_n^2},
\end{align*}
\]

where the parameters \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) are positive numbers and initial conditions \( x_0 \) and \( y_0 \) are arbitrary nonnegative numbers, considered in [4]. System (3) has between 1 and 9 equilibrium points and exhibits nine dynamic scenarios part of each is the Allee effect. In the case of the dynamic
scenario with nine equilibrium points system (3) exhibits both competitive exclusion and competitive coexistence as well as the Allee effect. Another system with quadratic terms is

\[ x_{n+1} = \frac{x_n^2}{B_1 x_n^2 + C_1 y_n^2}, \]
\[ y_{n+1} = \frac{y_n^3}{A_2 + B_2 x_n^2 + C_2 y_n^2}, \]

\[ n = 0, 1, \ldots, \]

where the parameters \( A_1, B_1, B_2, C_1, \) and \( C_2 \) are positive numbers and initial conditions \( x_0 \) and \( y_0 \) are arbitrary nonnegative numbers such that \( x_0 + y_0 > 0 \), considered in [5]. System (4) exhibits seven scenarios part of each is singular Allee’s effect, which means that the origin as the singular point of this system still has some basin of attraction. First systematic study for a system with quadratic terms was performed in [6] for system

\[ x_{n+1} = \frac{x_n}{A_1 + y_n^2}, \]
\[ y_{n+1} = \frac{y_n}{A_2 + x_n^2}, \]

\[ n = 0, 1, \ldots \]

which exhibits nine dynamic scenarios and whose dynamics is very similar to the corresponding system without quadratic terms considered in [7].

In general, it seems that an introduction of quadratic terms in equations of the Leslie-Gower model (2) generates the Allee effect. We will test this hypothesis in this paper by introducing the quadratic terms only in the second equation. System (1) can be considered as the competitive version of the decoupled system

\[ x_{n+1} = \frac{x_n}{A_1 + B_1 x_n}, \]
\[ y_{n+1} = \frac{y_n^2}{A_2 + C_2 y_n^2}, \]

\[ n = 0, 1, \ldots, \]

where the parameters \( A_1, A_2, B_1, \) and \( C_2 \) are positive numbers and initial conditions \( x_0 \) and \( y_0 \) are arbitrary nonnegative numbers, whose dynamics can be directly obtained from two separate equations. Unlike system (2) which has five regions of parameters with distinct local behavior system (1) has eighteen regions of parameters with distinct local behavior, which is caused by the geometry of the problem, that is, by the geometry of equilibrium curves. More precisely, the equilibrium curves of system (2) are lines while the equilibrium curves of system (1) are a line and a parabola. In the case when \( A_1 > 1 \), all equilibrium points are hyperbolic and all solutions are attracted to the three equilibrium points on the \( y \)-axis and we can describe this situation as competitive exclusion case. When \( A_1 = 1 \), the equilibrium point \( E_1 \) is nonhyperbolic and dynamics is analogous to the case when \( A_1 > 1 \). In both cases the Allee effect is present. When \( A_1 < 1 \), there exist II regions of parameters with different global dynamics. In nine of these regions the global dynamics is in competitive exclusion case, which means that all solutions converge to one of the equilibrium points on the axes and in only two situations we have competitive coexistence case, which means that the interior equilibrium points have substantial basin of attraction. In all II cases, the zero equilibrium has some basin of attraction which is a part of \( y \)-axis so we can say that in these cases system (1) exhibits weak Allee’s effect. Figure 3 gives the bifurcation diagram showing the transition from different global dynamics situations when \( A_1 \) is less than \( 1 \), since the cases \( A_1 \geq 1 \) are simple and do not need graphical interpretation.

The paper is organized as follows. Section 2 contains some necessary results on competitive systems in the plane. Section 3 provides some basic information about the number of equilibrium points. Section 4 contains local stability analysis of all equilibrium solutions. Section 5 contains some global results on injectivity of the map associated with system (1). Section 6 gives global dynamics of system (1) in all regions of the parameters.

2. Preliminaries

A first-order system of difference equations

\[ x_{n+1} = f(x_n, y_n), \]
\[ y_{n+1} = g(x_n, y_n), \]

\[ n = 0, 1, \ldots, \]

where \( \mathcal{S} \subset \mathbb{R}^2, (f, g) : \mathcal{S} \rightarrow \mathcal{S}, f, g \) are continuous functions is competitive if \( f(x, y) \) is nondecreasing in \( x \) and nonincreasing in \( y \), and \( g(x, y) \) is nonincreasing in \( x \) and nondecreasing in \( y \). If both \( f \) and \( g \) are nondecreasing in \( x \) and \( y \), system (7) is cooperative. Competitive and cooperative maps are defined similarly. Strongly competitive systems of difference equations or strongly competitive maps are those for which the functions \( f \) and \( g \) are coordinate-wise strictly monotone.

Competitive and cooperative systems have been investigated by many authors; see [1–3, 7–16]. Special attention to discrete competitive and cooperative systems in the plane was given in [1–3, 16, 17]. One of the reasons for paying special attention to two-dimensional discrete competitive and cooperative systems is their applicability and the fact that many examples of mathematical models in biology and economy which involve competition or cooperation are models which involve two species. Another reason is that the theory of two-dimensional discrete competitive and cooperative systems is very well developed, unlike such theory for three-dimensional and higher systems. Part of the reason for this situation is de Mottoni-Schiaffino theorem given below, which provides relatively simple scenarios for possible behavior of many two-dimensional discrete competitive and cooperative systems. However, this does not mean that one
can not encounter chaos in such systems as has been shown by Smith; see [16].

If \( \mathbf{v} = (u, v) \in \mathbb{R}^2 \), we denote with \( \mathcal{G}_\mathbf{v}(\mathbf{v}) \), \( \ell \in \{1, 2, 3, 4\} \), the four quadrants in \( \mathbb{R}^2 \) relative to \( \mathbf{v} \), that is, \( \mathcal{G}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u, y \geq v\} \), \( \mathcal{G}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u, y \geq v\} \), and so on. Define the South-East partial order \( \succeq_{se} \) on \( \mathbb{R}^2 \) by \( (x, y) \succeq_{se} (s, t) \) if and only if \( x \leq s \) and \( y \geq t \). Similarly, we define the North-East partial order \( \succeq_{ne} \) on \( \mathbb{R}^2 \) by \( (x, y) \succeq_{ne} (s, t) \) if and only if \( x \leq s \) and \( y \geq t \). For \( \mathcal{A} \subset \mathbb{R}^2 \) and \( x \in \mathbb{R}^2 \), define the distance from \( x \) to \( \mathcal{A} \) as \( \text{dist}(x, \mathcal{A}) := \inf\{(x - y) : y \in \mathcal{A}\} \). By int \( \mathcal{A} \) we denote the interior of a set \( \mathcal{A} \).

It is easy to show that a map \( F \) is competitive if it is nondecreasing with respect to the South-East partial order, that is, if the following holds:

\[
\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \succeq_{se} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \succeq_{se} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.
\]

For standard definitions of attracting fixed point, saddle point, stable manifold, and related notions see [10].

We now state three results for competitive maps in the plane. The following definition is from [16].

Definition 1. Let \( \mathcal{A} \) be a nonempty subset of \( \mathbb{R}^2 \). A competitive map \( T : \mathcal{A} \rightarrow \mathcal{A} \) is said to satisfy condition \((O+)\) if for every \( x, y \in \mathcal{A}, T(x) \succeq_{ne} T(y) \) implies \( x \succeq_{ne} y \), and \( T \) is said to satisfy condition \((O-)\) if for every \( x, y \in \mathcal{A}, T(x) \succeq_{ne} T(y) \) implies \( y \succeq_{ne} x \).

The following theorem was proved by de Mottoni-Schiaffino [17] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [13, 14].

Theorem 2. Let \( \mathcal{A} \) be a nonempty subset of \( \mathbb{R}^2 \). If \( T \) is a competitive map for which \((O+)\) holds then for all \( x \in \mathcal{A}, T^k(x) \) is eventually componentwise monotone. If the orbit of \( x \) has compact closure, then it converges to a fixed point of \( T \).

The situation where the endpoints of \( \mathcal{C} \) are boundary points of \( \mathcal{R} \) of interest. The following result gives a sufficient condition for this case.

Theorem 7. For the curve \( \mathcal{C} \) of Theorem 6 to have endpoints in \( \partial \mathcal{R} \), it is sufficient that at least one of the following conditions is satisfied

\begin{itemize}
  \item [(i)] The map \( T \) has no fixed points or periodic points of minimal period-two in \( \Delta \).
  \item [(ii)] The map \( T \) has no fixed points in \( \Delta \), \( \det J_T(\mathcal{X}) > 0 \), and \( T(x) = \mathcal{X} \) has no solutions \( x \in \Delta \).
  \item [(iii)] The map \( T \) has no points of minimal period-two in \( \Delta \), \( \det J_T(\mathcal{X}) < 0 \), and \( T(x) = \mathcal{X} \) has no solutions \( x \in \Delta \).
\end{itemize}
The next result is useful for determining basins of attraction of fixed points of competitive maps.

**Theorem 8.** (A) Assume the hypotheses of Theorem 6, and let \( \mathcal{C} \) be the curve whose existence is guaranteed by Theorem 6. If the endpoints of \( \mathcal{C} \) belong to \( \partial R \), then \( \mathcal{C} \) separates \( R \) into two connected components, namely,

\[
\mathcal{W}^- := \{ x \in R \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \leq_{sc} y \}, \\
\mathcal{W}^+ := \{ x \in R \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \leq_{sc} x \},
\]

such that the following statements are true:

(i) \( \mathcal{W}^- \) is invariant, and \( \text{dist}(T^n(x), \mathcal{Q}_2(\overline{x})) \to 0 \) as \( n \to \infty \) for every \( x \in \mathcal{W}^- \).

(ii) \( \mathcal{W}^+ \) is invariant, and \( \text{dist}(T^n(x), \mathcal{Q}_4(\overline{x})) \to 0 \) as \( n \to \infty \) for every \( x \in \mathcal{W}^+ \).

(B) If, in addition to the hypotheses of part (A), \( \overline{x} \) is an interior point of \( R \) and \( T \) is \( C^2 \) and strongly competitive in a neighborhood of \( \overline{x} \), then \( T \) has no periodic points in the boundary of \( \mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x}) \) except for \( \overline{x} \), and the following statements are true:

(iii) For every \( x \in \mathcal{W}^- \) there exists \( n_0 \in \mathbb{N} \) such that \( T^n(x) \in \text{int}\mathcal{Q}_2(\overline{x}) \) for \( n \geq n_0 \).

(iv) For every \( x \in \mathcal{W}^+ \) there exists \( n_0 \in \mathbb{N} \) such that \( T^n(x) \in \text{int}\mathcal{Q}_4(\overline{x}) \) for \( n \geq n_0 \).

If \( T \) is a map on a set \( R \) and if \( \overline{x} \) is a fixed point of \( T \), the stable set \( W^s(\overline{x}) \) of \( \overline{x} \) is the set \( \{ x \in R : T^n(x) \to \overline{x} \} \) and unstable set \( W^u(\overline{x}) \) of \( \overline{x} \) is the set

\[
\left\{ x \in R : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset R \text{ s.t. } T(x_n) = x_{n+1}, \ x_0 = x, \ \lim_{n \to -\infty} x_n = \overline{x} \right\}.
\]

When \( T \) is noninvertible, the set \( W^u(\overline{x}) \) may not be connected and made up of infinitely many curves, or \( W^s(\overline{x}) \) may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on \( R \), the sets \( W^s(\overline{x}) \) and \( W^u(\overline{x}) \) are the stable and unstable manifolds of \( \overline{x} \).

**Theorem 9.** In addition to the hypotheses of part (B) of Theorem 8, suppose that \( \mu > 1 \) and that the eigenspace \( E^\mu \) associated with \( \mu \) is not a coordinate axis. If the curve \( \mathcal{C} \) of Theorem 6 has endpoints in \( \partial R \), then \( \mathcal{C} \) is the stable set \( W^s(\overline{x}) \) of \( \overline{x} \), and the unstable set \( W^u(\overline{x}) \) of \( \overline{x} \) is a curve in \( R \) that is tangential to \( E^\mu \) at \( \overline{x} \) and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of \( W^u(\overline{x}) \) in \( R \) are fixed points of \( T \).

**3. Number of Equilibria**

In this section we give some basic facts which are used later. Let \( T \) be the map associated with system (1) given by

\[
T(x, y) = (f(x, y), g(x, y))
= \left( \frac{x}{A_1 + B_1 x + C_1 y}, \frac{y^2}{A_2 + B_2 x + C_2 y^2} \right).
\]

Let \( R = \mathbb{R}^2_+ \). The equilibrium points \((\overline{x}, \overline{y})\) of system (1) satisfy equations

\[
\overline{x} = \frac{A_1 + B_1 \overline{x} + C_1 \overline{y}}{A_2 + B_2 \overline{x} + C_2 \overline{y}^2} = \overline{x},
\]

\[
\overline{y}^2 = \frac{A_2 + B_2 \overline{x} + C_2 \overline{y}^2}{A_1 + B_1 \overline{x} + C_1 \overline{y}} = \overline{y}.
\]

For \( \overline{x} = 0 \) we have

\[
\overline{y} = \overline{y}^2 - A_2 \overline{y} - C_2 \overline{y}^3
\]

from which we obtain three equilibrium points

\[
E_1 = (0, 0), \quad E_2 = \left(0, 1 - \frac{\sqrt{\Delta_1}}{2C_2}\right), \quad E_3 = \left(0, 1 + \frac{\sqrt{\Delta_1}}{2C_2}\right),
\]

where \( \Delta_1 = 1 - 4A_2C_2 \).

Assume that \( \overline{x} \neq 0 \). Then, from the first equation of system (12) we have

\[
\overline{y} = -\frac{A_1 + B_1 \overline{x} - 1}{C_1}. \quad (15)
\]

By substituting this into the second equation we obtain

\[
A_1 + B_1 \overline{x} - 1 = 0 \quad (16)
\]

or

\[
\overline{y}(x) \equiv B_1^2 C_2 x^2 + x \left(B_1 (2(A_1 - 1)C_2 + C_1) + B_2 C_1^2\right) + \left((A_1 - 1)^2 C_2 + (A_1 - 1)C_1 + A_2 C_1^2\right) = 0.
\]
The following holds:

Lemma 10. The following hold:

(i) The equilibrium points $E_2$ and $E_3$ exist if and only if $\Delta_1 \geq 0$ and $E_4 = E_5$ if and only if $\Delta_1 = 0$.

(ii) The equilibrium point $E_4$, exists if and only if $A_1 \leq 1$ and $E_4 = E_5$ if and only if $A_1 = 1$.

(iii) Assume that $\Delta_2 \geq 0$. The equilibrium point $E_5$ exists if and only if $A_1 < 1$ and

\[
C_1 \leq \frac{(1 - A_1) B_1}{(1 - A_1) B_2 + 2 A_2 B_1},
\]

\[
C_2 \leq \frac{(B_2 C_1 + B_1)^2}{4 B_1 (A_2 B_1 + (1 - A_1) B_2)}
\]

or

\[
C_1 > \frac{(1 - A_1) B_1}{(1 - A_1) B_2 + 2 A_2 B_1},
\]

\[
C_2 \leq \frac{C_1 (1 - A_1 - A_2 C_1)}{(1 - A_1)^2}.
\]

(iv) Assume that $\Delta_2 \geq 0$. The equilibrium point $E_6$ exists if and only if $A_1 < 1$ and

\[
C_1 \leq \frac{(1 - A_1) B_1}{(1 - A_1) B_2 + 2 A_2 B_1},
\]

\[
\frac{C_1 (1 - A_1 - A_2 C_1)}{(1 - A_1)^2} \leq C_2 \leq \frac{(B_2 C_1 + B_1)^2}{4 B_1 (A_2 B_1 + (1 - A_1) B_2)}.
\]

Proof. The proof of the statements (i) and (ii) is trivial and we skip it. Now we prove the statement (iii). In view of Descartes’ rule of signs we obtain that (17) has no positive solutions if $A_1 \geq 1$. Now, we suppose that $A_1 < 1$. One can see that $\bar{y}_5 > 0$ for all values of parameters. We consider two cases:

(1) Assume that

\[
-2A_1 B_1 C_2 - B_2 C_1^2 - B_1 C_1 + 2B_1 C_2 \geq 0,
\]

which is equivalent to

\[
C_2 \geq \frac{C_1 (B_2 C_1 + B_1)}{2 (1 - A_1) B_1}.
\]

Since

\[
\Delta_2 \geq 0 \iff
\]

\[
C_2 \leq \frac{(B_2 C_1 + B_1)^2}{4 B_1 (A_2 B_1 + (1 - A_1) B_2)}
\]

we have that $\bar{y}_5 \geq 0$ if and only if

\[
\frac{C_1 (B_2 C_1 + B_1)}{2 (1 - A_1) B_1} \leq C_2 \leq \frac{(B_2 C_1 + B_1)^2}{4 B_1 (A_2 B_1 + (1 - A_1) B_2)}.
\]

(2) Assume that

\[
-2A_1 B_1 C_2 - B_2 C_1^2 - B_1 C_1 + 2B_1 C_2 < 0
\]

which is equivalent to

\[
C_2 < \frac{C_1 (B_2 C_1 + B_1)}{2 (1 - A_1) B_1}.
\]
Then \( \bar{x}_5 \geq 0 \) if and only if
\[
\Delta_2 C_1^2 - \left( - B_1 \left( 2(A_1 - 1)C_2 + C_1 \right) - B_2 C_2^2 \right)^2
= -4B_2^2 C_2 \left( (A_1 - 1)^2 C_2^2 + C_1 (A_2 C_1 + A_1 - 1) \right)
\geq 0,
\]
which is equivalent to
\[
C_2 \leq \frac{C_1 (-A_2 C_1 - A_1 + 1)}{(1 - A_1)^2}, \tag{33}
\]
\[
\Delta_2 \geq 0 \iff
C_2 \leq \frac{(B_2 C_1 + B_1)^2}{4B_1 (A_2 B_1 + (1 - A_1) B_2)}. \tag{34}
\]

Since
\[
\frac{(B_2 C_1 + B_1)^2}{4B_1 (A_2 B_1 + (1 - A_1) B_2)} - \frac{C_1 (-A_2 C_1 - A_1 + 1)}{(1 - A_1)^2}
= \frac{((-1 - A_1) B_2 C_1 + B_1 (-2A_2 C_1 - A_1 + 1))^2}{4(1 - A_1)^2 B_1 (A_2 B_1 + (1 - A_1) B_2)} \geq 0
\]
then from (33) and \( \Delta_2 \geq 0 \) we have
\[
C_2 \leq \frac{C_1 (-A_2 C_1 - A_1)}{(1 - A_1)^2}. \tag{36}
\]

Since
\[
\frac{C_1 (1 - A_2 C_1 - A_1)}{(1 - A_1)^2} - \frac{C_1 (B_2 C_1 + B_1)^2}{2(1 - A_1) B_1}
= \frac{C_1 ((1 - A_1) B_2 C_1 + B_1 (2A_2 C_1 + A_1 - 1))}{2(A_1 - 1)^2 B_1}
\]
we have that (31) and (36) are equivalent to
\[
C_1 > \frac{(1 - A_1) B_1}{2A_2 B_1 + (1 - A_1) B_2}, \tag{38}
\]
or
\[
C_1 \leq \frac{(1 - A_1) B_1}{2A_2 B_1 + (1 - A_1) B_2}, \tag{39}
\]

Now, the proof of the statement (iii) follows from (28), (38), and (39). The proof of the statement (iv) is similar and we skip it. \[\square\]

We now introduce the following notation for regions in parameter space \((C_1, C_2)\) (see Figure 1):
\[
\mathcal{R}_{24} = \left\{ (C_1, C_2) : A_1 < 1, \ A_1 > 0, \ C_1 \right\}
\]
\[
< \frac{(1 - A_1) B_1}{(1 - A_1) B_2 + 2A_2 B_1}, \quad \frac{C_1 (1 - A_1 - A_2 C_1)}{(1 - A_1)^2}
< \frac{(B_2 C_1 + B_1)^2}{4B_1 (A_2 B_1 + (1 - A_1) B_2)} \right\},
\]
\[
\mathcal{R}_{22} = \left\{ (C_1, C_2) : A_1 < 1, \ A_1 > 0, \ C_2 \right\}
\]
\[
< \frac{C_1 (1 - A_1 - A_2 C_1)}{(1 - A_1)^2} \right\},
\]
\[
\mathcal{R}_{21} = \left\{ (C_1, C_2) : A_1 < 1, \ A_1 > 0, \ C_2 \right\}
\]
\[
< \frac{(B_2 C_1 + B_1)^2}{4B_1 (A_2 B_1 + (1 - A_1) B_2)} \right\}.
\]
Figure 1 gives a graphical representation of above sets. The following result gives a complete classification for the number of equilibrium solutions of system (1).

**Proposition 11.** Let \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) be positive real numbers. Then, the number of positive equilibrium solutions of system (1) with parameters \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) can be from 1 to 6. The different cases are given in Table 1.

**Proof.** The proof follows from Lemma 10. \( \square \)

### 4. Linearized Stability Analysis

The Jacobian matrix of the map \( T \) has the form

\[
J_T = \begin{pmatrix}
\frac{A_1 + yC_1}{(A_1 + xB_1 + yC_1)^2} & -\frac{xC_1}{(A_1 + xB_1 + yC_1)^2} \\
\frac{y^2 B_2}{(A_2 + xB_2 + y^2 C_2)^2} & \frac{2y (A_2 + xB_2)}{(A_2 + xB_2 + y^2 C_2)^2}
\end{pmatrix}
\]
The determinant of (41) at the equilibrium point is given by

\[
\det J_T(\bar{x}, \bar{y}) = \frac{\bar{y}(2A_1(A_2 + B_2\bar{x}) + \bar{y}(2A_2 + B_2\bar{x})C_1)}{(A_1 + B_1\bar{x} + C_1\bar{y})^2 (A_2 + B_2\bar{x} + C_2\bar{y})^2} \tag{42}
\]

and the trace of (41) at the equilibrium point is given by

\[
\text{tr} J_T(\bar{x}, \bar{y}) = \frac{A_1 + \bar{y}C_1}{(A_1 + B_1\bar{x} + C_1\bar{y})^2} + \frac{2\bar{y}(A_2 + B_2\bar{x})}{(A_2 + B_2\bar{x} + C_2\bar{y})^2} \tag{43}
\]
The characteristic equation has the form
\[
\lambda^2 - \lambda \left( \frac{A_1 + \gamma C_1}{A_1 + \beta x + C_1 \gamma} + \frac{2 \gamma (A_2 + B_2 x)}{(A_2 + B_2 x + C_2 \gamma)} \right) \\
+ \frac{\gamma (2A_1 (A_2 + B_2 x) + \gamma (2A_2 + B_2 x) C_1)}{(A_1 + \beta x + C_1 \gamma)^2 (A_2 + B_2 x + C_2 \gamma)^2} = 0.
\] (44)

**Lemma 12.** The following statements hold:

(a) $E_1$ is locally asymptotically stable if $A_1 > 1$.

(b) $E_1$ is a saddle point if $A_1 < 1$.

(c) $E_1$ is a nonhyperbolic equilibrium point if $A_1 = 1$.

**Proof.** We have that, for the equilibrium point $E_1$, $\text{tr} J_f(E_1) = 1/A_1$ and $\det J_f(E_1) = 0$. The characteristic equation of (50) at $E_1$ has the form $\lambda^2 - (1/A_1) \lambda = 0$, from which the proof follows.

**Lemma 13.** The following statements hold:

(a) $E_4$ is locally asymptotically stable if $A_4 < 1$.

(b) $E_4$ is a nonhyperbolic equilibrium point if $A_4 = 1$.

**Proof.** We have that, for the equilibrium point $E_4$, $\text{tr} J_f(E_4) = A_1$ and $\det J_f(E_4) = 0$. The characteristic equation of (50) at $E_4$ has the form $\lambda^2 - A_1 \lambda = 0$, from which the proof follows.

The equilibrium points $E_5$ and $E_6$ are intersection points of the curves
\[
x_f(y) = \frac{1 - A_1 - C_1 y}{B_1},
\]
\[
x_g(y) = \frac{y - A_2 - C_2 y^2}{B_2}.
\] (45)

Let $\tilde{x}(y) = x_f(y) - x_g(y)$ for $y \in [(1 - \sqrt{1 - 4A_2 C_2})/2C_2, (1 + \sqrt{1 - 4A_2 C_2})/2C_2]$.

**Lemma 14.** Let $T = (f, g)$ be the map defined by (II). Then $f_x^i(E_i) < 1$, $f_y^i(E_i) < 1$, $g_x^i(E_i) = 0$ for $i = 1, 2$. Let
\[
\tilde{f}(y) = B_1 C_2 y^2 - (B_2 C_1 + B_1) y + A_2 B_1 + B_2 (1 - A_1).
\] (46)

Then, $\tilde{x}$ and $\tilde{y}$ are zeros of $\tilde{f}(y)$ and $\text{sign}(\tilde{x}(y)) = \text{sign}(\tilde{f}(y))$ for $y \in [(1 - \sqrt{1 - 4A_2 C_2})/2C_2, (1 + \sqrt{1 - 4A_2 C_2})/2C_2]$.

**Proof.** The first derivative of $x_f(y)$ is given by
\[
x_f'(y) = \frac{f_y^i(E_i)}{1 - f_x^i(E_i)} = -\frac{C_1}{B_1} < 0, \quad i = 5, 6.
\] (47)

Since $f_y^i(E_i) < 0$, $i = 5, 6$, we get $f_x^i(E_i) < 1$, $i = 5, 6$. Similarly, one can see that
\[
x_f'(\tilde{y}) = \frac{1 - g_y^i(E_i)}{g_x^i(E_i)}
\]
\[
= -\frac{1}{B_1} \frac{\sqrt{(B_4 C_4 + B_4 y)^2 - 4B_4 C_4 (A_1 B_1 - (A_1 - 1) B_2) + B_4 C_4}}{B_1 B_2}
\]
\[
< 0.
\] (48)

Since $g_y^i(E_i) < 0$, we get $g_x^i(E_i) < 1$. Further,
\[
\tilde{x}(y) = \frac{1 - A_1 - C_1 y - y - A_2 - C_2 y^2}{B_1} = B_1 C_2 y^2 - (B_2 C_1 + B_1) y + A_2 B_1 + B_2
\]
\[
= \frac{B_1 C_2 y^2 - (B_2 C_1 + B_1) y + A_2 B_1 - A_1 B_2 + B_2}{B_1 B_2}
\]
\[
= \frac{\tilde{f}(y)}{B_1 B_2},
\] (49)

from which the proof follows.

**Lemma 15.** Let $T$ be the map associated with system (1) and
\[
J_T(x, y) = \begin{pmatrix}
a_i & b_i \\
c_i & d_i
\end{pmatrix}
\] (50)
be the Jacobian matrix of $T$ at fixed point $E_i(i = 5, 6)$. Then the Jacobian matrix (50) has real and distinct eigenvalues $\lambda_1$ and $\lambda_2$ such that $0 < \lambda_1 < \lambda_2$. Furthermore, the following hold:
\[
\text{sign}(\tilde{x}(y)) = \text{sign}(1 - \lambda_1^2), \quad \lambda_2^2 < 1,
\]
\[
\text{sign}(\tilde{x}(y)) = \text{sign}(1 - \lambda_1^2)(1 - \lambda_2^2).
\] (51) (52)

**Proof.** Implicit differentiation of the equations defining $C_f$ and $C_g$ at $E_i$ gives
\[
x_f'(\tilde{y}) = \frac{f_y^i(E_i)}{1 - f_x^i(E_i)},
\]
\[
x_g'(\tilde{y}) = 1 - \frac{g_y^i(E_i)}{g_x^i(E_i)}.
\] (53)

Characteristic equations associated with the Jacobian matrix of $T$ at $E_i$ are given by
\[
p(\lambda) = \lambda^2 - \left[ f_x^i(E_i) + g_y^i(E_i) \right] \lambda
\]
\[
+ \left[ f_x^i(E_i) g_y^i(E_i) - f_y^i(E_i) g_x^i(E_i) \right]
\]
\[
= \lambda^2 - (a_i + d_i) \lambda + (a_i d_i - b_i c_i).
\] (54)
Since the map $T$ is competitive, then the eigenvalues of the Jacobian matrix of the map $T$, at the equilibrium $E_i$, are real and distinct and furthermore $0 < \lambda_1^{(i)} < \lambda_2^{(i)}$. By (53), we have

\[
\bar{x}'(\bar{y}) = x'_f(\bar{y}) - x'_g(\bar{y}) = \frac{f'_x(E_i)}{1 - f'_x(E_i)} - \frac{g'_x(E_i)}{g'_x(E_i)} = \frac{b_i}{1 - a_i} - \frac{1 - d_i}{c_i} = \frac{-1 + (a_i + d_i) \lambda_1^{(i)} - (a_id_i - b_ic_i)}{c_i(1 - a_i)} = \frac{1 - \lambda_1^{(i)} \lambda_2^{(i)}}{c_i(1 - a_i)}. \tag{55}
\]

In view of Lemma 14 and from $\text{tr} J_T(E_6) = a_6 + d_6 = f'_x(E_6) + g'_x(E_6) = \lambda_1^{(6)} + \lambda_2^{(6)} < 2$ we get $\lambda_1^{(6)} < 1$. The map $T$ is competitive, which implies $c_6 = g'_x(E_6) < 0$. In view of Lemma 14 we get $a_6 = f'_x(E_6) < 1$ from which it follows (51). Similarly, from $c_5 = g'_x(E_5) < 0$ and $a_5 = f'_x(E_5) < 1$ we obtain (52).

The following lemma describes the local stability of the equilibrium points $E_5$ and $E_6$.

**Lemma 16.** Assume that $A_1 < 1$ and $\Delta_2 \geq 0$. Then the following hold:

(i) If $\Delta_2 > 0$ and $E_6$ exists then it is locally asymptotically stable.

(ii) If $\Delta_2 > 0$ and $E_5$ exists then it is a saddle point.

(iii) If $\Delta_2 = 0$ then $E_5 = E_6$. Furthermore, if $E_5 = E_6$ then it is nonhyperbolic equilibrium point. The eigenvalues of $J_T(E_5 \equiv E_6)$ are given by $\lambda_1 = 1$ and $\lambda_2 < 1$.

**Proof.** Assuming that $\Delta_2 > 0$, then $\bar{y}_5$ and $\bar{y}_6$ are zeros of multiplicity one of

\[
\bar{f}(y) = B_1C_2y^2 - (B_2C_1 + B_1)y + A_2B_1 + B_2(1 - A_1) \tag{56}
\]

and $\bar{y}_6 > \bar{y}_5 > 0$. From this we have $\bar{f}(y) > 0$ for $y \in (0, \bar{y}_5) \cup (\bar{y}_5, \infty)$ and $\bar{f}(y) < 0$ for $y \in (\bar{y}_5, \bar{y}_6)$.

By Lemmas 6 and 7 from [19] the equilibrium curves $x_i(y)$ and $x_i'(y)$ intersect transversally at $E_5$ and $E_6$, that is, $\bar{x}'(\bar{y}) \neq 0 \quad i = 5, 6$. By this and Lemma 14 and by continuity of function $\bar{x}(y)$ there exists a neighborhood $U_{\bar{y}_5}$ of $\bar{y}_5$ such that $\bar{x}'(y) > 0$ for $y \in U_{\bar{y}_5}$ and $x_i'(y) < 0$ for $y \in U_{\bar{y}_5}$. This implies that $\bar{x}'(\bar{y}_5) > 0$ and $x_i'(\bar{y}_5) < 0$. By Lemma 15 we have that $E_6$ is locally asymptotically stable and $E_5$ is a saddle point whenever equilibrium points $E_5$ and $E_6$ exist.

Assume that $\Delta_2 = 0$. Then $\bar{y}_5 = \bar{y}_6$ is zero of $\bar{f}(y)$ of multiplicity two. In view of Lemmas 6 and 7 from [19] we have that $\bar{x}'(\bar{y}_5) = 0$. The rest of the proof follows from the proof of Lemma 15. \qed

**Lemma 17.** Assume that $\Delta_1 \geq 0$. The following statements are true:

(a) $E_2$ is a saddle point if $\Delta_1 > 0$, and $A_1 \geq 1$ or $A_1 < 1$ and $C_1 > (1 - A_1)(1 + \sqrt{1 - 4A_2C_2})/2A_2$.

(b) $E_2$ is a repeller if $\Delta_1 > 0$, $A_1 < 1$, and $C_1 < (1 - A_1)(1 + \sqrt{1 - 4A_2C_2})/2A_2$.

(c) $E_2$ is a nonhyperbolic equilibrium point if $\Delta_1 = 0$ or $\Delta_1 < 0$, $A_1 < 1$;

\[
C_1 = \frac{(1 - A_1)(1 + \sqrt{1 - 4A_2C_2})}{2A_2}. \tag{57}
\]

If $\Delta_1 = 0$ then the eigenvalues of $J_T(E_2)$ are given by

\[
\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2A_2C_1 + A_1}. \tag{58}
\]

with corresponding eigenvectors

\[
v_1 = (0, 1)^T, \quad v_2 = \left(\frac{2A_2C_1 + A_1 - 1}{B_2(2A_2C_1 + A_1)}, 1\right)^T. \tag{59}
\]

If (57) holds then the eigenvalues of $J_T(E_2)$ are given by

\[
\lambda_1 = 1, \quad \lambda_2 = 1 + \sqrt{1 - 4A_2C_2} > 1 \tag{60}
\]

with corresponding eigenvectors

\[
v_1 = \left(\frac{\sqrt{1 - 4A_2C_2}}{B_2}, 1\right)^T, \quad v_2 = (0, 1)^T. \tag{61}
\]

**Proof.** One can see that

\[
1 + \det J_T(E_2) - \text{tr} J_T(E_2) = 2 \frac{(1 - A_1)(1 - \sqrt{1 - 4A_2C_2}) + C_1(1 - 4A_2C_2 - \sqrt{1 - 4A_2C_2})}{2A_2C_1 + C_1(1 - \sqrt{1 - 4A_2C_2})}. \tag{62}
\]

\[
1 - \det J_T(E_2) = C_1(1 - \sqrt{1 - 4A_2C_2}) + 2C_2(1 - 1 - \sqrt{1 - 4A_2C_2}) \quad 2A_2C_1 + C_1(1 - \sqrt{1 - 4A_2C_2})
\]
(a) Since \( \det I_T(E_2) > 0 \) and \( \text{tr} I_T(E_2) > 0 \), the equilibrium \( E_2 \) is a saddle point if and only if \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) < 0 \). If \( A_1 \geq 1 \) it is obvious that \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) < 0 \). Assume that \( A_1 < 1 \). Then \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) < 0 \) if and only if
\[
C_1 > \frac{2(1-A_1)C_2\sqrt{1-4A_2C_2}}{4A_2C_2 - 1 + \sqrt{1-4A_2C_2}} \quad \text{(63)}
\]

from which the proof of the statement follows.

(b) Since \( \det I_T(E_2) > 0 \) and \( \text{tr} I_T(E_2) > 0 \), the equilibrium \( E_2 \) is repeller if and only if \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) > 0 \) and \( 1 - \det I_T(E_2) < 0 \). The proof of the statement follows from the facts
\[
1 - \det I_T(E_2) < 0 \iff C_1 < \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1 - \sqrt{1-4A_2C_2}}.
\]
\[
\frac{2(1-A_1)C_2\sqrt{1-4A_2C_2}}{4A_2C_2 - 1 + \sqrt{1-4A_2C_2}} - \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1 - \sqrt{1-4A_2C_2}} = -\frac{4A_2C_2 + \sqrt{1-4A_2C_2} + 1}{2A_2} < 0. \quad \text{(64)}
\]

(c) Since \( \det I_T(E_2) > 0 \) and \( \text{tr} I_T(E_2) > 0 \), the equilibrium \( E_2 \) is nonhyperbolic if and only if \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) = 0 \) or \( \det I_T(E_2) = 1 \) and \( \text{tr} I_T(E_2) \leq 2 \). From the proof of the statements (a) and (b) if
\[
\Delta_1 = 0
\]
with \( A_1 < 1 \),
\[
C_1 = \frac{(1-A_1)(\sqrt{1-4A_2C_2} + 1)}{2A_2} \quad \text{(65)}
\]
we obtain \( 1 + \det I_T(E_2) - \text{tr} I_T(E_2) = 0 \). Now, assume that \( A_1 > 0 \) and
\[
\det I_T(E_2) = 1 \iff C_1 = \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1 - \sqrt{1-4A_2C_2}} \quad \text{(66)}
\]

This implies \( \text{tr} I_T(E_2) - 2 = (1 - 4A_2C_2)/(\sqrt{1-4A_2C_2} + 1) > 0 \). The rest of the proof follows from the fact that if \( \Delta_1 = 0 \) then
\[
J_T(E_2) = \begin{pmatrix} 1 & 0 \\ A_1 + 2A_2C_1 & 1 \end{pmatrix} \quad \text{(67)}
\]
and if (57) holds then
\[
J_T(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-4A_2C_2} + 1 \end{pmatrix}. \quad \text{(68)}
\]

**Lemma 18.** Assume that \( \Delta_1 \geq 0 \). The following statements are true:

(a) \( E_3 \) is locally asymptotically stable if \( \Delta_1 > 0 \), and \( A_1 \geq 1 \) or \( A_1 < 1 \) and \( C_1 > (1-A_1)(1 - \sqrt{1-4A_2C_2})/2A_2 \).

(b) \( E_3 \) is a saddle point if \( \Delta_1 > 0 \), and \( A_1 < 1 \), \( C_1 < (1-A_1)(1 - \sqrt{1-4A_2C_2})/2A_2 \).

(c) \( E_3 \) is a nonhyperbolic equilibrium point if \( \Delta_1 = 0 \) or \( \Delta_1 > 0 \), \( A_1 < 1 \),
\[
C_1 = \frac{(1-A_1)(\sqrt{1-4A_2C_2})}{2A_2}. \quad \text{(69)}
\]

If \( \Delta_1 = 0 \) then the eigenvalues of \( J_T(E_3) \) are given by
\[
\lambda_1 = 1,
\]
\[
\lambda_2 = \frac{1}{2A_2C_1 + A_1} \quad \text{(70)}
\]
with corresponding eigenvectors
\[
v_1 = (0,1)^T,
\]
\[
v_2 = \left(\frac{2A_2C_1 + A_1 - \sqrt{1-4A_2C_2}}{B_2(2A_2C_1 + A_1)}, 1\right)^T. \quad \text{(71)}
\]

If (57) holds then the eigenvalues of \( J_T(E_3) \) are given by
\[
\lambda_1 = 1,
\]
\[
\lambda_2 = 1 - \sqrt{1-4A_2C_2} < 1 \quad \text{(72)}
\]
with corresponding eigenvectors
\[
v_1 = \left(-\frac{\sqrt{1-4A_2C_2}}{B_2}, 1\right)^T \quad \text{(73)}
\]
\[
v_2 = (0,1)^T.
\]

**Proof.** Since the proof of this lemma is similar to the proof of Lemma 17, it is omitted.

We summarize results about local stability in the following theorem.

**Theorem 19.** Let \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) be positive real numbers. Then, local stability of the equilibrium points for different parameter regions is given by Table 2.

**Proof.** The proof follows from Theorem 8 and Lemmas 17 and 18.

Figure 2 illustrates visually local stability of all equilibrium points of system (1).
Table 2: The local stability of the equilibrium points.

<table>
<thead>
<tr>
<th>Case</th>
<th>The local stability</th>
<th>Parameter region</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$, $E_6$; repeller: $E_2$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_1B_1}$, $C_2 = \frac{(B_1C_1 + B_1)^2}{4B_1(A_2B_1 + (1 - A_1)B_2)}$</td>
</tr>
<tr>
<td>(ii)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$; nonhyperbolic: $E_2 = E_6$; repeller: $E_2$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_1B_1}$, $C_2 = \frac{(B_1C_1 + B_1)^2}{4B_1(A_2B_1 + (1 - A_1)B_2)}$</td>
</tr>
<tr>
<td>(iii)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$, $E_5$; repeller: $E_2$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_2 &lt; \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
</tr>
<tr>
<td>(iv)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$; repeller: $E_2$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_2 &gt; \frac{(B_1C_1 + B_1)^2}{4B_1(A_2B_1 + (1 - A_1)B_2)}$, or $A_1 &lt; 1, \Delta_1 &gt; 0,$ $C_1 &gt; \frac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_1B_1}$, $C_2 &lt; \frac{(B_1C_1 + B_1)^2}{4B_1(A_2B_1 + (1 - A_1)B_2)}$</td>
</tr>
<tr>
<td>(v)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$, $E_5$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 &gt; \frac{1 - A_1}{2A_2}$, $C_2 &lt; \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
</tr>
<tr>
<td>(vi)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$; repeller: $E_2$; Nonhyperbolic: $E_3 = E_6$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_2 = \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
</tr>
<tr>
<td>(vii)</td>
<td>Saddle: $E_1$; LAS: $E_4$; repeller: $E_2$; Nonhyperbolic: $E_3 = E_6$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 = \frac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_1B_1}$, $C_2 = \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
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<tr>
<td>(viii)</td>
<td>Saddle: $E_1$; LAS: $E_4$; repeller: $E_2$; Nonhyperbolic: $E_3 = E_5$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 &gt; \frac{1 - A_1}{2A_2}$, $C_1 = \frac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_1B_1}$, $C_2 = \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
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<tr>
<td>(ix)</td>
<td>Saddle: $E_1$; LAS: $E_3$, $E_4$; nonhyperbolic: $E_2 = E_5$</td>
<td>$A_1 &lt; 1, \Delta_1 &gt; 0, C_1 &gt; \frac{1 - A_1}{2A_2}$, $C_2 = \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
</tr>
<tr>
<td>(x)</td>
<td>Saddle: $E_1$; LAS: $E_3$, $E_4$; nonhyperbolic: $E_2 = E_3$</td>
<td>$A_1 &lt; 1, \Delta_1 = 0, C_1 &gt; \frac{1 - A_1}{2A_2}$, $C_2 = \frac{C_1(1 - A_1 - A_3C_1)}{(1 - A_1)^2}$</td>
</tr>
<tr>
<td>(xi)</td>
<td>Saddle: $E_1$; LAS: $E_3$, $E_4$; nonhyperbolic: $E_2 = E_3$</td>
<td>$A_1 &lt; 1, \Delta_1 &lt; 0$</td>
</tr>
<tr>
<td>(xii)</td>
<td>Saddle: $E_1$; LAS: $E_4$</td>
<td>$A_1 &lt; 1, \Delta_1 &lt; 0$</td>
</tr>
<tr>
<td>(xiii)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$; nonhyperbolic: $E_2 = E_3$</td>
<td>$A_1 = 1, \Delta_1 &gt; 0$</td>
</tr>
<tr>
<td>(xiv)</td>
<td>Nonhyperbolic: $E_1 = E_4$, $E_3$</td>
<td>$A_1 = 1, \Delta_1 = 0$</td>
</tr>
<tr>
<td>(xv)</td>
<td>Nonhyperbolic: $E_1 = E_4$</td>
<td>$A_1 = 1, \Delta_1 &lt; 0$</td>
</tr>
<tr>
<td>(xvi)</td>
<td>Saddle: $E_1$, $E_3$; LAS: $E_4$</td>
<td>$A_1 &gt; 1, \Delta_1 &gt; 0$</td>
</tr>
<tr>
<td>(xvii)</td>
<td>LAS: $E_1$; nonhyperbolic: $E_2 = E_3$</td>
<td>$A_1 &gt; 1, \Delta_1 = 0$</td>
</tr>
<tr>
<td>(xviii)</td>
<td>LAS: $E_1$</td>
<td>$A_1 &gt; 1, \Delta_1 &lt; 0$</td>
</tr>
</tbody>
</table>
5. Injectivity and Convergence to Equilibrium Points

In this section we prove some global properties of the map $T$ such as injectivity and $(O+)$ property and give global behavior on the coordinate axes.

Lemma 20. The map $T$ is injective.

Proof. Assume that $T(x_1, y_1) = T(x_2, y_2)$. Then, we have

$$\frac{(y_1 - y_2)(A_1A_2y_1 + A_1B_2y_1 + A_1A_2y_2 + A_1B_2y_2 + A_2C_1y_1y_2 + B_2C_1y_1y_2 + A_2C_1y_2^2)}{A_1 + C_1y_2} = 0,$$

from which it follows that $y_1 = y_2$. From (77) we have $x_1 = x_2$, which complete the proof.

The global behavior of $T$ on the coordinate axes is described with the following result.

Lemma 21. The following statements hold:

(i) If $\Delta_1 \geq 0$ and $(x_n', y_n') = T^n(0, 0)$ then $x_n = 0$ and $(x_n, y_n) \to E_3$ for $y_0 \in (1 - \sqrt{1 - 4A_3C_2}/2C_2, \infty)$ and $(x_n', y_n') \to E_1$ for $y_0 \in (0, 1 + \sqrt{1 - 4A_3C_2}/2C_2)$.

(ii) If $\Delta_1 < 0$ and $(x_n, y_n) = T^n(0, 0)$ then $x_n = 0$ and $(x_n, y_n) \to E_1$ for $y_0 \in (0, \infty)$.

(iii) If $A_1 < 1$ and $(x_n', y_n') = T^n(x_0, 0)$ then $y_n' = 0$ and $(x_n', y_n') \to E_4$ for $x_0 \in (0, \infty)$.

(iv) If $A_1 \geq 1$ and $(x_n', y_n') = T^n(x_0, 0)$ then $y_n' = 0$ and $(x_n', y_n') \to E_1$ for $x_0 \in (0, \infty)$.

(v) $T([0, \infty) \times [0, \infty)) \subset [0, 1/B_1] \times [0, 1/C_2]$. 

Equation (74) is equivalent to

$$A_1x_1 - A_1x_2 - C_1x_2y_1 + C_1x_1y_2 = 0,$$

$$A_2y_1^2 + B_2x_2y_1^2 - A_2y_2^2 - B_2x_1y_2^2 = 0.$$
Proof.

(i) From (11) it is easy to see that if \( x_0 = 0 \) then \( x_n = 0 \) for \( n > 0 \). Since

\[
T(0, y_0) - (0, y_0) = \left( 0, -\frac{y_0 (C_2 y_0^2 - y_0 + A_2)}{A_2 + C_2 y_0^2} \right)
\]

we obtain that

\[
(0, y_0) \leq \frac{1}{C_2} T(0, y_0) \leq (0, y_0)
\]

for \( y_0 \in \left( 0, \frac{1 - \sqrt{1 - 4A_2 C_2}}{2C_2} \right) \cup \left( 1 + \sqrt{1 - 4A_2 C_2}, \infty \right).
\]

(ii) The proof of the statements (iii) and (iv) is similar to the proof of (i). Let \( E_1 \) be an equilibrium point. The main result of this paper is the following.

\[\text{Theorem 23. Every solution of system (1) converges to an equilibrium point.}\]

Proof. The map \( T \) associated with the system is injective. Relation (42) implies that determinant of Jacobian (41) is positive for all \( x \in [0, \infty) \times [0, \infty) \). By using Lemma 20 we have that condition \((O+)\) of Theorem 3 is satisfied for the map \( T \) (competitive). Then there exists a compact set \( \mathcal{R} \) such that for all \( x \in \mathcal{R} \) the map \( T \) is eventually componentwise monotone and \( T^n(x) \) is eventually monotone for all \( x \in [0, \infty)^2 \). The statement (v) of Lemma 21 implies that every solution enters in compact set \( \{ 0, 1/B_1 \} \times [0, 1/C_2] \), from which the proof follows.

Remark 24. In view of Theorem 23 the main objective in determining the global dynamics of system (1) is to characterize the basins of attractions of all equilibrium points. As we will see in Theorem 25 the boundaries of these basins of attraction will be the global stable manifolds of the saddle or nonhyperbolic equilibrium points, whose existence is guaranteed by Theorems 7, 8, and 9.

6. Global Behavior

In this section we give results which precisely describe global dynamics of system (1) including precise characterization of basins of attraction of different equilibrium points. The main result of this paper is the following.

\[\text{Theorem 25. The global behavior of system (1) is given by Table 3. See Figure 3 for visual illustration of dynamic scenarios.}\]

Proof. We will prove statements (i)–(x) listed in the second column of Table 3 in the given order. The proof of other statements is similar. Let \( \mathcal{R} = [0, \infty) \times [0, \infty) \).

(i) Suppose \((C_1, C_2) \in \mathcal{R}_{24}\). By Proposition 11, in \( Q_1(0, 0) \) there exist six equilibria \( E_1, E_2, E_3, E_4, E_5, \) and \( E_6 \). By Theorem 19 all equilibria \( E_3 \) and \( E_5 \) are locally asymptotically stable; \( E_1, E_4, \) and \( E_6 \) are saddle points and \( E_2 \) is repellor. In view of (41) the map \( T \) is competitive on \( \mathcal{R} \) and strongly competitive on \( \text{int}(\mathcal{R}) \). It follows from the Perron-Frobenius Theorem and a change of variables [16] that, at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis. Hence, all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the global stable manifold \( \mathcal{W}^u(E_2) \), with endpoint at point \( E_2 \), which is graph of an increasing function. Let \( \mathcal{W} = \{ (x, y) \mid x, y \leq \bar{x}, y \leq \bar{y} \} \) for some \( (\bar{x}, \bar{y}) \in \mathcal{W}^u(E_2) \) and \( \mathcal{W}^u = \{ (x, y) \mid (x, y) \leq (\bar{x}, \bar{y}) \} \) for some \( (\bar{x}, \bar{y}) \in \mathcal{W}^u(E_2) \). By Lemma 21 and uniqueness of the global stable manifold we have \( \mathcal{W}^u(E_1) = \{ (0, y) : 0 \leq y < \bar{y} \} \)
Table 3: The global behavior of system (1).

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameter region</th>
<th>Global behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( \mathcal{R}_2: A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist six equilibrium points ( E_1, E_2, E_3, E_4, E_5, ) and ( E_6, ) where ( E_4 ) and ( E_5 ) are locally asymptotically stable, ( E_1, E_2, ) and ( E_6 ) are saddle points, and ( E_3 ) is repeller. The stable manifold ( \mathcal{W}(E_3) ) of the saddle point ( E_3 ) is an increasing separatrix with endpoint at ( E_6, ) and solutions with initial point above the ( \mathcal{W}(E_3) ) converge to ( E_6, ) while solutions with initial point below the ( \mathcal{W}(E_3) ) converge to ( E_6. ) All orbits that start on ( \mathcal{W}(E_3) ) are attracted to ( E_6. ) The basins of attraction of ( E_1, E_2, ) and ( E_6, ) are respectively, ( \mathcal{B}(E_1) = {(0, y) : y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 } ) and ( \mathcal{B}(E_6) = {(0, y) : y &gt; (1 - 4A_2^2C_1^2)/2A_1^2 }. )</td>
</tr>
<tr>
<td></td>
<td>( C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 &lt; \frac{(1 - A_1)C_1}{(1 - A_1)^2(B_1C_1 + B_1)^2}, )</td>
<td>( 4B_1(A_2B_1 + (1 - A_1)B_2) )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( \mathcal{R}_3: A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist five equilibrium points ( E_1, E_2, E_3, E_4, ) and ( E_5, ) where ( E_4 ) is locally asymptotically stable, ( E_1 ) and ( E_3 ) are saddle points, and ( E_2 ) is repeller, and ( E_5 = E_6 ) is nonhyperbolic. There exists a continuous increasing curve ( \mathcal{C} ) with endpoint at ( E_2, ) which is a subset of the basin of attraction of ( E_5 = E_6. ) All solutions with initial point above ( \mathcal{C} ) converge to ( E_5 = E_6, ) while solutions with initial point below ( \mathcal{C} ) converge to ( E_6. ) The basins of attraction of ( E_1, E_2, ) and ( E_3, ) are respectively, ( \mathcal{B}(E_1) = {(0, y) : y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 } ) and ( \mathcal{B}(E_6) = {(0, y) : y &gt; (1 - 4A_2^2C_1^2)/2A_1^2 }. )</td>
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<tr>
<td></td>
<td>( C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 = \frac{1}{4B_1(A_2B_1 + (1 - A_1)B_2)} )</td>
<td></td>
</tr>
<tr>
<td>(iii)</td>
<td>( \mathcal{R}_3 : A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist five equilibrium points ( E_1, E_2, E_3, E_4, ) and ( E_5, ) where ( E_3 ) and ( E_4 ) are locally asymptotically stable, ( E_1 ) and ( E_2 ) are saddle points, and ( E_5 ) is a repeller. The stable manifold ( \mathcal{W}(E_5) ) of the saddle point ( E_5 ) is an increasing separatrix with endpoint at ( E_2, ) and solutions with initial point above ( \mathcal{W}(E_5) ) converge to ( E_2, ) while solutions with initial point below ( \mathcal{W}(E_5) ) converge to ( E_1. ) All orbits that start on ( \mathcal{W}(E_5) ) are attracted to ( E_1. ) The basin of attraction of ( E_5 ) is ( \mathcal{B}(E_5) = {(0, y) : 0 &lt; y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 }. )</td>
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<td></td>
<td>( C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 &lt; \frac{1}{4B_1(A_2B_1 + (1 - A_1)B_2)} )</td>
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<tr>
<td>(iv)</td>
<td>( \mathcal{R}_2 \cup \mathcal{R}_5 \cup \mathcal{R}_3 : A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist four equilibrium points ( E_1, E_2, E_5, ) and ( E_4, ) where ( E_4 ) is globally asymptotically stable, ( E_1 ) and ( E_2 ) are saddle points, and ( E_5 ) is a repeller. The basins of attraction of ( E_1, E_2, ) and ( E_5, ) are respectively, ( \mathcal{B}(E_1) = {(0, y) : y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 } ) and ( \mathcal{B}(E_5) = {(0, y) : y &gt; (1 - 4A_2^2C_1^2)/2A_1^2 }. )</td>
</tr>
<tr>
<td></td>
<td>( C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 = \frac{1}{4B_1(A_2B_1 + (1 - A_1)B_2)} )</td>
<td></td>
</tr>
<tr>
<td>(v)</td>
<td>( \mathcal{R}_2 : A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist five equilibrium points ( E_1, E_2, E_3, E_4, ) and ( E_5, ) where ( E_5, ) ( E_6 ) are locally asymptotically stable, ( E_1, E_2, ) and ( E_6 ) are saddle points, and ( E_3 ) and ( E_4 ) are saddle points. The stable manifold ( \mathcal{W}(E_3) ) for five saddle points is the stable manifold ( \mathcal{W}(E_4) ) is an increasing separatrix, and solutions with initial point above ( \mathcal{W}(E_3) ) converge to ( E_3, ) while solutions with initial point below ( \mathcal{W}(E_3) ) converge to ( E_4. ) All orbits that start on ( \mathcal{W}(E_3) ) are attracted to ( E_3. ) The basin of attraction of ( E_5 ) is given by ( \mathcal{B}(E_5) = {(0, y) : 0 &lt; y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 } )</td>
</tr>
<tr>
<td></td>
<td>( C_1 &lt; \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 &lt; \frac{1}{4B_1(A_2B_1 + (1 - A_1)B_2)} )</td>
<td></td>
</tr>
<tr>
<td>(vi)</td>
<td>( \mathcal{R}_2 : A_1 &lt; 1, \Delta_1 &gt; 0, )</td>
<td>There exist four equilibrium points ( E_1, E_2, E_3, E_4, ) and ( E_5, ) where ( E_4 ) is locally asymptotically stable, ( E_1 ) is a saddle point, ( E_3 ) ( E_5 = E_6 ) is a nonhyperbolic, and ( E_3 ) is a repeller. There exists a strictly increasing curve ( \mathcal{C} ) with endpoint at ( E_1 = E_5 = E_6, ) which is the subset of the basin of attraction of ( E_1 = E_5 = E_6. ) All solutions with initial point above ( \mathcal{C} ) converge to ( E_1 = E_5 = E_6, ) while solutions with initial point below ( \mathcal{C} ) converge to ( E_3. ) The basin of attraction of ( E_4 ) is given by ( \mathcal{B}(E_4) = {(0, y) : 0 &lt; y &lt; (1 - 4A_2^2C_1^2)/2A_1^2 }. )</td>
</tr>
<tr>
<td></td>
<td>( C_1 = \frac{(1 - A_1)B_1}{(1 - A_1)B_1^2 + 2A_1^2B_1^2}, ) ( C_2 = \frac{1}{4B_1(A_2B_1 + (1 - A_1)B_2)} )</td>
<td></td>
</tr>
<tr>
<td>Case</td>
<td>Parameter region</td>
<td>Global behavior</td>
</tr>
<tr>
<td>------</td>
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</tr>
<tr>
<td>(viii)</td>
<td>$C_2 : A_1 &lt; 0, A_1 &gt; 0, C_1 (1 - A_1 - A_2 C_1) &gt; 0, 1 - A_1 &gt; C_1 &gt; (1 - A_1) B_1 + 2 A_2 B_1$</td>
<td>There exist four equilibrium points $E_1, E_2, E_3,$ and $E_4$, where $E_3$ and $E_4$ are locally asymptotically stable, $E_1$ is a saddle point, and $E_2$ is nonhyperbolic. There exist continuous increasing curves $C_1$ and $C_2$ with endpoint at $E_2 = E_5$, which are the subsets of the basin of attraction of $E_2 = E_5$. Further, all solutions with initial point above $C_1$ converge to $E_3$ and all solutions with initial point below $C_1$ converge to $E_2$, while solutions with initial point below $C_1$ converge to $E_5$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = {(y, x) : y &gt; (1 - \sqrt{1 - 4 A_1 C_1}) / (2 A_2)}$.</td>
</tr>
</tbody>
</table>
| (ix) | $A_1 < 1, A_1 = 0, C_1 
eq 1 - A_1 / 2 A_2$ | There exist three equilibrium points $E_1, E_2, E_3,$ and $E_4$, where $E_1$ is locally asymptotically stable, $E_2$ is a saddle point, and $E_3$ is nonhyperbolic. There exists a continuous increasing curve $C$ with endpoint at $E_2 = E_5$, which is a subset of the basin of attraction of $E_2 = E_5$. All solutions with initial point above $C$ converge to $E_3 = E_6$, while solutions with initial point below $C$ converge to $E_4$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = \{(y, x) : 0 < y < (1 - \sqrt{1 - 4 A_1 C_1}) / (2 A_2)\}$. |
| (x) | $A_1 < 1, A_1 = 0$ | There exist two equilibrium points $E_1$, which is a saddle point, and $E_4$, which is globally asymptotically stable, where $\mathcal{B}(E_1) = (0, \infty) \times (0, \infty)$ and $\mathcal{B}(E_4) = \{(y, x) : y \geq 0\}$. |
| (xi) | $A_1 = 1, A_1 > 0$ | There exist two equilibrium points $E_1 = E_4$ and $E_2 = E_3$ which are nonhyperbolic. There exists a continuous increasing curve $C$ with endpoint at $E_2 = E_5$ which is a subset of the basin of attraction of $E_2 = E_5$. All solutions with initial point above $C$ converge to $E_4 = E_6$, while solutions with initial point below $C$ converge to $E_3 = E_7$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = \{(y, x) : 0 < y < (1 - \sqrt{1 - 4 A_1 C_1}) / (2 A_2)\}$. |
| (xii) | $A_1 = 1, A_1 = 0$ | There exists a continuous increasing curve $C$ with endpoint at $E_2 = E_5$ which is a subset of the basin of attraction of $E_2 = E_5$. All solutions with initial point above $C$ converge to $E_4 = E_6$, while solutions with initial point below $C$ converge to $E_3 = E_7$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = \{(y, x) : 0 < y < (1 - \sqrt{1 - 4 A_1 C_1}) / (2 A_2)\}$. |
| (xiii) | $A_1 = 1, A_1 < 0$ | There exists one equilibrium point $E_1 = E_4$ which is nonhyperbolic and global attractor. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = (0, \infty) \times (0, \infty)$. |
| (xiv) | $A_1 = 1, A_1 < 0$ | There exist three equilibrium points $E_1, E_2$, and $E_3$, where $E_1$ and $E_3$ are locally asymptotically stable and $E_2$ is a saddle point. The stable manifold $\mathcal{W}(E_3)$ of the saddle point $E_2$ is an increasing separatrix with endpoint at $E_2$, and solutions with initial point above $\mathcal{W}(E_2)$ converge to $E_3$, while solutions with initial point below $\mathcal{W}(E_2)$ converge to $E_3$. All orbits that start on $\mathcal{W}(E_2)$ are attracted to $E_2$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = \{(y, x) : 0 < y < (1 - \sqrt{1 - 4 A_1 C_1}) / (2 A_2)\}$. |
| (xv) | $A_1 > 1, A_1 > 0$ | There exist two equilibrium points $E_1$ and $E_2 = E_3$ which are locally asymptotically stable and $E_2 = E_3$ is nonhyperbolic. There exists a continuous increasing curve $C$ with endpoint at $E_2 = E_3$ which is a subset of the basin of attraction of $E_2 = E_3$. All solutions with initial point above $C$ converge to $E_4 = E_6$, while solutions with initial point below $C$ converge to $E_5 = E_7$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = (0, \infty) \times (0, \infty)$. |
| (xvi) | $A_1 > 1, A_1 < 0$ | There exists one equilibrium point $E_1$ which is globally asymptotically stable. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = (0, \infty) \times (0, \infty)$. |
| (xvii) | $A_1 > 1, A_1 = 0$ | There exist two equilibrium points $E_1$ and $E_2 = E_3$ which are locally asymptotically stable and $E_2 = E_3$ is nonhyperbolic. There exists a continuous increasing curve $C$ with endpoint at $E_2 = E_3$ which is a subset of the basin of attraction of $E_2 = E_3$. All solutions with initial point above $C$ converge to $E_4 = E_6$, while solutions with initial point below $C$ converge to $E_5 = E_7$. The basin of attraction of $E_1$ is $\mathcal{B}(E_1) = (0, \infty) \times (0, \infty)$. |
(ii) Suppose $C_3 \subseteq E_3$. By Proposition 11, in $Q_1(0,0)$ there exist five equilibria $E_1, E_2, E_3, E_4$, and $E_5 = E_6$. By Theorem 19 $E_1$ and $E_3$ are the saddle points, $E_2$ is repeller, and $E_4 = E_5$ is nonhyperbolic. Similarly as in the proof of the statement (i), all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the invariant curve $\mathcal{C}$ with one endpoint at $E_2$ and which is passing through $E_5 = E_6$, and it is graph of an increasing function. Let $\mathcal{W}^- = \{(x, y) : (x, y) \leq \infty (x_0, y_0) \text{ for some } (x_0, y_0) \in \mathcal{C}\}$ and $\mathcal{W}^+ = \{(x, y) : (x_0, y_0) \geq \infty (x, y) \text{ for some } (x_0, y_0) \in \mathcal{C}\}$. By Lemma 21 and uniqueness of the global stable manifold we have $\mathcal{W}^+(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_5C_5})/(2A_5)\}$ and $\mathcal{W}^-(E_5) = \{(0, y) : y > (1 - \sqrt{1 - 4A_5C_5})/(2A_5)\}$. Take $(x_0, y_0) \in \mathcal{W}^- \cap \text{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0 > 0$ such that $T^n(x_0, y_0) \in \text{int}(Q_1(E_5) \cap Q_2(E_5)) = [(E_3, E_2)] \subseteq \mathcal{R}(E_4)$. This completes the proof of statement (i).

(ii) Suppose $(C_1, C_2) \in \mathcal{C}_{31}$. By Proposition 11, in $Q_1(0,0)$ there exist five equilibria $E_1, E_2, E_3, E_4$, and $E_5 = E_6$. By Theorem 19 $E_1$ and $E_3$ are the saddle points, $E_2$ is repeller, and $E_4 = E_5$ is nonhyperbolic. Similarly as in the proof of the statement (i), all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the invariant curve $\mathcal{C}$ with one endpoint at $E_2$ and which is passing through $E_5 = E_6$, and it is graph of an increasing function. Let $\mathcal{W}^- = \{(x, y) : (x, y) \leq \infty (x_0, y_0) \text{ for some } (x_0, y_0) \in \mathcal{C}\}$ and $\mathcal{W}^+ = \{(x, y) : (x_0, y_0) \geq \infty (x, y) \text{ for some } (x_0, y_0) \in \mathcal{C}\}$. By Lemma 21 and uniqueness of the global stable manifold we have $\mathcal{W}^+(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_5C_5})/(2A_5)\}$ and $\mathcal{W}^-(E_5) = \{(0, y) : y > (1 - \sqrt{1 - 4A_5C_5})/(2A_5)\}$. Take $(x_0, y_0) \in \mathcal{W}^- \cap \text{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0 > 0$ such that $T^n(x_0, y_0) \in \text{int}(Q_1(E_5) \cap Q_2(E_5)) = [(E_3, E_2)] \subseteq \mathcal{R}(E_4)$. This completes the proof of statement (i).

Figure 3: Parameter regions in terms of parameters $C_1$ and $C_2$ and corresponding dynamic scenarios for system (1) if $A_1 < 1$. 

\[ y < \frac{1 - \sqrt{1 - 4A_5C_5}}{(2A_5)} \] and $\mathcal{W}^+(E_5) = \{(0, y) : y > \frac{1 - \sqrt{1 - 4A_5C_5}}{(2A_5)}\}$. Take $(x_0, y_0) \in \mathcal{W}^- \cap \text{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0 > 0$ such that $T^n(x_0, y_0) \in \text{int}(Q_1(E_5) \cap Q_2(E_5)) = [(E_3, E_2)] \subseteq \mathcal{R}(E_4)$. This completes the proof of statement (i).
T is competitive, in view of Corollary 4 int(Q_4(E_2) ∩ Q_3(E_2)) = [(E_3, E_1)] ⊆ B(E_4). This completes the proof of statement (ii).

(iii) The proof is similar to the proof of case (i) and we skip it.

(iv) Suppose (C_1, C_2) ∈ C_21 ∪ C_22 ∪ C_32. By Proposition 11, in Q_1(0,0) there exist four equilibria E_1, E_2, E_3, and E_4. By Theorem 19 E_1 and E_2 are saddle points; E_3 is repeller; E_4 is locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have B(E_3) = W^s(E_3) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\} and B(E_4) = W^s(E_4) = \{(0,y) : y > (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. Since, by Theorem 23, every solution of system (1) converges to an equilibrium point, we have that B(E_3) = (0,∞) × [0,∞).

(v) Suppose (C_1, C_2) ∈ C_23. By Proposition 11, in Q_1(0,0) there exist four equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 E_1 and E_2 are the saddle points; E_3 and E_4 are locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have B(E_3) = W^s(E_3) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\} and B(E_4) = W^s(E_4) = \{(0,y) : y > (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. Similarly as in the proof of case (i), all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the global stable manifold W^s(E_2), which is a graph of an increasing function. The rest of the proof follows from the facts that W^- = \{(x,y) : (x,y) \leq_{se} (x_0,y_0)\} for some (x_0, y_0) ∈ W^s(E_2) and W^+ = \{(x,y) : (x,y) \leq_{se} (x_1,y_1)\} for some (x_1, y_1) ∈ W^s(E_2) are invariant sets, E_4 ∈ W^-, E_3 ∈ W^+, uniqueness of the global stable manifold W^s(E_2) and Theorem 23.

(vi) The proof is similar to the proof of case (i) and we skip it.

(vii) The proof is the same as the proof of case (viii) and we skip it.

(viii) Suppose (C_1, C_2) ∈ C_22. By Proposition 11, in Q_1(0,0) there exist four equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 we have that E_4 is locally asymptotically stable; E_1 is a saddle point; E_3 is nonhyperbolic; and E_2 is repeller. By Lemma 21 and uniqueness of the global stable manifold we have B(E_3) = W^s(E_3) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. In view of Lemma 18, for E_3 = E_2 we have that the eigenspace E_1^2 associated with λ_2 is a coordinate axis, so we can not use Theorem 6. By Lemma 21 we obtain \{(0,y) : y > (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\} ⊆ B(E_3 = E_2). Similarly as in the proof of Theorem 8 (see [1] for more details) one can prove that for every x ∈ int(Q_4(E_3)) there exists n_0 ∈ N such that T^n(x) ∈ int(Q_4(E_3)) for n ≥ n_0. By Lemmas 21 and 22 for (x_0, y_0) ∈ int(Q_4(E_3)), there exists t_0 and t''_0 such that E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} (x_0,y_0) \leq_{se} (t''_0,0). Since E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} T^{n-1}(M(t'_0)) \leq E_4 we have that T^n(M(t'_0)) → E_4. Since T^n(0,0) → E_4 we obtain T^n(x_0, y_0) → E_4. This implies int(Q_4(E_3)) ⊆ B(E_4), which completes the proof.

(ix) Suppose (C_1, C_2) ∈ C_23. By Proposition 11, there exist four equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 we have that E_3 and E_4 are locally asymptotically stable; E_1 is a saddle point; E_2 = E_3 is nonhyperbolic. By Lemma 21 and uniqueness of the global stable manifold we have B(E_3) = W^s(E_3) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. In view of Lemma 18, for E_3, we have that λ_1 = 1 and λ_2 > 1, so we can not use Theorem 6. By Lemmas 21 and 22 for (x_0, y_0) ∈ int(Q_4(E_3)), there exists t'_0 and t''_0 such that E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} (x_0,y_0) \leq_{se} (t''_0,0). Since E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} T^{n-1}(M(t'_0)) \leq E_4 we have that T^n(M(t'_0)) → E_4. Since T^n(0,0) → E_4 we obtain T^n(x_0, y_0) → E_4. This implies int(Q_4(E_3)) ⊆ B(E_4), which completes the proof.

(x) Suppose (C_1, C_2) ∈ C_24. By Proposition 11, in Q_1(0,0) there exist three equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 E_3 = E_2 is a saddle point, and E_4 is locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have B(E_4) = W^s(E_4) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. In view of Lemma 18 we have that λ_1 = 1 and λ_2 < 1 if C_1 > (1 - \lambda_1)/(2A_2) and λ_2 > 1 if C_1 < (1 - \lambda_1)/(2A_2), so we can use Theorem 6 if C_1 > (1 - \lambda_1)/(2A_2). In this case there exists strictly increasing curve C with endpoint at E_2 = E_3. The rest of the proof follows from Theorems 7, 8, and 9 and Lemma 21. Now, we assume that C_1 < (1 - \lambda_1)/(2A_2). By Lemmas 21 and 22 for (x_0, y_0) ∈ int(Q_4(E_3)), there exists t'_0 and t''_0 such that E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} (x_0,y_0) \leq_{se} (t''_0,0). Since E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} T^{n-1}(M(t'_0)) \leq E_4 we have that T^n(M(t'_0)) → E_4. Since T^n(0,0) → E_4 we obtain T^n(x_0, y_0) → E_4. This implies int(Q_4(E_3)) ⊆ B(E_4). Let C denote the boundary of B(E_4) considered as a subset of Q_4(E_3). It is easy to see by using Lemmas 21 and 22 that E_2 ∈ C_1 ∪ C_2. Since T(int(B)) ⊆ int(B), following the proof of Claims 1 and 2 [20], one can see that T(C) ⊆ E_2. By Proposition 11, in Q_1(0,0) there exist three equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 E_3 = E_2 is a saddle point, and E_4 is locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have B(E_4) = W^s(E_4) = \{(0,y) : 0 ≤ y < (1 - \sqrt{1 - 4A_2C_2)/(2A_2)}\}. In view of Lemma 18 we have that λ_1 = 1 and λ_2 > 1, so we can not use Theorem 6. By Lemmas 21 and 22 for (x_0, y_0) ∈ int(Q_4(E_3)), there exists t'_0 and t''_0 such that E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} (x_0,y_0) \leq_{se} (t''_0,0). Since E_3 \leq_{se} T^n(M(t'_0)) \leq_{se} T^{n-1}(M(t'_0)) \leq E_4 we have that T^n(M(t'_0)) → E_4. Since T^n(0,0) → E_4 we obtain T^n(x_0, y_0) → E_4. This implies int(Q_4(E_3)) ⊆ B(E_4). Let C denote the boundary of B(E_4) considered as a subset of Q_4(E_3). It is easy to see by using Lemmas 21 and 22 that E_2 ∈ C. Since T(int(B)) ⊆ int(B), following the proof of Claims 1 and 2 [20], one can see that T(C) ⊆ E_2. Further, C is graph of strictly
increasing function. By Theorem 23 if \((x_0, y_0) \in \mathcal{C}\) then \(T^n(x_0, y_0) \to E_3 \in \mathcal{C}\). If \((x_0, y_0)\) is point above the curve \(\mathcal{C}\) then there exists \((x'_0, y'_0)\) and \(y'_0 > y_0\) such that \((0, y'_0) \leq_{\infty} (x'_0, y'_0) \leq_{\infty} (x_0, y'_0)\). Since \(T^n(0, y'_0) \leq_{\infty} T^n(x'_0, y'_0) \leq_{\infty} T^n(x_0, y'_0)\), \(T^n(0, y'_0) \to E_3\) and \(T^n(x'_0, y'_0) \to E_3\) as \(n \to \infty\) we have \(T^n(x_0, y_0) \to E_3\) as \(n \to \infty\).

\(\square\)

Based on a series of numerical simulations we propose the following conjectures.

Conjecture 26. Suppose that all assumptions of the statement (ix) of Theorem 25 are satisfied; then the following holds:

\[ \mathcal{B}(E_2) = \mathcal{C}_1 = \mathcal{C}_2. \]  

Conjecture 27. Assume that \(C_1, C_2 \in \mathcal{P}_1\). There exist three equilibrium points \(E_1, E_2, E_3\), and \(E_4\) where \(E_2\) is locally asymptotically stable; \(E_3\) is a saddle point; and \(E_1 = E_2 = E_4\) is nonhyperbolic. The basins of attraction of \(E_1\) and \(E_3\) and \(E_2 = E_4\) are given by \(\mathcal{B}(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}\) and \(\mathcal{B}(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}\) and the basin of attraction of \(E_4\) is \(\mathcal{B}(E_4) = \{(x, y) : x > 0, \ y \geq 0\}\).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


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