

2021

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Graphs with few trivial characteristic ideals

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Abstract

We give a characterization of the graphs with at most three trivial characteristic ideals. This implies the complete characterization of the regular graphs whose critical groups have at most three invariant factors equal to 1 and the characterization of the graphs whose Smith groups have at most 3 invariant factors equal to 1. We also give an alternative and simpler way to obtain the characterization of the graphs whose Smith groups have at most 3 invariant factors equal to 1, and a list of minimal forbidden graphs for the family of graphs with Smith group having at most 4 invariant factors equal to 1.

Keywords: characteristic ideal, Smith group, critical group, forbidden induced subgraph

MSC Subject Classification: 05C25, 13C40, 05C50, 13F20

1 Introduction

By considering an $m \times n$ matrix M with integer entries as a linear map $M : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$, the *cokernel* of M is the quotient module $\mathbb{Z}^n / \text{Im } M$. This finitely generated Abelian group becomes a graph invariant when we take the matrix M to be a matrix associated with the graph, say, the adjacency or Laplacian matrix. The cokernel of the adjacency matrix $A(G)$ is known as the *Smith group* of G and is denoted $S(G)$, and the torsion part of the cokernel of the Laplacian matrix $L(G)$ is known as the *critical group* $K(G)$ of G .

Smith groups were introduced in [31]. Recently, the computation of the Smith group for several families of graphs has attracted attention, see [10, 13, 19, 20, 36]. The critical group is especially interesting for connected graphs, since its order is equal to the number of spanning trees of the graph. The critical group has been studied intensively over the last 30 years on several contexts: the *group of components* [25, 26], the *Picard group* [11, 12], the *Jacobian group* [11, 12], the *sandpile group* [5, 17], *chip-firing game* [12, 27], or *Laplacian unimodular equivalence* [22, 28]. The book of Klivans [24] is an excellent reference on the theory of sandpiles and its connections to other combinatorial objects like hyperplane arrangements, parking functions, dominoes, etc.

The computation of the Smith normal form (SNF) of a matrix is a standard technique to determine its cokernel. We might refer the reader to Stanley's survey [33] on SNFs in combinatorics for more details in the topic.

One way to compute the SNF of a matrix M is by means of elementary row and column operations over the integers. Let M and N be two $n \times n$ matrices with integer entries. We say that M and N are *equivalent*, denoted by $N \sim M$, if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$. That is, M can be transformed to N by applying elementary row and column operations which are invertible over the ring of integers:

1. Swapping any two rows or any two columns.
2. Adding integer multiples of one row/column to another row/column.
3. Multiplying any row/column by ± 1 .

Moreover, if $N \sim M$, then $\text{coker}(M) = \mathbb{Z}^n / \text{Im } M \cong \mathbb{Z}^n / \text{Im } N = \text{coker}(N)$. Therefore, as the fundamental theorem of finitely generated Abelian groups states, the cokernel of M can be described as: $\text{coker}(M) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r} \oplus \mathbb{Z}^{n-r}$, where d_1, d_2, \dots, d_r are positive integers with $d_i \mid d_j$ for all $i \leq j$. These integers are called *invariant factors* of M . Let $\phi(M)$ denote the number of invariant factors of M equal to 1.

The computation of the invariant factors of the Laplacian matrix is an important technique used in the understanding of the critical group. For instance, several researchers have addressed the question of how often the critical group is cyclic, that is, how often $\phi(L(G))$ is equal to $n - 2$ or $n - 1$? In [26] and [35] D. Lorenzini and D. Wagner, based on numerical data, suggest we could expect to find a substantial proportion of graphs having a cyclic critical group. Based

on this, D. Wagner conjectured [35] that almost every connected simple graph has a cyclic critical group. A recent study [37] concluded that the probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\dots \approx 0.7935212,$$

where ζ is the Riemann zeta function; differing from Wagner's conjecture. Besides, it is interesting [15] that for any given connected simple graph, there is an homeomorphic graph with cyclic critical group. The reader interested on this topic may consult [16, 26, 37] for more questions and results.

On the other hand, the characterization of the family \mathcal{K}_k of simple connected graphs having critical group with k invariant factors equal to 1 has been of great interest. Probably, it was initially posed by R. Cori*. However, the first result appeared when D. Lorenzini noticed in [25], and independently A. Vince in [38], that the graphs in \mathcal{K}_1 consist only of complete graphs. After, C. Merino in [27] posed interest on the characterization of \mathcal{K}_2 and \mathcal{K}_3 . In this sense, some advances have been done. For instance, in [29] it was characterized the graphs in \mathcal{K}_2 whose third invariant factor is equal to n , $n - 1$, $n - 2$, or $n - 3$. In [14] the characterizations of the graphs in \mathcal{K}_2 with a cut vertex and number of independent cycles equal to $n - 2$ are given.

Later, a complete characterization of \mathcal{K}_2 was obtained in [6]. On the other hand, the characterization of the graphs in \mathcal{K}_3 seems to be a hard open problem [7]. For digraphs case, the characterization of digraphs with at most 1 invariant factor equal to 1 was completely obtained in [8]. These characterizations were obtained by using the *critical ideals* of a graph G , that are determinantal ideals, defined in [18], of the matrix $\text{diag}(x_1, \dots, x_n) - A(G)$, where x_1, \dots, x_n are indeterminates. These ideals turned out [3] to be related with other parameters like the *minimum rank* and the *zero-forcing number*. Similar ideals for the distance and distance Laplacian matrices were introduced in [2] with the name of *distance ideals*. Therefore, for example, the family of graphs with 2 trivial distance ideals contains the family of graphs whose distance matrix has at most 2 invariant factors equal to 1. It is interesting that there is an infinite number of minimal forbidden graphs for the graphs with two trivial distance ideals, see [4].

In the context of the Smith groups of graphs, it would be also interesting to characterize graphs having Smith group with at most k invariant factors equal to 1. For this we introduce further notation, let $\mathcal{S}_{\leq k}$ denote the family of simple connected graphs whose adjacency matrix has at most k invariant factors equal to 1, that is, $\phi(A(G)) \leq k$. The characterization of the $\mathcal{S}_{\leq 1}$ and $\mathcal{S}_{\leq 2}$ can be derived from [6], and the characterization of the digraphs with $\phi(A(G)) \leq 1$ was obtained in [8]. However, nothing is known on the structure of $\mathcal{S}_{\leq k}$, for $k \geq 3$.

The manuscript is organized as follows. In Section 2, we introduce the concept of characteristic ideals which are determinantal ideals defined in [18] as a generalization of the critical group and the characteristic polynomial. Also, we give the characterization of the graphs with one and two trivial characteristic

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ideals, and by product the characterization of the regular graphs in $\mathcal{K}_{\leq 1}$ and $\mathcal{K}_{\leq 2}$. We give, in Section 3, the characterization of graphs with 3 trivial characteristic ideals, consequently, this is used to give a complete characterization of regular graphs in $\mathcal{K}_{\leq 3}$. The characterization of $\mathcal{S}_{\leq 1}$, $\mathcal{S}_{\leq 2}$, and $\mathcal{S}_{\leq 3}$ can be derived from the obtained results, however, in Section 4, we give an alternative and simpler way to characterize these graph families. We also give a list of 43 forbidden graphs for $\mathcal{S}_{\leq 4}$.

2 Characteristic ideals of graphs

Consider a $n \times n$ matrix M whose entries are in the polynomial ring $\mathbb{Z}[X]$ with $X = \{x_1, \dots, x_m\}$. For $k \in [n] := \{1, \dots, n\}$, let $\mathcal{I} = \{r_j\}_{j=1}^k$ and $\mathcal{J} = \{c_j\}_{j=1}^k$ be two sequences such that $1 \leq r_1 < r_2 < \dots < r_k \leq n$ and $1 \leq c_1 < c_2 < \dots < c_k \leq n$. Let $M[\mathcal{I}; \mathcal{J}]$ denote the submatrix of a matrix M induced by the rows with indices in \mathcal{I} and columns with indices in \mathcal{J} . Recall the determinant of $M[\mathcal{I}; \mathcal{J}]$ is called a k -minor of M . The set of all k -minors of M is denoted by $\text{minors}_k(M)$. The k -th determinantal ideal $I_k(M)$ of a matrix M is the ideal generated by all k -minors of M . Some properties of determinantal ideals of graphs can be found in [1]. For example, determinantal ideals of M satisfy

$$\langle 1 \rangle \supseteq I_1(M) \supseteq \dots \supseteq I_n(M) \supseteq \langle 0 \rangle. \quad (1)$$

An ideal is said to be *trivial* or *unit* if it is equal to $\langle 1 \rangle$, that is, the ideal is equal to $\mathbb{Z}[X]$.

Definition 1. The k -th characteristic ideal $A_k(G, t)$ of a graph G is the k -th determinantal ideal of the matrix $tI_n - A(G)$, that is, the ideal $\langle \text{minors}_k(tI_n - A(G)) \rangle \subseteq \mathbb{Z}[t]$. The algebraic co-rank $\gamma_A(G)$ of a graph G is the maximum integer k such that $A_k(G, t)$ is trivial.

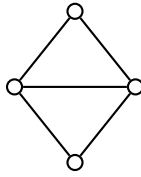


Figure 1: diamond graph

Example 2. Let G be the diamond graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$ such that each pair of vertices are adjacent, except for v_1 and v_3 , see Figure 1. Then

$$tI_4 - A(G) = \begin{bmatrix} t & -1 & 0 & -1 \\ -1 & t & -1 & -1 \\ 0 & -1 & t & -1 \\ -1 & -1 & -1 & t \end{bmatrix}$$

Since ± 1 is in $\text{minors}_1(tI_4 - A(G))$ and in $\text{minors}_2(tI_4 - A(G))$, then $A_1(G, t)$ and $A_2(G, t)$ are trivial. The different 3-minors of $tI_4 - A(G)$ are:

$$t^3 - 2t, -t^2 - 2t, t^2 + t, t^3 - 3t - 2, -2t - 2.$$

Note that $t = -(-t^2 - 2t) - (t^2 + t)$, then $t \in A_3(G, t)$, and similarly $2 \in A_3(G, t)$. Since all the 3-minors are a linear combination of t and 2, then $A_3(G, t) = \langle 2, t \rangle$. It is interesting to note that if $A_3(G, t)$ would be defined on $\mathbb{R}[t]$ instead, then $A_3(G, t)$ would be trivial. Finally, $A_4(G, t) = \langle \det(tI_4 - A(G)) \rangle = \langle t^4 - 5t^2 - 4t \rangle$.

Computing Gröbner basis of a characteristic ideal are an useful computational tool to find a minimal generating set. They can be computed in SAGE with the following code.

```

1 # G is a graph
2 def CharIdeals(G):
3     n = G.order()
4     R = macaulay2.ring("ZZ", '[t,x]').to_sage()
5     R.inject_variables(verbose=False);
6     L = diagonal_matrix([t for i in xrange(n)]) - G.adjacency_matrix()
7     Gamma = 0
8     for i in range(n+1):
9         M = L.minors(i)
10        I = R.ideal(M).groebner_basis()
11        print("Groebner basis of char ideal of size " + str(i))
12        print(str(I))
13        if I[0] == 1:
14            Gamma = i
15    print("gamma_A = " + str(Gamma))

```

Example 3. Thus, the Gröbner basis of the characteristic ideals and the algebraic co-rank of the diamond graph can be computed with the following SAGE code:

```

1 CharIdeals(Graph("C^"))

```

The connection of the characteristic ideals with the cokernel of the adjacency and Laplacian matrices is that the invariant factors can be recovered by evaluating the characteristic ideals. This relies on the theorem of elementary divisors, the details of which can be found in [23, Theorem 3.9].

Theorem 4 (Theorem of elementary divisors). *Let M a integer matrix of rank r with d_1, \dots, d_r its invariant factors. For $k \geq 1$, let Δ_k be the gcd of the k -minors of M , and $\Delta_0 = 1$. Then*

$$d_k = \frac{\Delta_k}{\Delta_{k-1}}.$$

Proposition 5. [1, Corollary 15] For $k \in [n]$,

$$A_k(G, 0) = \left\langle \prod_{j=1}^k d_j(A(G)) \right\rangle = \langle \Delta_k(A(G)) \rangle,$$

and if G is r -regular, then

$$A_k(G, r) = \left\langle \prod_{j=1}^k d_j(L(G)) \right\rangle = \langle \Delta_k(L(G)) \rangle,$$

where $\Delta_k(M)$ is the greatest common divisor of the k -minors of matrix M , and if $d_1(M) \mid \cdots \mid d_r(M)$ are the invariant factors in the Smith normal form of M , then $d_k(M) = \frac{\Delta_k(M)}{\Delta_{k-1}(M)}$ with $\Delta_0(M) = 1$.

Example 6. Continuing with the diamond graph. By evaluating, t at 0 in each characteristic ideal, we obtain that its SNF of the adjacency matrix is $\text{diag}(1, 1, 2, 0)$, meanwhile, for this case the SNF of its Laplacian matrix cannot be obtained since the diamond graph is not regular.

As consequence, if $A_k(G, t)$ is trivial, then the k -th invariant factor $d_k(A(G)) = 1$, and thus, $\gamma_A(G) \leq \phi(A(G))$. For the Laplacian matrix, we have the same when G is regular, that is, if G is regular, then $\gamma_A(G) \leq \phi(L(G))$. Then, the graphs in $\mathcal{S}_{\leq k}$ and the regular graphs in $\mathcal{K}_{\leq k}$ are contained in the family $\mathcal{C}_{\leq k}$ of graphs with at most k trivial characteristic ideals. By characterizing the graphs in $\mathcal{C}_{\leq k}$, we can use the containment to give a characterization of the regular graphs in $\mathcal{K}_{\leq k}$. Analogous ideas can be used to characterize $\mathcal{S}_{\leq k}$, however simpler ideas can be applied to obtain them, we will explore them in Section 4.

One advantage of characteristic ideals over critical group is that characteristic ideals are monotone on induced subgraphs.

Lemma 7. If H is an induced subgraph of G , then $A_k(H, t) \subseteq A_k(G, t)$.

Proof. Suppose G has n vertices, and H is an induced subgraph of G with m vertices. The result follows since any k -minor of $tI_m - A(H)$ is also a k -minor of $tI_n - A(G)$. Therefore, $\text{minors}_k(tI_m - A(H)) \subseteq \text{minors}_k(tI_n - A(G))$. \square

A similar result is not always true for the critical group, in fact, there are examples of graphs having different critical group, for instance, $K(K_4) \not\subseteq K(K_5)$. This is because, in general, it is not true that if H is an induced subgraph of G , then $L(H)$ is a submatrix of $L(G)$.

A graph G is *forbidden* for $\mathcal{C}_{\leq k}$ if $\gamma_A(G) \geq k + 1$. Thus, we can look for the minimal forbidden graphs to characterize the family $\mathcal{C}_{\leq k}$.

Lemma 8. The path P_k with k vertices is forbidden for $\mathcal{C}_{\leq k-2}$.

Proof. Let v_1, v_2, \dots, v_k be the vertices of P_k such that $v_i v_{i+1} \in E(P_k)$, for $i \in \{1, \dots, k-1\}$. Then, $A(P_k)$ is a tridiagonal matrix. Since $A(P_k)$ has a $(k-1)$ -minor equal to ± 1 , then P_k is forbidden for $\mathcal{C}_{\leq k-2}$. \square

Given a family \mathcal{F} of graphs, a graph G is called \mathcal{F} -free if no induced subgraph of G is isomorphic to a member of \mathcal{F} . The following theorem gives us the characterization of $\mathcal{C}_{\leq 1}$ and since the graphs in $\mathcal{K}_{\leq 1}$ are regular, we have $\mathcal{C}_{\leq 1} = \mathcal{K}_{\leq 1}$. Its proof is similar to Theorem 3.3 and Corollary 3.4 of [6].

Theorem 9. *Let G be connected simple graph. Then the following statements are equivalent.*

1. $G \in \mathcal{C}_{\leq 1}$,
2. $G \in \mathcal{K}_{\leq 1}$
3. G is P_3 -free
4. G is a complete graph

Now, before we give the characterizations of the graphs in $\mathcal{C}_{\leq 2}$, we give an explicit formula of the characteristic ideals of complete graphs and complete multipartite graphs, and few structural results needed for the characterization.

Lemma 10. [18, Proposition 3.15 & Theorem 3.16] *Let G be a complete graph with n vertices. Then*

$$A_j(G, t) = \begin{cases} \langle (t+1)^{j-1} \rangle & 1 \leq j \leq n-1, \\ \langle (t-n+1)(t+1)^{n-1} \rangle & j = n. \end{cases}$$

Lemma 11. [21, Theorem 3.2] *Let G be a complete multipartite graph with $m \geq 2$ parts of size $r_1, \dots, r_m \geq 2$. Let $n = \sum r_i$. Then*

$$A_j(G, t) = \begin{cases} \langle 1 \rangle & j \leq m-1, \\ \langle (m-1)t^{j-m}, t^{j-m+1} \rangle & m \leq j \leq n-m, \\ \langle t^{j-m+1} \prod_{a=1}^{m-n+j-1} (t+r_{i_a}), P \rangle & n-m < j < n \\ \langle \sum_{a=0}^m e_a(r_1, \dots, r_m) t^{n-a} \rangle & j = n, \end{cases}$$

where P is equal to $\left\{ \sum_{a=0}^{m-k} (k-1+a) e_a(r_{i_1}, \dots, r_{i_{m-k}}) t^{j-k-a} : k = n-j \geq 1 \text{ and } 1 \leq i_1 < \dots < i_{m-k} \leq m \right\}$,

and $e_a(s_1, \dots, s_l)$ is the elementary symmetric polynomial of degree a in l variables, i.e.,

$$e_a(s_1, \dots, s_l) = \sum_{1 \leq s_{i_1} < \dots < s_{i_a} \leq l} s_{i_1} \cdots s_{i_a}.$$

Let paw denote the four-vertex graph shown in Figure 2.

Lemma 12. [30, Theorem 1] *Let G be a paw-free connected graph. Then G is either K_3 -free or complete multipartite graph.*

Lemma 13. [9, Proposition 1] *Let G be a $\{P_4, K_3\}$ -free connected graph, then G is a complete bipartite graph.*

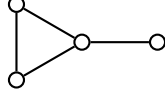


Figure 2: paw graph

Theorem 14. *Let G be connected simple graph. Then the following statements are equivalent:*

1. $G \in \mathcal{C}_{\leq 2}$,
2. G is $\{P_4, \text{paw}, K_5 - e\}$ -free,
3. G is complete graph or G is an induced subgraph of a complete tripartite graph.

Proof. (1) \implies (2) By Lemma 8, P_4 is forbidden for $\mathcal{C}_{\leq 2}$. Now considering

$$M = tI_4 - A(\text{paw}) = \begin{bmatrix} t & -1 & 0 & 0 \\ -1 & t & -1 & -1 \\ 0 & -1 & t & -1 \\ 0 & -1 & -1 & t \end{bmatrix},$$

we can obtain that $A_1(\text{paw}, t)$ and $A_2(\text{paw}, t)$ are trivial since there are appropriate minors of M equal to 1. Let

$$p(t) = \det(M[\{1, 2, 3\}; \{1, 2, 4\}]) = -t^2 - t + 1$$

and

$$q(t) = \det(M[\{1, 2, 3\}; \{1, 3, 4\}]) = t^2 + t.$$

Since $1 = p(t) + q(t) \in A_3(\text{paw}, t)$, then $A_3(\text{paw}, t)$ is trivial. Thus paw is forbidden for $\mathcal{C}_{\leq 2}$. Now, let

$$M = tI_5 - A(K_5 - e) = \begin{bmatrix} t & 0 & -1 & -1 & -1 \\ 0 & t & -1 & -1 & -1 \\ -1 & -1 & t & -1 & -1 \\ -1 & -1 & -1 & t & -1 \\ -1 & -1 & -1 & -1 & t \end{bmatrix}.$$

And, let

$$p(t) = \det(M[\{1, 2, 3\}; \{1, 2, 4\}]) = -t^2 - 2t$$

and

$$q(t) = \det(M[\{2, 3, 4\}; \{2, 3, 5\}]) = -t^2 - 2t - 1.$$

Since $1 = p(t) - q(t)$, then $A_3(K_5 - e, t)$ is trivial. From which follows that $K_5 - e$ is forbidden for $\mathcal{C}_{\leq 2}$.

(2) \implies (3) By Lemma 12, a paw-free graph is either K_3 or a complete multipartite graph. In the first case, considering that G is also P_4 -free, then by Lemma 13, G is a bipartite graph. On the other hand, let G be a complete multipartite graph with more than 3 partite sets. Since G is $\{K_5 - e\}$ -free, then each partite set has at most one vertex, that is, G is a complete graph.

(3) \implies (1) Lemma 10 states complete graphs have at most one trivial characteristic ideal. Now let G be a complete tripartite graph with each part of size at least 2. By Lemma 11, we have the third characteristic ideal is not trivial. Thus by Lemma 7, if H is an induced subgraph of G , then H has at most 2 trivial characteristic ideals. \square

The characterization of the regular graphs whose critical group have 2 invariant factors equal to 1 follows by evaluating the third characteristic ideal of these graphs at t equal the degree of any vertex.

Corollary 15. *Let G be a connected simple regular graph. Then $G \in \mathcal{K}_{\leq 2}$ if and only if G is either a complete graph K_r , a regular complete bipartite graph $K_{r,r}$ or a regular complete tripartite graph $K_{r,r,r}$.*

Proof. Since G is regular and $G \in \mathcal{C}_{\leq 2}$, then G is either a complete graph K_r , a regular complete bipartite graph $K_{r,r}$ or a regular complete tripartite graph $K_{r,r,r}$. On the other hand, let G be any of these graphs. By Lemmas 10 and 11, the third characteristic ideal of G is

$$A_3(G, t) = \begin{cases} \langle (t+1)^2(t-2) \rangle & K_{r+1} \text{ with } r = 2, \\ \langle (t+1)^2 \rangle & K_{r+1} \text{ with } r \geq 3, \\ \langle t^2, 2t \rangle & K_{r,r} \text{ with } r = 2, \\ \langle t \rangle & K_{r,r} \text{ with } r \geq 3, \\ \langle 2, t \rangle & K_{r,r,r} \text{ with } r \geq 2. \end{cases}$$

By evaluating $A_2(G, t)$ and $A_3(G, t)$ at t equal the degree of any vertex of G , we obtain that the third invariant factor of G is different than 1. \square

A characterization of the graphs with Smith groups having 2 invariant factors equal to 1 can also be obtained from the Theorem 14, however, we will use simpler ideas in Section 4.

3 Regular graphs with at most 3 trivial characteristic ideals

In this section we will characterize the graphs with at most 3 trivial characteristic ideals. As consequence, we will obtain a complete characterization of the regular graphs in $\mathcal{K}_{\leq 3}$.

Given a graph $G = (V, E)$ and a vector $\mathbf{d} \in \mathbb{Z}^V$, the graph $G^{\mathbf{d}}$ is constructed as follows. For each vertex $u \in V$, associate a new vertex set V_u , where V_u is a clique of cardinality $-\mathbf{d}_u$ when \mathbf{d}_u is negative, and V_u is a stable set of

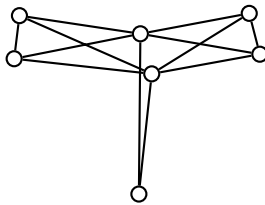


Figure 3: $S_4^{\mathbf{r}}$ with $\mathbf{r} = (2, 1, -2, -2)$

cardinality \mathbf{d}_u if \mathbf{d}_u when positive. Each vertex in V_u is adjacent with each vertex in V_v if and only if u and v are adjacent in G . Then the graph G is called the *underlying graph* of $G^{\mathbf{d}}$. For instance, let S_n denote the star graph with n vertices; with one apex vertex and $n - 1$ leaves. In Figure 3 there is a drawing of $S_4^{\mathbf{r}}$ with $\mathbf{r} = (2, 1, -2, -2)$, where the first entry of \mathbf{r} is associated with the apex vertex.

Let \mathcal{F} denote the collection of graphs shown in Figure 4. In the following, we seek to find a structural characterization for graphs containing none of the 14 given graphs in \mathcal{F} as an induced subgraph.

Lemma 16. *Let G be a connected graph in $\mathcal{C}_{\leq 3}$, then G is \mathcal{F} -free.*

Proof. It follows by computing the fourth characteristic ideals of the graphs in \mathcal{F} and see that they are trivial. Then, by Lemma 7, G cannot contain any graph in \mathcal{F} as induced subgraph. \square

Theorem 17. *Let G be a connected graph. If G is \mathcal{F} -free, then it is an induced subgraph of one of the following:*

- (1) C_5 ,
- (2) the triangular prism $K_3 \square K_2$,
- (3) a complete 4-partite graph,
- (4) $C_4^{\mathbf{r}}$, for some $-\mathbf{r} \in \mathbb{N}^4$, or
- (5) $S_4^{\mathbf{r}}$, for some $-\mathbf{r} \in \mathbb{N}^4$.

Proof. Suppose that G is a connected \mathcal{F} -free graph; we show that G has one of the five forms described above.

Since G is connected, the well known result of Seinsche [32] implies that G either contains P_4 as an induced subgraph, or G is the complement of a disconnected graph and hence is a join of two graphs with nonempty vertex sets.

Suppose first that G contains P_4 as an induced subgraph, and let w, x, y, z be the vertices, in order, of such an induced path.

Since G is $\{\text{fork, 4-pan, bull, } P_5, \text{co-4-pan, 3-fan, kite}\}$ -free, we conclude that any vertex of G not in $\{w, x, y, z\}$ is adjacent to either none of these four vertices,

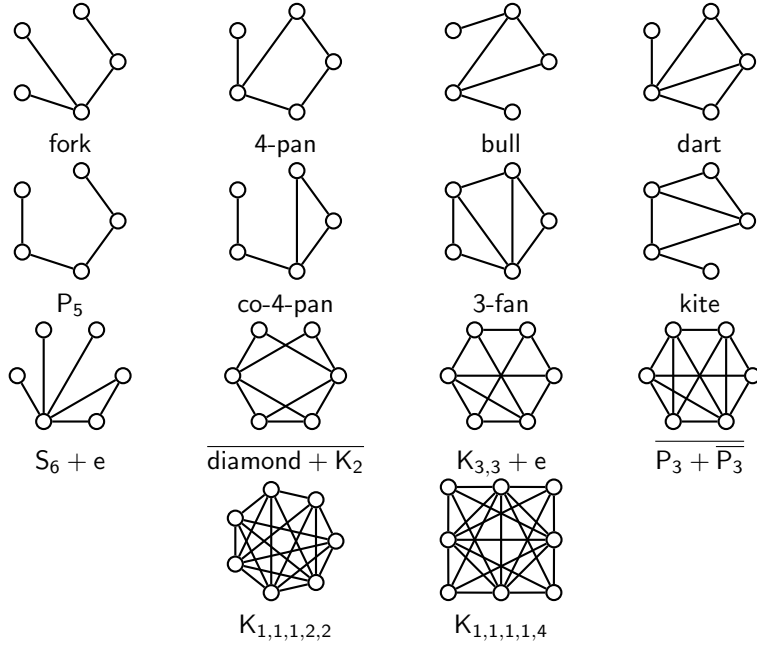


Figure 4: The family of graphs \mathcal{F} .

or it is adjacent to both endpoints w, z and at most one of the midpoints x, y . Hence we may partition the vertices of $G - \{w, x, y, z\}$ into three sets:

- V_{wz} : vertices adjacent to w, z and neither of x, y ;
- V_{wxz} : vertices adjacent to w, z and x but not y ;
- V_{wyz} : vertices adjacent to w, z and y but not x ;
- U : vertices adjacent to no vertex of $\{w, x, y, z\}$.

We illustrate these sets in Figure 5.

If there is a pair of non-adjacent vertices in V_{wz} then $G[\{w, y, z\} \cup V_{wz}]$ contains an induced copy of the **4-pan**. Moreover, if $|V_{wz}| \geq 2$ then $G[\{w, y, z\} \cup V_{wz}]$ contains an induced copy of the **dart**. Thus $|V_{wz}| \leq 1$.

Similarly, we have $|V_{wxz}| \leq 1$ and $|V_{wyz}| \leq 1$. If V_{wz} is nonempty, let us denote $V_{wz} = \{v_{wz}\}$ and so on.

Since the induced subgraph of G having vertex set $\{x, y, z, v_{wz}, v_{wxz}\}$ is not isomorphic to the **4-pan**, it must be the case that v_{wz} is adjacent to v_{wxz} . However, then the induced subgraph on $\{x, y, w, v_{wz}, v_{wxz}\}$ is isomorphic to the **kite**, a contradiction. Since a similar contradiction arises for vertices v_{wz} and v_{wyz} , we conclude that if V_{wz} is nonempty then both V_{wxz} and V_{wyz} are empty; if either V_{wxz} or V_{wyz} is nonempty, then V_{wz} is empty.

Let $E[A, B]$ be the set of edges between two sets of vertices A and B . If $E[V_{wxz}, V_{wyz}] \neq \emptyset$, then $G[\{w, x, z, v_{wxz}, v_{wyz}\}]$ contains the **3-fan** as an induced subgraph, a contradiction, so there are no edges between V_{wxz} and V_{wyz} .

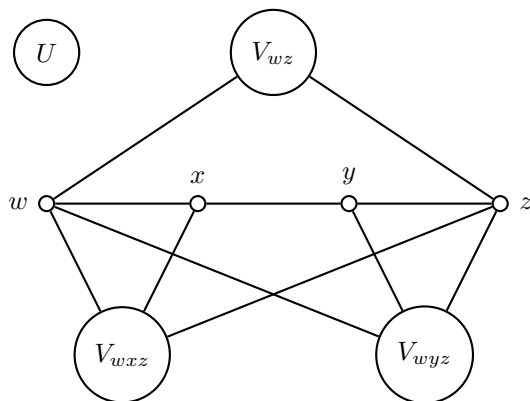


Figure 5: Diagram describing G .

name	alternative name
dart	$K_1 \vee (P_3 + K_1)$
3-fan	$P_4 \vee K_1$
$S_6 + e$	$K_1 \vee (K_2 + 3K_1)$
diamond + K_2	$2K_1 \vee (K_2 + 2K_1)$
$K_{3,3} + e$	$3K_1 \vee (K_2 + K_1)$
$P_3 + P_3$	$P_3 \vee (K_2 + K_1) = K_{1,2} \vee (K_2 + K_1)$
$K_{1,1,1,2,2}$	$K_3 \vee C_4$
$K_{1,1,1,1,4}$	$K_4 \vee 4K_1$

Table 1: Graphs in \mathcal{F} that are join of two graphs.

Now note that if any vertex in U has a neighbor in V_{wz} , then G induces P_5 , a contradiction. If U has any neighbor in V_{wxz} (or in V_{wyz}), then G induces both bull and 4-pan. Since G is connected, some vertex in U would have a neighbor in V_{wz} or V_{wxz} or V_{wyz} unless U were empty, so we conclude that U is empty.

We conclude that G is isomorphic to either P_4 , C_5 , the house graph, or the triangular prism. This completes the characterization of G when G induces P_4 .

Suppose henceforth that G is P_4 -free. As described previously, since G is a connected P_4 -free graph, then G can be written as $G = G_1 \vee G_2$, where G_1 and G_2 each have at least one vertex. Not every such graph is \mathcal{F} -free, as the graphs in Table 1 show.

If G is $K_2 + K_1$ -free, then, by Lemma 12, G is a complete multipartite graph. Since G is $\{K_{1,1,1,2,2}, K_{1,1,1,1,4}\}$ -free, if such a graph G has five or more partite sets, then no partite set can have four or more vertices, and at most one partite set can have two or three vertices. Thus, if G has five or more partite sets, then G is isomorphic to $3K_1 \vee K_m$ or to $2K_1 \vee K_m$ for some $m \geq 4$; which are included in the case (5).

If G contains $K_2 + K_1$, it must do so within G_1 or within G_2 . Without loss of generality, suppose that G_2 contains $K_2 + K_1$, and assume that G_2 cannot be written as a join of smaller graphs (if it could, we could redefine G_1 to include one of the vertex sets of this join). The forbidden subgraph assumptions imply that G_1 must be $\{P_3, 3K_1\}$ -free. Since G_1 is P_3 -free, it is a disjoint union of cliques. And since G_1 is $3K_1$ -free, there are at most two of these cliques. Hence G_1 has the form $K_p + K_q$, where $0 \leq p \leq q$ and $q \geq 1$ (by our assumption that the join $G = G_1 \vee G_2$ was nontrivial).

If $p \geq 1$ and $q \geq 2$, then G_1 contains $K_2 + K_1$ as an induced subgraph, and exchanging the roles of G_1 and G_2 in the arguments above imply that G_2 has the form $K_{p'} + K_{q'}$ for $p' \geq 1$ and $q' \geq 2$ and henceforth $G = (K_p + K_q) \vee (K_{p'} + K_{q'})$; which is included in case (4).

Next, we will consider the cases when $p = 1, q = 1$ and when $p = 0$ in detail. First we establish some further structure for G_2 .

Consider an induced copy of $K_2 + K_1$ within G_2 , and let v be a vertex of G_2 not in this induced subgraph. Since G is $K_1 \vee (P_3 + K_1)$ -free, we may assume that G_2 is $\{P_4, P_3 + K_1\}$ -free. And this implies that if v is adjacent to one endpoint of the K_2 -component in the $K_2 + K_1$ -subgraph, then it must be adjacent to the other endpoint.

Let ab be the edge and let c be the isolated vertex in an induced subgraph isomorphic to $K_2 + K_1$. Let X_d be the set of vertices in G_2 adjacent to none of a, b, c ; let X_{ab} be the set of vertices in G_2 adjacent to both a and b but not c ; let X_c be the set of vertices in G_2 adjacent to c but not a and b ; and let X_{abc} be the set of vertices in G_2 adjacent to all of a, b, c .

Now, if $p = q = 1$, then, since G is $\{2K_1 \vee (K_2 + 2K_1)\}$ -free, we may also conclude that G_2 is $K_2 + 2K_1$ -free, which implies that X_d is empty. And X_{abc} is empty as well, this is because, otherwise, we would have $P_3 \vee (K_2 + K_1)$ as an induced subgraph of G . The vertex sets X_{ab} and X_c must be cliques, otherwise, G would contain a $K_1 \vee (P_3 + K_1)$ or a $2K_1 \vee (K_2 + 2K_1)$, respectively. And $E[X_{ab}, X_c]$ is empty, since otherwise G would contain P_4 as induced subgraph. Therefore, G_2 is the disjoint union of two cliques, that is, $G = 2K_1 \vee (K_r + K_s)$ with $r \geq 2$ and $s \geq 1$. Which is contained in case (4).

On the other hand, let us consider the case $p = 0$ and $q \geq 1$. Then $G_1 = K_q$ and $V(G_2) = \{a, b, c\} \cup X$, where $X = X_{ab} \cup X_{abc} \cup X_c \cup X_d$.

The sets X_{ab} and X_c are cliques, since otherwise $G_2[\{a, c\} \cup X_{ab}]$ and $G_2[\{a, c\} \cup X_c]$ would, respectively, contain an induced copy of $(P_3 + K_1)$. Also X_d is a clique, since otherwise G would contain an induced copy of $K_1 \vee (K_2 + 3K_1)$.

Furthermore $E[X_c, X_d] = \emptyset = E[X_{ab}, X_d]$, otherwise $G_2[\{b, c\} \cup X_c \cup X_d]$ or $G_2[\{b, c\} \cup X_{ab} \cup X_d]$ would contain an induced copy of $P_3 + K_1$, respectively. Likewise, $E[X_{ab}, X_c] = \emptyset$ since otherwise P_4 would be an induced subgraph of $G_2[\{b, c\} \cup X_{ab} \cup X_c]$.

Moreover, $E[X_{abc}, X_d]$ is of maximum size, that is, every vertex of X_{abc} is adjacent to every vertex of X_d , since otherwise $G_2[\{b, c\} \cup X_{abc} \cup X_d]$ would contain an induced copy of $P_3 + K_1$. Also $E[X_{abc}, X_{ab}]$ and $E[X_{abc}, X_c]$ are of maximum size because G_2 is P_4 -free.

By the argument above and our assumption that G_2 cannot be written as a join of smaller graphs we can conclude that $X_{abc} = \emptyset$.

Finally, if $p = 0$ then $G = K_q \vee (K_r + K_s + K_t)$, where $r \geq 2, q, s \geq 1$ and $t \geq 0$. Which is included in case (5). \square

Lemma 18. *The third characteristic ideals of C_5 and $K_3 \square K_2$ are trivial and the fourth characteristic ideals of C_5 and $K_3 \square K_2$ are non trivial. In fact, $A_4(C_5, t) = \langle t^2 + t - 1 \rangle$ and $A_4(K_3 \square K_2, t) = \langle t + 2, 5 \rangle$.*

Observation 19. In the following, let $L_m = (t+1)I_m - J_m$. Note that, for any \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$, the 4-minors of the matrices $tl_{-(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} - A(C_4^{\mathbf{r}})$ and $tl_{-(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)} - A(S_4^{\mathbf{r}})$ are contained in the 4-minors of the matrices

$$tl_{16} - A\left(C_4^{(-4,-4,-4,-4)}\right) = \begin{bmatrix} L_4 & -J_4 & 0_4 & -J_4 \\ -J_4 & L_4 & -J_4 & 0_4 \\ 0_4 & -J_4 & L_4 & -J_4 \\ -J_4 & 0_4 & -J_4 & L_4 \end{bmatrix}$$

and

$$tl_{16} - A\left(S_4^{(-4,-4,-4,-4)}\right) = \begin{bmatrix} L_4 & -J_4 & -J_4 & -J_4 \\ -J_4 & L_4 & 0_4 & 0_4 \\ -J_4 & 0_4 & L_4 & 0_4 \\ -J_4 & 0_4 & 0_4 & L_4 \end{bmatrix},$$

respectively. Therefore, $A_4(C_4^{\mathbf{r}}, t) \subseteq A_4\left(C_4^{(-4,-4,-4,-4)}, t\right)$ and $A_4(S_4^{\mathbf{r}}, t) \subseteq A_4\left(S_4^{(-4,-4,-4,-4)}, t\right)$ for every \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$.

Lemma 20. *Let \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$. Then the fourth characteristic ideal of $C_4^{\mathbf{r}}$ is not trivial. Moreover, $A_4(C_4^{\mathbf{r}}, t) \subseteq \langle t + 1, 3 \rangle$.*

Proof. The Gröbner basis of the ideal generated by the 4-minors of the matrix $tl_{16} - A\left(C_4^{(-4,-4,-4,-4)}\right)$ is $\langle t + 1, 3 \rangle$, that is, $A_4\left(C_4^{(-4,-4,-4,-4)}, t\right) = \langle t + 1, 3 \rangle$. Hence, by the argument in Observation 19, we have that $A_4(C_4^{\mathbf{r}}, t) \subseteq \langle t + 1, 3 \rangle$ for any \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$. \square

Note that $A_4(C_4^{\mathbf{r}}, t) = \langle t + 1, 3 \rangle$ and $A_3(C_4^{\mathbf{r}}, t) = \langle 1 \rangle$ when $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4 \leq -4$. In a similar manner, given that the Gröbner basis of $A_4\left(S_4^{(-4,-4,-4,-4)}, t\right)$ is $\langle t + 1, 2 \rangle$, we have the following

Lemma 21. *Let \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$. Then the fourth characteristic ideal of $S_4^{\mathbf{r}}$ is not trivial. Moreover, $A_4(S_4^{\mathbf{r}}, t) \subseteq \langle t + 1, 2 \rangle$.*

Theorem 22. *A connected graph G is in $\mathcal{C}_{\leq 3}$ if and only if it is an induced subgraph of one of the following:*

- (1) C_5 ,

- (2) the triangular prism $K_3 \square K_2$,
- (3) a complete 4-partite graph,
- (4) $C_4^{\mathbf{r}}$, for some \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$, or
- (5) $S_4^{\mathbf{r}}$, for some \mathbf{r} such that $-\mathbf{r} \in \mathbb{N}^4$.

Proof. \Rightarrow) This follows from Lemma 16 and Theorem 17.

\Leftarrow) From Lemmas 11, 18, 20 and 21, we have that the 4-th characteristic ideals of the graphs C_5 , $K_3 \square K_2$, complete 4-partite graphs, $C_4^{\mathbf{r}}$ and $S_4^{\mathbf{r}}$ are not trivial. Then, by Lemma 7, the 4-th characteristic ideal of any induced subgraph of these graphs is non-trivial. \square

Now, we give the characterization of the regular graphs whose critical group has at most 3 invariant factors equal to 1.

Corollary 23. *Let G be a connected simple regular graph. Then $G \in \mathcal{K}_{\leq 3}$ if and only if G is one of the following:*

- (a) C_5 ,
- (b) $K_3 \square K_2$,
- (c) a complete graph K_r ,
- (d) a regular complete bipartite graph $K_{r,r}$,
- (e) a regular complete tripartite graph $K_{r,r,r}$,
- (f) a regular complete graph 4-partite graph $K_{r,r,r,r}$,
- (g) $C_4^{(-r,-r,-r,-r)}$, for any $r \in \mathbb{N}$.

Proof. \Rightarrow) This follows from the fact that G is a regular graph in $\mathcal{C}_{\leq 3}$.

\Leftarrow) It is clear from Lemma 18 that the fourth invariant factors of C_5 and $K_3 \square K_2$ are different than 1. The graphs in (c), (d) and (e) are precisely the graphs in $\mathcal{K}_{\leq 2} \subset \mathcal{K}_{\leq 3}$. For (f), if $r = 1$ we have the complete graph with four vertices. Therefore, we assume that $r \geq 2$. By Lemma 11, the fourth characteristic ideal of a 4-partite regular complete graph is $\langle 3, t \rangle$ and its third characteristic ideal is trivial. Then, evaluating at $t = 3r$, the degree of any vertex, we have that the fourth invariant factor is $\gcd(3, 3r) = 3$ and therefore $K_{r,r,r,r} \in \mathcal{K}_{\leq 3}$. Finally, for (g), note that the degree of any vertex of $C_4^{(-r,-r,-r,-r)}$ is $3r - 1$. By Lemma 20, when $r \geq 4$ the fourth invariant factor is the $\gcd(3r, 3) = 3$. Thus $C_4^{(-r,-r,-r,-r)} \in \mathcal{K}_{\geq 3}$ when $r \geq 4$. The lower cases can be explicitly computed to verify that $C_4^{(-r,-r,-r,-r)} \in \mathcal{K}_{\leq 3}$. \square

4 Graphs whose Smith group has at most 4 invariant factors equal to 1

In this section we give the characterizations of the graph families $\mathcal{S}_{\leq k}$ for $k \in \{1, 2, 3\}$. And for $k = 4$, we give a set of 43 minimal forbidden graphs for $\mathcal{S}_{\leq 4}$.

We have that $\mathcal{S}_{\leq k}$ is closed under induced subgraphs. This observation follows from next proposition.

Proposition 24. *If H is an induced subgraph of G , then $\phi(A(H)) \leq \phi(A(G))$.*

Proof. Let H be an induced subgraph of G . For any k such that $1 \leq k \leq |V(H)|$, the k -minors of $A(H)$ are contained in the k -minors of $A(G)$. Therefore, if $\Delta_k(A(H)) = 1$, then $\Delta_k(A(G)) = 1$. \square

We can define a graph G to be *forbidden* for $\mathcal{S}_{\leq k}$ when $\phi(A(G)) \geq k + 1$. Let $\mathbf{Forb}(\mathcal{S}_{\leq k})$ denote the set of minimal forbidden graphs for $\mathcal{S}_{\leq k}$ with respect to the induced subgraph order. Thus $G \in \mathcal{S}_{\leq k}$ if and only if G is $\mathbf{Forb}(\mathcal{S}_{\leq k})$ -free. Therefore, characterizing the minimal forbidden induced subgraphs for $\mathcal{S}_{\leq k}$ leads to a characterization of $\mathcal{S}_{\leq k}$. For instance, let P_2 denote the path with 2 vertices. We have that the Smith normal form of the adjacency matrix of P_2 has 2 invariant factors equal to 1. Then, $\mathcal{S}_{\leq 1}$ consists only of K_1 , and there is no graph G with $\phi(A(G)) = 1$.

Now, we are going to give an alternative proof of the characterization $\mathcal{S}_{\leq 2}$. For this, next result gives the SNF for complete k -partite graphs. A particular case of Lemma 11 is the following lemma, which also was noticed in [10].

Lemma 25. *Let G be a complete k -partite graph with n vertices. Then the Smith normal form of $A(G)$ is equal to $I_{k-1} \oplus (k-1) \oplus 0I_{n-k}$.*

Now we are ready to give the characterization of graphs whose Smith group have at most 2 invariant factors.

Theorem 26. *Let G be connected graph. Then the followings are equivalent.*

1. *the SNF of $A(G)$ has at most 2 invariant factors equal to 1,*
2. *G is $\{P_4, \mathbf{paw}, K_4\}$ -free,*
3. *G is an induced subgraph of a complete tripartite graph.*

Proof. (1) \implies (2) The SNF of the adjacency matrices of P_4 , \mathbf{paw} , K_4 are equal to $\text{diag}(1, 1, 1, 1)$, $\text{diag}(1, 1, 1, 1)$ and $\text{diag}(1, 1, 1, 3)$, respectively. Since any induced subgraph H of P_4 , \mathbf{paw} , or K_4 has $\phi(A(H)) \leq 2$, then $\{P_4, \mathbf{paw}, K_4\} \subseteq \mathbf{Forb}(\mathcal{S}_{\leq 2})$.

(2) \implies (3) By Lemma 12, G is either triangle free or a complete multipartite graph. In the first case by Lemma 13, G is a complete bipartite graph. And in the second case since G is K_4 -free, then G is complete tripartite graph.

(3) \implies (1) It follows by Lemma 25 that the SNF of $A(G)$ is at most 2. \square

An analogous reasoning give us the characterization of $\mathcal{S}_{\leq 3}$.

Theorem 27. *Let G be connected graph. Then the followings are equivalent.*

1. *the SNF of $A(G)$ has at most 3 invariant factors equal to 1*
2. *G is $\{P_4, \text{paw}, K_5\}$ -free*
3. *G is an induced subgraph of a complete four-partite graph*

Proof. (1) \implies (2) The SNF of the adjacency matrices of P_4 , paw, K_5 are equal to $\text{diag}(1, 1, 1, 1)$, $\text{diag}(1, 1, 1, 1)$ and $\text{diag}(1, 1, 1, 1, 4)$, respectively. Since any induced subgraph H of P_4 , paw, or K_5 has $\phi(A(H)) \leq 3$, then $\{P_4, \text{paw}, K_5\} \subseteq \text{Forb}(\mathcal{S}_{\leq 3})$.

(2) \implies (3) Since G is paw-free, then by Lemma 12, G is either triangle-free or a complete multipartite graph. Thus, in the first case, G is also K_3 -free, by Lemma 13, G is a complete bipartite graph. In the second case, since G is K_5 -free, then G is complete tripartite graph.

(3) \implies (1) It follows by Lemma 25 that the SNF of $A(G)$ is at most 3. \square

The next case is more complicated. With the use of SAGE [34], we found that there are 43 forbidden graphs for $\mathcal{S}_{\leq 4}$, see Figure 6. The following SAGE code computes the minimal forbidden graphs with at most m vertices for $\mathcal{S}_{\leq n}$.

```

1 import numpy as np
2
3 def MinForb(m,n):
4     Forbidden = []
5     for k in range(2,m):
6         for g in graphs(k):
7             if g.is_connected():
8                 SNF = g.adjacency_matrix().smith_form()[0].numpy().diagonal()
9                 num_ones = list(SNF).count(1)
10                if num_ones >= n+1:
11                    Forbidden.append([g.graph6_string(),num_ones])
12
13    Minimal = []
14    for g in range(0,len(Forbidden)):
15        flag = True
16        for h in Minimal:
17            if
18                Graph(Forbidden[g][0]).subgraph_search(Graph(h),induced=True)
19                != None:
20                    flag = False
21                    break
22        if flag == True:
23            Minimal.append(Forbidden[g][0])
24    return Minimal

```

The problem of characterizing graphs in $\mathcal{S}_{\leq 4}$ is not straightforward. However, it is interesting that if $G \in \mathcal{S}_{\leq 4}$, then any graph obtained by replacing its vertices by stable sets will be also in $\mathcal{S}_{\leq 4}$. This will be shown next.

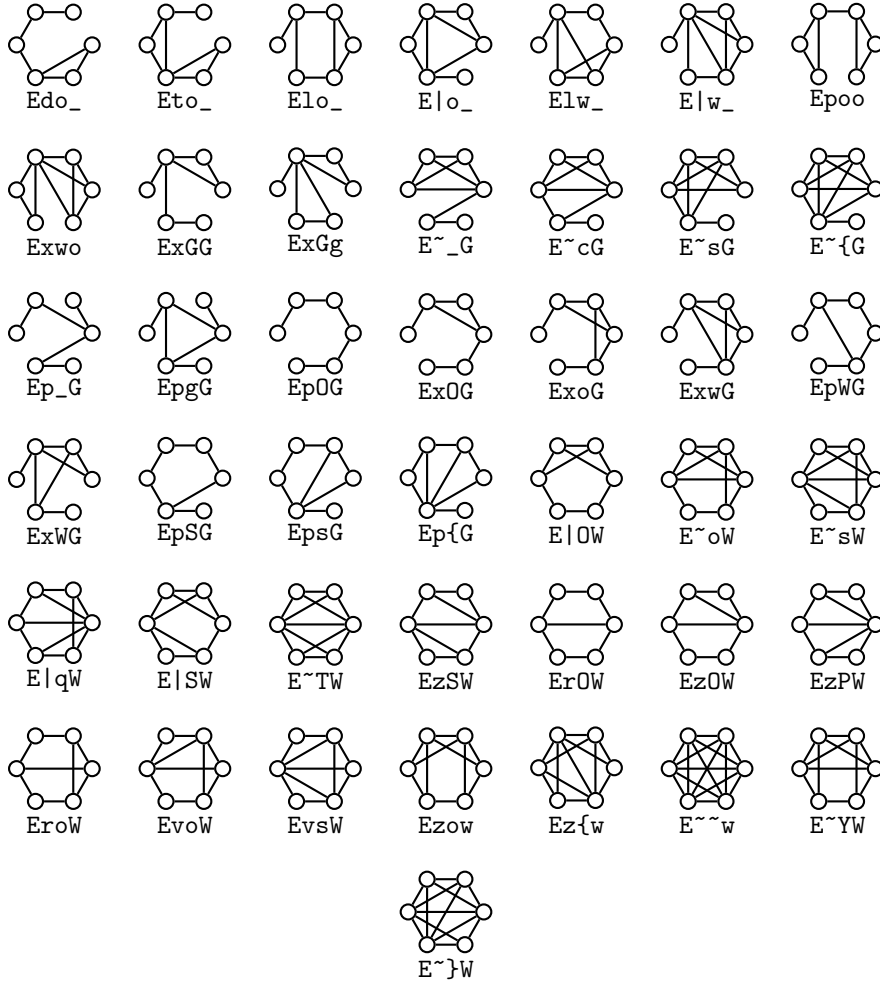


Figure 6: Some forbidden for $\mathcal{S}_{\leq 4}$.

Lemma 28. *For any $\mathbf{d} \in \mathbb{N}^V$, the non-zero invariant factors of $A(G)$ are equal to the non-zero invariant factors of $A(G^{\mathbf{d}})$.*

Proof. Given $u \in V$. Let G^u denote the graph obtained after duplicating vertex u . Since the adjacency matrix of G^u is equivalent to $\begin{bmatrix} A(G) & \mathbf{0}^T \\ \mathbf{0} & 0 \end{bmatrix}$, then the non-zero invariant factors of $A(G)$ and $A(G^u)$ are the same. From which the result follows. \square

Previous lemma help us in computing the Smith normal form of the adjacency matrix of graphs with duplicated vertices. In particular, it bound the

number of non-zero invariant factors.

Corollary 29. *Let G be a graph in $\mathcal{S}_{\leq k}$, then $G^{\mathbf{d}} \in \mathcal{S}_{\leq k}$ for any $\mathbf{d} \in \mathbb{N}^V$.*

Acknowledgement

This research was initiated during GRWC 2018 (Graduate Research Workshop in Combinatorics) at Iowa State University. Carlos A. Alfaro was supported by SNI and Ralihe R. Villagrán was supported by CONACyT.

References

- [1] A. Abiad, C.A. Alfaro, K. Heysse, M.C. Vargas. Eigenvalues, Smith normal form and determinantal ideals. Preprint arXiv:1910.12502.
- [2] C.A. Alfaro. On graphs with 2 trivial distance ideals. *Linear Algebra Appl.* 597 (2020) 69–85.
- [3] C.A. Alfaro and J. C.-H. Lin. Critical ideals, minimum rank and zero forcing number. *Appl. Math. Comput.* 358 (2019), 305–313.
- [4] C.A. Alfaro and L. Taylor. Distance ideals of graphs. *Linear Algebra Appl.* 584 (2020) 127–144.
- [5] C.A. Alfaro, C.E. Valencia. On the sandpile group of the cone of a graph. *Linear Algebra Appl.* 436 (2012), 1154–1176.
- [6] C.A. Alfaro and C.E. Valencia. Graphs with two trivial critical ideals. *Discrete Appl. Math.* 167 (2014), 33–44.
- [7] C.A. Alfaro and C.E. Valencia. Small clique number graphs with three trivial critical ideals. *Special Matrices* 6 (2018), 122–154.
- [8] C.A. Alfaro, C.E. Valencia and A. Vázquez-Ávila. Digraphs with at most one trivial critical ideal. *Linear and Multilinear Algebra* 66 (2018), 2036–2048.
- [9] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two. *Electron. J. Linear Algebra* 11 (2004), 258–280.
- [10] R.B. Bapat, M. Karimi. Integral complete multipartite graphs. *Linear Algebra Appl.* 549 (2018), 1–11.
- [11] R. Bacher, P. de la Harpe, T. Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. *Bull. Soc. Math. France.* 125 (1997), 167–198.
- [12] N. Biggs. Chip-firing and the critical group of a graph. *J. Alg. Combin.* 9 (1999), 25–46.

- [13] D.B. Chandler, P. Sin, Q. Xiang. The Smith group of the hypercube graph. *Des. Codes Cryptogr.* 84 (2017), 283–294.
- [14] W.H. Chan, Y. Hou, W.C. Shiu. Graphs whose critical groups have larger rank. *Acta Math. Sinica* 27 (2011), 1663–1670.
- [15] S. Chen and S.K. Ye, Critical groups for homeomorphism classes of graphs. *Discrete Math.* 309 (2008), 1 255–258.
- [16] J. Clancy, N. Kaplan, T. Leake, S. Payne and M.M. Wood, On a Cohen-Lenstra heuristic for Jacobians of random graphs. *J. Algebraic Combin.* 42 (2015), 3 701–723.
- [17] R. Cori and D. Rossin. On the sandpile group of dual graphs. *European J. Combin.* 21 (2000), 447–459.
- [18] H. Corrales and C.E. Valencia. On the critical ideals of graphs. *Linear Algebra Appl.* 439 (2013), 3870–3892.
- [19] J.E. Ducey, J. Gerhard, N. Watson. The Smith and Critical Groups of the Square Rook’s Graph and its Complement. *Electron. J. Combin.* 23 (2016), paper 4.9.
- [20] J.E. Ducey, D.M. Jalil. Integer invariants of abelian Cayley graphs. *Linear Algebra Appl.* 445 (2014), 316–325.
- [21] Y. Gao. On the critical ideals of complete multipartite graphs. *Electron. J. Linear Algebra* 36 (2020), 94–105.
- [22] R. Grone, R. Merris, W. Watkins. Laplacian unimodular equivalence of graphs. In: R. Brualdi, S. Friedland and V. Klee (Eds.) *Combinatorial and Graph-Theoretical Problems in Linear Algebra*. New York (NY): Springer-Verlag (1993), 175–180.
- [23] N. Jacobson. *Basic Algebra I, Second Edition*, W. H. Freeman and Company, New York, 1985.
- [24] C.J. Klivans. *The Mathematics of Chip-Firing*. CRC Press, Taylor & Francis Group (2018) ISBN: 978-1-138-63409-1
- [25] D.J. Lorenzini. A finite group attached to the Laplacian of a graph. *Discrete Math.* 91 (1991), 277–282.
- [26] D.J. Lorenzini. Smith normal form and Laplacians. *J. Combin. Theory B* 98 (2008), 1271–1300.
- [27] C. Merino, The chip-firing game. *Discrete Math.* 302 (2005), 188–210.
- [28] R. Merris. Unimodular Equivalence of Graphs. *Linear Algebra Appl.* 173 (1992), 181–189

- [29] Y. Pan, J. Wang. A note on the third invariant factor of the Laplacian matrix of a graph. *Journal of University of Science and Technology of China* 6 (2011), 471–476.
- [30] S. Olariu. Paw-free graphs. *Inform. Process. Lett.* 28 (1988), 53–54.
- [31] J.J. Rushanan. Topics in integral matrices and abelian group codes [dissertation]. ProQuest LLC, Ann Arbor (MI): California Institute of Technology; 1986.
- [32] D. Seinsche, On a Property of the Class of n-Colorable Graphs. *J. Combin. Theory B.* 16 (1974), 191–193.
- [33] R.P. Stanley. Smith normal form in combinatorics. *J. Combin. Theory A* 144 (2016), 476–495.
- [34] The developers. Sage Tutorial (Release 7.6). 2017. Number of pages 111. Located at: <http://www.sagemath.org>.
- [35] D.G. Wagner, The critical group of a directed graph. Preprint arXiv:math/0010241v1.
- [36] G. Williams. Smith forms for adjacency matrices of circulant graphs. *Linear Algebra Appl.* 443 (2014), 21–33.
- [37] M.M. Wood. The distribution of sandpile groups of random graphs. *J. Amer. Math. Soc.* 30 (2017), 915–958.
- [38] A. Vince. Elementary Divisors of Graphs and Matroids. *Europ. J. Combinatorics* 12 (1991), 445–453.