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E. Magyari

H. Thomas

R. Weber

C. Kaufman

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

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Integrable and nonintegrable classical spin clusters: integrability criteria and analytic structure of invariants

E. Magyari, H. Thomas, and R. Weber
Institut für Physik der Universität Basel, Basel, Switzerland

C. Kaufman and G. Müller
Department of Physics, University of Rhode Island, Kingston RI 02881, USA

The nonlinear dynamics is investigated for a system of N classical spins. This represents a Hamiltonian system with N degrees of freedom. According to the Liouville theorem, the complete integrability of such a system requires the existence of N independent integrals of the motion which are mutually in involution. As a basis for the investigation of regular and chaotic spin motions, we have examined in detail the problem of integrability of a two-spin system. It represents the simplest autonomous spin system for which the integrability problem is nontrivial. We have shown that a pair of spins coupled by an anisotropic exchange interaction represents a completely integrable system for any values of the coupling constants. The second integral of the motion (in addition to the Hamiltonian), which ensures the complete integrability, turns out to be quadratic in the spin variables. If, in addition to the exchange anisotropy also single-site anisotropy terms are included in the two-spin Hamiltonian, a second integral of the motion quadratic in the spin variables exists and thus guarantees integrability, only if the model constants satisfy a certain condition. Our numerical calculations strongly suggest that the violation of this condition implies not only the nonexistence of a quadratic integral, but the nonexistence of a second independent integral of motion in general. Finally, as an example of a completely integrable N-spin system we present the Kittel-Shore model of uniformly interacting spins, for which we have constructed the N independent integrals in involution as well as the action-angle variables explicitly.

I. INTRODUCTION

The question of integrability and nonintegrability of classical and quantum model systems, which draws increasing attention in many areas of physical research, is being investigated in four major contexts:

(i) In the context of classical dynamical systems with few degrees of freedom, the textbook examples of classical mechanics embody the hallmark of integrability. They have left the imprint of a paradigm, in spite of the realization a long time ago that the general formalism for the solution of such systems (the elegant Hamilton-Jacobi theory) breaks down as a practical tool for the majority of systems with as few as two degrees of freedom. The theory of Hamiltonian chaos, which emerged from that crisis, has now reached a fair degree of maturity [1]. Nevertheless, many important questions have remained open. Perhaps the most intriguing unsolved problem for nonintegrable Hamiltonian systems is the calculation of dynamic correlation functions, which in a large number of applications serves as the link to dynamical experiments in physical realizations.

(ii) In the context of classical dynamical many-body systems, nonintegrability is a prerequisite of ergodicity and mixing behavior, which in turn are the bedrock of statistical mechanics. However, powerful theorems and exactly solved models have given us ample warning that a large or infinite number of degrees of freedom does not guarantee these properties, even for strongly coupled systems. In fact, we have knowledge of a growing number of integrable classical many-body systems solvable by the inverse scattering transform, whose time evolution is governed by a few types of nonlinear excitations only [2]. However, dynamic correlation functions are nontrivial even for integrable models, and the interesting long-time asymptotic behavior is, in general, not known.

(iii) In the context of quantum few-body systems, there is convincing evidence that such systems have qualitatively different properties in the vicinity of the classical limit, depending on whether the associated classical system is dynamically integrable or nonintegrable [3]. To what extent such effects can be regarded as manifestations of quantum chaos is an unsettled issue.

(iv) Finally, in the context of quantum many-body systems integrable models have been studied in great detail, for example the class of Bethe ansatz solvable models. However, the dynamical properties of such systems, which are of primary importance for experimental comparisons, are in general highly nontrivial and not exactly known except for very special models. The study of nonintegrability effects in quantum many-body systems, on the other hand, is new territory in physical research, which is currently gaining momentum at a rapid rate [4, 5].

All these aspects of integrability and nonintegrability can be analyzed by a study of classical or quantum spin systems. Model systems containing a finite number N of quantum spins, each with quantum number s, are always integrable. Effects of nonintegrability can therefore be expected only in either one of the following two limits:

- classical limit: N finite, $s \to \infty$,
- thermodynamic limit: s finite, $N \to \infty$.

In the classical limit, one expects to observe manifestations of classical dynamical chaos if the corresponding classical spin system is nonintegrable. In the thermodynamic limit, on the other hand, one expects to observe manifestations of quantum chaos if the infinite quantum spin system is nonintegrable.

The present work represents a specific contribution to a detailed investigation of the various aspects of nonintegrability in classical and quantum spin systems: the study of the nonintegrability criteria and of the analytic structure of invariants for classical spin clusters. Pairs of interacting classical spins, which are the primary object of this paper, certainly belong to the simplest (autonomous) classical Hamiltonian systems in which deterministic chaos can be studied. Chaos in classical spin systems was previously investigated by Feingold, Moiseyev and Peres [6] (autonomous 2-spin system), by Nakamura, Nakahara and Bishop [7] (autonomous 3-spin system), and by Frahm and Mikeska [8] (nonautonomous 1-spin system), mainly in the context of studies of nonintegrability effects of the corresponding quantum spin clusters for $s \to \infty$.

In Sect. II we set up the formalism for the description of classical spin systems in the framework of Hamiltonian dynamics and establish the formal relationship to quantum spin systems. Furthermore, we briefly review the general integrability criteria for both autonomous and non-autonomous cases of such systems. The main theme of Sect. III is the investigation of the integrability criterion for autonomous 2-spin systems with anisotropic exchange coupling and single-site anisotropy, resulting in the explicit construction of a second integral of the motion (in addition to the energy) for a large class of such systems. We provide numerical evidence that the systems which violate our integrability criterion exhibit chaotic motion. For a single spin in a time-dependent external field we give a constructive criterion for the existence of a time-dependent integral of the motion, which guarantees integrability. Finally, we demonstrate the complete integrability of the Kittel-Shore model of N uniformly interacting spins by the explicit transformation of the Hamiltonian to action-angle variables.

II. DYNAMIC INTEGRABILITY OF CLASSICAL SPIN SYSTEMS

A. Description of classical spin systems

Consider a system of N identical classical spins, i.e. angular momentum vectors $\mathbf{S}_l$ of constant length $|\mathbf{S}_l| = S$, specified by a spin Hamiltonian $H(\mathbf{S}_1, \ldots, \mathbf{S}_N)$. The time evolution of this system is governed by equations of motion of the form

$$\frac{\mathrm{d}\mathbf{S}_l}{\mathrm{d}t} = \mathbf{S}_l \times \mathbf{h}_l \quad (l = 1, \ldots, N), \tag{II.1}$$

where

$$\mathbf{h}_l = -\partial H / \partial \mathbf{S}_l \tag{II.2}$$

is the effective field acting on spin $\mathbf{S}_l$, determined by the instantaneous value of the external field and the configuration consisting of the spin $\mathbf{S}_l$ and all spins $\mathbf{S}_{l'}$ in direct interaction with $\mathbf{S}_l$. The structure of the equations of motion (II.1) guarantees that the length $|\mathbf{S}_l|$ of each spin remains constant,

$$\mathbf{S}_l^2 = S^2 = \text{const.} \quad (l = 1, \ldots, N). \tag{II.3}$$

Thus, only $2N$ out of the $3N$ equations of motion (II.1) are independent. This may be taken into account by expressing the $\mathbf{S}_l$ in terms of spherical coordinates

$$\begin{aligned} \mathbf{S}_l &= (S_l^x, S_l^y, S_l^z) \\ &= S(\sin\vartheta_l \cos\phi_l, \sin\vartheta_l \sin\phi_l, \cos\vartheta_l) \end{aligned} \tag{II.4}$$

or by using some other representation (e.g. stereographic projection).

The third component of the vector equation (II.1) yields

$$\begin{aligned} \dot{S}_l^z &= S_l^y \frac{\partial H}{\partial S_l^x} - S_l^x \frac{\partial H}{\partial S_l^y} \\ &= -\sum_{\alpha=x,y,z} \frac{\partial H}{\partial S_l^\alpha} \frac{\partial S_l^\alpha}{\partial \phi_l} = -\left(\frac{\partial H}{\partial \phi_l}\right)_{S_l^z}. \end{aligned} \tag{II.5}$$

Similarly, the first two components of (II.1) can be combined into the equation

$$\begin{aligned} S \sin\vartheta_l \dot{\phi}_l &= -S\cos\vartheta_l \left(\frac{\partial H}{\partial S_l^x}\cos\phi_l + \frac{\partial H}{\partial S_l^y}\sin\phi_l\right) \\ &\quad + S\sin\vartheta_l \frac{\partial H}{\partial S_l^z} \\ &= -\sum_\alpha \frac{\partial H}{\partial S_l^\alpha}\frac{\partial S_l^\alpha}{\partial \vartheta_l} = -\left(\frac{\partial H}{\partial \vartheta_l}\right)_{\phi_l} \end{aligned}$$

whence

$$\dot{\phi}_l = \left(\frac{\partial H}{\partial S_l^z}\right)_{\phi_l}. \tag{II.6}$$

This proves that the equation of motion (II.1) for any given system of N classical spins (specified by some energy function H) represents, in fact, a Hamiltonian system with N degrees of freedom and with

$$p_l = S_l^z = S\cos\vartheta_l, \quad q_l = \phi_l; \quad l = 1, \ldots, N \tag{II.7}$$

being a set of canonical variables. Therefore, all the familiar results of Hamiltonian dynamics may be taken over. In particular, introducing Poisson brackets

$$\{A, B\} = \sum_l \left(\frac{\partial A}{\partial p_l}\frac{\partial B}{\partial q_l} - \frac{\partial B}{\partial p_l}\frac{\partial A}{\partial q_l}\right), \tag{II.8}$$

we can rewrite the equations of motion (II.5-6) in canonical form:

$$\dot{p}_l = \{H, p_l\}, \quad \dot{q}_l = \{H, q_l\} \quad (l = 1, \ldots, N). \tag{II.9}$$

More generally, the time evolution of any function F depending on the spin variables $\mathbf{S}_l$and in general on time, is described by the dynamical equation

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{H, F\}. \tag{II.10}$$

For $F = \mathbf{S}_l$ we obtain the equations of motion for the classical spin vectors

$$\frac{\mathrm{d}\mathbf{S}_l}{\mathrm{d}t} = \{H, \mathbf{S}_l\} \tag{II.11}$$

which are equivalent to (II.1). Notice that the Poisson brackets in (II.10) and (II.11) may be evaluated without specifying the canonical basis (p_l, q_l) by defining the Poisson brackets for the classical spin variables as

$$\{S_l^\alpha, S_{l'}^\beta\} = -S\,\delta_{ll'} \sum_\gamma \epsilon_{\alpha\beta\gamma} S_l^\gamma \quad (\alpha, \beta = x, y, z) \tag{II.12}$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. The compatibility of this definition with (II.8) results immediately from (II.4) and (II.7).

At this point, we should take note of the fact that the phase space of the classical spin system has a structure which differs essentially from that of the more familiar Hamiltonian systems representing the dynamics of particles. The "phase space" of a spin model such as (II.1) is, in fact, a compact manifold, a product of N spheres $\mathbf{S}_l^2 = S^2$; both canonical variables (II.7) for each degree of freedom are bounded by finite intervals: $-\pi < q_l \leq \pi$, $-S \leq p_l \leq S$. Also the total energy of the classical spin system, which is not expressible as the sum of a kinetic and a potential part, is bounded by a finite interval (except for certain pathological interactions, which are unphysical).

In the next Section we shall consider spin systems described by a Hamiltonian of the form

$$H = -\frac{1}{2}\sum_{ll'\alpha} J_{ll'}^\alpha S_l^\alpha S_{l'}^\alpha + \frac{1}{2}\sum_{l\alpha} A_\alpha (S_l^\alpha)^2 - \gamma_M \sum_{l\alpha} B_\alpha S_l^\alpha \tag{II.13}$$

whence

$$\mathbf{h}_l = \sum_\alpha \Big(\sum_{l'} J_{ll'}^\alpha S_{l'}^\alpha - A_\alpha S_l^\alpha + \gamma_M B_\alpha\Big)\mathbf{e}_\alpha. \tag{II.14}$$

Here, the $J_{ll'}^\alpha$ with $J_{ll'}^\alpha = J_{l'l}^\alpha$, $J_{ll}^\alpha = 0$ are two-spin interaction constants, the A_α are single-site (crystal-field) anisotropy coefficients, γ_M is the gyromagnetic ratio, $\mathbf{B}$ is an external magnetic field, and $\mathbf{e}_\alpha$, $\alpha = x, y, x$ are unit vectors along the coordinate axes in spin space.

Such a spin system is the classical counterpart of a quantum spin system represented in terms of spin operators $\hat{\mathbf{S}}_l = \hbar\hat{\sigma}_l$ with spin quantum number σ ($\sigma = 1/2, 1, 3/2, \ldots$). The operators $\hat{\sigma}_l^\alpha$ satisfy commutation relations

$$[\hat{\sigma}_l^\alpha, \hat{\sigma}_{l'}^\beta] = \mathrm{i}\,\delta_{ll'} \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{\sigma}_l^\gamma. \tag{II.15}$$

The quantum-mechanical spin length is given by the square root of $\sigma(\sigma+1)$, the eigenvalue of the operator $\hat{\sigma}_l^2$. The system is specified by a spin Hamiltonian $\hat{H}(\hat{\sigma}_1, \ldots, \hat{\sigma}_N)$, and its time evolution is governed by the Heisenberg equations of motion

$$\frac{\mathrm{d}\hat{\sigma}_l}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\sigma}_l], \quad l = 1, \ldots, N. \tag{II.16}$$

For the Hamiltonian

$$\hat{H} = \sum_\alpha \Big\{ -\frac{1}{2}\hbar^2 \sum_{l,l'}^N \hat{\sigma}_l^\alpha \hat{\sigma}_{l'}^\alpha + \frac{1}{2}\hbar^2 \sum_{l=1}^N A_\alpha (\hat{\sigma}_l^\alpha)^2 - \hbar\gamma_M B_\alpha \sum_{l=1}^N \hat{\sigma}_l^\alpha \Big\} \tag{II.17}$$

whose classical counterpart is (II.13), they read

$$\frac{\mathrm{d}\hat{\sigma}_l}{\mathrm{d}t} = \frac{1}{2}[\hat{\sigma}_l \times \hat{\mathbf{h}}_l + (\hat{\sigma}_l \times \hat{\mathbf{h}}_l)^\dagger],$$
$$\hat{\mathbf{h}}_l = \hbar \sum_\alpha \Big(\sum_{l'} J_{ll'}^\alpha \hat{\sigma}_{l'}^\alpha - A_\alpha \hat{\sigma}_l^\alpha\Big)\mathbf{e}_\alpha + \gamma_M \mathbf{B}. \tag{II.18}$$

The classical spin model can be obtained from its quantum-mechanical counterpart by taking the limit [9]

$$\hbar \to 0, \quad \sigma \to \infty \tag{II.19}$$

such that

$$\hbar\sqrt{\sigma(\sigma+1)} \to S, \quad \hat{\mathbf{S}}_l \to \mathbf{S}_l, \tag{II.20}$$

which implies

$$[\hat{S}_l^\alpha, \hat{S}_{l'}^\beta] = \hbar\,\delta_{ll'} \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{S}_l^\gamma. \tag{II.21}$$

The quantities $\mathbf{S}_l$ can therefore be reinterpreted as classical 3-component spin vectors of constant length S. This prescription transforms the quantum spin Hamiltonian (II.17) into the classical energy function (II.13), and the equations of motion (II.18) into the classical equations of motion (II.1) with $\mathbf{h}_l$ given by (II.14).

B. Dynamical integrability

Thus far, all our conclusions hold even if the classical spin Hamiltonian is time-dependent, for example, due to a time-dependent external magnetic field $\mathbf{B}(t)$. In the context of the integrability question, however, this explicit time dependence results in special consequences which call for a separate treatment. Let us first discuss the concept of integrability for autonomous systems, i.e. systems specified by a time-independent Hamiltonian.

According to the Liouville theorem on integrable dynamical systems [10], an autonomous system of N classical spins is integrable by quadratures (i.e. "completely integrable") if there exist N independent integrals of the motion which are mutually in involution:

$$I_k(\mathbf{S}_1, \ldots, \mathbf{S}_N) = \text{const.}, \quad k = 1, \ldots, N \qquad \text{(II.22)}$$

$$\{I_k, I_{k'}\} = 0, \quad k, k' = 1, \ldots, N. \qquad \text{(II.23)}$$

Equation (II.10) with $F = H$ shows that the Hamiltonian of an autonomous spin system is a conserved quantity. The same equation with $F = I_k$ shows that any integral of the motion is in involution with the Hamiltonian. Therefore, the Hamiltonian itself may be chosen as one of the integrals I_k.

The Liouville theorem implies that the N-dimensional hypersurface obtained by the intersection of the N $(2N-1)$-dimensional hypersurfaces I_k =const is diffeomorphic to an N-torus if the level set $M = \{I_k = \text{const}, k = 1.....N\}$ is a connected manifold (or to a set of disjoint N-tori if M is disconnected). Any individual phase-space trajectory of the system is confined to one N-toms. Thus, the motion of the phase point of a completely integrable N-spin system is characterized by at most N independent frequencies. If, in addition to the I_k's, $k = l.....N$, there exist further independent integrals of motion which are not in involution with the first N I_k's and which do not depend on time explicitly the number of independent frequencies is reduced.

The involution condition guarantees the existence of a canonical transformation to action-angle variables (J_l, ϕ_l) such that the new Hamiltonian H' becomes a function of the action variables J_l alone,

$$H' = H'(J_1, \ldots, J_N). \qquad \text{(II.25)}$$

The angle variables ψ_l (defined modulo 2π) are, therefore, cyclic coordinates, and the new canonical equations

$$\dot{J}_l = 0, \quad \dot{\psi}_l = \partial H'/\partial J_l \equiv \omega_l \qquad \text{(II.26)}$$

can be solved by quadratures.

The numerical values of the (conserved) action variables on a given N-torus are determined by the N action integrals

$$J_l = \frac{1}{2\pi} \oint_{C_l} p_l \mathrm{d}q_l, \quad l = 1, \ldots, N, \qquad \text{(II.27)}$$

where the C_l's are N topologically independent cycles on the N-torus.

In conclusion, a completely integrable N-spin system is characterized by the property that all trajectories are *regular*, i.e. confined to N-dimensional tori and described by a discrete Fourier spectrum. If fewer than N independent integrals of the motion in involution exist, at least part of the phase space is no longer foliated by invariant tori, thus allowing the presence of new types of trajectories in addition to regular ones: trajectories whose course through phase space is strikingly erratic and extremely sensitive to slight changes in initial conditions, and whose Fourier spectrum is continuous. They are called *chaotic* trajectories.

C. Non-Autonomous Systems

The concept of complete integrability is readily adapted to non-autonomous systems, i.e. systems specified by time-dependent Hamiltonians of the general form

$$H(t) = H(q_1, \ldots, q_N; p_1, \ldots, p_N; t). \qquad \text{(II.28)}$$

Such systems can be transformed, quite generally, into autonomous systems with one additional degree of freedom [1]. This is accomplished by formally extending the $2N$-dimensional phase space of the non-autonomous system (2.28) to the $(2N+2)$-dimensional phase space

$$(\bar{q}_1 = q_1, \ldots, \bar{q}_N = q_N, \bar{q}_{N+1} = t; \\ \bar{p}_1 = p_1, \ldots, \bar{p}_N = p_N, \bar{P}_{N+1} = -H) \qquad \text{(II.29)}$$

of the autonomous system specified by the Hamiltonian

$$\bar{H}(\bar{q}_1, \ldots, \bar{q}_{N+1}; \bar{p}_1, \ldots, \bar{p}_{N+1}) \\ = H((\bar{q}_1, \ldots, \bar{q}_N; \bar{p}_1, \ldots, \bar{p}_N; \bar{q}_{N+1}) + \bar{p}_{N+1}. \qquad \text{(II.30)}$$

The Hamiltonian flow in the extended phase space is then parameterized by a new "time" τ, which, however, does not appear in $\bar{H}$ itself. Hence, we have

$$\bar{H} = \text{const} \ \ (= 0). \qquad \text{(II.31)}$$

The new set of canonical equations

$$\frac{\mathrm{d}\bar{q}_k}{\mathrm{d}\tau} = \frac{\partial \bar{H}}{\partial \bar{p}_k}, \quad \frac{\mathrm{d}\bar{p}_k}{\mathrm{d}\tau} = -\frac{\partial \bar{H}}{\partial \bar{q}_k} \qquad \text{(II.32)}$$

includes (for $k = 1, \ldots, N$) the complete set of canonical equations of the old system (II.28). The remaining two equations (for $k = N+1$) imply

$$t = \tau, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}. \qquad \text{(II.33)}$$

Hence, the complete integrability of any non-autonomous system with N degrees of freedom depends on the existence of $N+1$ integrals of the motion I_k, $k = 1, \ldots, N+1$, which are mutually in involution.

The transformed Hamiltonian can always be chosen to be one of them. The remaining N integrals of the extended (autonomous) system (II.30) correspond to N integrals I_k of the original (non-autonomous) system, which are in general explicitly time-dependent. They are conserved quantities by virtue of the property

$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = \{H, I_k\} + \frac{\partial I_k}{\partial t} = 0. \qquad \text{(II.34)}$$

Evidently, H itself can never satisfy this property if it is time-dependent.

III. INTEGRABLE AND NONINTEGRABLE CLASSICAL SPIN CLUSTERS

In this Section our goal is a detailed investigation, by both analytical and numerical methods, of the complete integrability of classical spin clusters described by various special cases of the general Hamiltonian (II.13).

A. A Single Classical Spin in a Time-Dependent Field

Let us first consider the single-spin case of (II.13):

$$H = \frac{1}{2}A_x(S_x)^2+\frac{1}{2}A_y(S_y)^2+\frac{1}{2}A_z(S_z)^2-\gamma_M \mathbf{B}\cdot\mathbf{S}. \quad \text{(III.1)}$$

For $\mathbf{B}$ = const., the system is autonomous and H is an integral of the motion, the only one needed for the complete integrability of this system with one degree of freedom. In the presence of a time-dependent field $\mathbf{B}(t)$, on the other hand, $H(t)$ cannot be conserved according to (II.34), and the existence of the one time-dependent integral of the motion required for complete integrability is, in general, not guaranteed. Thus, the motion of a single classical spin may be chaotic if it is subject to a time-dependent field.

For an illustrative example we briefly discuss the special case $A_x, = A_y = 0$ of (III.1) studied recently by Frahm and Mikeska [8]. An oscillating field

$$\mathbf{B}(t) = (B\cos\omega t, 0, 0) \quad \text{(III.2)}$$

appears to result in a nonintegrable model, evidenced by the observation of chaotic trajectories in numerical calculations. The case of a rotating field

$$\mathbf{B}(t) = (B\cos\omega t, B\sin\omega t, 0) \quad \text{(III.3)}$$

on the other hand, was proved to be integrable [8].

The time-dependent integral of the motion $I(q;p;t)$ required for complete integrability is readily constructed by taking note that in the reference frame which rotates in phase with the external field about the z-axis the energy of the spin is conserved [8]. Here, we show that this is a special case of a general class of systems whose integrability is guaranteed by the following theorem:

If there exists, on the sphere $\mathbf{S}^2 = S^2$, a time-independent source-free velocity field $\mathbf{v}(\mathbf{S})$,

$$\frac{\partial}{\partial \mathbf{S}}\cdot\mathbf{v}(\mathbf{S}) = 0, \quad \text{(III.4)}$$

such that the continuity equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial \mathbf{S}}\cdot(H\mathbf{v}) = 0 \quad \text{(III.5)}$$

holds with density H and current density $H\mathbf{v}$, then there exists an integral of the motion of the form

$$I(\mathbf{S},t) = H(\mathbf{S},t) + G(\mathbf{S}) \quad \text{(III.6)}$$

where $G(\mathbf{S})$ is given by

$$\mathbf{v} = \mathbf{S}\times\partial G/\partial\mathbf{S}. \quad \text{(III.7)}$$

Indeed, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \frac{\partial G}{\partial \mathbf{S}}\cdot\dot{\mathbf{S}} = \frac{\partial H}{\partial t} - \frac{\partial G}{\partial \mathbf{S}}\cdot\mathbf{S}\times\frac{\partial H}{\partial \mathbf{S}} + \mathbf{v}\cdot\frac{\partial H}{\partial \mathbf{S}}$$

by using the equation of motion (II.1), and

$$\mathbf{v}\cdot\frac{\partial H}{\partial \mathbf{S}} = \frac{\partial}{\partial \mathbf{S}}\cdot(H\mathbf{v}) \quad \text{(III.8)}$$

by using (III.4). Therefore, $\mathrm{d}I/\mathrm{d}t = 0$ as stated.

For the case of the rotating magnetic field (III.3), (III.5) is satisfied by

$$\mathbf{v} = (-\omega S^y, \omega S^x, 0) = \mathbf{S}\times\frac{\partial}{\partial \mathbf{S}}(-\omega S^z) \quad \text{(III.9)}$$

which yields $G = -\omega S^z$. Hence the first integral is

$$I = H(\mathbf{S},t) - \omega S^z. \quad \text{(III.10)}$$

B. A Pair of Interacting Spins

We now turn to the two-spin case of the general model (II.13), on which the calculations for the present work have been focused. We have studied, in particular, the zero-field case, specified by the Hamiltonian

$$H = \sum_\alpha \left(-J_\alpha S_1^\alpha S_2^\alpha + \frac{1}{2}A_\alpha\left[(S_1^\alpha)^2 + (S_2^\alpha)^2\right]\right). \quad \text{(III.11)}$$

Since this is an autonomous system, H itself is an integral of the motion, and the criterion of complete integrability reduces in this case to the existence or nonexistence of a second integral of the motion I which is independent of H. A search for this second integral of the motion can be successfully carried out by means of analytical trial methods assisted by symmetry arguments as will be demonstrated in the following. Similar methods have been applied to different types of classical dynamical systems with two degrees of freedom, mainly particles in two-dimensional scalar potentials [11].

In order to exploit the xyz-permutation symmetries of the general structure of the underlying model Hamiltonian, which we expect to be manifest also in the general structure of the second invariant I, we shall operate with the classical spin variables S_l^α directly rather than with the canonical variables (II.7) even though the components (S_l^x, S_l^y, S_l^z) are not independent from one another.

In order to represent an integral of the motion, the function $I(\mathbf{S}_1, \mathbf{S}_2)$ must satisfy the equation

$$\dot{I} = \frac{\partial I}{\partial \mathbf{S}_1}\cdot\dot{\mathbf{S}}_1 + \frac{\partial I}{\partial \mathbf{S}_2}\cdot\dot{\mathbf{S}}_2 = 0. \quad \text{(III.12)}$$

where the time derivatives on the right-hand side are determined by the equation of motion (II.11). The condition (III.12) for the invariance of I thus becomes the following:

$$\frac{\partial I}{\partial \mathbf{S}_1}\cdot\left(\mathbf{S}_1\times\frac{\partial I}{\partial \mathbf{S}_1}\right) + \frac{\partial I}{\partial \mathbf{S}_2}\cdot\left(\mathbf{S}_2\times\frac{\partial I}{\partial \mathbf{S}_2}\right) = 0. \quad \text{(III.13)}$$

In the special case where the Hamiltonian is invariant under a continuous group of transformations, there always exists an independent integral of the motion (Noether's theorem). In particular, for

$$J_x = J_y, \quad A_x = A_y \tag{III.14}$$

the Hamiltonian (III.11) is rotationally invariant about the z-axis, resulting in the conservation of the z-component of the total spin

$$S_T^z = S_1^z + S_2^z = \text{const.} \tag{III.15}$$

Here, we are interested in the conditions under which there exists a second integral in the absence of continuous symmetries. In the following we present a complete solution of this problem for integrals which are quadratic in the spin variables.

We restrict the search to invariants which have the same symmetry as the Hamiltonian,

$$I = \sum_{\alpha=x,y,z} \left(-g_\alpha S_1^\alpha S_2^\alpha + \frac{1}{2} K_\alpha \left[(S_1^\alpha)^2 + (S_2^\alpha)^2 \right] \right). \tag{III.16}$$

The condition (III.13) then leads to the following set of equations for the parameters g_α, K_α:

$$\sum_{\text{cycl}} (A_\alpha - A_\beta) K_\gamma = 0 \tag{III.17}$$

and

$$\sum_\beta M_{\alpha\beta} g_\beta = N_\alpha, \quad \alpha = x,y,z, \tag{III.18}$$

where

$$M = \begin{pmatrix} A_y - A_z & -J_z & J_y \\ J_z & A_z - A_x & -J_x \\ -J_y & J_x & A_x - A_y \end{pmatrix} \tag{III.19}$$

and

$$N_\alpha = J_\alpha (K_\beta - K_\gamma), \quad \alpha\beta\gamma = \text{cycl}(xyz). \tag{III.20}$$

For the solution, we have to distinguish two cases: (i) pure exchange anisotropy ($A_x = A_y = A_z$) and (ii) nonzero site anisotropy (not all three A_α equal).

(i) For *pure exchange anisotropy* ($A_x = A_y = A_z$), (III.17) is identically satisfied and $\det M = 0$ for arbitrary J_α. The null eigenvector of N is given by

$$g_\alpha^{(0)} = a J_\alpha, \quad \alpha = x,y,z, \tag{III.21}$$

where a is an arbitrary constant. The solvability condition of (III.18),

$$\sum_\alpha g_\alpha^{(0)} N_\alpha = a \sum_{\text{cycl}} J_\alpha^2 (K_\beta - K_\gamma) = 0, \tag{III.22}$$

results in the two-parameter family of solutions

$$K_\alpha = b J_\alpha^2 + c, \quad \alpha = x,y,z, \tag{III.23}$$

where b and c are arbitrary constants. Finally, a particular solution of (III.18) with its right-hand side determined by (III.23) is given by

$$g_\alpha^{(1)} = b J_\beta J_\gamma, \quad \alpha\beta\gamma = \text{cycl}(xyz). \tag{III.24}$$

Upon insertion of these solutions into (3.16), we thus obtain

$$I = aH + bI_1 + \text{const}, \tag{III.25}$$

with

$$I_1 = \sum_{\text{cycl}} J_\alpha J_\beta S_1^\gamma S_2^\gamma + \frac{1}{2} \sum_\alpha J_\alpha^2 \left[(S_1^\alpha)^2 + (S_2^\alpha)^2 \right], \tag{III.26}$$

which is clearly independent of H except in the isotropic case $J_x = J_y = J_z$ (where the two independent integrals are H and S_T^z). In the unaxial case $J_x = J_y \neq J_z$, the invariant I_1 is a linear combination of $(S_T^z)^2$ and H,

$$I_1 = \frac{1}{2}(J_z^2 - J_x^2)(S_T^z)^2 + J_z H + \text{const.}$$

We thus conclude that the two-spin XYZ model described by the Hamiltonian

$$H = -J_x S_1^x S_2^x - J_y S_1^y S_2^y - J_z S_1^z S_2^z \tag{III.27}$$

is completely integrable for arbitrary values of the parameters J_x, J_y, J_z, and the second independent integral of the motion is given by (3.26).

(ii) For *nonzero single-site anisotropy* (not all A_α equal), (III.17) has the two-parameter family of solutions

$$K_\alpha = a A_\alpha + c, \quad \alpha = x,y,z, \tag{III.28}$$

where a and c are arbitrary constants.

This determines the right-hand side of (III.18):

$$N_\alpha = a J_\alpha (A_\beta - A_\gamma), \quad \alpha\beta\gamma = \text{cycl}(xyz). \tag{III.29}$$

We now have to distinguish the two subcases $\det M \neq 0$ and $\det M = 0$. If $\det M \neq 0$ then the only solution of (III.18) is

$$g_\alpha^{(1)} = a J_\alpha, \quad \alpha = x,y,z. \tag{III.30}$$

In that case, the resulting invariant

$$I = aH + \text{const.}$$

is not independent of H. In fact, we shall present numerical evidence suggesting that the classical two-spin model (III.11) is, in general, nonintegrable if $\det M \neq 0$. This implies not only the nonexistence of a quadratic invariant (III.14) other than H itself, but the nonexistence of any analytic function of the S_l^α which is an integral of the motion and independent of H.

An independent quadratic integral of the motion can exist only if

$$\det M = (A_x - A_y)(A_y - A_z)(A_z - A_x) + \sum_{\text{cycl}} J_\alpha^2 (A_\beta - A_\gamma) = 0 \quad \text{(III.31)}$$

and if the three other (3×3) subdeterminants of the augmented (3×4) matrix (M, N) also vanish. This latter solvability condition of (III.18) is automatically satisfied for any parameter set $\{J\alpha, A\alpha\}$ satisfying (III.31), since

$$\det \begin{pmatrix} N_x & M_{xy} & M_{xz} \\ N_y & M_{yy} & M_{yz} \\ N_z & M_{zy} & M_{zz} \end{pmatrix} = aJ_x \det M, \quad \text{etc.} \quad \text{(III.32)}$$

If $\det M = 0$, the general solution of (III.18) is then a linear superposition of the particular solution (III.30) and the null eigenvector of M,

$$g_\alpha^{(0)} = JJ_\alpha(A_\alpha - A_\beta)J_\gamma + (A_\alpha - A_\gamma)J_\beta - (A_\alpha - A_\beta)(A_\alpha - A_\gamma), \quad \text{(III.33)}$$

where $J = J_x + J_y + J_z$, and $\alpha\beta\gamma = \text{cycl}(xyz)$. Insertion of this general solution into (III.16) yields an invariant of the form

$$I = aH + bI_1 + \text{const}, \quad \text{(III.34)}$$

where a and b are arbitrary constants, and where

$$I_1 = -\sum_\alpha g_\alpha^{(0)} S_1^\alpha S_2^\alpha \quad \text{(III.35)}$$

is clearly independent of H. We thus conclude that the two-spin system with exchange and single-site anisotropy described by the Hamiltonian (III.11) is completely integrable if the parameters J_α, A_α satisfy the condition (III.31); the second integral is given by expression (III.35).

For an example illustrating the above results, we consider an XY-type special case of (III.11) corresponding to $J_\alpha = 0$, $J_x = J(1+\gamma)$, $J_y = J(1-\gamma)$, $A_z = 0$, $A_x = -A_y = \alpha J$:

$$H = J\big\{ -(1+\gamma)S_1^x S_2^x - (1-\gamma)S_1^y S_2^y + \frac{1}{2}\alpha\big[(S_1^x)^2 - (S_1^y)^2 + (S_2^x)^2 - (S_2^y)^2\big]\big\}. \quad \text{(III.36)}$$

The condition (III.31) for the existence of a quadratic invariant requires in this case

$$\alpha\big[\alpha^2 - (1+\gamma^2)\big] = 0. \quad \text{(III.37)}$$

On the (α, γ) parameter plane, this condition is represented by three nonintersecting curves as shown in Fig. 1. The line $\alpha = 0$ represents the completely integrable XY model with pure exchange anisotropy for which the second independent invariant is given by (III.26). The two hyperbolic curves $\alpha = \pm\sqrt{1+\gamma^2}$, on the other hand, represent completely integrable XY models with both exchange and single-site anisotropy for which the second independent invariant is given by (III.35).

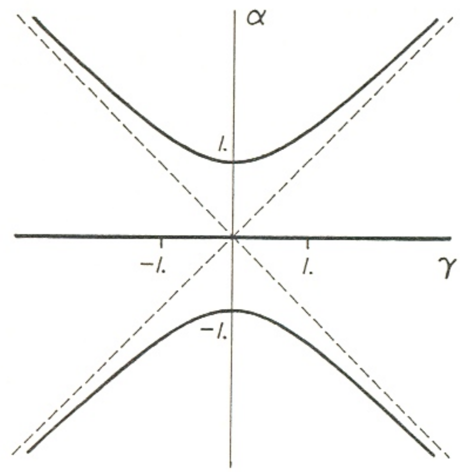

FIG. 1: The parameter values (α, γ), defined by condition (III.37), for which the system described by (III.36) is completely integrable, are located on the γ axis and on two hyperbolic branches

C. Regular and Chaotic Trajectories

For an illustration of the impact which the complete integrability or its absence have on the dynamical properties of classical spin clusters, we study, by numerical calculations, the time evolution of two special cases of the general two-spin model (III.11). The Hamiltonians

$$H_\gamma = -J(1+\gamma)S_1^x S_2^x - (1-\gamma)S_1^y S_2^y \quad \text{(III.38)}$$

$$H_A = -J(S_1^x S_2^x + S_1^y S_2^y) + \frac{1}{2}A\big[(S_1^x)^2 + (S_2^x)^2\big], \quad \text{(III.39)}$$

describe two simple models of a pair of classical spins interacting via a planar exchange interaction. In either model, the rotational symmetry about the z-axis is removed by anisotropy, in H_γ by an *exchange* anisotropy and in H_A by a *single-site* anisotropy. Although both models have exactly the same symmetry, only one of them, H_γ, satisfies the integrability condition discussed in Sect. III.2. If we express the classical spin $\mathbf{S}_l$ in terms of the two angular coordinates ϑ_l, ϕ_l as in (II.4) and normalize the length of each spin to $S = 1$, the equations of motion for the two models inferred from (II.5) and (II.6) are the following for H_γ and H_A, respectively:

$$\dot\vartheta_1 = J\sin\vartheta_2[\sin(\phi_1 - \phi_2) + \gamma\sin(\phi_1 + \phi_2)]$$
$$\dot\phi_1 = J\cot\vartheta_1 \sin\vartheta_2[\cos(\phi_1 - \phi_2) + \gamma(\phi_1 + \phi_2)] \quad \text{(III.40)}$$

$$\dot\vartheta_1 = J\sin\vartheta_2\sin(\phi_1 - \phi_2) - A\sin\vartheta_1\sin\phi_1\cos\phi_2$$
$$\dot\phi_1 = J\cot\vartheta_1\sin\vartheta_2\cos(\phi_1 - \phi_2) - A\cos\vartheta_1\cos^2\phi_1 \quad \text{(III.41)}$$

The remaining two equations for each model are obtained by interchanging subscripts 1 and 2. For given initial conditions $(\vartheta_1^{(0)}, \vartheta_2^{(0)}, \phi_1^{(0)}, \phi_2^{(0)})$, each set of four (highly nonlinear) equations then determines a trajectory in the 4-dimensional phase space.

In either model, the total energy

$$H_\gamma = -J \sin\vartheta_1 \sin\vartheta_2 \big[\cos(\phi_1 - \phi_2) + \gamma\cos(\phi_1 + \phi_2)\big] \tag{III.42}$$

$$\begin{aligned} H_A =& - J\sin\vartheta_1 \sin\vartheta_2 \cos(\phi_1 - \phi_2) \\ &+ \frac{1}{2}A\big(\sin^2\vartheta_1\cos^2\phi_1 + \sin^2\vartheta_2\cos^2\phi_2\big) \end{aligned} \tag{III.43}$$

is an invariant. Therefore, all trajectories in 4D phase are, in fact, confined to 3D hypersurfaces $H = \text{const}$.

A convenient way to visualize trajectories in higher dimensional spaces is by means of a Poincaré cut, i.e. a Poincaré surface of section. It is realized by a plot of all those points of a given trajectory for which one of the dynamical variables assumes a particular value [12]. In the following, we choose $\vartheta_2 = \pi/2$. The resulting set of phase points of any trajectory for a given energy is thus confined to a 2D surface in the 3D reduced space spanned by the remaining variables $\vartheta_1, \phi_1, \phi_2$, which is obtained as the intersection of the 3D energy hypersurface $H =$ const with the 3D Poincaré surface of section $\vartheta_2 = \pi/2$. This 2D set of phase points can then conveniently be represented by the projections onto the three coordinate planes (ϑ_1, ϕ_1), $\vartheta_1, \phi_2)$, (ϕ_1, ϕ_2).

As shown previously, the model H_γ is completely integrable, implying the existence of a second integral of the motion. It is given by (III.26),

$$\begin{aligned} I_\gamma =& J^2(1+\gamma^2)(\sin^2\vartheta_1 + \sin^2\vartheta_2) \\ &- 2J^2(1-\gamma^2)\cos\vartheta_1\cos\vartheta_2 \\ &+ 2\gamma J^2\big[\sin^2\vartheta_1\cos 2\phi_1 + \sin^2\vartheta_2\cos 2\phi_2\big]. \end{aligned} \tag{III.44}$$

The existence of this second invariant has the consequence that the trajectories in 4D phase are confined to the intersection of the two 3D hypersurfaces $H_\gamma = \text{const}$ and $I_\gamma = \text{const}$, which is a 2D surface in 4D phase space. This surface is an invariant torus. Trajectories which are confined to invariant tori are called regular trajectories. Since the model is completely integrable, the entire phase space is foliated by invariant tori, and all its trajectories are regular. On further intersection with the 3D Poincaré surface of section $\vartheta_2 = \pi/2$, the set of phase points resulting from any trajectory for given values of H_γ and I_γ is then reduced to a set of lines, i.e. a set of 1D objects in 3D $(\vartheta_1, \phi_1, \phi_2)$-space.

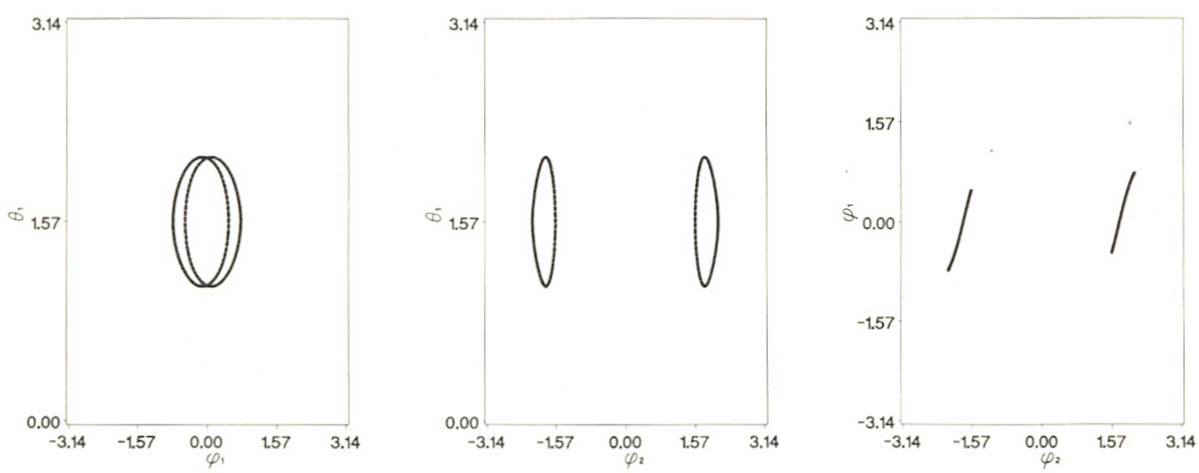

FIG. 2: Trajectory of the integrable classical two-spin model H_γ with $J = 1, \gamma = 0.5$ for initial conditions $\vartheta_1^{(0)} = 1.0876037$, $\vartheta_2^{(0)} = \pi/2$, $\phi_1^{(0)} = 0.3$, $\phi_2^{(0)} = \pi/2 + 0.3$; $E = 0.25$. Shown are the projections onto the three coordinate planes (ϑ_1, ϕ_1), (ϑ_1, ϕ_2), (ϕ_1, ϕ_2) of the phase points belonging to the Poincaré cut at $\vartheta_2 = \pi/2$

In Figs. 2 and 3 we show the three projections of two such trajectories of the completely integrable model H_γ or, more precisely, the projections of their intersection with the Poincaré surface of section $vartheta_2 = \pi/2$. Both trajectories are located on the same H_γ-hypersurface but on different I_γ-ypersurfaces. The reader will find it easy to reconstruct the 1D objects in 3D $(\vartheta_1, \phi_1, \phi_2)$-space.

We now turn to the model specified by the Hamiltonian H_A. For this model, a quadratic second invariant does not exist because condition (III.31) is violated. It appears that an analytic second invariant does not exist at all.

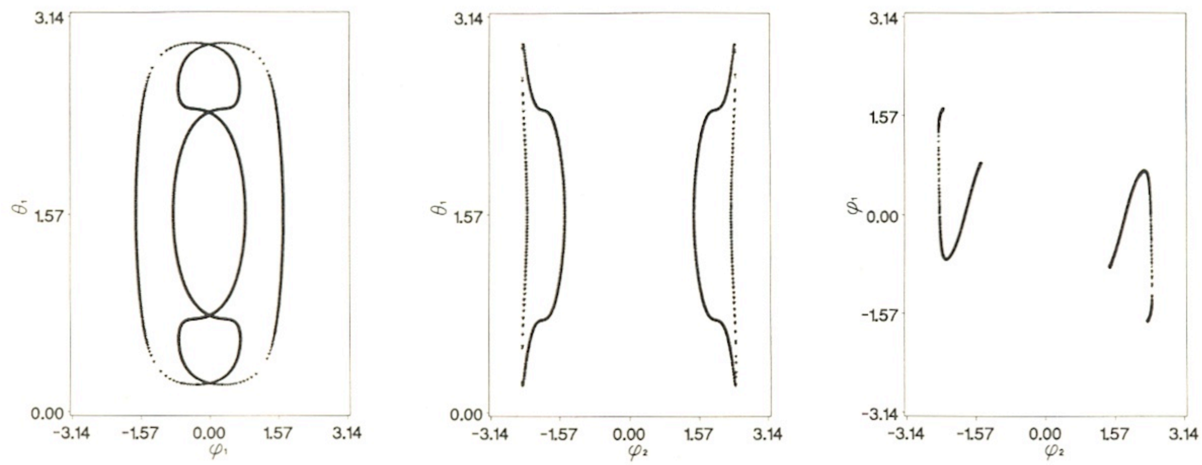

FIG. 3: Trajectory of the integrable classical two-spin model H_γ with $J = 1, \gamma = 0.5$ for initial conditions $\vartheta_1^{(0)} = 0.5321446$, $\vartheta_2^{(0)} = \pi/2$, $\phi_1^{(0)} = 0.7$, $\phi_2^{(0)} = \pi/2 + 0.7$; $E = 0.25$. Shown are the projections onto the three coordinate planes (ϑ_1, ϕ_1), (ϑ_1, ϕ_2), (ϕ_1, ϕ_2) of the phase points belonging to the Poincaré cut at $\vartheta_2 = \pi/2$

The consequence is that not all trajectories are confined to invariant tori, i.e. represented by sets of lines in the Poincaré surface of section $\vartheta_2 = \pi/2$. A single trajectory may "spread" over a nonzero fraction of the arch of the energy hypersurface in the sense that the set of points on the energy hypersurface which are approached arbitrarily by that trajectory has nonzero measure. Trajectories of this type are called chaotic. In an ergodic system, almost all trajectories spread over the entire hypersurface. In general, one has to expect, however, the presence of a set of regular trajectories, i.e. the presence of invariant tori in parts of the phase space, of nonzero measure, even though their existence is no longer associated with the existence of analytic invariants (integrals of the motion). The three projections of a regular trajectory (intersected by the hyperplane $\vartheta_2 = \pi/2$) of the nonintegrable model H_A are shown in Fig. 4.

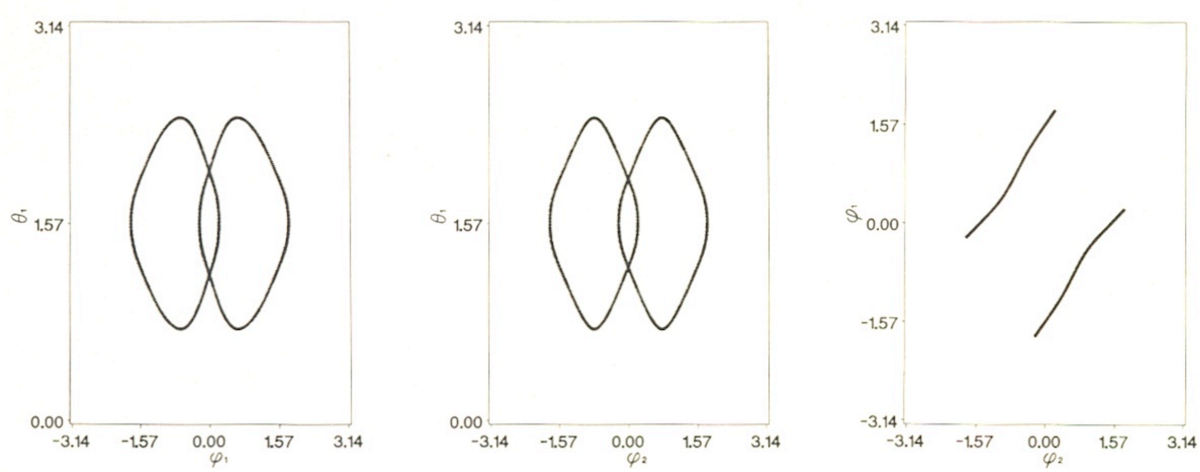

FIG. 4: Regular trajectory of the nonintegrable classical two-spin model H_A with $J = -A = 1$ for initial conditions $\vartheta_1^{(0)} = \pi/2$, $\vartheta_2^{(0)} = \pi/2$, $\phi_1^{(0)} = 0.22$, $\phi_2^{(0)} = 0.22 + \pi/2$; $E = -0.5$. Shown are the projections onto the three coordinate planes (ϑ_1, ϕ_1), (ϑ_1, ϕ_2), (ϕ_1, ϕ_2) of the phase points belonging to the Poincaré cut at $\vartheta_2 = \pi/2$

Figure 5 shows the three projections of a chaotic trajectory, corresponding to the same energy as the regular

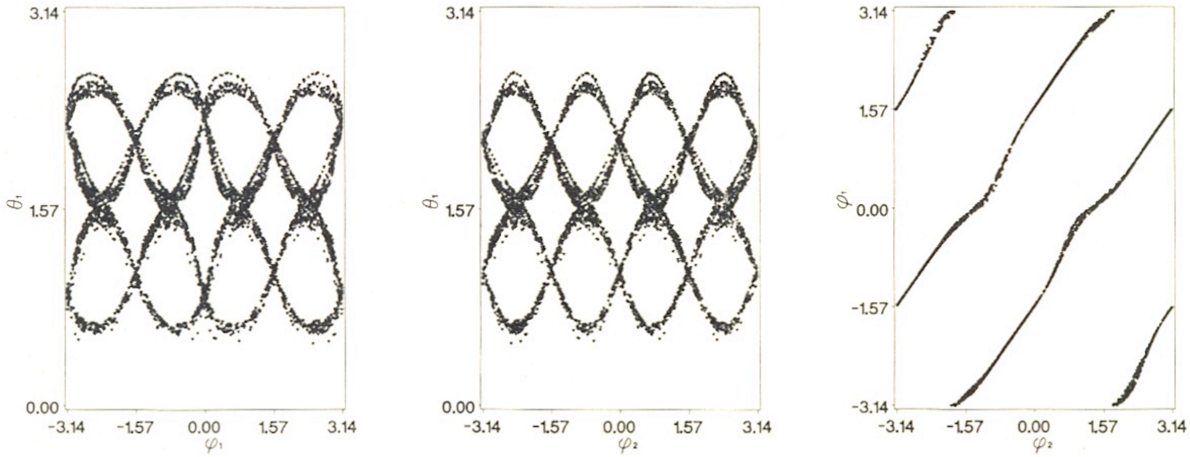

FIG. 5: Chaotic trajectory of the nonintegrable classical two-spin model H_A with $J = -A = 1$ for initial conditions $\vartheta_1^{(0)} = 1.54$, $\vartheta_2^{(0)} = 1.5718$, $\phi_1^{(0)} = 0.62$, $\phi_2^{(0)} = 2.1902$; $E = -0.5$. Shown are the projections onto the three coordinate planes (ϑ_1, ϕ_1), (ϑ_1, ϕ_2), (ϕ_1, ϕ_2) of the phase points belonging to the Poincaré cut at $\vartheta_2 = \pi/2$

trajectory shown in Fig. 4. It gives compelling evidence that no second integral of the motion exists for this 2-spin model. In the (ϑ_1, ϕ_1)- and (ϑ_2, ϕ_2)-projections, the set of phase points spreads considerably over the energy surface. The empty spaces of various size are attributable to the presence of invariant tori on that energy surface. The regular trajectory shown in Fig. 4 is one example. In systems with not more than two degrees of freedom, invariant tori divide phase space into disjoint parts. It is noteworthy that in the (ϕ_1, ϕ_2) projection, the chaotic trajectory is very much more constrained by the presence of invariant tori than in the other two projections. The chaotic trajectory is represented by a set of "thick" lines as compared to the "thin" lines representing the regular trajectories in the same projection.

D. A Completely Integrable N-Spin Cluster

We conclude this study of integrability in classical spin clusters with the discussion of a completely integrable system of N spins, all pairwise coupled by a Heisenberg interaction of uniform strength:

$$H = -\frac{1}{2}\sum_{l \neq l'} \mathbf{S}_l \cdot \mathbf{S}_{l'}. \qquad \text{(III.45)}$$

It is the Hamiltonian of a spin cluster with spins located at the vertices of an $(N-1)$-dimensional simplex (model of "uniformly interacting spins", [-13]). The equations of motion (II.11) can be written in the form

$$\dot{\mathbf{S}}_l = \mathbf{S}_l \times (\mathbf{S}_T - \mathbf{S}_l),$$

i.e.

$$\dot{\mathbf{S}}_l = \mathbf{S}_l \times \mathbf{S}_T, \qquad \text{(III.46)}$$

where

$$\mathbf{S}_T = \sum_{l=1}^{N} \mathbf{S}_l \qquad \text{(III.47)}$$

is the total spin of the cluster. Equations (III.46) and (III.47) imply

$$\dot{\mathbf{S}}_T = \sum_l \dot{\mathbf{S}}_l = \sum_l \mathbf{S}_l \times \mathbf{S}_T = \mathbf{S}_T \times \mathbf{S}_T = 0 \qquad \text{(III.48)}$$

i.e. the total spin is conserved (as a consequence of the rotational invariance of (III.45)). It follows that the equations of motion (III.46) are linear, and the time evolution of each individual spin depends, via $\mathbf{S}_T$, only on the initial conditions of the other spins, but not on their time evolution. For $\mathbf{S}_T = 0$ all spins are at rest. For $\mathbf{S}_T \neq 0$, N integrals of the motion in involution are, for example, the projections of the individual spins onto the direction of $\mathbf{S}_T$:

$$I_l = \mathbf{S}_l \cdot \mathbf{S}_T = \text{const}, \quad l = 1, 2, \ldots, N. \qquad \text{(III.49)}$$

The Hamiltonian (III.45), expressed in terms of these invariants, reads

$$H = -\frac{1}{2}\sum_{l=1}^{N} I_l + \frac{1}{2} N S^2. \qquad \text{(III.50)}$$

Note that all spins process around the direction of $\mathbf{S}_T$ with the same frequency

$$\omega = \left| \sum_{l=1}^{N} \mathbf{S}_l \right| = |\mathbf{S}_T|. \qquad \text{(III.51)}$$

The canonical transformation to action-angle variables consists in a rotation $R(\mathbf{S}_T)$ to a coordinate system with the polar axis $\tilde{z}$ parallel to $\mathbf{S}_T$:

$$\begin{aligned}\tilde{\mathbf{S}}_l &= R(\mathbf{S}_T)\mathbf{S}_l \\ &= S(\sin\tilde{\vartheta}_l\cos\tilde{\phi}_l, \sin\tilde{\vartheta}_l\sin\tilde{\phi}_l, \cos\tilde{\vartheta}_l),\end{aligned} \qquad \text{(III.52)}$$

where $(\tilde{\vartheta}_l, \tilde{\phi}_l)$ are the polar coordinates of the spins in the rotated coordinate system. The action variables are the new canonical momenta

$$\tilde{p}_l = \tilde{S}^z_l = S\cos\tilde{\vartheta}_l, \qquad \text{(III.53)}$$

and the angle-variables are the new azimuthal angles $\tilde{\phi}_l$ of the spins. The Hamiltonian in terms of these variables is

$$H = -\frac{1}{2}\sum_{l,l'}\tilde{p}_l\tilde{p}_{l'} + \frac{1}{2}NS^2, \qquad \text{(III.54)}$$

and thus the canonical equations yield

$$-\dot{\tilde{\phi}}_l = \sum_{l'=1}^{N}\tilde{p}_{l'} = \omega, \quad l = 1,\ldots,N. \qquad \text{(III.55)}$$

The fact that the time evolution of all N angle variables $\tilde{\phi}_l$ is determined by a single frequency is due to the existence of $N-1$ additional time-independent integrals of the motion

$$K_l = \tilde{\phi}_l - \tilde{\phi}_1, \quad l = 2,\ldots,N, \qquad \text{(III.56)}$$

which are independent of the I_l's, but not in involution with them. The N-th integral of the motion of our N-spin cluster,

$$K_1 = \tilde{\phi}_1 + \omega t \qquad \text{(III.57)}$$

is explicitly time-dependent.

IV. SUMMARY AND CONCLUSIONS

We have investigated the nonlinear dynamics forvarious model systems of N classical spins. Suchsystems are Hamiltonian systems with N degrees of freedom. For their complete integrability, the existence of N independent integrals of the motion in involution is required. The specific difference between the N-spin system and the more familiar classical particle systems with N degrees of freedom results from the facts that (i) the spin Hamiltonian is not of the type "kinetic energy plus potential energy," and (ii) the spin equations of motion describe a Hamiltonian flow on a $2N$-dimensional *compact* manifold S_2^N (consisting of the product of N spheres $\mathbf{S}_l^2 = S^2$). If the system is completely integrable, the spin motion is multiply periodic in time, characterized by a discrete spectrum, and each trajectory on S_2^N is confined to an N-dimensional submanifold which is diffeomorphic to an N-torus (regular motion). If fewer than N integrals of motion exists, then there occurs a new type of motion with a continuous frequency spectrum, on trajectories whose course through phase space is strikingly erratic and extremely sensitive to changes in initial conditions (chaotic motion). The coexistence of regular trajectories and chaotic trajectories in phase space is a characteristic feature of nonintegrable Hamiltonian systems.

As a basis for the investigation of these general aspects of regular and chaotic spin motion, we have examined in the present paper the problem of integrability of a two-spin system in detail. We have shown that a pair of spins coupled by an anisotropic exchange interaction (III.27) is completely integrable for any values of the coupling constants. The second independent integral of the motion which guarantees this dynamical property has the explicit form (III.26). If the two-spin Hamiltonian also includes single-site anisotropy terms (see (III.11)), a second independent quadratic integral of the motion (see (III.35)) exists only if the model parameters satisfy the condition (III.31). The results of our numerical calculations (as demonstrated, for example, by Fig. 5) strongly indicate that the violation of condition (III.31) implies not only the nonexistence of the quadratic invariant, but the nonexistence of a second independent analytic invariant in general.

In addition to the two spin cluster, we have discussed in this paper also two other remarkable spin systems – a single spin in a time-dependent external field and the Kittet-Shore model of N uniformly interacting spins, respectively. In the first case we have given a contributive criterion for the existence of a time-dependent integral of the motion which guarantees the complete integrability of the system. The complete integrability of the Kittel-Shore model, on the other hand, has been demonstrated by the explicit construction of N independent integrals of the motion in involution and the transformation to action-angle variables. The existence of $N-1$ additional independent integrals of the motion which are not explicitly time-dependent (and not in involution with the first N integrals) leads to the reduction of the number of independent frequencies in this system from N to $N-(N-1)=1$.

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