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ERGODIC THEORY AND THE FUNCTIONAL EQUATION $(I - T)x = y$

MICHAEL LIN and ROBERT SINE

The problem of solving the functional equation $(I - T)x = y$, for a given linear operator T on a Banach space X and a given $y \in X$, appears in many areas of analysis and probability. The well-known Neumann series gives $(I - T)^{-1}$ when $\|T\| < 1$. When $\|T\| = 1$, the problem is first to know if $y \in (I - T)X$, and then to find the solution x . The solution is usually found using an iterative procedure (see [4], [5], [6], [16]). We are interested in the convergence of $x_n = n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j y$ to the solution x , and obtain the precise *necessary and sufficient*

conditions (Corollary 3). The necessary condition $\sup_{k \geq 1} \left\| \sum_{j=0}^{k-1} T^j y \right\| < \infty$ is shown to be sufficient if T^m (for some $m > 0$) is weakly compact. An example shows that otherwise the condition need not be sufficient. The reflexive case appears in [1], [2], [3].

We then solve the problem of existence in the case of a dual operator on a dual space, obtaining as a corollary an application to Markov operators.

Next, we look at the same problem for $Tf(s) = f(\theta s)$, where T is induced on a suitable function space by a measurable map θ . A new “ergodic” proof for θ a minimal continuous map of a Hausdorff space is given.

Finally, we obtain results for positive conservative contractions (Markov operators) on $L_1(S, \Sigma, \mu)$. In that case we look also at solutions which are finite a.e., though not necessary in L_1 .

For the general Banach space approach, we need the *mean ergodic theorem*:

If $T^n/n \rightarrow 0$ strongly, and $\sup_n \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j \right\| < \infty$, then

$$\left\{ x : \frac{1}{n} \sum_{j=0}^{n-1} T^j x \text{ converges} \right\} = \{y : Ty = y\} \oplus \overline{(I - T)X}.$$

We call T mean ergodic if the above subspace is all of X . We mention the *uniform ergodic theorem* [19]:

$$(I - T)X \text{ is closed} \Leftrightarrow n^{-1} \sum_{k=0}^{n-1} T^k \text{ converges uniformly.}$$

In that case, $I - T$ is invertible on $(I - T)X$, and $\frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j$ converges uniformly to $(I - T)^{-1}$ (on $(I - T)X$), which is a generalization of the Neumann series theorem.

THEOREM 1. *Let T be mean ergodic. The following conditions are equivalent for $y \in X$:*

- (i) $y \in (I - T)X$;
- (ii) $x_n = \frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j y$ has a weakly convergent subsequence;
- (iii) $\{x_n\}$ converges strongly (and $x = \lim x_n$ satisfies $(I - T)x = y$).

Proof. (i) \Rightarrow (iii). Let $y = (I - T)x'$. By the mean ergodic theorem, $x' = x + z$, with $x \in \overline{(I - T)X}$ and $(I - T)z = 0$. Hence $y = (I - T)x$ with $x \in \overline{(I - T)X}$.

$$x_n = n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j (I - T)x = n^{-1} \sum_{k=1}^n (I - T^k)x = x - n^{-1} \sum_{k=1}^n T^k x.$$

But $\left\| n^{-1} \sum_{k=1}^n T^k x \right\| \rightarrow 0$, since $x \in \overline{(I - T)X}$, so $\|x_n - x\| \rightarrow 0$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i). Let $x_{n_i} \rightarrow x$ weakly. Then

$$(I - T)x = \lim (I - T)x_{n_i} = \lim n_i^{-1} \sum_{k=1}^{n_i} (I - T^k)y = y - \lim n_i^{-1} \sum_{k=1}^{n_i} T^k y.$$

By the mean ergodic theorem the limit satisfies

$$Ex_n = \frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j Ey = \frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} Ey = \frac{(n + 1)}{2} Ey$$

so $Ex_{n_i} \rightarrow Ex$ is possible only if $Ey = 0$. Hence $(I - T)x = y$.

REMARK. The solution x of $(I - T)x = y$, obtained in (iii), is always in $\text{clm}\{T^j y : j > 0\}$.

COROLLARY 2. Let T be power-bounded, and assume that for some $m > 0$, T^m is weakly compact. Let $y \in X$. Then the condition (iv) below is equivalent to the three conditions of Theorem 1:

$$(iv) \sup_{k > 0} \left\| \sum_{j=0}^{k-1} T^j y \right\| < \infty.$$

Proof. (i) \Rightarrow (iv).

$$y = (I - T)x \Rightarrow \left\| \sum_{j=0}^{k-1} T^j y \right\| = \|(I - T^k)x\| \leq \|x\|(1 + \sup \|T^n\|).$$

(iv) \Rightarrow (i). By (iv), $\left\| \frac{1}{k} \sum_{j=0}^{k-1} T^j y \right\| \rightarrow 0$. We restrict ourselves to $\text{clm}\{T^j y : j \geq 0\}$, on which T is now mean ergodic (in fact, T is mean ergodic on X). By (iv) and weak compactness of T^m , $\left\{ \sum_{j=0}^{k-1} T^j(T^m y) \right\}$ is weakly sequentially compact, and so is

$$z_n = \frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j T^m y, \text{ so, by Theorem 1 (iii), } z_n \rightarrow z \text{ which satisfies } (I - T)z = T^m y.$$

Now $x = z + \sum_{j=0}^{m-1} T^j y$ satisfies $(I - T)x = y$.

EXAMPLE 1. T may be a mean ergodic contraction, but, in general, (iv) does not imply the conditions of Theorem 1.

Let Y be a non-reflexive Banach space and T a contraction which is not mean ergodic (e.g., $Y = \ell_1$, T the shift to the right). Take $z \in Y$ such that $n^{-1} \sum_{j=0}^{n-1} T^j z$ does not converge (i.e., $z \notin \overline{(I - T)Y} \oplus \{Tx = x\}$). Let $y = (I - T)z$, and $X = \text{clm}\{T^j y : j \geq 0\}$. X is an invariant subspace for T , and T on X is mean ergodic (with no fixed points). Clearly y satisfies (iv). If there were $x \in X$ with $(I - T)x = y$, then

$$(I - T)(z - x) = 0, \text{ so } n^{-1} \sum_{k=1}^n T^k z = n^{-1} \sum_{k=1}^n T^k(z - x) + n^{-1} \sum_{k=1}^n T^k x \rightarrow z - x,$$

contradicting the choice of z . Hence $(I - T)x = y$ has no solution in X .

REMARK. The previous example shows also that without ergodicity in Theorem 1, (i) need not imply (ii): The $\{x_n\}$ is always in $\overline{(I - T)Y}$ (in fact, in X), while the solution is in Y , and if x_{n_i} converges weakly, the limit must be a solution. Hence $\{x_n\}$ has no weakly convergent subsequence.

COROLLARY 3. Let T satisfy:

- (a) $\sup_N \left\| N^{-1} \sum_{i=0}^{n-1} T^i \right\| < \infty$;
- (b) $T^n/n \rightarrow 0$ strongly.

Then the following conditions are equivalent for $y \in X$.

- (i) $y \in (I - T) \overline{(I - T)X}$;
- (ii) as in Theorem 1;
- (iii) as in Theorem 1.

Proof. Let $Y = \overline{(I - T)X}$. On Y , T is mean ergodic.

- (i) \Rightarrow (iii). $y = (I - T)x$, with $x \in Y$.
- (iii) follows from Theorem 1, applied in Y .
- (ii) \Rightarrow (i). If $x_{n_i} \xrightarrow{w} x$, the computation in the proof of Theorem 1 yields

$$n_i^{-1} \sum_{k=1}^{n_i} T^k y \xrightarrow{w} y - (I - T)x.$$

Hence $y \in Y \oplus \{Tz = z\} \equiv Z$. Apply Theorem 1 to T on Z to obtain $y \in (I - T)Z = (I - T)Y$.

COROLLARY 4. Let T be as in Corollary 2. Then the following conditions are equivalent for $y \in X$:

- (1) $\sum_{j=0}^k T^j y$ converges weakly (to $x \in X$, and then $(I - T)x = y$);
- (2) $T^n y \xrightarrow{w} 0$, and $\liminf_{k \rightarrow \infty} \left\| \sum_{j=0}^k T^j y \right\| < \infty$.

Proof. (1) \Rightarrow (2) is easy.

(2) \Rightarrow (1). If $\left\| \sum_{j=0}^{k_i} T^j y \right\| \leq M$, then $\sum_{j=0}^{k_i} T^j T^{m_j} y$ is weakly sequentially compact. Take a subsequence of $\{k_i\}$ (called still $\{k_i\}$) with $\sum_{j=0}^{k_i} T^j T^{m_j} y \xrightarrow{w} z$. Then

$$(I - T)z = T^m y - \lim_{i \rightarrow \infty} T^{m+k_i+1} y = T^m y.$$

Hence $x = z + \sum_{j=0}^{m-1} T^j y$ is in $\text{clm}\{T^n y\}$ with $(I - T)x = y$. Now also $T^n x \rightarrow 0$ weakly, so (1) holds.

REMARK. For strong convergence in (1) we put strong convergence in (2). If we know that $y \in (I - T)X$ and $T^n y$ converges (necessarily to 0) then $\sum_{j=0}^k T^j y$ will converge to x (in the same topology that $T^n y \rightarrow 0$), assuming only mean ergodicity, instead of weak compactness, for T power-bounded (see also [2]). However, (2) does not imply that $y \in (I - T)X$ (even when $\|T^n y\| \rightarrow 0$): see the beginning of Example 3.

EXAMPLE 2. The condition that $\left\{ \sum_{j=0}^{k-1} T^j y \right\}_{k=1}^\infty$ be weakly sequentially compact, though sufficient to imply the other conditions in Theorem 1, is not necessary.

In [17] there is an example of a real Banach space X and an isometry T , such that for some vector $x_0 \in X$ we have $\sup_{\|x\|=1} \frac{1}{N} \sum_{k=0}^{N-1} |\langle x^*, T^k x_0 \rangle| \rightarrow 0$, but for no subsequence n_j does $T^{n_j} x_0$ converge weakly to 0. Since clearly $\left\| \frac{1}{N} \sum_{k=1}^N T^k x_0 \right\| \rightarrow 0$, by restricting ourselves to $\text{clm}\{T^j x_0 : j \geq 0\}$ we have T mean ergodic. Let $y = (I - T)x_0$. Then $\sum_{j=0}^{k-1} T^j y = x_0 - T^k x_0$. The choice of x_0 shows that 0 is in the weak closure of $\{T^k x_0\}$. If this closure were weakly compact, some subsequence of $\{T^k x_0\}$ would converge weakly to zero, (since the weak topology on a weakly compact set in a separable Banach space is metrizable [7, V.6.3]) — a contradiction. Hence the closure is not weakly compact, and $\{T^k x_0\}$ is not weakly sequentially compact [7, V.6.1].

REMARKS. 1. Examples 1 and 2 show that we cannot, in general, reverse any of the implications $\left\{ \sum_{j=0}^{k-1} T^j y \right\}_{k \geq 1}$ is w.s. compact $\Rightarrow y \in (I - T)X \Rightarrow \left\{ \sum_{j=0}^{k-1} T^j y \right\}_{k \geq 1}$ bounded. Example 2 is new, and shows how remarks on compactness made by previous authors should be understood in relation to Theorem 1. Special examples of the kind of Example 1, for the shift in ℓ_∞ , appear in [10] (expressed in different terms).

Corollary 2 improves the result of Butzer and Westphal [3] (for Cesàro averages). In that connection they too consider the linear manifold $(I - T) \overline{(I - T)X}$. However, Corollary 3 is new. Theorem 1 is essentially given in [4].

In many cases, we may have to identify if $y^* \in (I - T)X^*$ when T is a contraction on X . Here condition (iv) works, because of weak-* compactness. For completeness, we repeat the first author's proof from [17].

THEOREM 5. Let $\sup \|T^n\| < \infty$. The following conditions are equivalent for $y^* \in X^*$.

- (i) $y^* \in (I - T^*)X^*$;
- (ii) $\sup_{k > 0} \left\| \sum_{j=0}^{k-1} T^{*j} y^* \right\| < \infty$.

Proof. (ii) \Rightarrow (i). Let $x_n^* = n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} T^{*j} y^*$. Then $\{x_n^*\}$ is bounded, hence is relatively compact in the weak-* topology. Let x^* be a weak-* closure point of $\{x_n^*\}$. For $y \in X$ there is a sequence $\{n_j\}$ with

$$\begin{aligned} \langle (I - T^*)x^*, y \rangle &= \langle x^*, (I - T)y \rangle = \lim \langle x_{n_j}^*, (I - T)y \rangle = \\ &= \lim \langle (I - T^*)x_{n_j}^*, y \rangle = \lim \langle y^* - n_j^{-1} \sum_{k=1}^{n_j} T^{*k} y^*, y \rangle = \langle y^*, y \rangle. \end{aligned}$$

Hence $(I - T^*)x^* = y^*$.

As an application of Theorem 5 we have the following corollary, which, in the measure preserving case, was proved by Browder [1, Theorem 2] by using a different method.

COROLLARY 6. *Let (S, Σ, μ) be a σ -finite measure space, and θ a non-singular measurable transformation of S . Then $f \in L_\infty$ is of the form $f(s) = g(s) - g(\theta s)$, with $g \in L_\infty$, if and only if $\sup_{k \geq 1} \left\| \sum_{j=0}^{k-1} f \circ \theta^j \right\|_\infty < \infty$.*

Proof. On $X = M(S, \Sigma, \mu)$, the space of finite signed measures absolutely continuous with respect to μ , define Tv by $Tv(A) = v(\theta^{-1}A)$. Then $X^* = L_\infty$, and $T^*f(s) = f(\theta s)$, and Theorem 5 applies.

The following result was conjectured by M. Keane and J. Aaronson for T positive.

THEOREM 7. *Let (S, Σ, μ) be a σ -finite measure space, and let T be a contraction on $L_1(S, \Sigma, \mu)$. Then $f \in L_1$ is of the form $f = (I - T)g$ with $g \in L_1$ if and only if $\sup_{k \geq 1} \left\| \sum_{j=0}^{k-1} T^j f \right\|_1 < \infty$.*

Proof. We identify $L_1(S, \Sigma, \mu)$, via the Radon-Nikodym theorem, with the space $M(S, \Sigma, \mu)$ of countably additive measures $\ll \mu$. Then we have $\sup_{k \geq 1} \left\| \sum_{j=0}^{k-1} T^j v \right\| < \infty$, with $dv = f d\mu$.

T^{**} acts on $L_\infty(S, \Sigma, \mu)^* = \text{ba}(S, \Sigma, \mu)$, the space of bounded finitely additive measures (= charges). By Theorem 5 (applied to v in L_∞^* and T^{**}), there exists $\eta \in \text{ba}(S, \Sigma, \mu)$ with $(I - T^{**})\eta = v$. Decompose [21] $\eta = \eta_1 + \eta_2$, with η_1

countably additive and η_2 a pure charge (i.e., $|\eta_2|$ does not bound any non-negative measure). Then

$$v = (I - T^{**})\eta = \eta_1 - T^{**}\eta_1 + \eta_2 - T^{**}\eta_2.$$

Since $T^{**}\eta_1 = T\eta_1 \in M(S, \Sigma, \mu)$, we obtain that $v_1 = \eta_2 - T^{**}\eta_2$ is countably additive. Hence $\|\eta_2\| \geq \|T^{**}\eta_2\| = \|\eta_2 - v_1\| = \|\eta_2\| + \|v_1\|$ since $\|T^{**}\| \leq 1$, while η_2 (a pure charge) and v_1 (a measure) are mutually singular [21]. Thus $v_1 = 0$ and $v = (I - T^{**})\eta_1 = (I - T)\eta_1$, yielding $g = \frac{d\eta_1}{d\mu}$ as a required solution.

In the next proposition, Theorem 5 cannot be applied, since the space $B(S, \Sigma)$ of bounded measurable functions is not a dual space, in general.

PROPOSITION 8. *Let (S, Σ) be a measurable space, and θ a measurable transformation of S into itself. Then $f \in B(S, \Sigma)$ is of the form $f(s) = g(s) - g(\theta s)$, with $g \in B(S, \Sigma)$, if and only if $\sup_{k>1} \left\| \sum_{j=0}^{k-1} f(\theta^j s) \right\| < \infty$.*

Proof. For f satisfying the condition, define

$$g(s) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j=0}^{k-1} f(\theta^j s).$$

Since $\left\| \frac{1}{n} \sum_{j=0}^{n-1} f \circ \theta^j \right\|_{\infty} \rightarrow 0$, we obtain

$$g(\theta s) = g(s) - f(s).$$

REMARKS. 1. The previous proof gives also a direct proof of Corollary 6.
 2. In Corollary 6, if θ is recurrent, a function g can be obtained by setting $g(s) = \sup_{k \geq 0} \sum_{j=0}^k f(\theta^j s)$ (see the first and last paragraphs of the proof of Theorem 9).

EXAMPLE 3. There exists a compact metric space S , a uniquely ergodic continuous map φ such that $\varphi^n s$ converges for every $s \in S$, and a function $f \in C(S)$ with $\sup_k \left\| \sum_{j=0}^{k-1} f(\varphi^j s) \right\|_{\infty} < \infty$, such that for every $g \in C(S)$, $g(s) - g(\varphi s) \neq f(s)$.

Proof. Let T' be an operator as in Example 1, on Y . Let $T = \frac{1}{2}(I + T')$. Then $I - T = \frac{1}{2}(I - T')$, so T is mean ergodic too, on X , and T^n converges strongly

to zero on X ($\|T^n(I - T)\| = \|2^{-n+1}(I + T)^n(I - T)\| \rightarrow 0$). Now T yields also an example of (iv) $\not\Rightarrow$ (i). Let S be the unit ball of X^* and the weak- $*$ topology, φ is the restriction of T^* to S and for $s \in S \subset X^*$, $f(s) = \langle s, y \rangle$, where y satisfies (iv).

Hence $\sup \left\| \sum_{j=0}^{k-1} f(\varphi^j s) \right\| = \sup \left\| \sum_{j=0}^{k-1} T^j y \right\| < \infty$. Now $\|T^n x\| \rightarrow 0$ for every $x \in X$ yields $\varphi^n(s) \rightarrow 0$ for every $s \in S$. Hence φ is uniquely ergodic and the operator $Ah(s) = h(\varphi s)$ is mean ergodic on $C(S)$, since $A^n h \rightarrow h(0)$ weakly (= pointwise).

If $f \in (I - A)C(S)$, we must have, by Theorem 1(iii), that $g_n = n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} A^j f$ converges strongly. But

$$\begin{aligned} g_n(s) &= n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} f(\varphi^j s) = n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} \langle y, T^{*j} s \rangle = \\ &= \left\langle n^{-1} \sum_{k=1}^n \sum_{j=0}^{k-1} T^j y, s \right\rangle, \end{aligned}$$

and the right-hand side does not converge uniformly on S , by the choice of T and y . Hence $f \notin (I - A)C(S)$.

THEOREM 9. *Let φ be a continuous map of a topological Hausdorff space S into itself, such that $\{\varphi^n s : n > 0\}$ is dense in S for every $s \in S$. Then $f \in C(S)$ is of the form $f(s) = g(s) - g(\varphi s)$, with $g \in C(S)$, if and only if $\sup_{k \geq 0} \left\| \sum_{j=0}^k f(\varphi^j s) \right\| < \infty$.*

Proof. We have to prove only the “if” part. Define $g(s) = \sup_{k \geq 0} \sum_{j=0}^k f(\varphi^j s)$. Then

$$g(\varphi s) = \sup_{k \geq 0} \sum_{j=1}^{k+1} f(\varphi^j s) = \sup_{k \geq 1} \sum_{j=0}^k f(\varphi^j s) - f(s).$$

If $g(s) = f(s)$, then $g(\varphi s) \leq 0$, so $g^+(\varphi s) = 0 = g(s) - f(s)$. If $g(s) > f(s)$, then $g(\varphi s) = g(s) - f(s) > 0$, so in any case we have $g^+(\varphi s) = g(s) - f(s)$.

Our purpose now is to show the continuity of g . We say that a function h has a jump of at least δ at s_0 if for every $\varepsilon > 0$ and U open containing s_0 there are s', s'' in U with $|h(s') - h(s'')| > \delta - \varepsilon$. If $J_\delta(h)$ is the set of points where h has a jump of least δ , then $J_\delta(h)$ is clearly closed. It is easy to show that $J_\delta(h^+) \subset J_\delta(h)$.

CLAIM 1. $\varphi(J_\delta(g)) \subset J_\delta(g)$.

We show that for $s_0 \in J_\delta(g)$, $\varphi s_0 \in J_\delta(g^+)$, which is enough. Let U be open with $\varphi s_0 \in U$, and let $\varepsilon > 0$. Since f is continuous, there is V open with $|f(s) - f(s_0)| < \frac{\varepsilon}{4}$ for $s \in V$. Let $W = \varphi^{-1}(U) \cap V$. It contains s_0 , so there are $s',$

s'' in W with $|g(s') - g(s'')| > \delta - \frac{\epsilon}{2}$. But $\varphi s'$ and $\varphi s''$ are in U , and using $g^+(\varphi s) = g(s) - f(s)$ we obtain

$$|g^+(\varphi s') - g^+(\varphi s'')| = |g(s') - g(s'') - [f(s) - f(s'')]| > \delta - \frac{\epsilon}{2} - 2\frac{\epsilon}{4} = \delta - \epsilon.$$

CLAIM 2. $J_\delta(g) = \emptyset$.

By Claim 1 J_δ is closed invariant for φ . If $J_\delta \neq \emptyset$, there is $s_0 \in J_\delta$ and $\{\varphi^n s_0\} \subset J_\delta$, so $J_\delta = S$. By definition, g is lower semicontinuous, i.e., $\{g > \alpha\}$ is open for every α . Let $\alpha_0 = \inf\{g(s) : s \in S\}$, $0 < \beta < \delta$. If $J_\delta = S$, then every open set $\neq \emptyset$ contains two points s', s'' with $|g(s') - g(s'')| > \beta$. Now $\{g > \alpha_0\}$ is open and non-empty (or $g \equiv \alpha_0$ and $J_\delta = \emptyset$). Hence there are points $s', s'' \in \{g > \alpha_0\}$. Hence $\{g > \alpha_0 + \beta\}$ is not empty. Similarly $\{g > \alpha_0 + n\beta\} \neq \emptyset$ for every n , contradicting the boundedness of g .

We have $J_\delta(g) = \emptyset$ for every $\delta > 0$, hence g is continuous. Now $g(\varphi s) \leq g^+(\varphi s) = g(s) - f(s)$, so that $h(s) \equiv g(s) - g(\varphi s) - f(s) \geq 0$ is continuous non-negative. But

$$\sum_{j=0}^k h(\varphi^j s) = g(s) - g(\varphi^{k+1} s) - \sum_{j=0}^k f(\varphi^j s),$$

so that $\sum_{j=0}^\infty h(\varphi^j s) < \infty$ for every $s \in S$. But our condition on φ implies that $\varphi^n s$ enters every non-empty open set infinitely many times. If $\left\{h > \frac{1}{n}\right\}$ is entered infinitely many times, $\sum_{j=0}^\infty h(\varphi^j s) = \infty$, a contradiction. Hence $\left\{h > \frac{1}{n}\right\} = \emptyset$ and $h \equiv 0$, so that $f(s) = g(s) - g(\varphi s)$.

COROLLARY 10. Let φ be as in the previous theorem and $f \in C(S)$. If $\sup_{k > 0} \left| \sum_{j=0}^k f(\varphi^j s_0) \right| \neq \infty$, then there is a $g \in C(S)$ with $f(s) = g(s) - g(\varphi s)$.

Proof. We prove $\sup_{k > 0} \left| \sum_{j=0}^k f(\varphi^j s) \right| < \infty$. Let $s_0 \in S$ satisfy $\sup_{k > 0} \left| \sum_{j=0}^k f(\varphi^j s_0) \right| = \alpha < \infty$. Then, for every m and n , we have $\left| \sum_{j=m}^n f(\varphi^j s_0) \right| \leq 2\alpha$. Now $\left\{s \in S : \sup_{n,m} \left| \sum_{j=m}^n f(\varphi^j s) \right| \leq \alpha\right\}$ is closed, φ -invariant, and non-empty. Hence it is all of S .

REMARKS. 1. Theorem 9 for the compact case appears in Gottschalk and Hedlund [15, 14.11] with a different proof. Browder [1] generalized their approach in order to obtain it in the general case treated here. The problem is treated (in disguise) also by Furstenberg [10, p. 162].

2. A result of Gottschalk [14] shows that if S is locally compact and φ is minimal, then in fact S must be compact.

3. Corollary 10 for the compact case, with a proof which generalizes that of [15], appears in Furstenberg, Keynes and Shapiro [13, Lemma 2.2], and in Shapiro 20, Theorem 2.3].

4. Our proof is more direct, since it is based on the fact that if $f(s) = g(s) - g(\varphi s)$, with $\inf\{g(s) : s \in S\} = 0$, then the minimality of φ implies that

$$\max_{0 \leq k \leq n} \sum_{j=0}^k f(\varphi^j s) = \max_{0 \leq k \leq n} [g(s) - g(\varphi^{k+1} s)] = g(s) - \min_{0 \leq k \leq n} g(\varphi^{k+1} s)$$

must converge everywhere to g . If S is compact the convergence is uniform, by Dini's theorem.

Claim 1 in our proof of continuity in Theorem 9 is a simplification of a method used by Furstenberg [11] for a different functional equation (which he attributes to Kakutani in [12]). Claim 2 avoids Baire's theorem (used in [11]), and allows general spaces.

The analogue of the previous corollary for non-singular transformations is easier:

THEOREM 11. *Let (S, Σ, μ) be a σ -finite measure space, and θ a non-singular transformation of S , which is conservative and ergodic (i.e., $\theta(A) \subset A$ implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$). If f is a.e. finite and satisfies $\mu \left\{ s : \sup_{k \geq 0} \left| \sum_{j=0}^k f(\theta^j s) \right| < \infty \right\} > 0$, then there is a $g \in L_\infty$ with $f(s) = g(s) - g(\theta s)$ a.e. (hence $f \in L_\infty$).*

Proof. Let $g_k(s) = \sum_{j=0}^{k-1} f(\theta^j s)$. We show $\sup_{k \geq 1} |g_k(s)|$ finite a.e. .

Let $A = \{s : \sup_k |g_k(s)| < \infty\}$. Then $g_k(\theta s) = g_{k+1}(s) - f(s)$ shows that $\theta s \in A$ for $s \in A$, and $\mu(A) > 0$ implies $\mu(S \setminus A) = 0$. Hence for a.e. s , $g_k(s)$ is bounded, yielding $g_{k-1} g_k(s) \rightarrow 0$ a.e. . Now let $g(s) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_k(s)$. Then

$$\begin{aligned} g(\theta s) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_k(\theta s) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_{k+1}(s) - f(s) = \\ &= g(s) - f(s) + \lim_{n \rightarrow \infty} [g_{n+1}(s) - g(s)]/n = g(s) - f(s). \end{aligned}$$

We have to show now that g is bounded. We have $g(s) - g(\theta^k s) = \sum_{j=0}^{k-1} f(\theta^j s)$, hence $\sup_k |g(\theta^k s)| < \infty$ a.e. .

Let $A_N = \{s : \sup_{k \geq 0} |g(\theta^k s)| \leq N\}$. Then $S = \bigcup_{N=1}^{\infty} A_N \pmod{\mu}$, and $\mu(A_N) > 0$ for some N . But $\theta(A_N) \subset A_N$, hence $S = A_N \pmod{\mu}$, and $|g(s)| \leq N$ a.e. .

REMARKS. 1. The previous theorem may fail for a general conservative and ergodic Markov operator on L_∞ . Let $\mu(S) = 1$, and define $Tf = \int f d\mu$, for $f \in L_1$.

If $f \in L_1$ with $\int f d\mu = 0$, then $\left| \sum_{j=0}^{k-1} T^j f \right| = |f|$. But we may take $f \notin L_\infty$.

2. If θ is only conservative (i.e., $\theta^{-1}(A) \supset A \Rightarrow \theta^{-1}(A) = A$), the theorem may fail. (Examples are easy to construct.)

For the general set-up of Theorem 1, if $(I - T)x_i = y$, then $T(x_1 - x_2) = x_1 - x_2$, so uniqueness of solutions in the Banach space depends on the fixed points of T . We now look at a Markov operator on L_1 , and study the finite solutions (not necessarily integrable) in a special case (see [8] for the extension of T).

DEFINITION. A positive contraction of $L_1(S, \Sigma, \mu)$ is called *conservative* if for $u > 0$ a.e., $u \in L_1$, we have $\sum_{j=0}^{\infty} T^j u(s) = \infty$ a.e. .

THEOREM 12. Let T be a conservative positive contraction on $L_1(S, \Sigma, \mu)$, and let $f \in L_1$. Let g_1 and g_2 be a.e. finite (measurable) functions satisfying $(I-T)g_i = f$. If

$$(*) \quad \lim_{n \rightarrow \infty} \frac{T^n |g_i|}{\sum_{j=0}^n T^j u} = 0 \quad \text{a.e.} \quad \text{for some } 0 < u \in L_1,$$

then

$$T(g_2 - g_1)^\pm = (g_2 - g_1)^\pm, \quad \text{and} \quad \left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\|_1 \rightarrow 0.$$

Proof. Let $g = g_2 - g_1$. Then $Tg = g$, hence $T|g| \geq |g|$, and $Tg^\pm \geq g^\pm$. (Since g need not be integrable, we cannot conclude equality immediately.)

Then $Tg^+ = g^+ + h$, with $0 < h < \infty$ a.e. . (By assumption, $T^n |g| < \infty$ a.e. for every n .) Hence also $Tg^- = g^- + h$.

Without loss of generality, we may and do assume $\mu(S) = 1$.

$$\sum_{i=0}^{n-1} T^i g^+ + \sum_{i=0}^{n-1} T^i h = \sum_{i=1}^n T^i g^+ \Rightarrow \sum_{i=0}^{n-1} T^i h + g^+ = T^n g^+.$$

We take the $u \in L_1$ with $u > 0$ a.e., for which (*) holds. Then

$$\frac{g^+ + \sum_{i=0}^{n-1} T^i h}{\sum_{i=0}^n T^i u} = \frac{T^n g^+}{\sum_{i=0}^n T^i u} \rightarrow 0 \text{ a.e. .}$$

Let $\Sigma_I = \{A \in \Sigma : T^* 1_A = 1_A \text{ a.e.}\}$. By the Chacon-Ornstein theorem (see [8])

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n T^i v}{\sum_{i=0}^n T^i u} = \frac{E(v | \Sigma_I)}{E(u | \Sigma_I)} \text{ a.e., for } v \in L_1,$$

and therefore also for any finite $v \geq 0$. We conclude that $\frac{E(h | \Sigma_I)}{E(u | \Sigma_I)} = 0$, so $h = 0$ a.e., since $h \geq 0$. Hence $Tg^\pm = g^\pm$.

Now $(I - T)g_1 = f$ implies, using (*), that

$$0 = \lim_{n \rightarrow \infty} \frac{g - T^{n+1}g}{\sum_{i=0}^n T^i u} = \lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n T^i f}{\sum_{i=0}^n T^i u} = \frac{E(f | \Sigma_I)}{E(g | \Sigma_I)}.$$

Hence $E(f | \Sigma_I) = 0$. Since all T^* -invariant functions in the conservative case are Σ_I -measurable, f is orthogonal to all T^* -invariant functions, hence is in $\overline{(I - T)L_1}$. Thus $\left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\| \rightarrow 0$.

COROLLARY 13. *Let T be as above. Let $f \in L_1$ satisfy $\left\| n^{-1} \sum_{j=0}^{n-1} T^j f \right\|_1 \rightarrow 0$. If $g_1 \geq 0, g_2 \geq 0$ satisfy $(I - T)g_i = f$, then $T(g_1 - g_2)^\pm = (g_1 - g_2)^\pm$.*

Proof. We show that (*) is satisfied for g_i :

$$\frac{g_i - T^n g_i}{\sum_{j=0}^{n-1} T^j u} = \frac{\sum_{j=0}^{n-1} T^j f}{\sum_{j=0}^{n-1} T^j u} \xrightarrow{n \rightarrow \infty} \frac{E(f | \Sigma_I)}{E(u | \Sigma_I)} = 0.$$

(Since $E(f | \Sigma_I)$ must be zero.)

REMARKS. 1. Condition (*) in Theorem 12 is a necessary and sufficient condition for obtaining $T|g| = |g|$ from $Tg = g$, for T conservative. If $T|g| = |g|$, then the proof of Corollary 13 shows that (*) holds. The following example shows that $Tg = g$ does not imply $T|g| = |g|$. Define T on $\ell_1(Z)$ by $(Tu)_i = \frac{1}{2}(u_{i-1} + u_{i+1})$. Then $g_i = i$ defines an invariant function, but $T|g| \neq |g|$, since $(T|g|)_0 = 1$.

2. In Corollary 13 we have looked at the uniqueness of positive solutions g to $(I - T)g = f$, when $f \in \overline{(I - T)L_1}$. Fong and Sucheston [9, Theorem 2.4] proved that (in the conservative case) positive *integrable* solutions exist for a dense subset of $\overline{(I - T)L_1}$.

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