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COUPLED FIELD DYNAMIC MODEL OF FATIGUE EVOLUTION

BY

SON HAI NGUYEN

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

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IN

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DOCTOR OF PHILOSOPHY DISSERTATION
OF
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ABSTRACT

Predicting fatigue life is still an unresolved challenge in engineering practice. One of the problems in developing an effective damage predictive model is that damage variables are hard to measure. In this dissertation, a reliable and practical methodology for estimating damage variables from easily measurable variables is presented and validated by simulations and experiments. The tracking method is based on the application of the smooth orthogonal decomposition and new characteristics of a dynamical system called *Characteristic Lengths* and *Distances*. These features which are derived from the Birkhoff Ergodic Theorem, are fast to calculate and do not require large data or computational resources.

The fatigue life prediction problem is further complicated by the nonlinear coupling between applied loads and fatigue life. Conventional fatigue damage models consider only basic load statistical quantities (e.g., mean, variance) which do not capture nonlinear behaviors. A new experimental system—coupling structural and crack growth dynamics—is used to show that fatigue damage accumulation is different under chaotic (i.e., deterministic) and stochastic (i.e., random) loading, even when both excitations possess the same spectral and statistical signatures. Furthermore, the conventional rain-flow counting method considerably overestimates damage in case of chaotic forcing. Important nonlinear loading characteristics, which can explain the observed discrepancies, are identified and suggested to be included as loading parameters in new macroscopic fatigue models.

An analytical approach to model fatigue damage accumulation is also considered. A coupled field dynamic model is derived using Hamilton's principle for a simply supported uniform Euler-Bernoulli beam containing a single-edge crack. In this model, the fatigue crack length is treated as a generalized coordinate of a mechanical system. The fatigue accumulation is a result of the interaction between

the beam oscillations and the crack propagation dynamics. Nonlinear characteristics of the beam motion are introduced as loading parameters to the fatigue model to match experimentally observed dynamics. The method of averaging is utilized as an analytical and numerical tool to compare the accumulation of fatigue damage in the system predicted by our model with Paris' law and the experimental data.

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DEDICATION

This dissertation is dedicated in the loving memory of my father Nguyen Van Hao.

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CHAPTER 1

Introduction

1.1 Motivation and Objective

Fatigue failure in material occurs when a material is subjected to the action of repeated fluctuating loads. It is estimated that 90% of all mechanical failures are due to fatigue damage [1]. Hence, fatigue failure has been a subject of various experimental and theoretical studies since the middle of 20th century. The primary goal of those studies was to predict failures and to improve fatigue life of machines and structures. It is impossible to predict the remaining useful life when the current damage state and its model of evolution are unknown. Therefore, there is a need for fatigue damage identification and a fatigue accumulation model.

Despite significant progress in material fatigue, the modeling of fatigue crack growth in structures is still a major challenge of engineering practice. Various solutions to this problem have been proposed and are summarized in Refs. [2, 3]. These approaches vary from continuum damage theories to linear damage rules. However, the industrial state of the art relies on the rain-flow counting method and the Palmgren-Miner rule [4, 5, 6], which are not physics-based. This fact suggests that the essential physics of fatigue damage is not completely understood or taken into account by currently available models. In particular, traditional fatigue studies ignore the interactions between the damage accumulation process and structural dynamics by neglecting damping and inertial forces. On the other hand, research of dynamic behavior of a damaged structure [7] usually assumes damage is a constant factor in their models. However, the fatigue process is not only driven by environmental factors and structural dynamics, but it also affects the structural dynamics by altering structural parameters. In addition, modeling of fatigue damage is further complicated by the nonlinear coupling between load

factors and fatigue life. For example, altering the application sequence of the same large and small amplitude loads results in a different time to failure [2]. The aim of this work is to investigate the coupling between the damage evolution and structural dynamics as well as to identify essential nonlinear load factors that contribute to fatigue life.

This dissertation consists of six chapters. Background and literature review of fatigue damage models and damage identification are given in Chapter 1. Chapter 2 presents basic tools used to characterize nonlinear dynamical systems. A new method for the selection of the time delay is also introduced. In Chapter 3, a reliable and practical methodology for estimating observable damage variables from easily measurable variables is developed and validated by simulations and experiments. Chapter 4 describes fatigue experiments with the use of a novel fatigue testing rig. Important nonlinear loading characteristics are identified to explain the observed discrepancies. Then, an analytical approach to model fatigue damage accumulation in one dimensional cracked beams is presented in Chapter 5. A coupled field dynamic model is derived for a simply supported uniform Euler-Bernoulli beam containing a single-edge crack. In the fatigue model, new nonlinear characteristics of beam motion are introduced as load parameters. Chapter 6 summarizes the contributions of the dissertation and suggestions for future work.

1.2 Literature Review of Fatigue Damage Research

1.2.1 Background

Fatigue accumulation occurs in several stages that have different physical mechanisms. Many investigators consider the fatigue life under cyclic loading to consist of three stages of development[1, 8]: crack initiation, crack growth period and fracture as shown in Fig. 1. Crack initiation involves cyclic plastic deformation and crack nucleation followed by micro crack propagation. The second stage

Figure 1. Different stages of the fatigue life

progresses from microcrack to macrocrack accumulation. When the strength of the remaining material is smaller than the applied loads, a fast fracture occurs in stage III.

Fatigue failure is a complex physical process which depends on a great number of factors which include geometry, material properties of structures and environmental conditions. It is impossible for a physical model to account for all fatigue influencing factors. Although many fatigue damage models have been proposed, most of them are incomplete to various degrees. In general, there are two basic approaches which can be used in fatigue life predictions, cumulative fatigue damage models and fatigue crack propagation models. In the following section, a brief overview is given, the more comprehensive recent survey can be found in Refs. [2, 3].

1.2.2 Cumulative fatigue damage models

The Palmgren-Miner rule [4, 5] is the most widely used method for estimating fatigue life. It is a stress-base approach which relies on the aid of stress-life (S - N) curves [1]. An S - N curve is a plot of stress (S) against the number of cycles to failure (N), and is obtained by testing a number of specimens at various stress levels under sinusoidal loading conditions. For some materials, an S - N curve approaches a horizontal line for large number of cycles as stress amplitude is reduced. This limit is called fatigue limit or fatigue strength.

The Palmgren-Miner rule describes damage as a cumulative sum $D = \sum_i n_i/N_i$ where n_i and N_i are the number of applied cycles and the total cycles to failure under the i -th constant amplitude loading (N_i is obtained from an S - N curve of the testing material). This simple rule states that the damage accumulated at each i -th stress level is independent of load interaction effects and can be superimposed to calculate the total damage. The failure will occur when the sum $D = 1$. Numerous improvements to this rule have been proposed and described in [2]. For example, Manson *et al.* [9, 10] introduced the double linear damage rule to account for the load interaction effects. However, it is currently understood that the fatigue life is nonlinearly coupled with load factors [11, 12]. Thus, all methods based on a superposition principle are inadequate in practice or of limited use.

1.2.3 Fatigue Crack Propagation Models Constant Amplitude Loading Models

The pioneering work of Paris [13, 14] relates the amplitude of the applied stress intensity factor (ΔK) and the crack growth rate (da/dN) in an empirical relationship

$$\frac{da}{dN} = C(\Delta K)^m, \quad (1)$$

Figure 2. Crack growth rate curve

where a is crack length, C and m are material constants. $K = Y\sigma\sqrt{\pi a}$ is a stress intensity factor which was originally used to describe the stress state near the tip of a crack [15]. In the formula, Y is a geometry factor. The relationship between the crack length and number of cycles can be obtained by integrating Eq. (1). Then the number of cycles to failure can be predicted if the initial and final crack sizes are provided.

Note that Eq. (1) is an empirical law established from fatigue experiments at various stress levels. A typical crack growth rate curve for most engineering alloys exhibits three distinct regions [8], as shown in Fig. 2. In the region I, the crack growth rate is of the order 10^{-6} mm/cycle, and asymptotically equal to zero at a threshold value ΔK_{th} which corresponds to the fatigue limit on an S - N curve. The fatigue crack growth in this region is extremely sensitive to environmental effects.

The microstructural properties of the material and the load ratio R^1 have large influences on the crack growth. Region II exhibits a linear relationship on a log-log scale between da/dN and ΔK . The growth rates in the range of 10^{-6} mm/cycle to 10^{-3} mm/cycle. In this region, the plastic zone size is much smaller than the material microstructure so that the microstructural effects do not play such an important role in crack growth. The crack growth rate in region III increases rapidly, causing fracture. The plot of $\log da/dN$ versus $\log \Delta K$ asymptotically approaches the fracture toughness K_c of the material. The crack propagation is very complex in this region due to the influence of the nonlinear properties of the material. However, it is usually neglected because the high crack growth rate results in short fatigue life.

Paris' law is validated under conditions of constant amplitude, zero mean loading (the load ratio $R = -1$), and is only applicable in region II (long cracks). However, when these rigorous requirements are not met, Paris' law loses its predictive ability. A multitude of modifications of Paris' law have been developed [3] to suit differences from these conditions such as R -effects, threshold limits, and small cracks. For example, the log-log plots of da/dN versus ΔK at various stress ratios are parallel to each other. Walker [16] improved the Paris model by including the effect of the stress ratio R . The Forman model [17] and the Collipriest model [18] are proposed to describe fatigue crack growth in region I and III. These models vary only in complexity and in the number of parameters required to fit experimental data. However, the fitting parameters in these models have obscure physical meaning, they do not depend on the material tensile properties. In practical situations, when these parameters cannot be found, these equations are completely useless. Another approach based on Continuum Damage Mechanics [19, 20] attempts to

¹ $R = \sigma_{\min}/\sigma_{\max}$, where $\sigma_{\min}$ is the minimum peak stress and $\sigma_{\max}$ is the maximum peak stress

derive physics-based fatigue crack growth equation. Current continuum damage models [21] involve sophisticated analyses of microstructure, plasticity, crack initiation and growth that are useful in predicting the crack growth rate. However, these approaches have not yet been utilized in fatigue life prediction because of practical difficulties such as insufficient experimental validation and unknown initial conditions.

Variable Amplitude Loading Models

The fatigue life prediction in variable amplitude loading conditions has received considerable attention, since many structures in service are not subject to constant amplitude cyclic loads. In contrast to a constant amplitude load, where no load history needs to be considered in a variable amplitude loading scenario, the load interactions have significant effects on the fatigue crack growth rate. Current models for the lifetime prediction under variable amplitude loads are based on different interpretation load interaction effects. These models can be divided into two approaches, global analysis and cycle-by-cycle analysis [22]. The global analysis tries to characterize entire loading cycles to predict the fatigue crack growth. The cycle-by-cycle analysis estimates the crack growth increment for each load cycle. Then the overall crack growth life is determined by an accumulation of the separate increments.

The Barsom model [23] is one of notable global analysis models which describes the effective stress intensity factor range ΔK_{rms} in terms of the root mean square value of stress intensity factor range,

$$\Delta K_{rms} = \sqrt{\frac{1}{n} \sum_{i=1}^n \Delta K_i^2}, \quad (2)$$

where ΔK_i is the stress intensity factor in the i th cycle for a load consisting of n

cycles. Similar to Paris' law, Barsom proposed the relationship

$$\frac{da}{dN} = C_B (\Delta K_{rms})^{m_B}, \quad (3)$$

where C_B and m_B are data fitting parameters. In this model, the load interaction effects generated by the load sequence are ignored. Therefore, the results are only acceptable if the load is uniformly stochastically distributed. Numerous modifications of Barsom model have been proposed in Ref. [3]. However, none of them improve the accuracy of the basic model.

The simplest form of the cycle-by-cycle analysis predicts the crack length a for the whole load spectrum by the sum:

$$a = a_0 + \sum_{i=1}^n f(\Delta K, R, \dots) = a_0 + \sum_{i=1}^n \Delta a_i, \quad (4)$$

where crack growth increment Δa_i for each cycle is calculated by integration. To account for the interaction effects by the plastic zone in front of the crack tip, Wheeler [24] introduced a retardation parameter C_P to Eq. (4)

$$a = a_0 + \sum_{i=1}^n C_P f(\Delta K, R, \dots). \quad (5)$$

C_P varies from 0 to 1, and depends on the size of the current plastic zone. The more complex and advanced models are crack closure models [25] and strip yield models [26, 3].

All of the methods consider fatigue dynamics in term of crack length change over each tensile fatigue cycle, da/dN . However, N is not mathematically defined for arbitrary loading conditions. In addition, the variety of current fatigue models confirms that there is no universal methodology for predicting fatigue life under arbitrary loads. In practical situations, the selection of the appropriate fatigue model is relied upon the analyst's experience.

1.3 Fatigue Damage Identification

Damage identification is one of the critical tasks in the field of structural health monitoring. In engineering practice, damage can be viewed as any physical process that gradually deteriorates the system's performance and leads to the failure of the system. The most comprehensive review can be found in Refs. [27, 28].

Most of the approaches are based on some feature metric that is sensitive to the changes in the systems dynamic behavior. For example, data-based approaches focus on the changes of statistics in time or frequency domain [29, 30, 31, 32]. For nonlinear systems, conventional methods rely on long-time invariant quantities such as Lyapunov exponents or fractal dimensions. The ability of these invariant measures to classify different faults is presented in Refs [33, 34, 35]. However, estimating Lyapunov exponents requires large experimental data and time consuming computation [36]. To deal with these factors, Clement *et al.* proposed the Jacobian Feature Vector [37] which is based on Lyapunov exponents algorithm but has faster computation. Other interesting feature extracted from measured data has been employed to identify damage by Todd *et al.* [38], where the local attractor variance ratio is described as a change in geometric properties of an attractor. Hively *et al.* [39] divided a reconstructed phase space into several small cells and introduced phase-space dissimilarity measures for damage detection.

All these quantities describe the long-time behavior of a dynamical system. Although they are able to detect sudden changes in a system caused by damage, they are ill suited for continuous damage tracking. In Refs [40, 41], a concept of phase space warping (PSW) was proposed to characterize changes in slow-time variability in the fast-time flow. A one-to-one relationship between PSW-based tracking vectors and actual damage states causing these changes has been demonstrated [42]. However, this procedure requires considerable computational time for

estimating metrics. Obviously, there is a need for a damage identification algorithm which is robust to the size of data but also fast to calculate and easy to implement.

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CHAPTER 2

Characteristics of Nonlinear Dynamical Systems

Nonlinear phenomena are very common in material fatigue. Fong *et al.* [1] showed that the propagation of the fatigue crack can change the vibration of a beam from periodic to chaotic, even when the excitation is harmonic. Fatigue processes are not only nonlinearly coupled with structural dynamics but also with load factors [2, 3]. Therefore, nonlinear analysis should be an appropriate approach to investigate a material fatigue process. In this chapter, some basic tools for analyzing nonlinear systems such as Lyapunov exponents, fractal dimensions are presented. Moreover, a new class of features based on Birkhof Ergodic Theorem called *Characteristic Length* and *Characteristic Distance* [4] are proposed. The existence, uniqueness and invariance properties are demonstrated using this theorem. Using simple simulations, these measures can be seen as nonlinear statistical quantities characterizing a deterministic dissipative dynamical system. An application of the characteristic length spectrum for choosing a time-delay in phase space reconstruction is described and the results for some dynamical systems are compared to those from the most widely used method (the average mutual information method).

2.1 Phase space reconstruction

Nonlinear time series analysis methods usually rely on the phase space of a system [5, 6]. The phase space describes or visualizes the evolution of the dynamical variables of a system. Each point in the phase space defines a state of a system. For a given system of equations, the phase space is well established. In an experimental context, the analytical equations of a system is quite hard to obtain, we are only provided a scalar time series $\{x_i\}_{i=1}^N$ represented as a measured quantity of the

dynamical variables. However, a topologically equivalent structure of the phase space can be constructed from the scalar time series by the method of time delay coordinate embedding [7, 8]. In this method, a point in the reconstructed phase space is described by a vector

$$\mathbf{y}(i) = [x_i, x_{i-\tau}, \dots, x_{i-(d-1)\tau}]^T, \quad (6)$$

where τ is a time-delay, d is an embedding dimension, T is a matrix transpose. Here, x is uniformly sampled using the sampling time t_s .

2.1.1 Embedding dimension

An embedding dimension d is usually determined by the *false nearest neighbors* (FNN) method [9, 10]. If the vector $\mathbf{y}(i)$ is the nearest neighbor of a reconstructed vector $\mathbf{y}(j)$ in d dimension but they are far away in $d + 1$ embedding dimension, then $\mathbf{y}(j)$ is considered to have a false nearest neighbor in d dimension. By counting the number of false nearest neighbors while increasing the embedding dimension, one can determine the appropriate embedding dimension if in this dimension there is a small number of false nearest neighbors.

2.1.2 Time delay

In phase space reconstruction theory, the value of time delay τ is not provided since it considers ideal, noise-free and infinitely long time series. However, in practice, the shape of reconstructed phase space depends strongly on the value of τ [11]. When choosing a time delay, the idea is to find τ which spreads the phase space as much as possible without destroying the structure. The maximally separated coordinates will minimize the sharp changes of the trajectory, hence, the evolution of the dynamics will be maximally smooth. An appropriate value of τ can be estimated by the first zero-crossing of the auto correlation function [5, 6]. However, this approach does not account for the nonlinearity in the data [5] and is

not appropriate for analyzing a nonlinear time series [12]. The most widely used approach is to choose τ from the first minimum of the average mutual information [12] which calculates the mutual dependence between two vectors $x(t)$ and $x(t + \tau)$ over a range of time delays.

$$I(\tau) = \sum_{i,j} p_{i,j}(\tau) \ln p_{i,j}(\tau) - 2 \sum_i p_i \ln p_i, \quad (7)$$

where p_i is the probability of a value of the signal $x(t)$ inside the i^{th} bin of a histogram of $x(t)$. $p_{i,j}$ is the joint probability of $x(k)$ in i^{th} bin and $x(k + \tau)$ in j^{th} bin.

2.2 Nonlinear Invariant Measures

The two important characteristics of nonlinear dynamical systems are Lyapunov exponents and fractal dimensions. The Lyapunov exponents are the average exponential rates of separation of trajectories that are infinitesimally close initially [13, 14]. Fractal dimensions [15] relate to the complexity of the system's attractor, and can provide a lower bound on the active degrees-of-freedom in a dynamical system. Both are invariant measures of the system motion and thus are independent of the initial conditions of the orbit. In addition, they are invariant under smooth transformations of the state space. This implies we can evaluate them reliably from experimental data.

2.2.1 Lyapunov Exponents

In an d -dimensional system, there are d Lyapunov exponents, which describe the divergence (or convergence) rate of nearby trajectories along d principal axes in the system's state space. The maximum Lyapunov exponent λ receives a lot of attention since it is the indicator of chaos in a system.

In the phase space of a dynamical system, two trajectories with initial sepa-

Figure 3. Divergence of close trajectories after n iterations.

ration δ_0 diverge after n time steps as

$$\delta_n = \delta_0 e^{\lambda t_n}, \quad (8)$$

where δ_n is the Euclidean distance between two neighboring trajectories after t_n time, $t_n = n\Delta t$ (see Fig. 3). Taking the log on both sides of Eq. (8) yields

$$\ln \delta_n = \lambda t_n + \ln(\delta_0) \quad (9)$$

In what follows we describe an algorithm introduced by Kantz [5] to estimate $\ln \delta_n$.

Randomly choose a point $\mathbf{x}_k$ on the attractor $\mathbf{x}$ and select N_ϵ nearest neighbors. Calculate the logarithm of the average distance of all neighbors to the reference point after n time steps. Repeating this process N times to get

$$\ln \delta_n = \frac{1}{N} \sum_{k=1}^N \ln \left[\frac{1}{N_\epsilon} \sum_{\mathbf{x}_m \in \mathbf{B}_\epsilon(\mathbf{x}_k)} \|\mathbf{x}_{k+n} - \mathbf{x}_{m+n}\| \right], \quad (10)$$

where n is number of steps ahead, $\mathbf{B}_\epsilon(\mathbf{x}_k)$ is a set of N_ϵ nearest points of point $\mathbf{x}_k$. λ can then be estimated from the slope of the plot of $\ln \delta_n$ against t_n .

2.2.2 Correlation Dimension

Among fractal dimensions, the correlation dimension is of greatest interest in nonlinear analysis [5, 6]. It is due to the fact that this measure can be calculated

more easily and reliably than other dimensions. The correlation dimension relates to the probability of two points on an attractor being in the same area [16].

Let a set of points $\{\mathbf{x}_i\}_{i=1}^N$ uniformly sample an embedded attractor, the correlation dimension D_2 is estimated by calculating the correlation sum $C(\epsilon)$ [16, 17]. $C(\epsilon)$ is computed by counting number of pairs $\{i, j\}$ with $\|\mathbf{x}_i - \mathbf{x}_j\| < \epsilon$:

$$C(\epsilon) = \frac{2}{(N-s)(N-s-1)} \sum_{i=1}^N \sum_{j=i+s+1}^N \Theta(\epsilon - \|\mathbf{x}_i - \mathbf{x}_j\|), \quad (11)$$

where N is the number of embedded points in d -dimensional space, Θ is the Heaviside function, $\Theta(x) = 0$ if $x \leq 0$ and $\Theta(x) = 1$ for $x > 0$. s is the time interval needed to remove any time correlations in the pairs of points [18, 19]. The reason for this is we only consider points which are close in the geometry. We don't want to include those points which are close in time. A good choice for s is $s = d\tau$, where τ is time delay [18]. For small ϵ

$$C(\epsilon) \approx \epsilon^{D_2}. \quad (12)$$

Therefore, D_2 can be estimated by a least-squares fit of the slope of $\log C(\epsilon)$ versus $\log \epsilon$ around small value ϵ .

2.3 New Class of Characteristics for Nonlinear Dynamics

Lyapunov exponents and fractal dimensions are rooted in Ergodic theory [20] which is originally developed to identify and classify invariant measures under the time evolution. In the following, one of the important theorems in Ergodic theory, Birkhoff Ergodic Theorem, is stated. Then a new class of characteristics of nonlinear dynamics based on this theorem is proposed.

2.3.1 Birkhoff Ergodic Theorem

Consider the time evolution of a dynamical system, measured in discrete time units, given by a transformation $\mathcal{T} : X \rightarrow X$, so that if $\mathbf{x} \in X \subset \mathbb{R}^n$ is the

current state then $\mathcal{T}(\mathbf{x})$ is the state of the system after one time unit. If the system is at the steady state, by the invariant property $\mathcal{T}(X) \subseteq X$ then $\mathcal{T}$ is a measure preserving transformation. For example, let X be the phase space of a mechanical system. Every point of X represents the values of position and momentum variables. A measurement of the system (e.g., velocity, acceleration) can be defined by a function $f : X \rightarrow \mathbb{R}$. To measure a quantity of the system, one usually takes n successive measurements $f(\mathbf{x}), f(\mathcal{T}(\mathbf{x})), \dots, f(\mathcal{T}^n(\mathbf{x}))$ and looks at their average. The question is if the average exists and is invariant when $n \rightarrow \infty$. Ergodic theory is originally developed to answer this question.

Birkhoff Ergodic Theorem for measure preserving transformation [20]: Let (X, B, μ) be a finite measure space. Let $\mathcal{T} : X \rightarrow X$ be a measure preserving transformation. For any $f \in L^1(X, B, \mu)$ the following limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\mathcal{T}^k(\mathbf{x})) \quad (13)$$

exists and is invariant.

Using this theorem, we can define useful invariant metrics for a dynamical system's attractor which are fast, easy to calculate, robust to noise and do not require large data or computational resources. In what follows, by choosing the Euclidean distance as the function f , we present two of such metrics .

2.3.2 Characteristic Lengths

Let $f_k : X \rightarrow \mathbb{R}$ be a measurable function, which is a Euclidean distance between the current state of the system $\mathbf{x}$ and the state after k time units $\mathcal{T}^k(\mathbf{x})$.

$$f_k(\mathbf{x}) = \|\mathbf{x} - \mathcal{T}^k(\mathbf{x})\|_2 \quad (14)$$

Now we can define a characteristic length $L(k)$ as the limit

$$L(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_k(\mathcal{T}^i(\mathbf{x})) . \quad (15)$$

Figure 4. Illustration of Characteristic Distances

As we can see, one can obtain a full spectrum of these characteristic lengths for different k . However, the information in this metric becomes redundant once the characteristic length reaches the attractor size (similar to the largest Lyapunov exponent, or a correlation integral).

2.3.3 Characteristic Distances

Let $f_y : X \rightarrow \mathbb{R}$ be a measurable function, which is the Euclidean distance between a point $\mathbf{x} \in X$ and a fixed point $\mathbf{y} \in R^n$ (see Fig. 4)

$$f_y(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|_2. \quad (16)$$

Then the characteristic distance $D(\mathbf{y})$ can be defined as a limit

$$D(\mathbf{y}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_y(\mathcal{T}^i(\mathbf{x})). \quad (17)$$

By the Birkhoff Ergodic Theorem, the characteristic length and the characteristic distance are invariant measures on an attractor. From Eq. (15), the charac-

teristic length $L(k)$ can be seen as an average length between any two points on an attractor that are separated by k time units. On the other hand, the characteristic distance is an average distance from an arbitrary fixed point in the phase space to an attractor.

For a periodic attractor, there exist an N such that $\mathbf{x} = \mathcal{T}^N(\mathbf{x})$. In this case, from the definition of characteristic lengths, we will have $L(k) = L(k+N)$ for all k , thus $L(k)$ would be N -periodic. On a quasi-periodic or chaotic orbit, the system will return arbitrarily close to its starting point for some large N . Therefore, $L(N) \approx 0$ for some large N for a quasiperiodic orbit due to its regularity. However due to the existence of a positive Lyapunov exponent, it is not expected that $L(N) \approx 0$ in chaotic motion even for some large N .

2.3.4 Illustration Using Rössler Model

To show how characteristic lengths behave in different situations, the Rössler model is chosen to generate well sampled, clean data. In particular, the following flow is considered:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}\tag{18}$$

In the simulations, b is fixed at 0.6, c is fixed at 6.0, and a is used as a bifurcation parameter. To understand how the dynamic system changes as the parameter varies, the bifurcation diagram is computed and demonstrated in Fig. 5. The diagram shows the local maxima of y as a function of a . Filled-in regions of the plot indicate chaotic regions. For each particular value of a , 100,000 steady state trajectory points are generated with time step $t_s = 0.06$ to estimate characteristic lengths and distances.

Figure 5. Bifurcation diagram for Rössler equation

Figure 6. The characteristic lengths. P-1, P-2, P-3, P-4 and chaotic solution (a,b) occur at $a = 0.1, 0.16, 0.28, 0.18, 0.3$ and 0.4 , respectively.

Figure 6 shows plot of characteristic length versus time. The dynamical system's periodicity can be characterized by the behavior of characteristic lengths by

Figure 7. The characteristic distances of several random points.

observing their periodicity. If the system exhibits a periodic orbit, the function $L(k)$ also has the same period with respect to k . For the chaotic motion, the value of $L(k)$ oscillates but does not come close to 0 even for large k .

Figure 7 shows the plot of characteristic distance as a function of a bifurcation parameter a . During bifurcations, the position of the attractor may change dramatically, so the characteristic distance is not expected to be a continuous function of a . However, the estimated characteristic distances may still provide valuable information about the parameter drifts in the system.

2.4 Choosing Time Delays in Phase Portrait Reconstructions

2.4.1 Basic Idea

Phase space reconstruction is the first step in analyzing a nonlinear system from observed time series. In this process, one must choose a time delay τ . For harmonic data, the appropriate time delay τ is one quarter of the period [21]. This value is approximately equal to the first zero of the autocorrelation function [21].

In the previous section, the dynamical system's periodicity can be characterized by the periodicity of characteristic lengths. An approximate period of the attractor X can be defined by $T_{min} \in \mathbb{N}$ so that $\|\mathbf{x}(i) - \mathbf{x}(i + T_{min})\|_2$ is minimized. The value of T_{min} is the first minimum of the characteristic length curve $L(k)$ defined by Eq. (14). If the phase space is reconstructed from a time series

$\{x_i\}$, T_{min} can be determined from the curve $L_1(k)$ against k , where

$$L_1(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |(x_i - x_{i+k})|. \quad (19)$$

We can think L_1 is the characteristic length of an one-dimensional attractor.

By selecting the time delay τ as a quarter of T_{min} , the reconstructed phase space will be stretched to all directions, like a circle. This value of τ guarantees the smoothness in time evolution. Therefore, in this dissertation, τ is suggested to be a quarter of T_{min} .

2.4.2 Numerical Examples

The new algorithm is now compared with the average mutual information (AMI) method for a set of time series generated by the Lorenz equations, a three-dimensional irrational torus, and the Rössler equations. In each system, 5000 data points of x is used to estimate the time-delay τ .

Lorenz Equation

$$\begin{aligned} \dot{x} &= -a(x - y) \\ \dot{y} &= -xz + cx - y \\ \dot{z} &= xy - bz \end{aligned} \quad (20)$$

1. First, Eq. (20) is solved for $a = 16$, $b = 4$, and $c = 45.92$ with time step $t_s = 0.01$ to get 5000 trajectory points. Then $L_1(k)$ is estimated using Eq. (19), the result is shown in Figure 8(a). The time delay suggested by our method is $T_{min}/4 = 11$. This value is close to the time delay $\tau = 10$ calculated by the AMI method [6].
2. Now, an example in [22] is used where $a = 10$, $b = 28$, and $c = 8/3$ to obtain the spectrum $L_1(k)$ shown in Fig. 8(b). The time delay $\tau = T_{min}/4 \approx 17$ agrees with the value found in [22].

Figure 8. Estimation of $L_1(k)$ in Lorenz attractor

Figure 9. Estimation of $L_1(k)$ in the three-torus (a) and Rössler attractor (b)

The three-torus

$$x(t) = \sin\left(\frac{3t}{500}\right) + \sin\left(\frac{3\sqrt{2}t}{250}\right) + \sin\left(\frac{9\sqrt{3}t}{500}\right) \quad (21)$$

In this simulation, 5000 data points are used to estimate $L_1(k)$ with $t_s = 1$. Fig. 9(a) shows the result. Again, the time delay $\tau = T_{\min}/4 = 51$, is close to the time delay found by the AMI method [22] which is 54.

Rössler Equation Another example in [22] is used for estimating time delay. The Eq. (18) is solved for $a = 0.2$, $b = 0.4$, and $c = 5.7$ to generate 5000 data points with $t_s = 0.05$. Plot of $L_1(k)$ versus k is shown in Fig. 9(b). The delay

Figure 10. Reconstructed phase space of Rössler equation with $\tau = 20, 30$ (left, right)

time $\tau = T_{min}/4 = 30$, while the AMI method yielded $\tau = 20$ [22]. However, the reconstructed phase space in Fig. 10 show that both values give good embedding (phase space with $\tau = 30$ is more uniform and close to circular trajectory).

2.4.3 Noise Effects

To illustrate the robustness of the method to noise, 5000 normally distributed random numbers were generated, and were added to the chaotic signal x in the Lorenz equation in $1/5$, $1/2$, $4/5$ and $5/5$ amplitude ratios, which corresponds to 20%, 50%, 80% and 100% noise in the signal. Fig. 11 shows that the method based on characteristic length performed well, especially with very noisy data. T_{min} remains close to the value 71 despite the presence of noise. Therefore, the delay time is close to the correct value $\tau = 17$ even in the 100% noise level signal.

The characteristic lengths are relative to averages and the average of the noise tends to 0. This fact yields the robustness to noise of our method. On the other hand, the plot of average mutual information for the same data in Fig. 12 demonstrates the sensitivity to noise of the AMI method.

Figure 11. Estimation of $L_1(k)$ in the Lorenz equation with noise level of 20%, 50%, 80% and 100%

Figure 12. Average mutual information of $L_1(k)$ in the Lorenz equation with noise level of 20%, 50%, 80% and 100%

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CHAPTER 3

Identification of Fatigue Damage

In this chapter, a data-based damage identification method based on *Characteristic Distances* (CD) is presented, where an ensemble of CDs is used as a tracking feature vector. Then damage identification is achieved by using the smooth orthogonal decomposition. This method is verified by both numerical simulations and experimental data.

3.1 Theory

In a material, a fatigue crack usually grows over thousands of load cycles. Hence, the fatigue damage is viewed as a slow-time state variable evolving in a hierarchical dynamical system [1]:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}(\boldsymbol{\phi}), t), \quad (22a)$$

$$\dot{\boldsymbol{\phi}} = \epsilon \mathbf{g}(\boldsymbol{\phi}, \mathbf{x}), \quad (22b)$$

$$y = h(\mathbf{x}) \quad (22c)$$

where $\mathbf{x} \in X \subset \mathbb{R}^n$ is a fast-time dynamic variable, $\boldsymbol{\phi} \in P \subset \mathbb{R}^m$ is a hidden slow-time damage dynamic variable, which alters the parameter vector $\boldsymbol{\mu} \in \mathbb{R}^p$. $t \in \mathbb{T} \subset \mathbb{R}$ is time, overdots denote time differentiation, $0 < \epsilon \ll 1$ is a small rate constant defining the time scale separation. A time series y is a measurement of the fast time variable $\mathbf{x}$.

In practical situations, the functions $\mathbf{f}$ and $\mathbf{g}$ are not provided, we only have access to the scalar time series y . In the following, characteristic distances are utilized to characterize the variations of the slow damage variable $\boldsymbol{\phi}$ from the observed fast time variable y . It is assumed that the state variable $\boldsymbol{\phi}$ changes

Figure 13. Graphical illustration demonstrates the construction of feature vectors.

slowly. Thus, ϕ is considered as approximately constant for a fast-time data set collected over an intermediate time interval.

A general solution to the system can be written as $\mathbf{x} = \mathbf{X}(t, \mathbf{x}_0, \phi)$, where $\mathbf{x}_0$ is the initial condition. Hence, the continuous formulation of the characteristic distance of a fixed point $\mathbf{y}$ to the phase space X can be expressed as

$$D(\mathbf{y}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\mathbf{y} - \mathbf{X}(t, \mathbf{x}_0, \phi)\| \, dt. \quad (23)$$

This limit is approximated by the integral

$$D(\mathbf{y}) \approx \frac{1}{T_n} \int_0^{T_n} \|\mathbf{y} - \mathbf{X}(t, \mathbf{x}_0, \phi)\| \, dt, \quad (24)$$

where T_n is a large number. By the Mean Value Theorem for integrals, there is a $t_i \in [0, T_n]$ such that

$$\frac{1}{T_n} \int_0^{T_n} \|\mathbf{y} - \mathbf{X}(t, \mathbf{x}_0, \phi)\| \, dt = \|\mathbf{y} - \mathbf{X}(t_i, \mathbf{x}_0, \phi)\|. \quad (25)$$

Therefore, an estimated value of $D(\mathbf{y})$ is a function of ϕ . In what follows, $D(\mathbf{y})$ for different points $\mathbf{y}$ will be used to reconstruct ϕ .

For tracking purposes, the characteristic distances are calculated for each data set and are assembled together in a feature vector $[D(\phi, \mathbf{y}_1); D(\phi, \mathbf{y}_2); \dots; D(\phi, \mathbf{y}_n)]$

(see Fig. 13), where $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are randomly chosen fixed points in the phase space. These feature vectors describe the evolution of the slow state variable ϕ and are assembled together into the tracking matrix $\mathbf{Y}$ in time sequence. This matrix $\mathbf{Y}$ embeds the changes in the parameters of the fast-time subsystem and can be projected onto actual damage states. In particular, there is a projection $\mathbf{q}$ that yields a damage tracking coordinate $\varphi = \mathbf{Y}\mathbf{q}$. It is hypothesized that φ is a linear combination of components of ϕ . In practical situations, $\mathbf{q}$ and φ are not known. However, they can be estimated from $\mathbf{Y}$ by the *smooth orthogonal decomposition* (SOD) [2, 3] technique.

3.2 Smooth Orthogonal Decomposition

SOD is a multivariate data analysis method which can extract deterministic trends from a multivariate data set. SOD analysis is performed by looking for $\mathbf{q}$ so that the damage tracking coordinate φ has maximum smoothness and variation

$$\max_{\mathbf{q}} \frac{\|\mathbf{Y}\mathbf{q}\|^2}{\|\mathbf{D}\mathbf{Y}\mathbf{q}\|^2}, \quad (26)$$

where $\mathbf{D}$ is a discrete differential operator,

$$\mathbf{D} := \frac{1}{t_s} \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}.$$

The solution to Eq. (26) is obtained by generalized singular value decomposition of matrix $\mathbf{Y}$ and its time derivative $\mathbf{D}\mathbf{Y}$

$$\mathbf{Y} = \mathbf{U}\mathbf{C}\mathbf{X}^T, \quad \mathbf{D}\mathbf{Y} = \mathbf{V}\mathbf{S}\mathbf{X}^T \quad (27)$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, $\mathbf{C}$ and $\mathbf{S}$ are diagonal matrices, and $\mathbf{X}$ is a square matrix. The *Smooth Orthogonal Coordinates* (SOCs) φ are given by the columns of $\mathbf{U}\mathbf{C}$, *Smooth Projective Modes* (SPMs) $\mathbf{q}$ are provided by columns

of $\mathbf{X}^{-T}$, and *Smooth Orthogonal Values* (SOVs) are $\boldsymbol{\sigma} = \text{diag}(\mathbf{C}^T \mathbf{C}) / \text{diag}(\mathbf{S}^T \mathbf{S})$. The greater the magnitude of the SOV the smoother in time is its corresponding SOC. Since the slow-time state variable is a product of a smooth deterministic process, these variables are expected to be embedded in the smoothest SOCs. In the presence of an m -dimensional damage evolution process, the damage identification method assumes that the m smoothest SOCs are linear combinations of the actual damage states.

3.3 Numerical Validation

In the following, I will present two examples with the time series derived from a Rössler and a Duffing equation to validate the CD-based damage identification algorithm. In the simulations, a damage process is introduced by slowly changing a parameter in the equations.

Rössler Equation

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}\tag{28}$$

1. Using the phase space from previous simulations of Eq. (28) in Chapter 2, where b is fixed at 0.6, c is fixed at 6.0, and a is varied slowly and linearly from 0.1 to 0.4 (see Fig. 14(b)). Here, a is considered a damage variable. For each value of a , 200 characteristic distances to 200 arbitrary fixed points are estimated. Finally, the 601×200 tracking matrix $\mathbf{Y}$ is assembled. Tracking results after applying SOD to this matrix are shown in Fig. 14. We can see in Fig. 14a, the first SOV is much larger than the rest SOV, which is consistent with the fact that there is only one damage process in the system. The SOC corresponding with the largest SOV is depicted in Fig. 14c. To indicate the

Figure 14. First ten SOVs (a), parameter a versus time (b), dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus parameter a (red dots) with least square linear fit (black line) (d). The correlation between a and SOC_1 is $r = 0.9998$.

strength of the linear relationship between real damage states and the first SOC, the correlation coefficient r is calculated. $r \approx 1$ shows that there is a strong linear relationship between two variables (see Fig. 14d).

2. a is now changed sinusoidally from 0.1 to 0.4 (see Fig. 15(b)). SOD is applied to 786×200 tracking matrix $\mathbf{Y}$. Figure 15 shows the tracking results. Again, we obtain the linear relationship between the smoothest SOC_1 and a (see Fig. 15d).

Figure 15. First ten SOVs (a), parameter a versus time (b), dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus parameter a (red dots) with least square linear fit (black line) (d). The correlation between a and SOC_1 is $r = 0.9998$.

Duffing Equation

In the previous simulations, the real phase space is used to calculate the characteristic distances. In this example, only a time series x from a two-well Duffings equation is considered:

$$\ddot{x} + \gamma\dot{x} + \alpha x + \beta x^3 = f \cos(\omega t) \quad (29)$$

where the system's parameters and forcing frequency are fixed to $\gamma = .25$, $\alpha = -1$, $\beta = 1$, and $\omega = 1$. f is used as a slow damage variable or a bifurcation parameter. The bifurcation diagram is shown in the Fig. 16. Consider the case when f is changed slowly and linearly from 0.35 to 0.4. We can see that for this range of f , the response of the system is in the chaotic region.

Figure 16. Bifurcation diagram for Duffing oscillator

In the calculations, the first 50 cycles of data were dropped, and 100,000 steady state points are recorded for each forcing amplitude using a sampling time $t = \pi/36$. The phase space is then reconstructed using a time delay $\tau = 53$ and embedding dimension $d = 4$. Figure 17 shows the characteristic distances to the reconstructed phase space for three randomly chosen points.

400 randomly chosen points in the phase space are used to assemble 1001×400 tracking matrix $\mathbf{Y}$. Then the SOD tracking result is shown in Fig. 18. Again, the

Figure 17. The characteristic distances of several random points in the phase space of the Duffing oscillator

Figure 18. First ten SOVs (a), parameter f versus time (b), dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus parameter f (red dots) with least square linear fit (black line) (d). The correlation between f and SOC_1 is $r = 0.9991$.

CD-based damage identification method gave a linear relationship between the smoothest SOC_1 and the real damage variable (see Fig. 18d). Please note that these tracking results are still possible even when each component of matrix $\mathbf{Y}$ is discontinuous as shown in Fig. 17.

3.4 Empirical Evaluation

A modified version of the well known two-well magneto-elastic oscillator is used to validate this new approach to slow-parameter tracking (see Fig. 19). Description of this experiment is given in Ref. [3]. In this experiment, a couple of electromagnets powered by a computer-controlled power supply are used to cause perturbations in the magnetic potential at the free end of a vibrating cantilever

Figure 19. Experimental apparatus of the two-well magneto-elastic oscillator. 1. Electromagnetic shaker; 2. Sliding frame; 3. Laser vibrometer CH1; 4. Laser vibrometer CH2; 5. Cantilever beam; and 6. Two permanent/electromagnet stacks.

beam. The vibration of the beam is measured by two laser vibrometers (CH1, CH2) mounted near the clamped beam end. The position of the beam is calculated by the difference between CH1 and CH2. Vibration data is low-pass filtered with 50 Hz cut off frequency and collected at a 160 Hz sampling frequency.

The voltages supplied to the electromagnets ($v_1(t)$ and $v_2(t)$) are altered harmonically and independently as shown in Fig. 21b (top) and Fig. 21b (bottom). These voltages play a role of damage variables in the system.

The first test lasts about 12 hours and 6.6 million data points are recorded [3]. The 5-dimensional fast-time phase space is reconstructed using a delay time of six time samples ($t_s = 1/160$ s). The time series are split into 800 data records, and each record contains 2^{13} points. Here, the characteristic distances are calculated for 300 fixed points randomly chosen in the reconstructed phase space. Then the 800×300 tracking matrix $\mathbf{Y}$ is assembled. Several columns of the tracking matrix are shown in Fig. 20. The SOD-based tracking result is shown in Fig. 21(top).

Figure 20. Three randomly chosen columns of the tracking matrix

The two largest SOVs are several orders-of-magnitude larger than the rest SOV, which is consistent with 2-dimensional damage process in the system. The SOC₁ and SOC₂ corresponding with these two largest SOVs are depicted in Fig. 21c. The scaled SOC₁ and SOC₂ are compared with $v_1 + v_2$ and $v_1 - v_2$ (see Fig. 21d). The correlation between $v_1 + v_2$, $v_1 - v_2$ and SOC₁, SOC₂ are 0.9951 and 0.9834, respectively. The result confirms that these SOC₁ and SOC₂ are linear combinations of real damage states.

The second test runs about 20 hours and 12 million data points are recorded [4]. The data is divided into 1480 records. There are nearly 2^{12} points in each record. Then the tracking matrix is assembled by calculating characteristic distances for 300 randomly chosen phase-space points. The tracking results are shown in Fig. 21 (bottom). The correlation between $v_1 + v_2$, $v_1 - v_2$ and SOC₁, SOC₂ are 0.9961 and 0.9891, respectively. Again, the two smoothest SOC₁ and SOC₂ are linear combinations of the real damage states.

Figure 21. Top: First test; Bottom: Second test. Plot of ten largest SOCs (a); Actual damage states v_1 (—) & v_2 (---) versus time (b); Plot of the first (—) and second (---) SOCs corresponding to the two largest SOVs (c); Plots of $v_1 + v_2$ (—), $v_1 - v_2$ (---), and the scaled first two SOCs (.) (d).

3.5 Discussion

The application of the SOD is prompted by the assumptions that feature vectors are in smooth relationship with respect to slow-time damage variables. In the simulation examples, the systems exhibited complicated bifurcations while one of the parameters was varied slowly and played a role of damage variable. The estimated characteristic distances are not smooth functions of the damage variable. However, when we combined the characteristic distances from different fixed points in the phase space and applied SOD, we still recovered the corresponding smooth deterministic trend, which can be used for one-to-one damage tracking.

List of References

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CHAPTER 4

Fatigue Testing Apparatus and Experiments

It is clear that fatigue in materials is a nonlinear process in which the fatigue life is nonlinearly coupled with load factor [1, 2]. Current fatigue damage models only take into account basic load statistical quantities such as mean, variance, maximum and minimum. These simple features do not capture important nonlinear behaviors of the applied loads. In this chapter, new nonlinear load factors that contribute to fatigue life are identified.

In the next section, a novel fatigue testing rig based on inertial forces is described. In our experiment, fatigue tests are conducted using chaotic and random loading, which have the similar power spectral densities and the probability distribution functions. Experimental data are used to demonstrate the ability of CD-based damage identification method to track damage. Since the mean and variance of the excitation are kept constant throughout the experiment, superposition based cycle counting methods should be applicable. Then the rain-flow counting method and Palmgren-Miner rule are applied to quantify the damage variables. Nonlinear loading characteristics are proposed to explain the discrepancies in the results.

4.1 New Fatigue Testing Apparatus

4.1.1 Mechanical setup

The experimental system was first introduced in [3]. A schematic of the test rig is shown in Fig. 22. The mechanical backbone of the system is a slip table guided by four linear bearings on two parallel rails mounted to a granite base. An LDS V722 electromagnetic shaker, whose maximum frequency is about 4kHz, is used to drive the slip table under various loading conditions. The specimen is a single-edge notched beam which is simply supported by pins on each end. Two

Figure 22. Schematic of the apparatus. 1. Shaker; 2. Granite base; 3. Slip table; 4. Back mass; 5. Specimen supports; 6. Pneumatic cylinder supports; 7. Rails; 8. Front cylinder; 9. Front mass; 10. The specimen; 11. Linear bearings for the masses; 12. Central rail for the masses; 13. Linear bearings for the slip table; 14. Back cylinder; 15. Flexible axial coupling.

inertial masses (5.8 kg each), guided by linear bearings mounted to the slip table provide dynamic loads. The masses are kept in contact with the specimen by using two pneumatic cylinders. Different R -ratios can be obtained by adjusting the pressure in each cylinder and the amplitude of the signal supplied to the shaker. Pressures within the cylinders, however, must be large enough to keep the masses in contact with the specimen at all times. This requirement allows for the transfer of inertial forces to the specimen. When the slip table moves, the specimen drives the masses to follow the motion of the slip table. A photo of the system is included in Fig. 23.

The specimens are designed to follow the fracture toughness test standard ASTM E1820-08a [4] and their model is shown in Fig. 24. The specimen is made of 6061 aluminum bar stock, with dimensions $314.70 \times 20.32 \times 6.35$ mm (length $\times$ height $\times$ width), and the fatigue is initiated by a machined 'V' notch. The notch

Figure 23. Photograph of the system

Figure 24. Model of the specimen. The machined notch can be seen in the center, the round hole is on the left and the oblong slot on the right.

is cut to a depth of 7.62 mm by a blade 0.762 mm thick with a 60° V-shaped tip. The initial crack length was examined under a Stocker and Yale optical micrometer at 40X magnification. The crack lengths of all the fatigue tests recorded were measured to be $7.823 \text{ mm} \pm 0.069 \text{ mm}$. The specimen drawing is provided in Appendix B.

In order to get rid of residual and machining stresses, the specimens are fully annealed prior to testing. First, the specimens are loaded into a heat treating oven, which is preheated to 775° F and remains there for two hours. Then the oven is shut off and the specimens are allowed to slowly cool. The rate of cooling is kept

lower than 50° F per hour, which is achieved by the insulative properties of the oven.

4.1.2 Instrumentation

The structural response is measured using accelerometers and an eddy current displacement sensor. Two single-axis accelerometers from PCB Piezoelectronics, model number 333B42, are used. One accelerometer is mounted to the slip table, the other is mounted to one of the two masses. The sensitive axis of each accelerometer is always kept parallel with the horizontal x -axis. An eddy current sensor from Lion Precision, model number U5 with an ECL202 driver, is used to measure the relative displacement between the beam and the slip table. The crack growth is monitored by an Alternating Current Potential Difference (ACPD) crack growth monitor. ACPD is a nondestructive testing technique which utilize the skin effect of current flow on the surface of metallic objects. First, a high-frequency AC voltage (20kHz) is injected into the beam. Then a voltage (potential difference) across the notch is measured. If the crack grows, the current flowing along the beam's surface will travel further and the potential difference will be greater.

Drive signals of the testing rig are generated by a program controlled function generator (Tektronix AFG 30222), which can generate different types of excitation signals based on either internal functions (such as random and harmonic) or stored arbitrary time histories (such as chaotic). The function generator is connected to a PC using a USB cable in order to have full control of the excitation signals. The preload is measured by two pressure sensors which are directly connected to the pneumatic cylinders. The dynamical forces between the pneumatic cylinders and the masses are measured by two piezoelectric force sensors (Model 208C02 from PCB Piezoelectronics). All data from the sensors are recorded at 1kHz sampling rate by a data acquisition (DAQ) card from National Instrument on a PC; and all

control and DAQ tasks are implemented through a LabView program.

The new design has two major advantages over a similar inertial force based fatigue apparatus described in [5, 6]. Firstly, the horizontal setup reduces the excitation force necessary to drive the system. Therefore, a similar load capacity can be achieved by using a smaller, and lower cost electromagnetic shaker. Secondly, the slip table is built using a standard mounting surface, which allows for simple modifications to accommodate various types of specimens and configurations.

4.2 Description of Experiment and Results

4.2.1 Signal Generation

In our experiment, chaotically and stochastically excited fatigue tests are conducted. Both excitations exhibit similar spectral and statistical properties, but they are fundamentally different in their temporal structures. Random or stochastic source is non-deterministic, while chaotic loading is generated by simulating a deterministic dynamical system. In particular, we have used a steady state response of the following double-well Duffing oscillator to generate the chaotic excitation signal supplied to the shaker,

$$\ddot{x} + 0.25 \dot{x} - 0.6 x + x^3 = 0.2 \cos t, \quad (30)$$

where the over-dot indicates differentiation with respect to dimensionless time. The response of the Eq. (30) is sampled uniformly using a dimensionless time increment $\Delta t = 0.2$. The chaotic signal is then obtained by subtracting the mean from the Duffing signal. To generate spectrally and statistically similar stochastic excitation, the chaotic signal is modified using the Iterated Amplitude Adjusted Fourier Transformed Surrogates method as described in Ref [7]. This method is originally utilized for testing some null hypothesis in an observed time series. First, a sorted list of the values of the solution of Eq. (30) called $\{x_n\}$ and the squared amplitudes of its Fourier transform $X_k^2 = \left| \sum_{n=0}^{N-1} x_n \exp \frac{i2\pi kn}{N} \right|^2$ are stored. $\{x_n^0\}$ is

Figure 25. Histogram and power spectrum of the original chaotic (left) and stochastic (right) signals

a random shuffle of $\{x_n\}$, the superscript indicates the iteration. Then the iterative procedure follows this sequence:

1. Taking the Fourier transform of $\{x_n^i\}$, replace the squared amplitudes $\{X_k^{2(i)}\}$ by $\{X_k^2\}$ and then performing the inverse Fourier transform yielding $\{y_n^i\}$.
2. Rank-order the resulting series $\{y_n^i\}$ according to $\{x_n\}$ to get $\{x_n^{i+1}\}$.

The two steps are repeated several times and Fig. 25 shows the comparison of the histograms and the power spectrums of the original chaotic and stochastic signals.

The topologically equivalent phase space of the chaotic attractor is also reconstructed from the recorded x time series using delay coordinate embedding as described in Chapter 2. In particular, a delay time $\tau = 8$ is used to generate the chaotic phase portrait shown on the right plot in Fig. 26. The same delay

Figure 26. Delay coordinate reconstruction of phase portraits for the stochastic (left) and original chaotic (right) signals

was used to reconstruct the corresponding surrogate time-series phase portrait as shown in the left plot of Fig. 26, which shows the clear difference in time evolution of these trajectories: the deterministic chaotic signal has an attractor, while stochastic surrogate data does not.

4.2.2 Results of the Experiment

These two signals supplied to the shaker are henceforth called Chaotic and Random. The experimental design is very straightforward. The nominal excitation amplitude is set to keep the test time manageably short for quick turnaround time. In our setup, the break of a specimen is indicated by the saturation of the eddy current sensor which usually happens after a crack reaches its fracture stage. The specimen at the end of the experiment is shown in Fig. 27. The results in a total of ten sets of data are shown in Table 1. It should be noted that all names are indicative of the order the tests are run and no further inference should be made.

Table 1. Test results. Mean, variance, skewness and kurtosis are calculated from table acceleration data. Damage is estimated based on rain-flow counting and the Palmgren-Miner rule.

Random Signal	TTF hrs:min	Damage	Mean V	Variance V ²	Skewness	Kurtosis
1-A	0:53	0.97	-0.0020±0.0014	0.0740±0.0038	0.0012±0.0060	2.5850±0.0053
2-A	1:07	1.06	-0.0021±0.0014	0.0716±0.0038	0.0081±0.0049	2.5827±0.0066
3-A	0:44	1.00	-0.0020±0.0015	0.0742±0.0038	0.0092±0.0076	2.5980±0.0094
4-A	0:52	1.07	-0.0019±0.0015	0.0747±0.0041	0.0073±0.0069	2.5898±0.0096
5-A	0:45	0.86	-0.0038±0.0006	0.0724±0.0036	0.0081±0.0052	2.5941±0.0062
Average	0:52	0.99	-0.0023±0.0015	0.0733±0.0018	0.0067±0.0067	2.5892±0.0094
Chaotic Signal	TTF hrs:min	Damage	Mean V	Variance V ²	Skewness	Kurtosis
1-B	1:44	1.48	-0.0011±0.0014	0.0717±0.0010	-0.0086±0.0052	2.6878±0.0770
2-B	1:50	2.08	-0.0031±0.0014	0.0717±0.0011	-0.0091±0.0062	2.6636±0.0878
3-B	1:46	1.51	-0.0019±0.0015	0.0725±0.0008	-0.0075±0.0068	2.6259±0.0775
4-B	1:42	1.39	-0.0025±0.0015	0.0721±0.0010	-0.0051±0.0041	2.6927±0.0503
5-B	1:13	1.03	-0.0017±0.0014	0.0717±0.0010	-0.0062±0.0050	2.6996±0.0699
Average	1:39	1.50	-0.0021±0.0016	0.0719±0.0010	-0.0074±0.0057	2.6720±0.0785

Figure 27. Specimen at the end of the experiment.

Some of the loads' metrics are included in Table 1 to show that the basic statistical qualities of both signals match well and do not show any significant deviation from experiment to experiment. Any drift in these statistics, from test to test, is due to the small differences between the specimens and/or the lack of a sophisticated shaker output controller. However, these tests have proven very repeatable and the variations in the statistics of the excitation signals are negligible

Figure 28. Histogram and power spectrum of the table acceleration data of the random (left) and chaotic (right) signals

at this amplitude level.

The actual load histograms and power spectral densities from the slip table acceleration signals (during experiments 1-A and 1-B, respectively) are shown in Fig. 28. The acceleration signals were collected over ten minutes. The probability distributions for the voltage signals supplied to the shaker are almost identically (Fig. 25). However, these signals then pass through the shaker and couplings to reach the table. The table acceleration probability distributions are not then entirely identical as seen in Fig. 28, while the power spectral densities remain fairly similar for both stochastic and chaotic loads. Furthermore, the differences in the histograms are only observed for small amplitudes.

The corresponding phase portraits are reconstructed and shown in Fig. 29

Figure 29. Reconstructed phase portrait of the table acceleration data for the random (left) and chaotic (right) signals

(time delay $\tau = 6$), where again we see the characteristics observed for the original input signals. It would be a very difficult task to build input surrogate data which would match the table acceleration properties of the chaotic signal. As long as the main factors, such as mean, minimum, and maximum acceleration are preserved, the differences are deemed acceptable.

4.3 Damage Tracking

In the experiments, the evolution of the fatigue crack is recorded by the ACPD measurement equipment. The time series of the acceleration of the front mass and the displacement of the specimen describe the fast-time dynamics of the system. However, the embedding theory [8, 9] does not suggest which time series is optimal for phase space reconstruction. In what follows, CD-based damage identification method is applied to both time series to identify the hidden damage and to track its evolution.

The time series of the displacement of the specimen is split into small records. Each record contains 20,000 points. The 5-dimensional fast-time phase space is reconstructed using a delay time of seven time samples ($t_s = 0.001$ s). The charac-

Figure 30. Three columns of the tracking matrix for test 1A (random)

teristic distances are calculated for 160 fixed points randomly chosen in the reconstructed phase space. Then the tracking matrix $\mathbf{Y}$ is assembled. Several columns of the tracking matrix for test 1A and 1B are shown in Fig. 30 and Fig. 31, respectively. The SOD-based tracking results for these tests are shown in Fig. 32 and Fig. 33, where plot (d) shows that the dominant SOC₁ of the tracking matrix map directly onto the ACPD signals. The results for test 2-5 are in Appendix C, and the correlation coefficients between the smoothest SOC₁ and the ACPD signal are shown in Table 2.

Figure 31. Three columns of the tracking matrix for test 1B (chaotic)

Figure 32. Tracking result from the displacement of the specimen in Test 1A. First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure 33. Tracking result from the displacement of the specimen in Test 1B. First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Table 2. The correlation coefficient between the smoothest SOC_1 calculated from the measures of the displacement of the beam and the ACPD signal.

Test	1	2	3	4	5
Random	0.9968	0.9933	0.9919	0.9867	0.9929
Chaotic	0.9951	0.9880	0.9838	0.9855	0.9871

Now the same process is applied for the time series of the acceleration of the front mass. Figure 34 and 35 show the tracking results for test 1A and 1B, respectively. Then the correlation coefficient between SOC_1 and ACPD signal are determined and summarized in Table 3. The value $r \approx 1$ in Table 2, 3 demonstrates that the fatigue damage has been captured from the fast-time dynamics.

Figure 34. Tracking result from the acceleration of the front mass in Test 1A. First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure 35. Tracking result from the acceleration of the front mass in Test 1B. First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Table 3. The correlation coefficient between the smoothest SOC_1 calculated from the measures of the acceleration of the front mass and the ACPD signal.

Test	1	2	3	4	5
Random	0.9748	0.9928	0.9695	0.9886	0.9947
Chaotic	0.9932	0.9765	0.9925	0.9921	0.9890

4.4 Cumulative Fatigue Damage

The first and most obvious piece of data that gives an indication of a fracture process is the *time to failure* (TTF). A clear trend emerges in the TTF data shown in Table 1. It is obvious that the random excitation is much more damaging to the specimen than the chaotic excitation. These test results show that the

Figure 36. Beam diagram with applied forces

system under chaotic excitation outlasts the system under random excitation by nearly a factor of two in all accounts. The question then is if these differences can be explained solely by the differences observed in the table excitation histograms shown in Fig. 28. Since the mean, variance, skewness and kurtosis of the excitation do not vary significantly during the experiment, superposition principle based cycle counting methods should account for the differences in the histograms.

Rain-flow counting method [10, 11] and the Palmgren-Miner rule are used here to quantify the damage based on stress cycles. Theoretically, the criteria for failure would occur when the damage variable, $0 \leq D \leq 1$, reaches one.

The specimen is regarded as a simply supported beam with applied loads, as shown in Fig. 36 where P is the static mean force due to the pressure within the cylinders, and a is the relative acceleration between the masses and the slip table: $a = a_{mass} - a_{table}$. In this testing, both cylinders have equal pressure. The stress

at the crack tip is approximated by simple beam theory,

$$\sigma = \frac{Mb}{2I}, \quad (31)$$

where M is the bending moment, $M = \frac{1}{4}[ma(L + 2L_2) + P(L - 2L_2)]$, I is the second moment of area of the beam's cross-section and b is the height of the beam.

4.4.1 Rain-flow Cycle Counting Method

The rain-flow cycle counting method is applied to the time series of the maximum stress at the notch of the beam. The objective of this method is to reduce a variable load history signal into a sequence of constant amplitude cycles. The cycle counting algorithm described in [11] is as follows:

Let X be the range under consideration, Y be the previous range adjacent to X and S be the starting point of the history

1. Read the next peak or valley. If out of data go to step 6.
2. If there are less than three points go to step 1. If there are three or more peaks and valleys, form ranges X and Y using the three most recent peaks and valleys that have not been discarded, taking X as the most recent range.
3. Compare the absolute values of ranges X and Y
 - (a) If $X < Y$, go to Step 1
 - (b) If $X \geq Y$, go to Step 4.
4. If range Y contains the starting point S , go to Step 5; otherwise, count range Y as one cycle; discard the peak and valley of Y ; and go to Step 2.
5. Count range Y as one-half cycle; discard the first point (peak or valley) in range Y ; move the starting point to the second point in range Y ; and go to Step 2

Figure 37. Histograms of cycle amplitudes counted by the rain-flow counting method in random (left) and chaotic (right) tests.

6. Count each range that has not been previously counted as one-half cycle.

Figure 37 shows the histograms of the resulting cycle amplitudes from stress histories of test 1-A and 1-B. In general, the TTF depends on the sequence in which the different amplitude cycles are applied, in our tests the load profiles are not altered over time, and the same load sequence is applied repeatedly. Thus, any differences observed in TTF are only attributable to the difference in temporal structure of each load signal.

4.4.2 Palmgren-Miner Rule

The Palmgren-Miner rule uses an $S-N$ curve for the test specimen to evaluate and determine the damage variable. This curve indicates how many stress cycles a material can experience at each stress amplitude level. Then the damage variable corresponding to each amplitude of stress in Fig. 37 is the ratio of the actual cycles experienced over the maximum possible as given by the $S-N$ curve. Since D is simply a ratio, its exact value depends on the $S-N$ curve used and is inherently

nondescript. A simple S - N curve can be expressed by the equation

$$\log_{10} S = A + B \log_{10} N, \quad (32)$$

where S is the stress amplitude and N is the number of cycles until the failure at that stress level. For the comparison of chaotic and random loadings, the S - N curve is chosen to yield $D \approx 1$ for all random loadings. Hence, the coefficients A and B are determined to be 2.58 and -0.25 , respectively. Then the damage variable is calculated for each test. The results are shown in Table 1.

Typically, the Palmgren-Miner rule would be used to predict the failure by characterizing a typical stress history over some time and approximating the stress history over the expected lifetime of the fatiguing structure. The failure criteria would be met once the damage variable $D = 1$. In our analysis, the damage criteria is calculated over the actual stress history of the entire load time history. Therefore, since the failure criteria is equivalent for all tests it can be expected that for all tests D would be approximately equal if the linear damage law is applicable. This is true across tests of the same type, chaotic or random. However, in comparison between the two, the linear damage law does not reflect the observed effects of the time history on TTF. In particular, it significantly overestimates damage for the chaotic signal.

The fatigue testing using random and chaotic excitation that have nearly identical statistical and spectral properties showed that specimens failed twice as fast during chaotic forcing (Table 1). The actual stress loads at the crack had almost exact spectral characteristics, but the histograms showed that chaotic forcing has more low amplitude loads, while maintaining approximately the same loading for larger amplitudes (see Fig. 28). Everything being equal and keeping excitation amplitude constant over the whole experiments, the linear Palmgren-Miner damage law, which is independent of the loading history, should have accounted for the

differences in the histograms. However, while it was adequate to estimate damage during the random excitation, Palmgren-Miner rule significantly overestimated the damage variable for the chaotic excitation. This implies that the excitation time history is very important in fatigue accumulation, even when main statistical moments of the load at the crack do not change significantly, and both linear statistics and spectral properties of both signals are very similar. Thus, we need to look at nonlinear characteristics of the excitation time series.

4.5 Nonlinear Characterization of Loading

The two main features of dissipative nonlinear dynamical systems are Lyapunov exponents and fractal dimensions. These quantities are characteristics of a deterministic dynamical system, and are not defined or appropriate for the stochastic systems. Here we will not focus on estimating actual Lyapunov exponents or fractal dimensions, but instead look at features extracted by applying the appropriate algorithms to the load time histories. The hope is that these features will capture the differences that can be used to explain the discrepancies in the observed TTF.

We use the algorithm described in Chapter 2 to calculate the maximal Lyapunov exponent to obtain feature curves that describe average divergence of nearby trajectories in the phase space over finite time. Since crack propagation can make the system structurally unstable, stress data is divided into data records of 100,000 points in each. A delay time of 7 time steps is used for the delay coordinate embedding of the time series. Embedding dimension is determined for the random signal using the first minimum of false nearest neighbors (FNNs) curve [12]. Figure 38 shows that an embedding dimension of 5 is sufficient. The average trajectory divergence is estimated in each data records. The results for test 1-A and 1-B are shown in Fig. 39. As expected, there is a clear difference between the divergence

Figure 38. Estimation of embedding dimension for the random (left) and chaotic (right) signals

Figure 39. Average trajectory divergence rates for all records in random loading (left) and chaotic loading (right)

rates observed for the chaotic and random excitations for small time scales, while these features are fairly consistent during each experiment. This divergence rate at small scale for the random signal is much higher than for the chaotic, which could also explain differences observed in the histograms for the small amplitudes (larger divergence will cause fewer small distance scales in the data).

Figure 40 shows the average trajectory divergence rates for all ten tests in the first data record. These results are consistent for each test type and again show clear difference in the random and chaotic divergence, which is smaller for the chaotic excitation.

Figure 40. Average trajectory divergence rates for all tests using random loading (left) and chaotic loading (right)

Figure 41. Correlation integral for all ten tests

The correlation integral is usually used to estimate the fractal dimension of an attractor. However, since random signal does not have a finite-dimensional attractor, we use the correlation integral as a feature differentiating our signals. This integral is estimated using the method described in Chapter 2, and results are shown in Fig. 41. Again we see that for small length scales there are clear differences. For the chaotic signal there are considerably more smaller distances than

for the random signal. However, both signals have identical correlation integral for the large amplitudes or distances, which again correlates well with the similarity of the histograms for large amplitudes.

4.6 Discussion

Differences in excitation observed in the histograms did not prove sufficient to provide consistent estimates of TTF using Palmgren-Miner rule and the rain-flow counting method. Additional nonlinear characteristics (trajectory divergence rates and correlation integral) showed more drastic differences in the small time- and length-scale dynamics that can explain the differences in TTF. Larger divergence rates for small scales in the random signal clearly correlate well with shortened TTF in the random tests. For small length- and time-scales, this causes the load on the crack to change faster for the random signal than for the chaotic. Therefore, the crack under random load experiences fewer low stress cycles when compared to the chaotic load which can explain the differences in the TTF. In particular, this suggests that fatigue life depends not only on the cycles of maximum stress experienced, but also on the rate at which this maximum stress is achieved. The observed differences in divergence rates are also reflected in the deviations observed in the correlation integrals at small scale. Therefore, both of these features correlate well with the observed TTF and should be used as loading parameters for a new damage model that could account for the observed differences.

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CHAPTER 5

Coupled Field Equations of Motion for Damaged Beams

In this section, the coupled field model of a simply supported uniform Euler-Bernoulli beam containing a single-edge crack is developed via Hamilton's principle. Loading rate and divergence rate are introduced as loading parameters to account for the dependence of a fatigue process on the nonlinear characteristics identified in Chapter 4. To reduce the accumulation of damage in the system to the Paris' law or to a simple differential equation, the method of averaging is utilized as an analytical and numerical tool. Finally, to validate the proposed fatigue model, its predictions are contrasted with the experiments described in Chapter 4.

The main idea is to consider a fatigue crack as a generalized coordinate together with the common macroscopic structural displacement that describes the current configuration of a cracked beam. Instead of using the Principle of Virtual Work [1, 2], we focus on identifying internal and kinetic energies. Then Hamilton's principle is applied to obtain the coupled field equations. The approach here is based on the modification of the one-dimensional model of nonlinear damage dynamics described in Ref [3].

5.1 Equations of Motion for a Cracked Beam

A simply supported Euler-Bernoulli beam with a crack is shown in Fig. 42. Let $w(x, t)$ denote the deflection of the beam at spatial coordinate x and time t . The scalar damage field $\phi(x, t)$ is introduced to represent the current damage state, $0 \leq \phi \leq 1$ (0 for the undamaged state and 1 for fractured). At this point, the damage field is unspecified and is viewed as an abstract, slowly evolving variable [4]. However, $\phi(x, t)$ can be conceived as a field of microcracks. A more precise definition of the damage variable will be discussed in the next section. Although

the presence of the cracks changes a number of material properties, in this study we assume damage accumulation only affects in changing the potential energy by weakening of the beam's bending stiffness [5, 6]. The next assumption is that the variations of macroscopic structural displacement and the damage variables are independent. The following derivation follows closely the one in Ref. [3].

The total potential energy of our system can be written as:

$$V = \int_0^L \left(\frac{1}{2} \bar{E} \bar{I} w''^2 + \frac{1}{2} \kappa \phi'^2 \right) dx, \quad (33)$$

where the first term is the macroscopic elastic strain energy determined by the classical beam theory [7]. Here, $\bar{E} \bar{I} = \bar{E} \bar{I}(x, \phi)$ is the effective stiffness, and the prime denotes differentiation with respect to x . The second term of the integrand is the simplest form of a microstructural interface energy associated with the damage variable [8, 9]. However, the distribution of cracks in structures is not a smooth function of spatial coordinates because of stress concentration. Since this study considers an already cracked beam and in an effort to make sense of Eq. (33), we set $\kappa = 0$.

The kinetic energy is expressed in the following form:

$$T = \int_0^L \left(\frac{1}{2} m_\rho \dot{w}^2 + \frac{1}{2} \mu \dot{\phi}^2 \right) dx, \quad (34)$$

where $m_\rho(x)$ is the mass per unit length of the beam, $\mu(\phi, x)$ is micro-inertia per unit length, and the overdot denotes differentiation with respect to time, t . The second term in Eq. (34) represents the microkinetic energy. As mentioned above, we assume that the damage variable accumulates slowly (e.g., a crack grows over thousands of load cycles). The microkinetic energy is much smaller than the macroscopic kinetic energy. Therefore, it is reasonable to ignore the effect of micro-inertia by setting $\mu = 0$.

Figure 42. Geometry of a simply supported beam with an edge crack

The virtual work done by nonconservative forces can be written as

$$\delta W = \int_0^L \left(f(x, t) \delta w - c \dot{w} \delta w - \Phi(\phi, \dot{\phi}) \delta \phi \right) dx, \quad (35)$$

where $f(x, t)$ is the body force per unit length, c is a linear viscous damping coefficient, and $\Phi \geq 0$ is the generalized force associated with the damage variable ϕ . Since damage growth is expected to dissipate energy, the Φ term is kept negative.

Hamilton's principle for any two arbitrary times t_1 and t_2 is given in the form

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt = 0. \quad (36)$$

Taking the first variation of the kinetic energy component in Eq. (34) and integrating it over the length of the beam yields the following:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \left([m_\rho \dot{w} \delta w]_{x=0}^{x=L} - \int_0^L m_\rho \ddot{w} \delta w dx \right) dt.$$

Since $w(0, t) = w(L, t) = 0$ for a simply supported beam, we obtain

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \int_0^L (-m_\rho \ddot{w} \delta w) dx dt. \quad (37)$$

The integral of the variation of the potential energy in Eq. (33) is

$$\int_{t_1}^{t_2} \delta V dt = \int_{t_1}^{t_2} \int_0^L \left(\bar{E} \bar{I} w'' \delta w'' + \frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial \phi} \delta \phi \right) dx dt, \quad (38)$$

and integrating its first term by parts two times, we get

$$\int_{t_1}^{t_2} \int_0^L \bar{E} \bar{I} w'' \delta w'' dx dt = \int_{t_1}^{t_2} \int_0^L (\bar{E} \bar{I} w'')'' \delta w dx dt, \quad (39)$$

where we again used the fact that for simply supported beam $w(0, t) = w(L, t) = 0$ and $w''(0, t) = w''(L, t) = 0$. Hence, Eq. (38) becomes

$$\int_{t_1}^{t_2} \delta V dt = \int_{t_1}^{t_2} \int_0^L \left[(\bar{E} \bar{I} w'')'' \delta w + \frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial \phi} \delta \phi \right] dx dt. \quad (40)$$

Inserting Eqs. (35), (37), and (40) into Eq. (36) and collecting the corresponding variational terms, we get

$$\begin{aligned} & - \int_{t_1}^{t_2} \int_0^L \left[m_\rho \ddot{w} + (\bar{E} \bar{I} w'')'' + c \dot{w} - f(x, t) \right] \delta w dx dt \\ & - \int_{t_1}^{t_2} \int_0^L \left(\frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial \phi} + \Phi \right) \delta \phi dx dt = 0. \end{aligned} \quad (41)$$

Since the variations $\delta \phi$ and δw are arbitrary and independent of each other, Eq. (41) can be satisfied only if the coefficients of $\delta \phi$ and δw are zero. Therefore, we obtain the following coupled system of partial differential equations:

$$m_\rho \ddot{w} + (\bar{E} \bar{I} w'')'' + c \dot{w} = f(x, t), \quad (42a)$$

$$\frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial \phi} + \Phi = 0. \quad (42b)$$

The partial differential equations will also have the corresponding boundary conditions $w(0, t) = w''(0, t) = 0$ and $w(L, t) = w''(L, t) = 0$. Eq. (42a) is the transverse vibration of the beam, in which the stiffness can vary with the damage variable. Eq. (42b) describes the evolution of the damage ϕ . From Eq. (42b), the generalized force Φ has a similar expression to the crack driving force G described in Refs. [10, 11] which is the rate of change in potential energy with respect to crack area

$$G = - \frac{\partial U}{\partial a}, \quad (43)$$

where U is the potential energy, a is the crack length. At the location of the crack, U is given by

$$U = \frac{1}{2} \bar{E} \bar{I} w''^2. \quad (44)$$

Substituting Eq. (44) into Eq. (43), we get $G = \frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial a}$, which is similar to Φ in Eq. (42b).

5.2 Damage Variable and the Effective Stiffness

A general form of the damage evolution equation is given by the expression in Eq. (42b). However, the mechanism of damage is unspecified there. A variety of ways to define material damage have been proposed in the past, ranging from scalar to tensorial. In our problem, a V-notch is machined at the center of the beam. Therefore, the crack length is the most appropriate definition of the damage variable. Since the beam has only one crack at the center, $\phi(x, t) = 0$ for $\forall x \neq L_1$. At $x = L_1$, we normalize ϕ as

$$\phi = \frac{a}{b}, \quad (45)$$

where a is the crack length, and b is the width of the beam.

The simplest form of the beam stiffness [1] at the cracked cross-section is

$$\bar{E} \bar{I}(\phi, L_1) = EI(1 - \phi), \quad (46)$$

where E and I are the elastic modulus and the second moment of area of the undamaged beam, respectively.

The presence of a crack changes the stiffness distribution in the vicinity of the cracked cross-section. Therefore, we can define the effective stiffness in the form

$$\bar{E} \bar{I}(\phi, x) = EI[1 - \phi D(x - L_1)], \quad (47)$$

where $D(x)$ is a spatial function that decays exponentially along the length of the beam and has its maximum value at the crack tip ($D(L_1) = 1$). A particular expression of $D(x)$ can be found in Ref [5].

5.3 Damage Growth Rate Law

In Eq. (42b), the evolution law for ϕ is only derived by assuming Φ explicitly depends on $\dot{\phi}$. For materials without a fatigue limit (e.g., aluminum, copper), cracks always propagate as long as applied loads fluctuate. To characterize the dependance of Φ on a fluctuating stress σ , we define the loading rate parameter as

$$R_\sigma = \sqrt{\frac{\langle \dot{\sigma}^2 \rangle}{\langle \sigma^2 \rangle}}, \quad (48)$$

where $\langle x \rangle$ means the expectation value of x . Thus, for a cyclic stress $\sigma = \frac{\Delta\sigma}{2} \cos(\omega t)$, $R_\sigma = \omega$. The experiments in Chapter 4 suggested that the divergence rate and correlation sum of the excitation time series should be incorporated in a fatigue damage model. However, the differences in the correlation sum at small scales can be explained by the differences in the histograms at small amplitudes (see Fig. 28). In particular, chaotic and random loadings have the same number of large amplitudes but different number of small amplitudes. The difference can be recognized by the roughness of time series. Therefore, the loading rate can reflect the correlation sum of an applied load.

The first term of equation Eq. (42b) is always nonzero when $w'' \neq 0$. Therefore, if Φ is only a function of ϕ and $\dot{\phi}$, we will get $\dot{\phi} > 0$ as long as $w'' \neq 0$. This means the crack always propagates, even under static loads. We address this problem and take the nonlinear characteristics of the applied load into the account by making Φ explicit function of R_σ as follows:

$$\Phi(\phi, \dot{\phi}, R_\sigma) = \frac{1}{\eta(\lambda)} \frac{f(\phi) \dot{\phi}^p}{R_\sigma^q}. \quad (49)$$

where p, q are some positive exponents and λ is the divergence rate of the applied load. λ is determined by the slope of the average trajectory divergence curve defined in Chapter 2. η and f are some functions, and will be determined later.

Now, consider Eq. (42b) at the location of the crack, $x = L_1$

$$\frac{1}{2}w''^2(L_1, t) \frac{\partial \bar{E} \bar{I}}{\partial \phi(L_1, t)} + \Phi(\phi(L_1, t), \dot{\phi}(L_1, t), R_\sigma) = 0. \quad (50)$$

For simplicity, let $\phi \equiv \phi(L_1, t)$ to get

$$\Phi(\phi, \dot{\phi}, R_\sigma) = \frac{1}{2}w''^2(L_1)EI. \quad (51)$$

Substituting $w''(L_1) = \frac{2\sigma}{bE}$ into the above gives the following expression

$$\Phi = \frac{2I}{b^2E}\sigma^2. \quad (52)$$

Finally, solving the Eq. (49) for $\dot{\phi}$, we obtain a crack propagation model:

$$\dot{\phi} = \left[\frac{2\eta(\lambda)I}{b^2E} \right]^{\frac{1}{p}} R_\sigma^{\frac{q}{p}} f(\phi)^{-\frac{1}{p}} \sigma^{\frac{2}{p}}. \quad (53)$$

To check the validity of the damage model derived above, we compare Eq. (53) with Paris' law for macroscopic crack growth under cyclic load $\sigma = \Delta\sigma \cos(\omega t)/2$. In particular, we substitute $R_\sigma = \omega$ and $\lambda = 0$ into Eq. (53) to get

$$\dot{\phi} = \left[\frac{\eta(0)I}{2b^2E} \right]^{\frac{1}{p}} \omega^{\frac{q}{p}} f(\phi)^{-\frac{1}{p}} \Delta\sigma^{\frac{2}{p}} (\cos(\omega t)^2)^{\frac{1}{p}}, \quad (54)$$

or

$$\dot{\phi} = C_0 \omega^{\frac{q}{p}} f(\phi)^{-\frac{1}{p}} \Delta\sigma^{\frac{2}{p}} (\cos(\omega t)^2)^{\frac{1}{p}}, \quad (55)$$

where all material constants and systems parameters have been merged into the constant C_0 . As mentioned above, ϕ is a slow variable, therefore, C_0 is supposed to be a very small positive number. Therefore, we can apply the method of averaging [12] to Eq. (55) yielding

$$\dot{\phi} \approx C_0 \omega^{\frac{q}{p}} f(\phi)^{-\frac{1}{p}} \Delta\sigma^{\frac{2}{p}} \left(\frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} (\cos(\omega t)^2)^{\frac{1}{p}} dt \right), \quad (56)$$

or

$$\dot{\phi} \approx C_1 \omega^{\frac{q}{p}} f(\phi)^{-\frac{1}{p}} \Delta\sigma^{\frac{2}{p}}, \quad (57)$$

where $C_1 = C_0 I_d$, I_d is the value of the integral inside the brackets of Eq. (56). Changing to a new time variable $N = \omega t / (2\pi)$ and letting $\phi = a/b$, Eq. (57) can be rewritten to get

$$\frac{da}{dN} \approx C_2 \omega^{\frac{q}{p}-1} f(\phi)^{-\frac{1}{p}} \Delta \sigma^{\frac{2}{p}}, \quad (58)$$

where C_2 is a constant, $C_2 = 2\pi b C_1$.

On the other hand, Paris' law for macroscopic crack growth under cyclic load is given by the following equation:

$$\frac{da}{dN} = C (\Delta K_I)^m, \quad (59)$$

where N is the number of load cycles, C and m are material constants. ΔK_I is the amplitude of the change in the stress intensity factor during one forcing cycle. In literature, K_I is given in a variety of forms. However, in general, it can be expressed as $K_I = \sigma \sqrt{\pi a} F(\phi)$. Using this expression for K_I , Eq. (59) becomes

$$\frac{da}{dN} = C_3 \left[\sqrt{\phi} F(\phi) \right]^m \Delta \sigma^m, \quad (60)$$

where C_3 is a constant, $C_3 = C(\pi b)^{\frac{m}{2}}$.

Comparing the exponents of ω , $\Delta \sigma$, and the functions of ϕ in Eqs. (58) and (60) yields

$$\frac{q}{p} = 1 \rightarrow q = p, \quad (61)$$

$$\frac{2}{p} = m \rightarrow p = \frac{2}{m}, \quad (62)$$

$$f(\phi) = \left[\sqrt{\phi} F(\phi) \right]^{-mp}. \quad (63)$$

For metals, m is typically between 2 and 4 [10]. In the experiments, the specimen is made from aluminum ($m \approx 3$). By setting $m = 3$, we get $q = p = \frac{2}{3}$, and $f(\phi) = \left[\sqrt{\phi} F(\phi) \right]^{-2}$.

Figure 43. Beam diagram with applied forces

5.4 Experimental Validation

For cyclic loads, Eq. (53) recovers the Paris' law. Hence, in this section, the fatigue processes under irregular, non-cyclic loads described in Chapter 4 are utilized again to identify the function $\eta(\lambda)$ and to verify the new model.

The beam diagram and applied forces are shown in Fig. 43. The applied loads can be expressed as

$$\begin{aligned} P_1 &= P - m a, \\ P_2 &= \frac{P + m a}{2}, \end{aligned} \quad (64)$$

where P is the static mean force due to the pressure within the cylinders, and a_{rel} is the relative acceleration between the masses and the slip table: $a = a_{mass} - a_{table}$.

The coupled field equation of the cracked beam is given by

$$\begin{aligned} m_\rho \ddot{w} + (\bar{E} \bar{I} w'')'' + c \dot{w} &= P_1 \delta_D(x - L/2) - P_2 \delta_D(x - L_2) \\ &\quad - P_2 \delta_D(x - L + L_2), \end{aligned} \quad (65)$$

$$\frac{1}{2} w''^2 \frac{\partial \bar{E} \bar{I}}{\partial \phi} + \Phi(\phi, \dot{\phi}, R_\sigma) = 0, \quad (66)$$

Figure 44. Determination of K_I by the principle of superposition

where $\delta_D(x)$ is the Dirac delta function. In the experiments, the measurements of $w''(L/2, t)$ are not available. However, we can estimate $w''(L/2, t)$ by the recorded measurements of $w(L_0, t)$ (at $L_0 = 40$ mm) and assuming the solution of Eq. (65) can be expressed by separation of spatial and temporal variables

$$w(x, t) = q(t)\psi(x), \quad (67)$$

where $\psi(x)$ is the mode shape of the beam oscillations [7]. Then, we get

$$w''(L/2, t) = w(L_0, t) \frac{\psi''(L/2)}{\psi(L_0)}. \quad (68)$$

The ratio $\psi''(L/2)/\psi(L_0)$ is a constant, therefore, $w''(L/2, t)$ and $w(L_0, t)$ are linearly proportional.

The stress intensity factor is obtained from the principle of superposition [11] as shown in Fig. 44:

$$K_I^{(a)} = K_I^{(b)} - K_I^{(c)}, \quad (69)$$

where $K_I^{(b)}$ and $K_I^{(c)}$ are stress intensity factors for a single edge notched beam bending and a pure beam bending [13], respectively:

$$K_I^{(b)} = \sigma^{(b)} \sqrt{\pi a} F_I^{(b)}(\phi), \quad (70)$$

with $\sigma^{(b)} = 6M^{(b)}/hb^2$, $M^{(b)} = P_1 L/4$, and

$$F_I^{(b)}(\phi) = 1.106 - 1.552\phi + 7.71\phi^2 - 13.53\phi^3 + 14.23\phi^4; \quad (71)$$

and

$$K_I^{(c)} = \sigma^{(b)} \sqrt{\pi a} F_I^{(c)}(\phi), \quad (72)$$

with $\sigma^{(c)} = 6M^{(b)}/hb^2$, $M^{(b)} = P_2 L_1$, and

$$F_I^{(c)}(\phi) = 1.122 - 1.40\phi + 7.33\phi^2 - 13.08\phi^3 + 14.0\phi^4. \quad (73)$$

From Eq. (64), $\Delta P_1 = -m\Delta a$, $\Delta P_2 = m\Delta a/2$, and we obtain

$$\Delta K = m \frac{3\sqrt{\pi}L}{\sqrt{b}h^2} \Delta a \left(\frac{1}{2} \sqrt{\phi} F_I^{(b)} + \frac{L_2}{L} \sqrt{\phi} F_I^{(c)} \right). \quad (74)$$

The function $f(\phi)$ in Eq. (53) has the following form

$$f(\phi) = \left(\frac{1}{2} \sqrt{\phi} F_I^{(b)} + \frac{L_2}{L} \sqrt{\phi} F_I^{(c)} \right)^{-2}. \quad (75)$$

Using the relationship $w''(L/2) = \frac{2\sigma}{bE}$ and Eq. (68), the loading rate R_σ can be determined by:

$$R_\sigma = \sqrt{\frac{\langle \dot{\sigma}^2 \rangle}{\langle \sigma^2 \rangle}} = \sqrt{\frac{\langle \dot{w}_{x=L_0}^2 \rangle}{\langle w_{x=L_0}^2 \rangle}}. \quad (76)$$

Finally, the damage growth rate in Eq. (53) can be written as

$$\dot{\phi} = A(\lambda) \left(\frac{1}{2} \sqrt{\phi} F_I^{(b)} + \frac{L_2}{L} \sqrt{\phi} F_I^{(c)} \right)^3 R_\sigma \langle w_{x=L_0}^3 \rangle, \quad (77)$$

where $A(\lambda) = \left[\frac{EI\psi''(L/2)^2}{2\psi(L_0)^2} \eta(\lambda) \right]^{\frac{3}{2}}$ depends on material properties, geometry of the beam, the mode shape ψ , and the divergence rate λ . Instead of identifying the function $\eta(\lambda)$, we focus on estimating the function $A(\lambda)$. In what follows, the measurements of the displacement of the beam are used to advance and validate the model. In each test, the time series $w(L_0, t)$ are recorded at 1 kHz sampling rate, and are split into small data records, where each record contains 10,000 points. The estimated divergence rates are shown in Table 4. It is assumed that the changes in the damage variable over one data record (i.e., over 10 s time span) are negligible due to its slow-time nature. The method of averaging is applied to

Table 4. Divergence rates and values of function $A(\lambda)$

	Chaotic		Random	
	Divergence rate λ	$A(\lambda)$	Divergence rate λ	$A(\lambda)$
1	15.3 ± 2.4	29×10^{-7}	32.8 ± 1.4	50×10^{-7}
2	15.0 ± 2.6	30×10^{-7}	32.4 ± 1.4	44×10^{-7}
3	15.3 ± 3.3	32.5×10^{-7}	32.3 ± 1.3	45×10^{-7}
4	15.3 ± 2.7	26.5×10^{-7}	32.3 ± 1.2	42×10^{-7}
5	15.3 ± 3.6	32×10^{-7}	33.0 ± 1.0	47×10^{-7}

Figure 45. Comparison of the ACPD signal (red —) with numerical integration (blue - -) for chaotic test (left) and stochastic test (right)

simulate Eq. (77), but instead of integrating over one forcing period, we integrate over one data record. Thus, the fast time function $w(L_0, t)^3$ is replaced by its moving average $\langle w(L_0, t)^3 \rangle_t$.

In Eq. (77), the function $A(\lambda)$ is unknown. However, due to the notch, the initial damage $\phi_0 = 0.375$, and the experiments are programmed to stop when $\phi_f \approx 0.6$. Using these initial and final conditions, the value of A is determined through *ad hoc* matching of the observed trajectories and shown in Table 4. The resulting damage trajectories from numerical simulations of Eq. (77) and the corresponding ACPD signals are normalized and shown in Fig. 45 for two fatigue tests, 1A and 1B which correspond to $A = 29 \times 10^{-7}$ and 50×10^{-7} , respectively. The

Figure 46. Exponential curve fitting

comparisons of the real damage and the simulated damage of the rest of fatigue tests are in Appendix D. In the real world, engineering materials don't have self-healing properties. The function $A(\lambda) \geq 0$ for all λ . If the response of the system exhibits a super stable fixed point (e.g., a critically damped oscillator), the system decays to its equilibrium point as quickly as possible. In this situation, $\lambda = -\infty$ and $A(\lambda) = 0$ because there is no crack propagation. The analysis in Chapter 4 showed that the larger λ of the applied load results in the higher crack growth rate. Especially, when the system grows unbounded, λ is expected to be a large number, the failure of the system should happen immediately. Therefore, A is assumed to be an exponential function of λ :

$$A(\lambda) = \alpha e^{\beta\lambda}, \quad (78)$$

where α and β are constants. To determine the values of α and β , three fatigue tests under a periodic load ($\lambda = 0$) were conducted and $A(0)$ was estimated. Then applying the exponential curve fitting yields $\alpha = 20 \times 10^{-7}$ and $\beta = 0.02$ (see Fig. 46)

Table 5. Estimated Time to Failure for Large Amplitude

Chaotic			Random		
Actual TTF (hrs:min)	Estimated TTF (hrs:min)	Error	Actual TTF (hrs:min)	Estimated TTF (hrs:min)	Error
1:44	1:53	8%	0:53	1:10	32%
1:50	2:21	28%	1:07	1:06	0%
1:46	2:01	14%	0:44	0:55	25%
1:42	1:43	0%	0:52	1:00	15%
1:13	1:41	35%	0:45	0:54	20%

5.5 Predicting Time to Failure

The ultimate goal of all fatigue crack propagation models is to predict the fatigue life of a system. If the applied load is available, the proposed model can simulate the fatigue crack evolution in time, then the failure can be determined. However, it is not practical to monitor the response of a system for the whole fatigue life. The failure of a system should be estimated from the structural response during the beginning of the fatigue life. In the following, our model described by Eq. (77) is used to estimate the time to failure of the specimen. The loading rate R_σ , divergence rate λ , and $\langle w_{x=L_0}^3 \rangle$ are determined from the first 200,000 points (≈ 3 minutes of the fatigue life) of the displacement time series of the specimen at $x = 40mm$. Then the TTF is calculated by integrating Eq. (77) from $\phi = 0.375$ to $\phi = 0.6$.

Table 5 shows the estimated TTF for ten fatigue tests in Chapter 4. In these experiments, the amplitude is large so that the fatigue process mainly stays in the Paris regime [10]. Table 6 shows the estimated TTF for experiments with smaller amplitude.

Table 6. Estimated Time to Failure for Small Amplitude

Chaotic			Random		
Actual TTF (hrs:min)	Estimated TTF (hrs:min)	Error	Actual TTF (hrs:min)	Estimated TTF (hrs:min)	Error
3:56	3:11	19%	5:21	4:56	8%
5:45	4:36	20%	8:05	7:05	12%
4:38	4:50	4%	3:53	3:32	9%

5.6 Discussion

Figure 45 shows the comparison between the real crack growth trajectory and the damage simulated using Eq. (77). The plots in this figure suggests that the fatigue model proposed in this study provides accurate damage evolution predictions. The small differences between the real crack and simulated crack evolutions can be explained by the inherent variations/indeterminicity in the initial condition. In all tests reported here, we assume the initial damage to be the same, $\phi_0 = 0.375$. However, the actual initial cracks in each test are different due to the notch fabrication tolerances and material irregularities effecting the microstructure of the beam. In practice, it is difficult, if not impossible, to determine the accurate size of the initial crack. In addition, in Eq. (77), environmental factors are neglected (e.g., temperature, humidity). However, even in our controlled environment there were parameters that were not tracked or controlled such as ambient temperature variations.

For the fatigue life prediction, the difference between the TTF and the estimated TTF is mainly caused by the estimated load parameters. The dynamics of the system change the applied load to the specimen. However, the load parameters were not calculated from the whole time series. Particularly, in the experiment, the applied load increases when the fatigue crack grows (see Fig. 47). Hence, the

Figure 47. Dependence of $R_\sigma \langle w_{x=L_0}^3 \rangle$ in time (Chaotic signal 1-B)

model overestimated the TTF. Figure 47 also suggests that one should take into account the dependence of load parameter on the fatigue crack when estimating TTFs. In practical situations, a safety factor needs to be included in results.

In the dissertation, Paris' law is used as a special case to develop the fatigue model. Therefore, this model is only valid in Paris regime which requires long cracks and ignores the threshold limit and load ratio effects. The crack initiation were not taken into account in the model, thus the estimated TTF is smaller than the real TTF for small load amplitude. The inconsistency of TTF in Table 6 also describes the sensitivity of the crack initiation to the microstructure of the beam material.

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CHAPTER 6

Conclusion

New nonlinear characteristic features for dynamical systems based on Birkhoff Ergodic theorem were introduced. The characteristic lengths measure the average length of two points in a phase space separated by a fixed time interval, and yield characteristic length spectrum. This spectrum provides information about nonlinear correlations in the data and can be used in lieu of mutual information to estimate parameters for delay coordinate embedding. The characteristic distances are average distances from fixed points in a phase space to all points on the attractor. They are sensitive to small parameter changes and can be used to track or identify these changes in conjunction with SOD. Both metrics are simple and easy to calculate, making them suitable for online, real-time applications. Using these characteristics and SOD analysis, slowly evolving damage variables were continuously tracked and identified in simple simulations and experiments.

Fatigue process under chaotic and random excitations were conducted using a novel fatigue testing apparatus to investigate the coupling between applied loads and fatigue life. In the experiments, both loadings shared the same statistical and spectral properties. However, experimental results showed that random loading was more damaging than chaotic. Standard, linear Palmgren-Miner rule and the rain-flow counting method grossly overestimated damage for the chaotically excited system, while they were adequate to describe damage in randomly excited system. Nonlinear dynamical characteristics of the excitation time series were considered to explain the differences. These metrics clearly showed that while the signals had similar characteristics at large scale, for a small scale trajectory divergence rates were much higher in the random signal. In addition, there was

finer structure/detail in the chaotic signal as shown by the correlation sum. These observations stipulated that crack was experiencing fewer low stress cycles for the random excitation, which was also reflected in the excitation histograms. These results also suggested that fatigue life depends on the rate at which maximum stress is reached at the crack tip. Therefore, both of the considered nonlinear characteristics correlate well with the observed differences in the TTF, and should be used as loading parameters for the new damage laws providing more consistent TTF estimates.

The nonlinear characteristics were then used to develop a new dynamic model of fatigue evolution. The model can describe fatigue damage evolution under arbitrary loading, and also captures the interaction between the fatigue damage and structural dynamics. Using the method of averaging, the crack evolution was numerically simulated using the experimental measurements of the structural vibration as input. The damage model predictions closely followed the actual, experimentally measured, crack growth trajectories in both chaotically and randomly forced tests. Then the TTFs were estimated from the short-time response of the system in several fatigue experiments. The initial results have been shown that the proposed model can be used for estimating fatigue life.

While many materials have a fatigue limit (i.e., a loading threshold below which damage will not occur), the author restricted attention to materials without a fatigue limit. Future challenges will include incorporation of fatigue limits. This problem may be overcome by setting a threshold in the generalized force Φ . Fatigue accumulation occurs in several stages that have different physical mechanisms. Therefore, fatigue life prediction will improve considerably if for each of these stages corresponding predictive models are used. Future investigation is how to detect transitions from the initiation to the crack growth period. Finally, the damage

tracking results depends on the number of characteristic distances and locations of the fixed points. These factors are not suggested in the current tracking damage algorithm. Hence, the algorithm should be optimized for practical applications.

APPENDIX A

List of Acronyms

Table A.1. List of acronyms used throughout this dissertation

FNN	False Nearest Neighbor
AMI	Average Mutual Information
SOD	Smooth Orthogonal Decomposition
SOM	Smooth Orthogonal Mode
SPM	Smooth Projection Mode
SOC	Smooth Orthogonal Coordinate
CD	Characteristic Distance
ACPD	Alternating Current Potential Difference
TTF	Time To Failure

APPENDIX B Specimen Drawing

APPENDIX C

Damage Tracking Results

C.1 Tracking Result from the Displacement of the Specimen

Figure C.1. Top: test 2A; Bottom: test 2B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.2. Top: test 3A; Bottom: test 3B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.3. Top: test 4A; Bottom: test 4B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.4. Top: test 5A; Bottom: test 5B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

C.2 Tracking Result from the Acceleration of the front Mass

Figure C.5. Top: test 2A; Bottom: test 2B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.6. Top: test 3A; Bottom: test 3B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.7. Top: test 4A; Bottom: test 4B; First ten SOVs (a), ACPD signal (b), Dominant SOC_1 corresponding to the largest SOV (c), plot of SOC_1 versus ACPD signal (d).

Figure C.8. Top: test 5A; Bottom: test 5B; First ten SOVs (a), ACPD signal (b), Dominant SOC₁ corresponding to the largest SOV (c), plot of SOC₁ versus ACPD signal (d).

APPENDIX D
Simulation Results

Figure D.1. Comparison of the ACPD signal (red —) with numerical integration (blue - -) for chaotic test (left) and stochastic test (right)

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