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Robust Exponential Synchronization for a Class of Master-Slave Distributed Parameter Systems with Spatially Variable Coefficients and Nonlinear Perturbation

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This paper addresses the exponential synchronization problem of a class of master-slave distributed parameter systems (DPs) with spatially variable coefficients and spatiotemporally variable nonlinear perturbation, modeled by a couple of semilinear parabolic partial differential equations (PDEs). With a locally Lipschitz constraint, the perturbation is a continuous function of time, space, and system state. Firstly, a sufficient condition for the robust exponential synchronization of the unforced semilinear master-slave PDE systems is investigated for all admissible nonlinear perturbations. Secondly, a robust distributed proportional-spatial derivative (P-sD) state feedback controller is desired such that the closed-loop master-slave PDE systems achieve exponential synchronization. Using Lyapunov’s direct method and the technique of integration by parts, the main results of this paper are presented in terms of spatial differential linear matrix inequalities (SDLMIs). Finally, two numerical examples are provided to show the effectiveness of the proposed methods applied to the robust exponential synchronization problem of master-slave PDE systems with nonlinear perturbation.

1. Introduction

The study on master-slave systems has become more important for theoretical and practical points in many fields, including communication, mechanical systems, robotics, chemical reactions, and biological systems [1–9]. Ever since the discovery of Christian Huygens in 1665 on the synchronization of two pendulum clocks [10], synchronization has received considerable attention for a long time as a typical collective behavior and a basic motion in nature with potential applications in many different areas including secure communication, chaos generators design, chemical reactions, biological systems, and information science [11–26]. The theory of synchronization for master-slave systems, which aims to control the slave system so that the output of the slave system follows the output of the master system [27], is a recent research area extensively investigated nowadays in many industrial and technical processes, such as unmanned aerial vehicle (UAV) team, vehicular platoons, rendezvous of space shuttles, and many other practical control systems (see, e.g., [28, 29] and the references therein).

The existing works most considered dynamical behavior of master-slave systems described by ordinary differential equations (ODEs) or delay differential equations (DDEs), and
a variety of synchronization criteria have been presented [14–16]. In practice, however, the outputs, inputs, and process states with relevant parameters usually vary both temporally and spatially in nature, and thereby their behavior depending on time and spatial position could be described by distributed parameter systems (DPSs) modeled by partial differential equations (PDEs). Unfortunately, few works have investigated the synchronization of master-slave PDE systems.

As a result of the infinite-dimensional nature of master-slave PDE systems, the existing finite-dimensional control theory and techniques for the master-slave ODE systems are difficult to be directly employed for the control design of master-slave PDE systems. In this situation, it is important to study the synchronization problem of master-slave PDE systems. Therefore, some researchers have paid attention to the study of synchronization of master-slave PDE systems [17–29], where “design-then-reduce” approach was employed to take the full advantage of the original PDE model for the controller design [30–34]. References [35–37] researched synchronization of neural networks with reaction-diffusion terms. Yuan et al. proposed synchronization of the coupled synchronization of neural networks with reaction-diffusion.
stand for the partial derivatives with respect to \(x, t\), respectively, \(x \in [l_1, l_2] \subset \mathbb{R}\) and \(t \in [0, \infty)\) are the spatial position and time, respectively, and \(u(x, t) \in \mathbb{R}^m\) is the distributed control input. \(\Theta_1(x), \Theta_2(x), \) and \(A(x) \in \mathbb{R}^{m \times m}\), \(x \in [l_1, l_2]\), are real known matrix functions. \(G(x) \in \mathbb{R}^{m \times n}\), \(x \in [l_1, l_2]\), is a real known matrix function. \(f(y(x, t), x, t)\) is \(n\)-dimensional real vector-valued spatiotemporally variable nonlinear perturbation with \(f(0, x, t) = 0\) for all \(x \in [l_1, l_2]\) and \(t \geq 0\). Moreover, assume that \(f(y(x, t), x, t)\) satisfies the following Lipschitz condition as follows.

**Assumption I.** \(f(y(x, t), x, t)\) is continuous with respect to its arguments and is locally Lipschitz continuous in \(y(x, t)\) on \(\Omega\); that is, there exists a scalar \(\chi > 0\) such that the following inequality holds for all \(y(x, t), \tilde{y}(x, t) \in \Omega, t \geq 0\) and \(x \in [l_1, l_2]\):

\[
\|f(y(x, t), x, t) - f(\tilde{y}(x, t), x, t)\|_2^2 \\
\leq \chi \|y(x, t) - \tilde{y}(x, t)\|_2^2,
\]  

(4)

where \(\Omega = \{ y(x, t) \mid \sigma_{l_1}(x) \leq y(x, t) \leq \sigma_{l_2}(x), x \in [l_1, l_2], t \geq 0, i = 1, 2, \ldots, n \}\). Therefore, we have that

\[
\|f(y(x, t), x, t)\|_2^2 \leq \chi \|y(x, t)\|_2^2.
\]  

(5)

Define the synchronization error \(e(x, t) \equiv z(x, t) - y(x, t)\), and we have the following synchronization error system \(\mathcal{E}\):

\[e_i(x, t) = \left( \Theta_1(x) e_i(x, t) + \Theta_2(x) e_i(x, t) + A(x) e_i(x, t) + b(x, t) \right)
\]

(6)

Choose \(\mathcal{H}^m \equiv \mathcal{L}^2_2([l_1, l_2]; \mathbb{R}^m)\) as the state space and the trajectory segment \(e_i(x, t) = [e(x, t), x \in [l_1, l_2]]\) as the state. Define the spatial differential operator \(\mathcal{D}\) in \(\mathcal{H}^m\) as

\[
\mathcal{D}(\mathcal{E}) \equiv \frac{d}{dx} \left[ \Theta_1(x) \frac{d\mathcal{E}(t)}{dx} + \Theta_2(x) \frac{d\mathcal{E}(t)}{dx} \right]
\]  

(7)

with its domain

\[
\mathcal{D}(\mathcal{E}) = \{ \mathcal{E} \in \mathcal{H}^{l_2}_2([l_1, l_2]; \mathbb{R}^m) : \mathcal{E}_{x} |_{x=l_1} = 0, \mathcal{E}_{x} |_{x=l_2} = 0 \}.
\]  

(8)

Let \(\mathcal{U}^m \equiv \mathcal{L}^2_2([l_1, l_2]; \mathbb{R}^m)\) be control input space and the trajectory segment \(u_i(t) = [u(x, t), x \in [l_1, l_2]]\) as the control input. Then the state-space description of the synchronization error system (6) can be rewritten as the following nonlinear abstract differential equation on the Hilbert space \(\mathcal{H}^m\) [43]:

\[
e(t) = \mathcal{A}(t) + \mathcal{B}(t) + \mathcal{F}(e) + \mathcal{G}(u(t)),
\]

(9)

where \(\mathcal{A} \equiv \mathcal{A}(t), \mathcal{G} \equiv \mathcal{G}(t), \mathcal{F}(e) \equiv \mathcal{F}(e(t)), \) and \(e \equiv e(t) \in \mathcal{H}^m\).

Using (8), we get from (6) that

\[
e_i(x, t) = \mathcal{A}e_i(x, t) + \mathcal{B}e_i(x, t) + \mathcal{F}(e(x, t))
\]

(10)

\[
e_i(x, t) |_{x=l_1} = e_i(x, t) |_{x=l_2} = 0,\]

where \(e_i(x) \equiv z_i(x) - y_i(x)\).

We introduce the following definition of the exponential synchronization for the master-slave systems (2) and (3) in the sense of norm \(\| \cdot \|_2\)

**Definition 2.** For a given constant \(\rho > 0\), the master-slave systems (2)-(3) achieve \(\rho\)-exponential synchronization or exponential synchronization with a given decay rate \(\rho\), if there exists a constant \(\sigma > 0\) such that the following inequality holds for any initial condition \(e_i(x), x \in [l_1, l_2]\):

\[
\|e_i(t)\|_2^2 \leq \sigma \exp(-\rho t) \|e_i(0)\|_2^2, \quad \forall t \geq 0. \]

(11)

It is easily seen from Definition 2 that the master-slave systems (2)-(3) achieve exponential synchronization with a given decay rate \(\rho\) if and only if the error system (6) is exponentially stable with a given decay rate \(\rho\).

For simplicity, when \(u(x, t) \equiv 0\), the master-slave systems (2)-(3) are referred to as *unforced* master-slave systems.

**Lemma 3.** For any two square integrable vector functions \(a(x), b(x) \in \mathbb{R}^n, x \in [l_1, l_2] \subset \mathbb{R}\), the following inequality holds for any positive scalar function \(\alpha(x) \in \mathbb{R}\) defined on interval \([l_1, l_2]\):

\[
2 \langle a, b \rangle \leq \langle a, \alpha a \rangle + \langle b, \alpha^{-1} b \rangle.
\]

(12)

**Proof.** It is easily found that the inequality \([\alpha(x)a(x) - b(x)]^T[\alpha(x)a(x) - b(x)] \geq 0\) holds for any \(x \in [l_1, l_2]\). Hence,

\[
0 \leq [\alpha(x)a(x) - b(x)]^T[\alpha(x)a(x) - b(x)]
\]

(13)

\[
= \alpha^2(x)a^T(x)a(x) - 2\alpha(x)a^T(x)b(x)
\]

\[
+ b^T(x)b(x),
\]

which implies

\[
2a^T(x)b(x) \leq \alpha(x)a^T(x)a(x)
\]

(14)

\[
+ \alpha^{-1}(x)b^T(x)b(x), \quad x \in [l_1, l_2].
\]
Integrating inequality (14) from $l_1$ to $l_2$, we can obtain that

\[
2 \int_{l_1}^{l_2} a^T(x) b(x) \, dx \leq \int_{l_1}^{l_2} \alpha(x) a^T(x) a(x) \, dx + \int_{l_1}^{l_2} \alpha^{-1}(x) b^T(x) b(x) \, dx
\]

which means that inequality (12) holds. The proof is complete.

3. Exponential Synchronization Analysis

This section aims to analyze exponential synchronization for the unforced semilinear master-slave PDE systems (2)-(3).

We consider the following Lyapunov functional for the unforced synchronization error system (10):

\[
V(t) = \int_{l_1}^{l_2} e^T(x,t) P(x) e(x,t) \, dx,
\]

where $P(x) > 0$ is a real $n \times n$ matrix function to be determined. The time derivative of $V(t)$ along the solution of system (10) is given by

\[
\dot{V}(t) + 2\rho V(t) = 2 \int_{l_1}^{l_2} e^T(x,t) P(x) e(x,t) \, dx + 2\rho \int_{l_1}^{l_2} e^T(x,t) P(x) e(x,t) \, dx
\]

By Assumption 1 and Lemma 3, for any positive scalar function $\alpha(x) \in \mathcal{R}, x \in [l_1, l_2]$, we have

\[
2 \int_{l_1}^{l_2} e^T(x,t) P(x) \alpha(x) \, dx \leq \int_{l_1}^{l_2} \alpha^{-1}(x) \, dx
\]

By integrating by parts and taking into account the boundary condition in (10), we have

\[
\int_{l_1}^{l_2} e^T(x,t) P(x) \left( \Theta_1(x) e_x(x,t) \right) \, dx = e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \bigg|_{x=l_2}^{x=l_1}
\]

\[
- \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx
\]

\[
= - \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx + 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx
\]

\[
= - \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx + 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx
\]

According to the definition of $\mathcal{A}$ in (7) and considering (18), we have

\[
\int_{l_1}^{l_2} e^T(x,t) P(x) \mathcal{A} e(x,t) \, dx = - \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx + \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_2(x) e_x(x,t) \, dx
\]

Substituting (19), (20) into (17), we obtain

\[
\dot{V}(t) + 2\rho V(t) \leq - \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx + \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_2(x) e_x(x,t) \, dx
\]

\[
- 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx
\]

\[
+ 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_2(x) e_x(x,t) \, dx
\]

\[
+ 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \mathcal{A}(x) e(x,t) \, dx
\]

\[
+ \int_{l_1}^{l_2} e^T(x,t) \left[ \alpha^{-1}(x) P(x) P(x) + \chi \alpha(x) I \right] e(x,t) \, dx
\]

\[
\cdot e(x,t) \, dx + 2 \rho \int_{l_1}^{l_2} e^T(x,t) P(x) e(x,t) \, dx
\]

\[
- \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_1(x) e_x(x,t) \, dx
\]

\[
+ 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \Theta_2(x) e_x(x,t) \, dx
\]

\[
+ 2 \int_{l_1}^{l_2} e^T(x,t) P(x) \mathcal{A}(x) e(x,t) \, dx
\]

\[
+ \int_{l_1}^{l_2} e^T(x,t) \left[ \alpha^{-1}(x) P(x) P(x) + \chi \alpha(x) I \right] e(x,t) \, dx
\]
\[ \mathbf{e}_x(x,t) \, dx + \int_{l_1}^{l_2} \dot{e}_x(x,t) \left( [P(x) \, A(x) + \ast] \right) \, dx + \alpha^{-1}(x) \, P(x) \, P(x) I + 2 \rho P(x) \, e(x,t) \, dx = \int_{l_1}^{l_2} \dot{e}_x(x,t) \, \overline{\Psi}(x) \, e(x,t) \, dx, \]

where \( \overline{\Psi}(x) \triangleq [\overline{\Psi}_{11}(x) \, P(x) \, \Theta_2(x) - P_x(x) \, \Theta_1(x)] \ast - \left[ P(x) \, \Theta_1(x) + \ast \right] \ast \ast - \alpha(x) \, I \]

\[ < 0, \quad x \in [l_1, l_2], \]

\[ \overline{\Psi}_{11}(x) = [P(x) \, A(x) + \ast] + \chi \alpha(x) I + 2 \rho P(x). \]  

**Theorem 4.** Consider the unforced semilinear master-slave PDE systems (2) and (3) under Assumption 1. For a given scalar \( \rho > 0 \), the master-slave PDE systems (2) and (3) achieve exponential synchronization with a given decay rate \( \rho \), if there exist a matrix function \( P(x) > 0 \) and a scalar function \( \alpha(x) > 0 \) satisfying the following SDLMI in space:

\[ \overline{\Psi}(x) \]

\[ \begin{bmatrix} \overline{\Psi}_{11}(x) \quad P(x) \Theta_2(x) - P_x(x) \Theta_1(x) \quad P(x) \\ * \ast \ast - [P(x) \Theta_1(x) + \ast] \ast \ast - \alpha(x) I \end{bmatrix} < 0, \quad x \in [l_1, l_2]. \]  

\[ (24) \]

**Proof.** Assume that SDLMI (23) is satisfied for matrix functions \( P(x) > 0 \) and \( \alpha(x) > 0 \). Using Schur complement, the following inequality is achieved if SDLMI (23) is satisfied for each \( x \in [l_1, l_2] \):

\[ \overline{\Psi}(x) < 0, \quad x \in [l_1, l_2]. \]  

Substituting inequality (25) into (21) yields

\[ \dot{V}(t) + 2 \rho V(t) < 0, \quad \forall \text{non-zero } \mathbf{e}(\cdot, t). \]  

Integration of (26) from 0 to \( t \) derives

\[ V(t) \leq V(0) \exp(-2 \rho t). \]  

Since \( P(x) > 0 \) is a spatially continuous matrix function of \( x \) defined on \([l_1, l_2]\), it is easily observed that \( V(t) \) given by (18) satisfies the following inequality:

\[ p_1 \| e(\cdot, t) \|_2^2 \leq V(t) \leq p_2 \| e(\cdot, t) \|_2^2, \]  

\[ (28) \]

where \( p_1 \triangleq \min_{x \in [l_1, l_2]} \{ \min_{x \in [l_1, l_2]} |\lambda_j(P(x))| \} \) and \( p_2 \triangleq \max_{x \in [l_1, l_2]} \{ \max_{x \in [l_1, l_2]} |\lambda_j(P(x))| \} \) are two positive constants. Using (28), we can get the following relation:

\[ p_1 \| e(\cdot, t) \|_2^2 \leq V(t) \leq V(0) \exp(-2 \rho t) \leq p_2 \| e_0(\cdot) \|_2^2 \exp(-2 \rho t). \]  

**Remark 5.** The controller \( \mathcal{U} \) (31) of the slave system \( \delta \) needs distributed sensing and actuation, whereas it has become feasible nowadays to produce integrated control circuitry with a large number of microsensors and actuators. It has been pointed out in [33, 34, 38, 39] that the controller \( \mathcal{U} \) (31) with spatial derivative term provides more spatial performance and that is why the controller \( \mathcal{U} \) (31) is chosen in this paper.

**4. P-sD Control Design**

Once the semilinear master-slave PDE systems (2)-(3) are not exponentially synchronized by themselves, it is desired to design a distributed P-sD controller in the semilinear master-slave PDE systems (2) and (3). Moreover, the SDLMI feasibility problem is approximately solved via the finite difference method and the existing LMI optimization techniques [41, 42].

\[ \mathbf{u}(x,t) = \mathbf{K}(x) \mathbf{e}(x,t) + \mathbf{L}(x) \mathbf{e}_x(x,t), \]  

\[ (31) \]

where \( \mathbf{K}(x), \mathbf{L}(x) \in \mathbb{R}^{m \times n}, x \in [l_1, l_2], \) are continuous matrices to be determined.

\[ \mathcal{U} \] with spatial derivative term provides more spatial performance and that is why the controller \( \mathcal{U} \) (31) is chosen in this paper.
Substituting (31) into (10) gives the following closed-loop synchronization error system:

\[
\dot{e}_x(x, t) = A_x(x) e(x, t) + A_x(x) e(x, t) + F(e(x, t)) + G(x) L(x) e_x(x, t),
\]

\[
e_x(x, t) \big|_{x=\ell_1} = e_x(x, t) \big|_{x=\ell_2} = 0,
\]

\[
e(x, 0) = e_0(x),
\]

where \(A_x(x) \triangleq A(x) + G(x) K(x)\).

Hence, the objective of this section is to develop a design method of the distributed state feedback controller (31) to ensure exponential synchronization of the master-slave systems (2)-(3), that is, the exponential stability of the closed-loop system (32). To achieve the goal, the following assumption is necessary.

**Assumption 6.** The matrix \(\Theta_1(x)\) in (2) and (3) is diagonal; that is, \(\Theta_1(x) = \text{diag}(\theta_{11}(x), \ldots, \theta_{1m}(x)), x \in [l_1, l_2]\).

**Theorem 7.** Consider the closed-loop semilinear master-slave PDE systems (2) and (3) under Assumptions 1 and 6. For a given scalar \(\rho > 0\), the distributed P-sD controller (31) can guarantee that the master-slave PDE systems (2) and (3) achieve exponential synchronization with a given decay rate \(\rho\), if there exist \(n \times n\) matrix functions \(Q(x) \triangleq \text{diag}[q_{11}(x), q_{22}(x), \ldots, q_{nn}(x)] > 0\), a scalar function \(\beta(x) > 0\), and \(m \times n\) matrix functions \(M(x)\) and \(N(x)\) satisfying the following SDLMI in space:

\[
\Xi(x) \triangleq \begin{bmatrix} \Xi_{11}(x) & \Xi_{12}(x) \\ \Xi_{21}(x) & Q(x) \end{bmatrix},
\]

\[
< 0, \quad x \in [l_1, l_2],
\]

where

\[
\Xi_{11}(x) = [A(x) Q(x) + G(x) M(x) + \beta(x) I] + 2 \rho Q(x),
\]

\[
\Xi_{12}(x) = \Theta_2(x) Q(x) + G(x) N(x) - Q_x(x) \Theta_1(x).
\]
In this case, the gain matrix functions \( K(x), L(x), x \in [l_1, l_2] \), of the suitable controller (31) can be given as

\[
K(x) = M(x)Q^{-1}(x), \\
L(x) = N(x)Q^{-1}(x), \\
x \in [l_1, l_2].
\]  

(39)

**Proof.** Assume that SDLMI (37) is satisfied for \( Q(x) \equiv \text{diag}\{q_1(x), q_2(x), \ldots, q_n(x)\} > 0, \beta(x) > 0, M(x), \) and \( N(x) \). Using the Schur complement, SDLMI (37) is equivalent to the inequality

\[
\Xi(x) = \begin{bmatrix} \Xi_{11}(x) & \Xi_{12}(x) \\ * & -[\Theta_1(x)Q(x) + \star] \end{bmatrix} < 0,
\]

where

\[
\Xi_{11}(x) = [A(x)Q(x) + G(x)M(x) + \star] + \beta(x)I + \chi^\beta(x)Q(x)Q(x) + 2\rho Q(x).
\]

Letting

\[
P(x) = Q^{-1}(x), \\
M(x) = K(x)Q(x), \\
N(x) = L(x)Q(x),
\]

pre- and postmultiplying (40) by \( \text{diag}\{Q^{-1}(x), Q^{-1}(x)\} \), respectively, with application of Assumption 6, and considering the property \( \Theta_1(x)Q(x) = Q(x)\Theta_1(x), Q_2(x) = -Q(x)Q(x)Q(x), \) and the commutative law of matrix multiplication [47], we can obtain

\[
\Xi(x) = \text{diag}\{Q^{-1}(x), Q^{-1}(x)\} \cdot \Xi(x) \cdot \text{diag}\{Q^{-1}(x), Q^{-1}(x)\},
\]

(42)

\[x \in [l_1, l_2].\]

Since \( \text{diag}\{Q^{-1}(x), Q^{-1}(x)\} > 0, x \in [l_1, l_2] \), we can get the following inequality from (40) and (42):

\[
\Xi(x) < 0, \quad x \in [l_1, l_2].
\]

(43)

Substituting inequality (43) into (35) yields

\[
\dot{V}(t) + 2\rho V(t) < 0, \quad \forall \text{non-zero } \Xi(i, t).
\]

(44)

The rest of proof of Theorem 7 is similar to that of Theorem 4, and thus we can also get the conclusion that the semilinear master-slave PDE systems (2) and (3) achieve exponential synchronization with a given decay rate \( \rho \) with the suitable controller (31). Moreover, from (41), we have (39). The proof is complete. \( \square \)

Theorem 7 presents an SDLMI-based condition for the existence of a distributed controller (31) for the exponential synchronization of the semilinear master-slave PDE systems (2) and (3). Explicit expressions of a desired controller are proposed when the SDLMI (37) is feasible.

**Remark 8.** Notice that an SDLMI control design has been more recently proposed in [33, 34] for a class of PDE systems.

Different from the stable control design for PDE systems in [33, 34], this paper provides an SDLMI-based sufficient condition on the exponential synchronization for master-slave PDE systems.

**Remark 9.** Notice that [38–40] dealt with exponential synchronization of linear models, while this paper considers exponential synchronization of the model with nonlinear perturbation.

**Remark 10.** For a special case, when the master system \( \mathcal{M} \): \( y_i(x, t) \equiv 0 \), we have \( z(x, t) = e(x, t) \), and then the exponential synchronization of the master-slave systems is equal to exponential stability of the slave system \( \mathcal{S} \). In other words, Theorem 4 provides the sufficient condition of exponential stability of the slave system \( \mathcal{S} \) with a given decay rate \( \rho \) when the master system \( y_i(x, t) \equiv 0 \), while Theorem 7 also contains the way of the P-sD controller (31) design for exponential stability.

### 5. Numerical Simulation

In this section, in order to show the effectiveness of Theorems 4 and 7, we consider the following two examples.

**Example 1.** Consider semilinear master-slave PDE systems (2) and (3) with the following spatial variable coefficients:

\[
y(x, t) = \begin{bmatrix} y_1(x, t) \end{bmatrix}, \quad z(x, t) = \begin{bmatrix} z_1(x, t) \end{bmatrix},
\]

\[
\Theta_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \\
\Theta_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
A(x) = \begin{bmatrix} -2.7 - \exp\left(-\frac{x}{L}\right) & 0.5 \\ -2 \cos(x) & -3.7 \end{bmatrix}, \quad l_1 = 0, \\
L_2 = 10,
\]

\[
f(y(x, t), x, t) = \begin{bmatrix} -0.25y_1^5 \\ 0 \end{bmatrix}, \\
f(z(x, t), x, t) = \begin{bmatrix} -0.25z_1^5 \\ 0 \end{bmatrix},
\]

and the following initial conditions:

\[
y_0(x) = \begin{bmatrix} -0.2 + 0.6 \cos\left(\frac{\pi x}{L}\right) \\ -0.7 \sin\left(\frac{\pi x}{L} + \frac{\pi}{2}\right) \end{bmatrix}, \\
z_0(x) = \begin{bmatrix} 0.5 \cos\left(\frac{2\pi x}{L}\right) \\ -0.3 + 0.1 \cos\left(\frac{\pi x}{L}\right) \end{bmatrix},
\]

(46)

where \( L \equiv l_2 - l_1 = 10 \).
Let \( \Omega \triangleq \{ \tilde{y}(x, t) \mid -0.8 \leq \tilde{y}(x, t) \leq 0.8, \; \tilde{y}(x, t) \in \{ y_i(x, t), z_i(x, t) \}, \; x \in [0, 10], \; t \geq 0, \; i = 1, 2 \} \). The parameter \( \chi \) satisfying Assumption 1 is chosen as

\[
\chi \triangleq \max_{y(x,t),z(x,t) \in \Omega} \left\{ \left\| \frac{\partial f(y(x,t),x,t)}{\partial y(x,t)} \right\|^2, \left\| \frac{\partial f(z(x,t),x,t)}{\partial z(x,t)} \right\|^2 \right\} = 0.0625 \\
\max_{y(x,t),z(x,t) \in \Omega} \left\{ 25y_1^4(x,t), 25z_1^4(x,t) \right\} = 6.5536.
\]

Divide the spatial domain \([0, 10]\) into space instances \( \{ x_k, k \in \{ 0, 1, 2, \ldots, 100 \} \} \) of the same distance, where \( x_k - x_{k-1} = 0.1 \). With application of Theorem 4, employing Algorithm 1 in [34] to solve SDLMI (23) and setting \( Q(x_{-1}) = 1 \) and \( \rho = 0.1 \), we get feasible solutions of \( P(x) > 0 \) and \( \alpha(x) > 0 \). Therefore, the unforced semilinear master-slave PDE systems (2) and (3) achieve robust exponential synchronization with a given decay rate \( \rho \), which is apparent from Figure 2.

**Example 2.** Consider semilinear master-slave PDE systems (2) and (3) with the following spatially variable coefficients:

\[
y(x,t) = [y_1(x,t), y_2(x,t)]^T, \\
z(x,t) = [z_1(x,t), z_2(x,t)]^T, \\
\Theta_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \\
\Theta_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A(x) = \begin{bmatrix} 1 & 2 \exp\left(-\frac{x}{L}\right) \\ \sin(x) & -3 \end{bmatrix}, \\
l_1 = 0, \\
l_2 = 20, \\
G(x) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\
f(y(x,t),x,t) = \begin{bmatrix} -0.5y_1^3 \\ 0 \end{bmatrix}, \\
f(z(x,t),x,t) = \begin{bmatrix} -0.5z_1^3 \\ 0 \end{bmatrix},
\]

and the following initial conditions:

\[
y_0(x) = \begin{bmatrix} 0.6 \cos\left(\frac{\pi x}{L}\right) \\ -0.4 \end{bmatrix}, \\
z_0(x) = \begin{bmatrix} 0.5 \cos\left(\frac{2\pi x}{L}\right) \\ -0.2 \end{bmatrix},
\]

where \( L \triangleq l_2 - l_1 = 20 \).

Figure 3 shows the unforced profiles of evolution of the semilinear master-slave PDE systems (2) and (3) with the parameters given in (40) under the initial condition given in (41). It is apparent from Figure 3 that the operating steady
state $y(x, t)$ of the master PDE system (2) and the operating steady state $z(x, t)$ of the slave PDE system (3) are not synchronized.

Let $\Omega \doteq \{y(x, t) \mid -1.7 \leq y_i(x, t) \leq 1.7, y_i(x, t) \in \{y_i(x, t), z_i(x, t)\}, x \in [0, 20], t \geq 0, i = 1, 2\}$. The parameter $\chi$ satisfying Assumption 1 is chosen as

$$
\chi \doteq \max_{y(x, t), z(x, t) \in \Omega} \left\{ \frac{\|f(y(x, t), x, t)\|}{\|\partial_y f(y(x, t), x, t)\|^2}, \frac{\|f(z(x, t), x, t)\|}{\|\partial_z f(z(x, t), x, t)\|^2} \right\} = 0.09
$$

$$
\cdot \max_{y(x, t), z(x, t) \in \Omega} \left\{ 9y_i^4(x, t), 9z_i^4(x, t) \right\} = 6.5025.
$$

In this example, the distributed P-sD state feedback controller (31) is employed, where $K(x) \doteq [k_1(x) k_2(x)]$, $L(x) \doteq [l_1(x) l_2(x)]$, $x \in [0, 20]$, are $1 \times 2$ matrix functions to be determined. Divide the spatial domain $[0, 20]$ into space instances $\{x_k, k \in \{0, 1, 2, \ldots, 40\}, x_0 = 0, x_{40} = 20\}$ of the same distance, where $x_k - x_{k-1} = 0.5$. With application to Theorem 7, employing Algorithm 1 in [34] to solve SDLMI (37) and setting $Q(x_{-1}) = 0.3I$ and $\rho = 0.1$, we get feasible solutions of $Q(x) \doteq \text{diag}(q_1(x), q_2(x), \ldots, q_n(x)) > 0, \beta(x) > 0, M(x), a n d N(x)$. Thereby, we can get the approximated control gain matrix functions $K(x)$ and $L(x)$ by (39) as shown in Figures 4 and 5, respectively.

Applying the controller (31) with the approximated control gain matrices $K(x)$ and $L(x)$ shown in Figures 4 and 5 to the system (2) and (3), respectively.
to the semilinear master-slave PDE systems (2) and (3) with
the parameters given in (48) under the initial condition
given in (49), Figure 6 indicates the closed-loop profiles of
evolution of the semilinear master-slave PDE systems of
\( y(x, t) = [y_1(x, t), y_2(x, t)]^T \) and \( z(x, t) = [z_1(x, t), z_2(x, t)]^T \),
respectively.

It is clear from Figure 6 that the proposed distributed
state feedback controller (31) can ensure that the semilinear
master-slave PDE systems (2) and (3) with the parameters
given in (48) under the initial condition given in (49)
achieve exponential synchronization with a given decay rate \( \rho \). Moreover, the evolution profile of the distributed control
\( u(x, t) \) is shown in Figure 7.

6. Conclusions

In this paper, we have addressed the exponential synchroniza-
tion problem of semilinear master-slave PDE systems with
spatially variable coefficients. An SDLMI-based sufficient
condition for the robust exponential synchronization of the
unforced semilinear master-slave PDE systems has been pre-
sented. Moreover, when the master-slave PDE systems are not
ever synchronously synchronized, it has been proved in detail that
P-sD controllers with appropriate gain parameters can guide
the closed-loop master-slave PDE systems to exponentially
synchronize with a given decay rate \( \rho \) if a given SDLMI
criterion is fulfilled. Finally, numerical examples showed the
effectiveness of the proposed methods.

Conflict of Interests

The authors declare that there is no conflict of interests
regarding the publication of this paper.

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