

1991

# Oscillations of delay differential equations

K. Gopalsamy

G. Ladas

*University of Rhode Island*

Follow this and additional works at: [https://digitalcommons.uri.edu/math\\_facpubs](https://digitalcommons.uri.edu/math_facpubs)

Terms of Use

All rights reserved under copyright.

---

## Citation/Publisher Attribution

Gopalsamy, K., & Ladas, G. (1991). Oscillations of delay differential equations. *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 32(4), 377-381. doi:10.1017/S0334270000008493

Available at: <https://doi.org/10.1017/S0334270000008493>

This Article is brought to you for free and open access by the Mathematics at DigitalCommons@URI. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of DigitalCommons@URI. For more information, please contact [digitalcommons@etal.uri.edu](mailto:digitalcommons@etal.uri.edu).

## OSCILLATIONS OF DELAY DIFFERENTIAL EQUATIONS

K. GOPALSAMY<sup>1</sup> and G. LADAS<sup>2</sup>

(Received 30 July 1989; revised 28 February 1990)

### Abstract

Sufficient conditions are established for all solutions of the linear system

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^n q_{ij}y_j(t - \tau_{ij}) = 0, \quad i = 1, 2, \dots, n,$$

to be oscillatory, where  $q_{ij} \in (-\infty, \infty)$ ,  $\tau_{ij} \in (0, \infty)$ ,  $i, j = 1, 2, \dots, n$ .

### 1. Introduction

Consider the system of delay differential equations

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^n q_{ij}y_j(t - \tau_{ij}) = 0, \quad i = 1, 2, \dots, n \quad (1)$$

where the coefficients are real numbers and the delays are positive real numbers. We say that a solution

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \quad (2)$$

of (1) oscillates if for some  $i \in (1, 2, \dots, n)$ ,  $y_i(t)$  has arbitrarily large zeros. A solution  $y(t)$  of (1) is said to be nonoscillatory if there exists a  $t_0 \geq 0$  such that for each  $i = 1, 2, \dots, n$ ,  $y_i(t) \neq 0$  for  $t \geq t_0$ . The aim of this brief paper is to derive a set of sufficient conditions for all solutions

<sup>1</sup>School of Mathematics, Flinders University, Bedford Park, S. A. 5042.

<sup>2</sup>Department of Mathematics, University of Rhode Island, Kingston, R. I., U.S.A.

© Copyright Australian Mathematical Society 1991, Serial-fee code 0334-2700/91

of (1) to oscillate. Our result is an extension of a result of Gopalsamy in [2], where only bounded solutions of systems like (1) have been considered. For references concerning the oscillation of systems, the reader is referred to the references in [2].

## 2. Sufficient conditions for oscillation

The following lemma will be useful in the proof of our theorem below.

**LEMMA 1.** *Assume that (1) has a nonoscillatory solution (2). Then there are numbers*

$$\delta_i \in \{-1, 1\} \quad \text{for } i = 1, 2, \dots, n$$

*such that the system*

$$\frac{dz_i(t)}{dt} + \sum_{j=1}^n p_{ij} z_j(t - \tau_{jj}) = 0 \quad (3)$$

*where*

$$p_{ij} = \frac{\delta_j}{\delta_i} q_{ij} \quad \text{for } i, j = 1, 2, \dots, n \quad (4)$$

*has a nonoscillatory solution*  $[z_1(t), z_2(t), \dots, z_n(t)]^T$  *with eventually positive components*  $z_i(t)$ ,  $i = 1, 2, \dots, n$ .

**PROOF.** The components  $y(t)$  of (2) are positive or negative eventually. That is, there exists a  $T \geq 0$  such that  $y_i(t) \neq 0$  for  $t \geq T$  and  $i = 1, 2, \dots, n$ . Set  $\delta_i = \text{sign}[y_i(t)]$ ,  $i = 1, 2, \dots, n$  and  $t \geq T$ . It is now easy to see that

$$z(t) = [\delta_1 y_1(t), \delta_2 y_2(t), \dots, \delta_n y_n(t)]^T \quad (5)$$

satisfies (3) and  $\delta_i y_i(t) > 0$  for  $i = 1, 2, \dots, n$  and  $t \geq T$ .

The next result is concerned with the asymptotic behaviour of nonoscillatory solutions of (1).

**LEMMA 2.** *Consider the system (1) and suppose that the constant coefficients of (1) satisfy*

$$q = \min_{1 \leq i \leq n} \left[ q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| \right] > 0. \quad (6)$$

*Then every nonoscillatory solution*  $y(t) = (y_1, y_2, \dots, y_n)$  *satisfies*

$$\lim_{t \rightarrow \infty} y_i(t) = 0.$$

**PROOF.** Clearly (6) is also satisfied with the  $q_{ij}$  replaced by the respective  $p_{ij}$  of (4). From this and (5) it suffices to prove the lemma for nonoscillatory solutions of (2) with eventually positive components. Let us assume that there is a  $t_0 \geq 0$  such that  $y_i(t) > 0$  for  $t \geq t_0$ ,  $i = 1, 2, \dots, n$ . If we let

$$w(t) = \sum_{j=1}^n y_j(t), \quad t \geq t_0 \tag{7}$$

then

$$\frac{dw(t)}{dt} + \sum_{i=1}^n \sum_{j=1}^n q_{ij} y_j(t - \tau_{jj}) = 0$$

or

$$\begin{aligned} \frac{dw(t)}{dt} + \sum_{i=1}^n q_{ii} y_i(t - \tau_{ii}) &= - \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n q_{ij} y_j(t - \tau_{jj}) \\ &\leq \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| y_i(t - \tau_{ii}). \end{aligned} \tag{8}$$

It follows from (8) that

$$\frac{dw(t)}{dt} + \sum_{i=1}^n \left\{ q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| \right\} y_i(t - \tau_{ii}) \leq 0. \tag{9}$$

An integration of both sides of (9) leads to

$$w(t) + q \int_{t_0+\tau}^t \sum_{i=1}^n y_i(s - \tau_{ii}) ds \leq w(t_0 + \tau) \tag{10}$$

where  $\tau = \max_{1 \leq j \leq n} \tau_{jj}$ . A consequence of (10) is that  $w$  is bounded and  $y_i \in L_1(t_0 + \tau, \infty)$  for  $i = 1, 2, \dots, n$ . From the boundedness of  $w$  one can conclude that of  $y_i$  since  $w(t) = \sum_{i=1}^n y_i(t)$  and  $y_i(t) > 0$  eventually. It will now follow from (1) that  $\dot{y}_i$  is bounded for  $t \geq \tau$ , and therefore  $y_i$  is uniformly continuous on  $[0, \infty)$ . The uniform continuity of  $y_i$  on  $[0, \infty)$ , the eventual positivity of  $y_i$  and the integrability of  $y_i$  on a half-line together with a lemma of Barbalat [1], will imply that  $\lim_{t \rightarrow \infty} y_i(t) = 0$ ,  $i = 1, 2, \dots, n$  and this completes the proof.

**THEOREM.** Let  $q_{ij} \in (-\infty, \infty)$ ,  $\tau_{jj} \in (0, \infty)$ ,  $i, j = 1, 2, \dots, n$ . If

$$q\tau_* > \frac{1}{e} \quad \text{where } q = \min_{1 \leq i \leq n} \left( q_{ii} \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| \right), \quad \tau_* = \min_{1 \leq i \leq n} \tau_{ii} \tag{11}$$

then every solution of (1) oscillates.

**PROOF.** Assume for the sake of a contradiction that (1) has a nonoscillatory solution (2). In view of Lemma 1 we can assume that the components of  $y_i(t)$  are eventually positive for  $i = 1, 2, \dots, n$ . We have directly from (1) that

$$\sum_{i=1}^n \frac{dy_i(t)}{dt} + \sum_{j=1}^n \sum_{i=1}^n q_{ij}y_j(t - \tau_{jj}) = 0$$

which satisfies

$$\sum_{i=1}^n \left[ \frac{dy_i(t)}{dt} \right] + \sum_{i=1}^n \left( q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| \right) y_i(t - \tau_{ii}) \leq 0. \tag{12}$$

We have from (12) that  $w(t) = \sum_{i=1}^n y_i(t)$  satisfies

$$\frac{dw(t)}{dt} + \sum_{i=1}^n \left[ q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |q_{ji}| \right] y_i(t - \tau_{ii}) \leq 0. \tag{13}$$

Integrating both sides of (13) over  $(t, \infty)$  and using the fact

$$w(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (\text{since } y_i(t) \rightarrow 0, \quad i = 1, 2, \dots, n)$$

we derive that

$$-w(t) + q \int_t^\infty \sum_{i=1}^n y_i(s - \tau_{ii}) \leq 0 \tag{14}$$

and this leads to

$$w(t) \geq q \int_t^\infty \sum_{i=1}^n y_i(s - \tau_{ii}) ds. \tag{15}$$

It is found from (15) that

$$w(t) \geq q \int_{t-\tau_*}^\infty \sum_{i=1}^n y_i(s) ds, \quad \tau_* = \min_{1 \leq i \leq n} \tau_{ii} \tag{16}$$

or

$$w(t) \geq q \int_{t-\tau_*}^\infty w(s) ds. \tag{17}$$

Now we let

$$F(t) = \int_{t-\tau_*}^\infty w(s) ds \tag{18}$$

and derive from (17) and (18) that

$$\begin{aligned}\frac{dF(t)}{dt} &= -w(t - \tau_*) \\ &\leq -qF(t - \tau_*); \quad t > 2\tau_*. \end{aligned} \quad (19)$$

It follows from (19) that  $F$  is an eventually positive solution of

$$\frac{dF(t)}{dt} + qF(t - \tau_*) \leq 0; \quad t > 2\tau_*. \quad (20)$$

But it is well known (from Ladas and Stavroulakis [3]) that when (11) holds, (20) cannot have an eventually positive solution and this contradiction completes the proof.

### References

- [1] I. Barbalat, "Systemes d'equations differentielles d'oscillations nonlineaires," *Rev. Roumain. Math. Pures Appl.* **4** (1959) 267–270.
- [2] K. Gopalsamy, "Oscillatory properties of systems of first order delay differential inequalities," *Pacific J. Math.* **128** (1987) 299–305.
- [3] G. Ladas and I. P. Stavroulakis, "On delay differential inequalities of first order," *Funkcialaj Ekvacioj.* **25** (1982) 105–113.