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# Connectivity of some algebraically defined digraphs

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*Dedicated to the memory of Vasyl Dmytrenko (1961-2013)*

## Abstract

Let  $p$  be a prime,  $e$  a positive integer,  $q = p^e$ , and let  $\mathbb{F}_q$  denote the finite field of  $q$  elements. Let  $f_i: \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$  be arbitrary functions, where  $1 \leq i \leq l$ ,  $i$  and  $l$  are integers. The digraph  $D = D(q; \mathbf{f})$ , where  $\mathbf{f} = (f_1, \dots, f_l): \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^l$ , is defined as follows. The vertex set of  $D$  is  $\mathbb{F}_q^{l+1}$ . There is an arc from a vertex  $\mathbf{x} = (x_1, \dots, x_{l+1})$  to a vertex  $\mathbf{y} = (y_1, \dots, y_{l+1})$  if  $x_i + y_i = f_{i-1}(x_1, y_1)$  for all  $i$ ,  $2 \leq i \leq l+1$ . In this paper we study the strong connectivity of  $D$  and completely describe its strong components. The digraphs  $D$  are directed analogues of some algebraically defined graphs, which have been studied extensively and have many applications.

**Keywords:** Finite fields; Directed graphs; Strong connectivity

## 1 Introduction and Results

In this paper, by a *directed graph* (or simply *digraph*)  $D$  we mean a pair  $(V, A)$ , where  $V = V(D)$  is the set of vertices and  $A = A(D) \subseteq V \times V$  is the set of arcs. The *order* of  $D$  is the number of its vertices. For an arc  $(u, v)$ , the first vertex  $u$  is called its *tail* and the second vertex  $v$  is called its *head*; we denote such an arc by  $u \rightarrow v$ . For an integer  $k \geq 2$ , a *walk*  $W$  from  $x_1$  to  $x_k$  in  $D$  is an alternating sequence  $W = x_1 a_1 x_2 a_2 x_3 \dots x_{k-1} a_{k-1} x_k$  of vertices  $x_i \in V$  and arcs  $a_j \in A$  such that the tail of  $a_i$  is  $x_i$  and the head of  $a_i$  is  $x_{i+1}$  for every  $i$ ,  $1 \leq i \leq k-1$ . Whenever the labels of the arcs of a walk are not important, we use the notation  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_k$  for the walk. In a digraph  $D$ , a vertex  $y$  is *reachable* from a vertex  $x$  if  $D$  has a walk from  $x$  to  $y$ . In particular, a vertex is reachable from

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itself. A digraph  $D$  is *strongly connected* (or, just *strong*) if, for every pair  $x, y$  of distinct vertices in  $D$ ,  $y$  is reachable from  $x$  and  $x$  is reachable from  $y$ . A *strong component* of a digraph  $D$  is a maximal induced subdigraph of  $D$  that is strong. For all digraph terms not defined in this paper, see Bang-Jensen and Gutin [1].

Let  $p$  be a prime,  $e$  a positive integer, and  $q = p^e$ . Let  $\mathbb{F}_q$  denote the finite field of  $q$  elements, and  $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ . We write  $\mathbb{F}_q^n$  to denote the Cartesian product of  $n$  copies of  $\mathbb{F}_q$ . Let  $f_i: \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$  be arbitrary functions, where  $1 \leq i \leq l$ ,  $i$  and  $l$  are positive integers. The digraph  $D = D(q; f_1, \dots, f_l)$ , or just  $D(q; \mathbf{f})$ , where  $\mathbf{f} = (f_1, \dots, f_l): \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^l$ , is defined as follows. (Throughout all of the paper the bold font is used to distinguish elements of  $\mathbb{F}_q^j$ ,  $j \geq 2$ , from those of  $\mathbb{F}_q$ , and we simplify the notation  $\mathbf{f}((x, y))$  and  $f((x, y))$  to  $\mathbf{f}(x, y)$  and  $f(x, y)$ , respectively.) The vertex set of  $D$  is  $\mathbb{F}_q^{l+1}$ . There is an arc from a vertex  $\mathbf{x} = (x_1, \dots, x_{l+1})$  to a vertex  $\mathbf{y} = (y_1, \dots, y_{l+1})$  if and only if

$$x_i + y_i = f_{i-1}(x_1, y_1) \quad \text{for all } i, 2 \leq i \leq l + 1.$$

We call the functions  $f_i$ ,  $1 \leq i \leq l$ , the *defining functions* of  $D(q; \mathbf{f})$ .

If  $l = 1$  and  $\mathbf{f}(x, y) = f_1(x, y) = x^m y^n$ ,  $1 \leq m, n \leq q - 1$ , we call  $D$  a *monomial digraph*, and denote it by  $D(q; m, n)$ .

The digraphs  $D(q; \mathbf{f})$  and  $D(q; m, n)$  are directed analogues of some algebraically defined graphs, which have been studied extensively and have many applications. See Lazebnik and Woldar [11] and references therein; for some subsequent work see Viglione [15], Lazebnik and Mubayi [7], Lazebnik and Viglione [10], Lazebnik and Verstraëte [9], Lazebnik and Thomason [8], Dmytrenko, Lazebnik and Viglione [3], Dmytrenko, Lazebnik and Williford [4], Ustimenko [14], Viglione [16], Terlep and Williford [13], Kronenthal [6], Cioabă, Lazebnik and Li [2], and Kodess [5].

We note that  $\mathbb{F}_q$  and  $\mathbb{F}_q^l$  can be viewed as vector spaces over  $\mathbb{F}_p$  of dimensions  $e$  and  $el$ , respectively. For  $X \subseteq \mathbb{F}_q^l$ , by  $\langle X \rangle$  we denote the span of  $X$  over  $\mathbb{F}_p$ , which is the set of all finite linear combinations of elements of  $X$  with coefficients from  $\mathbb{F}_p$ . For any vector subspace  $W$  of  $\mathbb{F}_q^l$ ,  $\dim(W)$  denotes the dimension of  $W$  over  $\mathbb{F}_p$ . If  $X \subseteq \mathbb{F}_q^l$ , let  $\mathbf{v} + X = \{\mathbf{v} + \mathbf{x} : \mathbf{x} \in X\}$ . Finally, let  $\text{Im}(\mathbf{f}) = \{(f_1(x, y), \dots, f_l(x, y)) : (x, y) \in \mathbb{F}_q^2\}$  denote the image of function  $\mathbf{f}$ .

In this paper we study strong connectivity of  $D(q; \mathbf{f})$ . We mention that by Lagrange's interpolation (see, for example, Lidl, Niederreiter [12]), each  $f_i$  can be uniquely represented by a bivariate polynomial of degree at most  $q - 1$  in each of the variables. We therefore also call functions  $f_i$  *defining polynomials*.

In order to state our results, we need the following notation. For every  $\mathbf{f}: \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^l$ , we define

$$\begin{aligned} \mathbf{g}(t) &= \mathbf{f}(t, 0) - \mathbf{f}(0, 0), & \mathbf{h}(t) &= \mathbf{f}(0, t) - \mathbf{f}(0, 0), \\ \tilde{\mathbf{f}}(x, y) &= \mathbf{f}(x, y) - \mathbf{g}(y) - \mathbf{h}(x), \\ \mathbf{f}_0(x, y) &= \mathbf{f}(x, y) - \mathbf{f}(0, 0), & \text{and} \\ \tilde{\mathbf{f}}_0(x, y) &= \mathbf{f}_0(x, y) - \mathbf{g}(y) - \mathbf{h}(x). \end{aligned}$$

As  $\mathbf{g}(0) = \mathbf{h}(0) = \mathbf{0}$ , one can view the coordinate function  $g_i$  of  $\mathbf{g}$  (respectively,  $h_i$  of  $\mathbf{h}$ ),  $i = 1, \dots, l$ , as the sum of all terms of the polynomial  $f_i$  containing only indeterminate

$x$  (respectively,  $y$ ), and having zero constant term. We, however, wish to emphasise that in the definition of  $\tilde{\mathbf{f}}(x, y)$ ,  $\mathbf{g}$  is evaluated at  $y$ , and  $\mathbf{h}$  at  $x$ . Also, we will often write a vector  $(v_1, v_2, \dots, v_{l+1}) \in \mathbb{F}_q^{l+1} = V(D)$  as an ordered pair  $(v_1, \mathbf{v}) \in \mathbb{F}_q \times \mathbb{F}_q^l$ , where  $\mathbf{v} = (v_2, \dots, v_{l+1})$ .

The main result of this paper is the following theorem, which gives necessary and sufficient conditions for the strong connectivity of  $D(q; \mathbf{f})$  and provides a description of its strong components in terms of  $\langle \text{Im}(\tilde{\mathbf{f}}_0) \rangle$  over  $\mathbb{F}_p$ .

**Theorem 1.** *Let  $D = D(q; \mathbf{f})$ ,  $D_0 = D(q; \mathbf{f}_0)$ ,  $W_0 = \langle \text{Im}(\tilde{\mathbf{f}}_0) \rangle$  over  $\mathbb{F}_p$ , and  $d = \dim(W_0)$  over  $\mathbb{F}_p$ . Then the following statements hold.*

(i) *If  $q$  is odd, then the digraphs  $D$  and  $D_0$  are isomorphic. Furthermore, the vertex set of the strong component of  $D_0$  containing a vertex  $(u, \mathbf{v})$  is*

$$\begin{aligned} & \left\{ (a, \mathbf{v} + \mathbf{h}(a) - \mathbf{g}(u) + W_0) : a \in \mathbb{F}_q \right\} \cup \left\{ (b, -\mathbf{v} + \mathbf{h}(b) + \mathbf{g}(u) + W_0) : b \in \mathbb{F}_q \right\} \\ & = \left\{ (a, \pm \mathbf{v} + \mathbf{h}(a) \mp \mathbf{g}(u) + W_0) \right\}. \end{aligned} \quad (1)$$

*The vertex set of the strong component of  $D$  containing a vertex  $(u, \mathbf{v})$  is*

$$\left\{ (a, \mathbf{v} + \mathbf{h}(a) - \mathbf{g}(u) + W_0) : a \in \mathbb{F}_q \right\} \cup \left\{ (b, -\mathbf{v} + \mathbf{h}(b) + \mathbf{g}(u) + \mathbf{f}(0, 0) + W_0) : b \in \mathbb{F}_q \right\}. \quad (2)$$

*In particular,  $D \cong D_0$  is strong if and only if  $W_0 = \mathbb{F}_q^l$  or, equivalently,  $d = el$ .*

*If  $q$  is even, then the strong component of  $D$  containing a vertex  $(u, \mathbf{v})$  is*

$$\begin{aligned} & \left\{ (a, \mathbf{v} + \mathbf{h}(a) + \mathbf{g}(u) + W_0) : a \in \mathbb{F}_q \right\} \cup \left\{ (a, \mathbf{v} + \mathbf{h}(a) + \mathbf{g}(u) + \mathbf{f}(0, 0) + W_0) : a \in \mathbb{F}_q \right\} \\ & = \left\{ (a, \mathbf{v} + \mathbf{h}(a) + \mathbf{g}(u) + W) : a \in \mathbb{F}_q \right\}, \end{aligned} \quad (3)$$

*where  $W = W_0 + \langle \{f(0, 0)\} \rangle = \langle \text{Im}(\tilde{\mathbf{f}}) \rangle$ .*

(ii) *If  $q$  is odd, then  $D \cong D_0$  has  $(p^{el-d} + 1)/2$  strong components. One of them is of order  $p^{e+d}$ . All other  $(p^{el-d} - 1)/2$  strong components are isomorphic, and each is of order  $2p^{e+d}$ .*

*If  $q$  is even, then the number of strong components in  $D$  is  $2^{el-d}$ , provided  $\mathbf{f}(0, 0) \in W_0$ , and it is  $2^{el-d-1}$  otherwise. In each case, all strong components are isomorphic, and are of orders  $2^{e+d}$  and  $2^{e+d+1}$ , respectively.*

We note here that for  $q$  even the digraphs  $D$  and  $D_0$  are generally not isomorphic.

We apply this theorem to monomial digraphs  $D(q; m, n)$ . For these digraphs we can restate the connectivity results more explicitly.

**Theorem 2.** Let  $D = D(q; m, n)$  and let  $d = (q - 1, m, n)$  be the greatest common divisor of  $q - 1$ ,  $m$  and  $n$ . For each positive divisor  $e_i$  of  $e$ , let  $q_i := (q - 1)/(p^{e_i} - 1)$ , and let  $q_s$  be the largest of the  $q_i$  that divides  $d$ . Then the following statements hold.

(i) The vertex set of the strong component of  $D$  containing a vertex  $(u, v)$  is

$$\{(x, v + \mathbb{F}_{p^{e_s}}): x \in \mathbb{F}_q\} \cup \{(x, -v + \mathbb{F}_{p^{e_s}}): x \in \mathbb{F}_q\}. \quad (4)$$

In particular,  $D$  is strong if and only if  $q_s = 1$  or, equivalently,  $e_s = e$ .

(ii) If  $q$  is odd, then  $D$  has  $(p^{e-e_s} + 1)/2$  strong components. One of them is of order  $p^{e+e_s}$ . All other  $(p^{e-e_s} - 1)/2$  strong components are all isomorphic and each is of order  $2p^{e+e_s}$ .

If  $q$  is even, then  $D$  has  $2^{e-e_s}$  strong components, all isomorphic, and each is of order  $2^{e+e_s}$ .

Our proof of Theorem 1 is presented in Section 2, and the proof of Theorem 2 is in Section 3. In Section 4 we suggest two areas for further investigation.

## 2 Connectivity of $D(q; \mathbf{f})$

Theorem 1 and our proof below were inspired by the ideas from [15], where the components of similarly defined bipartite simple graphs were described.

We now prove Theorem 1.

*Proof.* Let  $q$  be odd. We first show that  $D \cong D_0$ . The map  $\phi: V(D) \rightarrow V(D_0)$  given by

$$(x, \mathbf{y}) \mapsto (x, \mathbf{y} - \frac{1}{2}\mathbf{f}(0, 0)) \quad (5)$$

is clearly a bijection. We check that  $\phi$  preserves adjacency. Assume that  $((x_1, \mathbf{x}_2), (y_1, \mathbf{y}_2))$  is an arc in  $D$ , that is,  $\mathbf{x}_2 + \mathbf{y}_2 = \mathbf{f}(x_1, y_1)$ . Then, since  $\phi((x_1, \mathbf{x}_2)) = (x_1, \mathbf{x}_2 - \frac{1}{2}\mathbf{f}(0, 0))$  and  $\phi((y_1, \mathbf{y}_2)) = (y_1, \mathbf{y}_2 - \frac{1}{2}\mathbf{f}(0, 0))$ , we have

$$(\mathbf{x}_2 - \frac{1}{2}\mathbf{f}(0, 0)) + (\mathbf{y}_2 - \frac{1}{2}\mathbf{f}(0, 0)) = \mathbf{f}(x_1, y_1) - \mathbf{f}(0, 0) = \mathbf{f}_0(x_1, y_1),$$

and so  $(\phi((x_1, \mathbf{x}_2)), \phi((y_1, \mathbf{y}_2)))$  is an arc in  $D_0$ . As the above steps are reversible,  $\phi$  preserves non-adjacency as well. Thus,  $D(q; \mathbf{f}) \cong D(q; \mathbf{f}_0)$ .

We now obtain the description (1) of the strong components of  $D_0$ , and then explain how the description (2) of the strong components of  $D$  follows from (1).

Note that as  $\mathbf{f}_0(0, 0) = \mathbf{0}$ , we have  $\mathbf{g}(t) = \mathbf{f}_0(t, 0)$ ,  $\mathbf{h}(t) = \mathbf{f}_0(0, t)$ ,  $\mathbf{g}(0) = \mathbf{h}(0) = \mathbf{0}$ , and  $\tilde{\mathbf{f}}_0(x, y) = \mathbf{f}_0(x, y) - \mathbf{g}(y) - \mathbf{h}(x)$ .

Let  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_d \in \text{Im}(\tilde{\mathbf{f}}_0)$  be a basis for  $W_0$ . Now, choose  $x_i, y_i \in \mathbb{F}_q$  be such that  $\tilde{\mathbf{f}}_0(x_i, y_i) = \tilde{\alpha}_i$ ,  $1 \leq i \leq d$ .

Let  $(u, \mathbf{v})$  be a vertex of  $D_0$ . We first show that a vertex  $(a, \mathbf{v} + \mathbf{y})$  is reachable from  $(u, \mathbf{v})$  if  $\mathbf{y} \in \mathbf{h}(a) - \mathbf{g}(u) + W_0$ . In order to do this, we write an arbitrary  $\mathbf{y} \in \mathbf{h}(a) - \mathbf{g}(u) + W_0$  as

$$\mathbf{y} = \mathbf{h}(a) - \mathbf{g}(u) + (a_1\tilde{\alpha}_1 + \dots + a_d\tilde{\alpha}_d),$$

for some  $a_1, \dots, a_d \in \mathbb{F}_p$ , and consider the following directed walk in  $D_0$ :

$$\begin{aligned} (u, \mathbf{v}) &\rightarrow (0, -\mathbf{v} + \mathbf{f}_0(u, 0)) = (0, -\mathbf{v} + \mathbf{g}(u)) \\ &\rightarrow (0, \mathbf{v} - \mathbf{g}(u)) \end{aligned} \tag{6}$$

$$\rightarrow (x_1, -\mathbf{v} + \mathbf{g}(u) + \mathbf{f}_0(0, x_1)) = (x_1, -\mathbf{v} + \mathbf{g}(u) + \mathbf{h}(x_1)) \tag{7}$$

$$\rightarrow (y_1, \mathbf{v} - \mathbf{g}(u) - \mathbf{h}(x_1) + \mathbf{f}_0(x_1, y_1)) \tag{8}$$

$$\rightarrow (0, -\mathbf{v} + \mathbf{g}(u) + \mathbf{h}(x_1) - \mathbf{f}_0(x_1, y_1) + \mathbf{g}(y_1)) \tag{9}$$

$$= (0, -\mathbf{v} + \mathbf{g}(u) - \tilde{\mathbf{f}}_0(x_1, y_1)) = (0, -\mathbf{v} + \mathbf{g}(u) - \tilde{\alpha}_1) \tag{10}$$

$$\rightarrow (0, \mathbf{v} - \mathbf{g}(u) + \tilde{\alpha}_1). \tag{11}$$

Traveling through vertices whose first coordinates are 0,  $x_1$ ,  $y_1$ , 0, 0, and 0 again (steps 6–11) as many times as needed, one can reach vertex  $(0, \mathbf{v} - \mathbf{g}(u) + a_1\tilde{\alpha}_1)$ . Continuing a similar walk through vertices whose first coordinates are 0,  $x_i$ ,  $y_i$ , 0, 0, and 0,  $2 \leq i \leq d$ , as many times as needed, one can reach vertex  $(0, \mathbf{v} - \mathbf{g}(u) + (a_1\tilde{\alpha}_1 + \dots + a_i\tilde{\alpha}_i))$ , and so on, until the vertex  $(0, -\mathbf{v} + \mathbf{g}(u) - (a_1\tilde{\alpha}_1 + \dots + a_d\tilde{\alpha}_d))$  is reached. The vertex  $(a, \mathbf{v} + \mathbf{y})$  will be its out-neighbor. Here we indicate just some of the vertices along this path:

$$\begin{aligned} &\rightarrow \dots \\ &\rightarrow (0, \mathbf{v} - \mathbf{g}(u) + a_1\tilde{\alpha}_1) \\ &\rightarrow (x_2, -\mathbf{v} + \mathbf{g}(u) - a_1\tilde{\alpha}_1 + \mathbf{h}(x_2)) \\ &\rightarrow (y_2, \mathbf{v} - \mathbf{g}(u) + a_1\tilde{\alpha}_1 - \mathbf{h}(x_2) + \mathbf{f}_0(x_2, y_2)) \\ &\rightarrow (0, -\mathbf{v} + \mathbf{g}(u) - a_1\tilde{\alpha}_1 + \mathbf{h}(x_2) - \mathbf{f}_0(x_2, y_2) + \mathbf{g}(y_2)) \\ &= (0, -\mathbf{v} + \mathbf{g}(u) - a_1\tilde{\alpha}_1 - \tilde{\alpha}_2) \\ &\rightarrow (0, \mathbf{v} - \mathbf{g}(u) + a_1\tilde{\alpha}_1 + \tilde{\alpha}_2) \\ &\rightarrow \dots \\ &= (0, -\mathbf{v} + \mathbf{g}(u) - a_1\tilde{\alpha}_1 - a_2\tilde{\alpha}_2) \\ &\rightarrow \dots \\ &= (0, -\mathbf{v} + \mathbf{g}(u) - (a_1\tilde{\alpha}_1 + \dots + a_d\tilde{\alpha}_d)) \\ &\rightarrow (a, \mathbf{v} - \mathbf{g}(u) + \mathbf{h}(a) + (a_1\tilde{\alpha}_1 + \dots + a_d\tilde{\alpha}_d)) \\ &= (a, \mathbf{v} + \mathbf{y}). \end{aligned}$$

Hence,  $(a, \mathbf{v} + \mathbf{y})$  is reachable from  $(u, \mathbf{v})$  for any  $a \in \mathbb{F}_q$  and any  $\mathbf{y} \in \mathbf{h}(a) - \mathbf{g}(u) + W_0$ , as claimed. A slight modification of this argument shows that  $(a, -\mathbf{v} + \mathbf{y})$  is reachable from  $(u, \mathbf{v})$  for any  $\mathbf{y} \in \mathbf{h}(a) + \mathbf{g}(u) + W_0$ .

Let us now explain that every vertex of  $D_0$  reachable from  $(u, \mathbf{v})$  is in the set

$$\{(a, \pm \mathbf{v} \mp \mathbf{g}(u) + \mathbf{h}(a) + W_0) : a \in \mathbb{F}_q\}.$$

We will need the following identities on  $\mathbb{F}_q$  and  $\mathbb{F}_q^2$ , respectively, which can be checked easily using the definition of  $\tilde{\mathbf{f}}$ :

$$\begin{aligned}\tilde{\mathbf{f}}_0(t, 0) &= \mathbf{g}(t) - \mathbf{h}(t) = -\tilde{\mathbf{f}}_0(0, t) \quad \text{and} \\ \mathbf{f}_0(x, y) &= \mathbf{g}(x) + \mathbf{h}(y) + \tilde{\mathbf{f}}_0(x, y) - \tilde{\mathbf{f}}_0(0, y) + \tilde{\mathbf{f}}_0(0, x).\end{aligned}$$

The identities immediately imply that for every  $t, x, y \in \mathbb{F}_q$ ,

$$\begin{aligned}\mathbf{g}(t) - \mathbf{h}(t) &\in W_0 \quad \text{and} \\ \mathbf{f}_0(x, y) &= \mathbf{g}(x) + \mathbf{h}(y) + w \quad \text{for some } w = w(x, y) \in W_0.\end{aligned}$$

Consider a path with  $k$  arcs, where  $k > 0$  and even, from  $(u, \mathbf{v})$  to  $(a, \mathbf{v} + \mathbf{y})$ :

$$(u, \mathbf{v}) = (x_0, \mathbf{v}) \rightarrow (x_1, \dots) \rightarrow (x_2, \dots) \rightarrow \dots \rightarrow (x_k, \mathbf{v} + \mathbf{y}) = (a, \mathbf{v} + \mathbf{y}).$$

Using the definition of an arc in  $D_0$ , and setting  $\mathbf{f}_0(x_i, x_{i+1}) = \mathbf{g}(x_i) + \mathbf{h}(x_{i+1}) + w_i$ , and  $\mathbf{g}(x_i) - \mathbf{h}(x_i) = w'_i$ , with all  $w_i, w'_i \in W_0$ , we obtain:

$$\begin{aligned}\mathbf{y} &= \mathbf{f}_0(x_{k-1}, x_k) - \mathbf{f}_0(x_{k-2}, x_{k-1}) + \dots + \mathbf{f}_0(x_1, x_2) - \mathbf{f}_0(x_0, x_1) \\ &= \sum_{i=0}^{k-1} (-1)^{i+1} \mathbf{f}_0(x_i, x_{i+1}) = \sum_{i=0}^{k-1} (-1)^{i+1} (\mathbf{g}(x_i) + \mathbf{h}(x_{i+1}) + w_i) \\ &= -\mathbf{g}(x_0) + \mathbf{h}(x_k) + \sum_{i=1}^{k-1} (-1)^{i-1} (\mathbf{g}(x_i) - \mathbf{h}(x_i)) + \sum_{i=0}^{k-1} (-1)^{i+1} w_i \\ &= -\mathbf{g}(x_0) + \mathbf{h}(x_k) + \sum_{i=1}^{k-1} (-1)^{i-1} w'_i + \sum_{i=0}^{k-1} (-1)^{i+1} w_i.\end{aligned}$$

Hence,  $\mathbf{y} \in -\mathbf{g}(x_0) + \mathbf{h}(x_k) + W_0$ . Similarly, for any path

$$(u, \mathbf{v}) = (x_0, \mathbf{v}) \rightarrow (x_1, \dots) \rightarrow (x_2, \dots) \rightarrow \dots \rightarrow (x_k, \mathbf{v} + \mathbf{y}) = (a, -\mathbf{v} + \mathbf{y}),$$

with  $k$  arcs, where  $k$  is odd and at least 1, we obtain  $\mathbf{y} \in \mathbf{g}(x_0) + \mathbf{h}(x_k) + W_0$ .

The digraph  $D_0$  is strong if and only if  $W_0 = \langle \text{Im}(\tilde{\mathbf{f}}_0) \rangle = \mathbb{F}_q^d$  or, equivalently,  $d = \text{el}$ . Hence part (i) of the theorem is proven for  $D_0$  and  $q$  odd.

Let  $(u, \mathbf{v})$  be an arbitrary vertex of a strong component of  $D$ . The image of this vertex under the isomorphism  $\phi$ , defined in (5), is  $(u, \mathbf{v} - \frac{1}{2}\mathbf{f}(0, 0))$ , which belongs to the strong component of  $D_0$  whose description is given by (1) with  $\mathbf{v}$  replaced by  $\mathbf{v} - \frac{1}{2}\mathbf{f}(0, 0)$ . Applying the inverse of  $\phi$  to each vertex of this component of  $D_0$  immediately yields the description of the component of  $D$  given by (2). This establishes the validity of part (i) of Theorem 1 for  $q$  odd.

For  $q$  even we first apply an argument similar to the one we used above for establishing components of  $D_0$  for  $q$  odd. As  $p = 2$ , the argument becomes much shorter, and we obtain (3). Then we note that if

$$(u, \mathbf{v}) = (x_0, \mathbf{v}) \rightarrow (x_1, \dots) \rightarrow (x_2, \dots) \rightarrow \dots \rightarrow (x_k, \mathbf{v} + \mathbf{y})$$

is a path in  $D$ , then

$$\mathbf{y} = \sum_{i=0}^{k-1} \mathbf{f}_0(x_i, x_{i+1}) + \delta \cdot \mathbf{f}(0, 0),$$

where  $\delta = 1$  if  $k$  is odd, and  $\delta = 0$  if  $k$  is even.

For (ii), we first recall that any two cosets of  $W_0$  in  $\mathbb{F}_p^{kl}$  are disjoint or coincide. It is clear that for  $q$  odd, the cosets (1) coincide if and only if  $\mathbf{v} \in \mathbf{g}(u) + W_0$ . The vertex set of this strong component is  $\{(a, \mathbf{h}(a) + W_0) : a \in \mathbb{F}_q\}$ , which shows that this is the unique component of such type. As  $|W_0| = p^d$ , the component contains  $q \cdot p^d = p^{e+d}$  vertices. In all other cases the cosets are disjoint, and their union is of order  $2qp^d = 2p^{e+d}$ . Therefore the number of strong components of  $D_0$ , which is isomorphic to  $D$ , is

$$\frac{|V(D)| - p^{e+d}}{2p^{e+d}} + 1 = \frac{p^{e(l+1)} - p^{e+d}}{2p^{e+d}} + 1 = \frac{p^{el-d} + 1}{2}.$$

For  $q$  even, our count follows the same ideas as for  $q$  odd, and the formulas giving the number of strongly connected components and the order of each component follow from (3).

For the isomorphism of strong components of the same order, let  $q$  be odd, and let  $D_1$  and  $D_2$  be two distinct strong components of  $D_0$  each of order  $2p^{e+d}$ . Then there exist  $(u_1, \mathbf{v}_1), (u_2, \mathbf{v}_2) \in V(D_0)$  with  $\mathbf{v}_1 \notin \mathbf{g}(u_1) + W_0$  and  $\mathbf{v}_2 \notin \mathbf{g}(u_2) + W_0$  such that  $V(D_1) = \{(a, \mathbf{v}_1 + \mathbf{h}(a) - \mathbf{g}(u_1) + W_0) : a \in \mathbb{F}_q\}$  and  $V(D_2) = \{(a, \mathbf{v}_2 + \mathbf{h}(a) - \mathbf{g}(u_2) + W_0) : a \in \mathbb{F}_q\}$ .

Consider a map  $\psi : V(D_1) \rightarrow V(D_2)$  defined by

$$(a, \pm \mathbf{v}_1 + \mathbf{h}(a) \mp \mathbf{g}(u_1) + \mathbf{y}) \mapsto (a, \pm \mathbf{v}_2 + \mathbf{h}(a) \mp \mathbf{g}(u_2) + \mathbf{y}),$$

for any  $a \in \mathbb{F}_q$  and any  $\mathbf{y} \in W_0$ . Clearly,  $\psi$  is a bijection. Consider an arc  $(\alpha, \beta)$  in  $D_1$ . If  $\alpha = (a, \mathbf{v}_1 + \mathbf{h}(a) - \mathbf{g}(u_1) + \mathbf{y})$ , then  $\beta = (b, -\mathbf{v}_1 - \mathbf{h}(a) + \mathbf{g}(u_1) - \mathbf{y} + \mathbf{f}_0(a, b))$  for some  $b \in \mathbb{F}_q$ . Let us check that  $(\psi(\alpha), \psi(\beta))$  is an arc in  $D_2$ . In order to find an expression for the second coordinate of  $\psi(\beta)$ , we first rewrite the second coordinate of  $\beta$  as  $-\mathbf{v}_1 + \mathbf{h}(a) + \mathbf{g}(u_1) + \mathbf{y}'$ , where  $\mathbf{y}' \in W_0$ . In order to do this, we use the definition of  $\mathbf{f}_0$  and the obvious equality  $\mathbf{g}(b) - \mathbf{h}(b) = \tilde{\mathbf{f}}_0(b, 0) \in W_0$ . So we have:

$$\begin{aligned} & -\mathbf{v}_1 - \mathbf{h}(a) + \mathbf{g}(u_1) - \mathbf{y} + \mathbf{f}(a, b) \\ &= -\mathbf{v}_1 - \mathbf{h}(a) + \mathbf{g}(u_1) - \mathbf{y} + \tilde{\mathbf{f}}_0(a, b) + \mathbf{g}(b) + \mathbf{h}(a) \\ &= -\mathbf{v}_1 + \mathbf{h}(b) + \mathbf{g}(u_1) + (\mathbf{g}(b) - \mathbf{h}(b)) - \mathbf{y} + \tilde{\mathbf{f}}_0(a, b) \\ &= -\mathbf{v}_1 + \mathbf{h}(b) + \mathbf{g}(u_1) + \mathbf{y}', \end{aligned}$$



where  $\mathbf{y}' = (\mathbf{g}(b) - \mathbf{h}(b)) - \mathbf{y} + \tilde{\mathbf{f}}_0(a, b) \in W_0$ . Now it is clear that  $\psi(\alpha) = (a, \mathbf{v}_2 + \mathbf{h}(a) - \mathbf{g}(u_2) + \mathbf{y})$  and  $\psi(\beta) = (b, -\mathbf{v}_2 + \mathbf{h}(b) + \mathbf{g}(u_2) + \mathbf{y}')$  are the tail and the head of an arc in  $D_2$ . Hence  $\psi$  is an isomorphism of digraphs  $D_1$  and  $D_2$ .

An argument for the isomorphism of all strong components for  $q$  even is absolutely similar. This ends the proof of the theorem.  $\square$

We illustrate Theorem 1 by the following example.

**Example 3.** Let  $p \geq 3$  be prime,  $q = p^2$ , and  $\mathbb{F}_q \cong \mathbb{F}_p(\xi)$ , where  $\xi$  is a primitive element in  $\mathbb{F}_q$ . Let us define  $f: \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$  by the following table:

$\begin{array}{c c} & x \\ \hline y & \end{array}$	0	1	$x \neq 0, 1$
0	0	$\xi$	1
1	$\xi$	$2\xi$	$\xi$
$y \neq 0, 1$	2	$\xi$	0

As 1 and  $\xi$  are values of  $f$ ,  $\langle \text{Im}(f) \rangle = \mathbb{F}_q^2$ . Nevertheless,  $D(q; f)$  is not strong as we show below.

In this example, since  $l = 1$ , the function  $\mathbf{f} = f$ . Since  $f(0, 0) = 0$ ,  $f_0 = f$ , and

$$\mathbf{g}(t) = g(t) = f(t, 0) = \begin{cases} 0, & t = 0, \\ \xi, & t = 1, \\ 1, & \text{otherwise} \end{cases}, \quad \mathbf{h}(t) = h(t) = f(0, t) = \begin{cases} 0, & t = 0, \\ \xi, & t = 1, \\ 2, & \text{otherwise} \end{cases}.$$

The function  $\tilde{\mathbf{f}}_0(x, y) = \tilde{f}(x, y) = f(x, y) - f(y, 0) - f(0, x)$  can be represented by the table

$\begin{array}{c c} & x \\ \hline y & \end{array}$	0	1	$x \neq 0, 1$
0	0	0	-1
1	0	0	-2
$y \neq 0, 1$	1	-1	-3

and so  $\langle \text{Im}(\tilde{f}_0) \rangle = \mathbb{F}_p \neq \langle \text{Im}(f) \rangle = \mathbb{F}_{p^2}$ .

As  $l = 1$ ,  $e = 2$ , and  $d = 1$ ,  $D(q; f)$  has  $(p^{le-d} + 1)/2 = (p + 1)/2$  strong components. For  $p = 5$ , there are three of them. If  $\mathbb{F}_{25} = \mathbb{F}_5[\xi]$ , where  $\xi$  is a root of  $X^2 + 4X + 2 \in \mathbb{F}_5[X]$ , these components can be presented as:

$$\begin{aligned} & \{(a, h(a) + \mathbb{F}_5) : a \in \mathbb{F}_{25}\}, \\ & \{(a, h(a) - \xi + \mathbb{F}_5) : a \in \mathbb{F}_{25}\} \cup \{(b, h(b) + \xi + \mathbb{F}_5) : b \in \mathbb{F}_{25}\}, \\ & \{(a, h(a) + 2\xi + \mathbb{F}_5) : a \in \mathbb{F}_{25}\} \cup \{(b, h(b) - 2\xi + \mathbb{F}_5) : b \in \mathbb{F}_{25}\}. \end{aligned}$$

### 3 Connectivity of $D(q, m, n)$

The goal of this section is to prove Theorem 2.

For any  $t \geq 2$  and integers  $a_1, \dots, a_t$ , not all zero, let  $(a_1, \dots, a_t)$  (respectively  $[a_1, \dots, a_t]$ ) denote the greatest common divisor (respectively, the least common multiple) of these numbers. Moreover, for an integer  $a$ , let  $\bar{a} = (q - 1, a)$ . Let  $\langle \xi \rangle = \mathbb{F}_q^*$ , i.e.,  $\xi$  is a generator of the cyclic group  $\mathbb{F}_q^*$ . (Note the difference between  $\langle \cdot \rangle$  and  $\langle \cdot \rangle$  in our notation.) Suppose  $A_k = \{x^k : x \in \mathbb{F}_q^*\}$ ,  $k \geq 1$ . It is well known (and easy to show) that  $A_k = \langle \xi^{\bar{k}} \rangle$  and  $|A_k| = (q - 1)/\bar{k}$ .

We recall that for each positive divisor  $e_i$  of  $e$ ,  $q_i = (q - 1)/(p^{e_i} - 1)$ .

**Lemma 4.** *Let  $q_s$  be the largest of the  $q_i$  dividing  $\bar{k}$ . Then  $\mathbb{F}_{p^{e_s}}$  is the smallest subfield of  $\mathbb{F}_q$  in which  $A_k$  is contained. Moreover,  $\langle A_k \rangle = \mathbb{F}_{p^{e_s}}$ .*

*Proof.* By definition of  $\bar{k}$ ,  $q_s$  divides  $k$ , so  $k = tq_s$  for some integer  $t$ . Thus for any  $x \in \mathbb{F}_q$ ,

$$x^k = x^{tq_s} = \left( x^{\frac{p^{e_s}-1}{p^{e_s}-1}} \right)^t \in \mathbb{F}_{p^{e_s}},$$

as  $x^{(p^e-1)/(p^{e_s}-1)}$  is the norm of  $x$  over  $\mathbb{F}_{p^{e_s}}$  and hence is in  $\mathbb{F}_{p^{e_s}}$ . Suppose now that  $A_k \subseteq \mathbb{F}_{p^{e_i}}$ , where  $e_i < e_s$ . Since  $A_k$  is a subgroup of  $\mathbb{F}_{p^{e_i}}^*$ , we have that  $|A_k|$  divides  $|\mathbb{F}_{p^{e_i}}^*|$ , that is,  $(q - 1)/\bar{k}$  divides  $p^{e_i} - 1$ . Then  $\bar{k} = r \cdot (q - 1)/(p^{e_i} - 1) = rq_i$  for some integer  $r$ . Hence,  $q_i$  divides  $\bar{k}$ , and a contradiction is obtained as  $q_i > q_s$ . This proves that  $\langle A_k \rangle$  is a subfield of  $\mathbb{F}_{p^{e_s}}$  not contained in any smaller subfield of  $\mathbb{F}_q$ . Thus  $\langle A_k \rangle = \mathbb{F}_{p^{e_s}}$ .  $\square$

Let  $A_{m,n} = \{x^m y^n : x, y \in \mathbb{F}_q^*\}$ ,  $m, n \geq 1$ . Then, obviously,  $A_{m,n}$  is a subgroup of  $\mathbb{F}_q^*$ , and  $A_{m,n} = A_m A_n$  – the product of subgroups  $A_m$  and  $A_n$ .

**Lemma 5.** *Let  $d = (q - 1, m, n)$ . Then  $A_{m,n} = A_d$ .*

*Proof.* As  $A_m$  and  $A_n$  are subgroups of  $\mathbb{F}_q^*$ , we have

$$|A_{m,n}| = |A_m A_n| = \frac{|A_m| |A_n|}{|A_m \cap A_n|}. \quad (12)$$

It is well known (and easy to show) that if  $x$  is a generator of a cyclic group, then for any integers  $a$  and  $b$ ,  $\langle x^a \rangle \cap \langle x^b \rangle = \langle x^{[a,b]} \rangle$ . Therefore,  $A_m \cap A_n = \langle \xi^{[\bar{m}, \bar{n}]} \rangle$  and  $|A_m \cap A_n| = (q - 1)/[\bar{m}, \bar{n}]$ .

We wish to show that  $|A_{m,n}| = |A_d|$ , and since in a cyclic group any two subgroups of equal order are equal, that would imply  $A_{m,n} = A_d$ .

From (12) we find

$$|A_{m,n}| = \frac{(q - 1)/\bar{m} \cdot (q - 1)/\bar{n}}{(q - 1)/[\bar{m}, \bar{n}]} = \frac{(q - 1) \cdot \overline{[\bar{m}, \bar{n}]}}{\bar{m} \cdot \bar{n}}. \quad (13)$$

We wish to simplify the last fraction in (13). Let  $M$  and  $N$  be such that  $q - 1 = M\bar{m} = N\bar{n}$ . As  $d = (q - 1, m, n) = (\bar{m}, \bar{n})$ , we have  $\bar{m} = dm'$  and  $\bar{n} = dn'$  for some co-prime integers

$m'$  and  $n'$ . Then  $q - 1 = dm'M = dn'N$  and  $(q - 1)/d = m'M = n'N$ . As  $(m', n') = 1$ , we have  $M = n't$  and  $N = m't$  for some integer  $t$ . This implies that  $q - 1 = dm'n't$ . For any integers  $a$  and  $b$ , both nonzero, it holds that  $[a, b] = ab/(a, b)$ . Therefore, we have

$$[\overline{m}, \overline{n}] = [dm', dn'] = \frac{dm'dn'}{(dm', dn')} = \frac{dm'dn'}{d(m', n')} = dm'n'.$$

Hence,  $[\overline{m}, \overline{n}] = (q - 1, [\overline{m}, \overline{n}]) = (dm'n't, dm'n') = dm'n'$ , and

$$|A_{m,n}| = \frac{(q - 1) \cdot dm'n'}{\overline{m} \cdot \overline{n}} = \frac{(q - 1) \cdot dm'n'}{dm' \cdot dn'} = \frac{q - 1}{d}.$$

Since  $\overline{d} = (q - 1, d) = d$  and  $|A_d| = (q - 1)/\overline{d}$ , we have  $|A_{m,n}| = |A_d|$  and so  $A_{m,n} = A_d$ .  $\square$

We are ready to prove Theorem 2.

*Proof.* For  $D = D(q; m, n)$ , we have

$$\langle \text{Im}(\tilde{\mathbf{f}}_0) \rangle = \langle \text{Im}(f) \rangle = \langle \text{Im}(x^m y^n) \rangle = \langle A_{m,n} \rangle = \langle A_d \rangle = \mathbb{F}_{p^{e_s}},$$

where the last two equalities are due to Lemma 5 and Lemma 4.

Part (i) follows immediately from applying Theorem 1 with  $W = \mathbb{F}_{p^{e_s}}$ ,  $\mathbf{g} = \mathbf{h} = 0$ . Also,  $D$  is strong if and only if  $\mathbb{F}_{p^{e_s}} = \mathbb{F}_q$ , that is, if and only if  $e_s = e$ , which is equivalent to  $q_s = 1$ .

The other statements of Theorem 2 follow directly from the corresponding parts of Theorem 1.  $\square$

## 4 Open problems

We would like to conclude this paper with two suggestions for further investigation.

**Problem 1.** Suppose the digraphs  $D(q; \mathbf{f})$  and  $D(q; m, n)$  are strong. What are their diameters?

**Problem 2.** Study the connectivity of graphs  $D(\mathbb{F}; \mathbf{f})$ , where  $\mathbf{f}: \mathbb{F}^2 \rightarrow \mathbb{F}^l$ , and  $\mathbb{F}$  is a finite extension of the field  $\mathbb{Q}$  of rational numbers.

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