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Research Article

Global Asymptotic Stability and Naimark-Sacker Bifurcation of Certain Mix Monotone Difference Equation

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We investigate the global asymptotic stability of the following second order rational difference equation of the form $x_{n+1} = (Bx_nx_{n-1} + F)/(bx_nx_{n-1} + cx_{n-1}^2)$, $n = 0, 1, \ldots$, where the parameters B, F, b, and c and initial conditions x_{-1} and x_0 are positive real numbers. The map associated with this equation is always decreasing in the second variable and can be either increasing or decreasing in the first variable depending on the parametric space. In some cases, we prove that local asymptotic stability of the unique equilibrium point implies global asymptotic stability. Also, we show that considered equation exhibits the Naimark-Sacker bifurcation resulting in the existence of the locally stable periodic solution of unknown period.

1. Introduction and Preliminaries

In this paper, we investigate the local and global dynamics of the following difference equation:

$$x_{n+1} = \frac{Bx_nx_{n-1} + F}{bx_nx_{n-1} + cx_{n-1}^2} \quad n = 0, 1, \ldots \tag{1}$$

where the parameters B, F, b, c are positive real numbers and initial conditions x_{-1} and x_0 are arbitrary positive real numbers. Equation (1) is the special case of a general second order quadratic fractional equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \ldots, \tag{2}$$

with nonnegative parameters and initial conditions such that $A+B+C > 0, a+b+c+d+e+f > 0$ and $ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n + ex_{n-1} + f > 0$, $n = 0, 1, \ldots$. Several global asymptotic results for some special cases of Equation (2) were obtained in [1–11]. Also, Equation (1) is a special case of the equation

$$x_{n+1} = \frac{Bx_nx_{n-1} + Cx_{n-1}^2 + F}{bx_nx_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{3}$$

with positive parameters and nonnegative initial conditions x_{-1}, x_0. Local and global dynamics of Equation (3) was investigated in [12].

The special case of Equation (3) when $B = C = 0$, i.e.,

$$x_{n+1} = \frac{F}{bx_nx_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots \tag{4}$$

was studied in [8]. The authors performed the local stability analysis of the unique equilibrium point and gave the necessary and sufficient conditions for the equilibrium to be locally asymptotically stable, a repeller or nonhyperbolic equilibrium. Also, it was shown that Equation (4) exhibits the Naimark-Sacker bifurcation.

The special case of Equation (3) (when $B = F = 0$ and $C = 1$) is the following equation:

$$x_{n+1} = \frac{x_{n-1}^2}{bx_nx_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{5}$$

where the parameters b, c, and f are nonnegative numbers with condition $b + c > 0$, $f \neq 0$ and the initial conditions x_{-1}, x_0 arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equation (5) is a perturbed Sigmoid Beverton-Holt difference equation and it was studied in [9]. The special case of Equation (5) for $b = 0$ is the well-known Thomson equation

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots, \tag{6}$$

where the parameters c and f are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers, is used in the modelling of fish population [13].

The dynamics of (6) is very interesting and follows from the dynamics of related equation

$$x_{n+1} = \frac{x_n^2}{cx_n^2 + f}, \quad n = 0, 1, \ldots. \tag{7}$$

Indeed (6) is delayed version of (7) and so it exhibits the existence of period-two solutions.

Two interesting special cases of Equation (2) are the following difference equations:

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \ldots, \tag{8}$$

studied in [14], and

$$x_{n+1} = \frac{x_n x_{n-1} + \alpha x_n + \beta x_{n-1}}{a x_n x_{n-1} + b x_{n-1}}, \quad n = 0, 1, \ldots, \tag{9}$$

studied in [5]. In both equations, (8) and (9), the associated map changes its monotonicity with respect to its variable.

In this paper, in some cases when the associated map changes its monotonicity with respect to the first variable in an invariant interval, we will use Theorems 1 and 2 below in order to obtain the convergence results. However, if $F = F_g = (B/b)^3 c$, we would not be able to use this method, so we will use the semicycle analysis; see [15] to show that each of the following four sequences $\{x_{4k}\}_{k=1}^{\infty}$, $\{x_{4k+1}\}_{k=0}^{\infty}$, $\{x_{4k+2}\}_{k=0}^{\infty}$, $\{x_{4k+3}\}_{k=0}^{\infty}$ converges to the unique equilibrium point.

Also, we will show that Equation (1) exhibits the Naimark-Sacker bifurcation resulting in the existence of the locally stable periodic solution of unknown period.

Note that the problem of determining invariant intervals in the case when the associated map changes its monotonicity with respect to its variable has been considered in [17, 18].

In this paper, we will use the following well-known results, Theorem 2.22, in [16], and Theorem 1.4.7 in [19].

Theorem 1. *Let $[a, b]$ be a compact interval of real numbers and assume that $f : [a, b] \times [a, b] \longrightarrow [a, b]$ is a continuous function satisfying the following properties:*

(a) *$f(x, y)$ is nondecreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each $x \in [a, b]$;*

(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$f(m, M) = m$$
$$\text{and } f(M, m) = M, \tag{10}$$

then $m = M$.

Then

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \cdots \tag{11}$$

has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Equation (11) converges to $\overline{x}$.

Theorem 2. *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b] \times [a, b] \longrightarrow [a, b]$ is a continuous function satisfying the following properties:*

(a) *$f(x, y)$ is nonincreasing in both variables*

(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$f(m, m) = M$$
$$\text{and } f(M, M) = m, \tag{12}$$

then $m = M$.

Then, (11) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Equation (11) converges to $\overline{x}$.

Remark 3. As is shown in [20] the unique equilibrium in Theorems 1 and 2 is globally asymptotically stable.

The rest of this paper is organized as follows. The second section presents the local stability of the unique positive equilibrium solution and the nonexistence of the minimal period-two solution. The third section gives global dynamics in certain regions of the parametric space. The results and techniques depend on monotonic character of the transition function $f(x, y)$ which is either decreasing in both arguments or increasing in first and decreasing in second argument. In simpler situations Theorems 1 and 2 are sufficient to prove global stability of the unique equilibrium. In more complicated situations we use the semicycle analysis, which is extensively used in [15, 19] for many linear fractional equations, to prove that every solution has four convergent subsequences, which leads to the conclusion that every solution converges to period-four solution. In some parts of parametric space we prove that there is no minimal period-four solution and so every solution converges to the equilibrium, while in other parts of parametric space we prove that the period-four solution exists. The semicycle analysis presented here uses innovative techniques based on analysis of systems of polynomial equations which coefficients depend on four parameters. Finally in the region of parameters complementary to the one where the period-four solution exists we prove that the Naimark-Sacker bifurcation takes place which produces locally stable periodic solution. All numerical simulations indicate that the equilibrium solution is globally asymptotically stable whenever it is locally asymptotically stable and that the dynamics is chaotic whenever the equilibrium is repeller. An interesting feature of Equation (1) is that it gives an example of second order difference equation with period-four solution for which period-two solution does not exist. The global dynamics of Equation (11) when the transition function $f(x, y)$ is either increasing in both arguments or decreasing in the first and increasing in the second argument is fairly simple as every solution $\{x_n\}$ breaks into two eventually monotonic subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$; see [21–23]. The global dynamics of Equation (11) when the transition function $f(x, y)$ is either decreasing in both arguments or increasing in the first and decreasing in the second argument could be quite complicated ranging from global asymptotic stability of the equilibrium, see [19, 21, 22, 24–26] to conservative and nonconservative chaos, see [3, 19, 26]. Interesting applications can be found in [27].

2. Linearized Stability

In this section, we present the local stability of the unique positive equilibrium of Equation (1) and the nonexistence of the minimal period-two solution of Equation (1).

In view of the above restriction on the initial conditions of Equation (1), the equilibrium points of Equation (1) are the positive solutions of the equation

$$\overline{x} = \frac{B\overline{x}^2 + F}{(b+c)\,\overline{x}^2}, \tag{13}$$

or equivalently

$$(b+c)\,\overline{x}^3 - B\overline{x}^2 - F = 0. \tag{14}$$

Equation (1) has the unique positive solution $\overline{x}$ given as

$$\overline{x} = \frac{B}{3(b+c)} + \frac{\sqrt[3]{2}B^2}{3(b+c)\sqrt[3]{\Delta + \sqrt{\Delta^2 - 4B^6}}} + \frac{\sqrt[3]{\Delta + \sqrt{\Delta^2 - 4B^6}}}{3\sqrt[3]{4}}, \tag{15}$$

where

$$\Delta = 2B^3 + 27F(b+c)^2. \tag{16}$$

Now, we investigate the stability of the positive equilibrium of Equation (1). Set

$$f(u,v) = \frac{Buv + F}{buv + cv^2} = \frac{Buv + F}{v(bu + cv)}, \tag{17}$$

and observe that

$$f_u' = \frac{Bcv^2 - bF}{v(bu+cv)^2}, \qquad f_v' = -\frac{Bcuv^2 + 2cFv + bFu}{v^2(bu+cv)^2} < 0. \tag{18}$$

The linearized equation associated with Equation (1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = sz_n + tz_{n-1} \tag{19}$$

where

$$s = f_u'(\overline{x},\overline{x}) \quad \text{and } t = f_v'(\overline{x},\overline{x}). \tag{20}$$

Theorem 4. *Let $F_0 = (B/c)^3 b$. The unique equilibrium point $\overline{x}$ of Equation (1) given by (15) is*

(i) *locally asymptotically stable if $F < F_0$,*
(ii) *a repeller if $F > F_0$,*
(iii) *a nonhyperbolic point of elliptic type if $F = F_0$.*

Proof. In view of

$$\begin{aligned} s &= f_u'(\overline{x},\overline{x}) = \frac{Bc\overline{x}^2 - bF}{\overline{x}^3(b+c)^2} = \frac{c}{b+c} - \frac{F}{(b+c)\overline{x}^3}, \\ t &= f_v'(\overline{x},\overline{x}) = -\frac{Bc\overline{x}^2 + 2cF + bF}{\overline{x}^3(b+c)^2} = -s - \frac{2F}{(b+c)\overline{x}^3} \\ &= -\frac{c}{b+c} - \frac{F}{(b+c)\overline{x}^3} < 0, \end{aligned} \tag{21}$$

we have that

$$s + t = -\frac{2F}{\overline{x}^3(b+c)} < 0 \tag{22}$$

and

$$\begin{aligned} s^2 - (1-t)^2 &= \left(\frac{Bc\overline{x}^2 - bF}{\overline{x}^3(b+c)^2}\right)^2 - \left(1 + \frac{Bc\overline{x}^2 + 2cF + bF}{\overline{x}^3(b+c)^2}\right)^2 \\ &= \frac{\left(Fb - Bc\overline{x}^2\right)^2}{\overline{x}^6(b+c)^4} - \frac{\left(b^2\overline{x}^3 + 2bc\overline{x}^3 + Fb + c^2\overline{x}^3 + Bc\overline{x}^2 + 2Fc\right)^2}{\overline{x}^6(b+c)^4} \\ &= -\frac{\left(2F + b\overline{x}^3 + c\overline{x}^3\right)\left(b^2\overline{x}^3 + 2bc\overline{x}^3 + c^2\overline{x}^3 + 2Bc\overline{x}^2 + 2Fc\right)}{\overline{x}^6(b+c)^3} \\ &< 0, \end{aligned} \tag{23}$$

and so $|s| < |1-t|$.

Also, we have

$$1 - t = 1 + \frac{Bc\overline{x}^2 + 2cF + bF}{\overline{x}^3(b+c)^2} > 0 \Longrightarrow t < 1. \tag{24}$$

Since $|s| < |1-t|$, the equilibrium point $\overline{x}$ will be nonhyperbolic if $t = -1$ and $|s| < 2$. From $t = -1$ we obtain

$$-\frac{c}{b+c} - \frac{F}{(b+c)\overline{x}^3} = -1 \Longleftrightarrow \overline{x} = \sqrt[3]{\frac{F}{b}}, \tag{25}$$

and by using (14), we have

$$(b+c)\left(\sqrt[3]{\frac{F}{b}}\right)^3 - B\left(\sqrt[3]{\frac{F}{b}}\right)^2 - F = 0 \Longleftrightarrow F = F_0 = \left(\frac{B}{c}\right)^3 b. \tag{26}$$

Now,

$$s = \frac{c}{b+c} - \frac{F}{(b+c)(F/b)} = -\frac{b-c}{b+c} \tag{27}$$

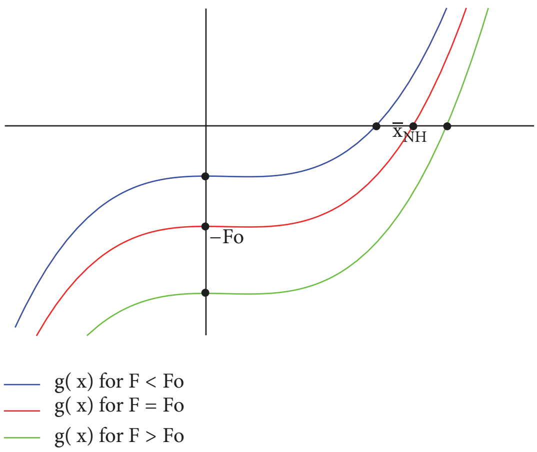


Figure 1: If $F < F_0$, then $\overline{x} < \overline{x}_{nh}$, i.e., $\overline{x}$ is LAS, and if $F > F_0$, then $\overline{x} > \overline{x}_{nh}$, i.e., $\overline{x}$ is repeller.

and the characteristic equation of (19) is of the form

$$\lambda^2 + \frac{b-c}{b+c}\lambda + 1 = 0, \tag{28}$$

from which

$$\lambda_{1,2} = \frac{c-b \pm i\sqrt{(b+3c)(3b+c)}}{2(b+c)} \quad \text{and} \quad |\lambda_{1,2}| = 1, \tag{29}$$

that is, $\overline{x}$ is nonhyperbolic equilibrium point. Let us denote

$$g(x) = (b+c)x^3 - Bx^2 - F. \tag{30}$$

Then,

$$g(\overline{x}_{nh}) = 0 \Longleftrightarrow F = F_0, \quad \text{and } g(0) = -F = -F_0. \tag{31}$$

The condition $|s| < |1-t| = 1-t$ is always satisfied. Hence, it holds: the equilibrium solution $\overline{x}$ is locally asymptotically stable if

$$t < -1 \Longleftrightarrow -\frac{c}{b+c} - \frac{F}{(b+c)\overline{x}^3} < -1 \Longleftrightarrow \overline{x} < \sqrt[3]{\frac{F}{b}} = \overline{x}_{nh}, \tag{32}$$

i.e., $F < F_0$ and a repeller if $t > -1$, which is equivalent with $\overline{x} > \sqrt[3]{F/b} = \overline{x}_{nh}$, i.e., $F > F_0$. See Figure 1. □

Lemma 5. *Equation (1) has no minimal period-two solution.*

Proof. Otherwise Equation (1) has a minimal period-two solution $\ldots x, y, x, y, \ldots$ which satisfies

$$x = \frac{Byx+F}{byx+cx^2}, \quad y = \frac{Bxy+F}{bxy+cy^2}. \tag{33}$$

Then,

$$bx^2y + cx^3 = Bxy + F, \quad bxy^2 + cy^3 = Bxy + F, \tag{34}$$

which yields

$$(x-y)\left(bxy + c\left(x^2+xy+y^2\right)\right) = 0, \tag{35}$$

which implies $x = y$. So, there is no a minimal period-two solution. □

3. Global Results

In this section, we prove several global attractivity results in the parts of parametric space.

We notice that the function $f(u,v)$ is always decreasing with respect to the second variable and can be either decreasing or increasing with respect to the first variable, depending on the sign of the nominator of f'_u. Therefore,

$$f'_u = 0 \Longleftrightarrow v = \sqrt{\frac{bF}{Bc}}, \tag{36}$$

and the function $f(u,v)$ is nonincreasing in both variables if $v \le \sqrt{bF/Bc}$, and nondecreasing with respect to the first variable and nonincreasing with respect to the second variable if $v \ge \sqrt{bF/Bc}$. Since

$$f\left(\sqrt{\frac{bF}{Bc}}, \sqrt{\frac{bF}{Bc}}\right) = \frac{B}{b}, \tag{37}$$

if we denote $F_g = (B/b)^3 c$, we can have three possible cases:

$$\frac{B}{b} > \sqrt{\frac{bF}{Bc}} \Longleftrightarrow F < F_g, \quad \frac{B}{b} = \sqrt{\frac{bF}{Bc}} \Longleftrightarrow F = F_g, \quad \frac{B}{b} < \sqrt{\frac{bF}{Bc}} \Longleftrightarrow F > F_g. \tag{38}$$

As we have been seen, the nature of the local stability of the equilibrium point depends on the parameter F_0, so we distinguish the following scenarios:

(1) $F_g \le F_0$,

(2) $F_g > F_0$.

Case 1 ($F_g \le F_0$). Notice first that $F_g < F_0$ implies $c < b$ and that $F_g = F_0$ implies $c = b$. Now, we observe three subcases.

(a) $F < F_g \le F_0$. If $F < F_g \le F_0$, the function $f(u, v)$ is nondecreasing with respect to the first variable and nonincreasing with respect to the second variable on the invariant interval of Equation (1) which is given by

$$[L, U] = \left[\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right], \quad (39)$$

i.e., it holds

$$f : \left[\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right]^2 \longrightarrow \left[\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right]. \quad (40)$$

Indeed, since

$$\max_{(x,y)\in[L,U]^2} f(x, y) = f(U, L) \quad \text{and} \quad \min_{(x,y)\in[L,U]^2} f(x, y) = f(L, U) \quad (41)$$

we have that

$$f(U, L) = f\left(\frac{B}{b}, \sqrt{\frac{bF}{Bc}}\right) = \frac{F + (B^2/b)\sqrt{(1/B)F(b/c)}}{B\sqrt{(1/B)F(b/c)} + (1/B)Fb} = \frac{B}{b} = U, \quad (42)$$

and

$$\begin{aligned} & f(L, U) \ge L \Longleftrightarrow \\ & f\left(\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right) = \frac{F + (B^2/b)\sqrt{(1/B)F(b/c)}}{B\sqrt{(1/B)F(b/c)} + (B^2/b^2)c} \\ & \ge \sqrt{\frac{bF}{Bc}} \Longleftrightarrow \\ & F + \frac{B^2}{b}\sqrt{\frac{1}{B}F\frac{b}{c}} \ge B\left(\frac{bF}{Bc}\right) + \frac{B^2}{b^2}c\sqrt{\frac{bF}{Bc}} \Longleftrightarrow \\ & Fb^2c + bcB^2\sqrt{\frac{bF}{Bc}} \ge b^3F + B^2c^2\sqrt{\frac{bF}{Bc}} \Longleftrightarrow \\ & Fb^2(c - b) \ge (c - b)cB^2\sqrt{\frac{bF}{Bc}} \end{aligned} \quad (43)$$

which is true for $c \le b$ and $F < F_g$.

Also, since $F < (B^3/b^3)c = F$, we obtain

$$g\left(\sqrt{\frac{bF}{Bc}}\right)g\left(\frac{B}{b}\right) = -\frac{(b + c)(B^3c - Fb^3)\left(F - c\sqrt{(bF/Bc)^3}\right)}{b^3c} < 0. \quad (44)$$

This means that the equilibrium point $\overline{x}$ belongs to the invariant interval $[L, U]$.

Theorem 6. *If $F < F_g \le F_0$, then the equilibrium point $\overline{x}$ is globally asymptotically stable.*

Proof. The system of algebraic equations

$$f(m, M) = m, \quad f(M, m) = M, \quad (45)$$

is reduced to the system

$$F + BMm = m(cM^2 + bmM), \quad F + BMm = M(cm^2 + Mbm), \quad (46)$$

which yields

$$Mm(b - c)(M - m) = 0. \quad (47)$$

Since $c \ne b$, then it implies that $m = M = \overline{x}$. Now, by using Theorems 1 and 4, the conclusion follows. □

For some numerical values of parameters we give a visual evidence for Theorem 6 which indicates that in the case when $F < F_g < F_0$, the corresponding orbit converges very quickly (see Figure 2(a)), and in the case when $F < F_g = F_0$, the corresponding orbit converges significantly slower (see Figure 2(b)).

(b) $F_g < F < F_0$

Lemma 7. *If $F < F_d = 4B^3/(b + c)^2$, then the system of algebraic equations*

$$f(m, m) = M \quad \text{and} \quad f(M, M) = m, \quad (48)$$

has the unique solution $(m, M) = (\overline{x}, \overline{x})$

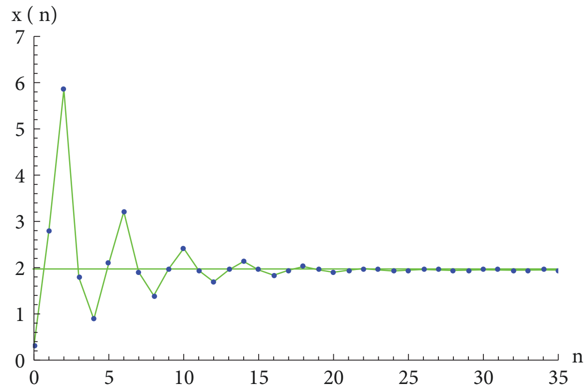
Proof. From (48) we have that

$$Bm^2 + F = Mm^2(b + c), \quad BM^2 + F = mM^2(b + c), \quad (49)$$

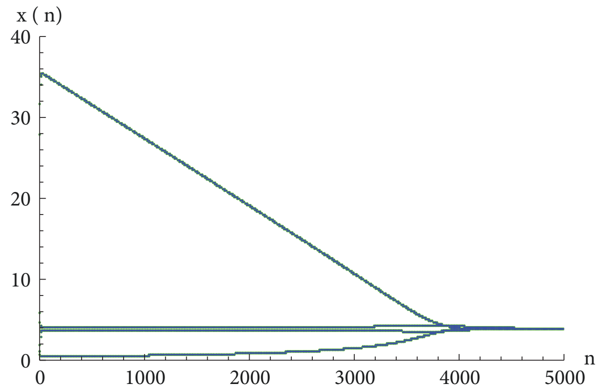
that is,

$$\begin{aligned} & M(bm^2 + cm^2) - Bm^2 - m(bM^2 + cM^2) + BM^2 = 0 \\ & (M - m)[B(m + M - (b + c)mM)] = 0, \end{aligned} \quad (50)$$

from which $m = M = \overline{x}$ or $B(m + M - (b + c)mM) = 0$.



(a) $F < F_g < F_0$



(b) $F < F_g = F_0$

Figure 2: The orbit for (a) $b = 2$, $c = 1$, $B = 4$, $F = 7$, $F_g = 8$, $F_0 = 128$, and $(x_0, x_{-1}) = (0.3, 2.8)$ and (b) $b = 0.5$, $c = 0.5$, $B = 2$, $F = 30$, $F_g = F_0 = 32$, and $(x_0, x_{-1}) = (1.1, 1)$ generated by Dynamica 4 [16].

If $B(m+M-(b+c)mM) = 0$, then $m = BM/(-B+M(b+c))$. Since $M = (Bm^2 + F)/m^2(b+c) > B/(b+c)$ (see the first equation of system (49)), then $m > 0$. After substituting m in the second equation of system (49), we get

$$BM^2 + F = M^2\left(b + c^2\right)\frac{BM}{-B + M(b+c)} \Longleftrightarrow$$
$$B^2M^2 + F(b+c)M + BF = 0, \tag{51}$$

from which we have that

$$M_{1,2} = \frac{1}{2B^2}\left(F(b+c) \pm \sqrt{F^2(b+c)^2 - 4B^3F}\right) > 0. \tag{52}$$

Straightforward calculation show that $m_1 = M_2$ and $m_2 = M_1$. Notice that the solution (m_2, M_2) is exactly the same as the solution (M_1, m_1), and that system (48) has a unique solution $m = M = \overline{x}$ if $F \leq 4B^3/(b+c)^2 = F_d$. □

Theorem 8. *If $F_g < F \leq F_d < F_0$, where $F_d = 4B^3/(b+c)^2$, then the equilibrium $\overline{x}$ is globally asymptotically stable.*

Proof. If $F_g < F < B^3/bc$, then

$$f:\left[\frac{B}{b},\sqrt{\frac{bF}{Bc}}\right]^2 \longrightarrow \left[\frac{B}{b},\sqrt{\frac{bF}{Bc}}\right], \tag{53}$$

which means that the interval $[L, U] = [B/b, \sqrt{bF/Bc}]$ is an invariant interval.

Indeed, since the function f is nonincreasing in both variables on the invariant interval, then

$$\max_{(x,y)\in[L,U]^2} f(x,y) = f(L,L)$$
$$\text{and} \min_{(x,y)\in[L,U]^2} f(x,y) = f(U,U), \tag{54}$$

and we obtain that

$$f(U,U) = f\left(\sqrt{\frac{bF}{Bc}},\sqrt{\frac{bF}{Bc}}\right)$$
$$= \frac{F(b/c) + F}{(1/B)F(b^2/c) + (1/B)Fb} = \frac{B}{b} = L, \tag{55}$$

and

$$f(L,L) \leq U \Longleftrightarrow$$
$$f\left(\frac{B}{b},\frac{B}{b}\right) = \frac{B^3 + Fb^2}{B^2(b+c)} \leq \sqrt{\frac{bF}{Bc}} \Longleftrightarrow$$
$$\left(\frac{B^3 + Fb^2}{B^2(b+c)}\right)^2 - \frac{bF}{Bc} < 0 \Longleftrightarrow \tag{56}$$
$$\left(B^3c - Fb^3\right)\left(B^3 - Fbc\right) < 0 \Longleftrightarrow$$
$$F_g = \frac{B^3}{b^3}c < F < \frac{B^3}{bc}.$$

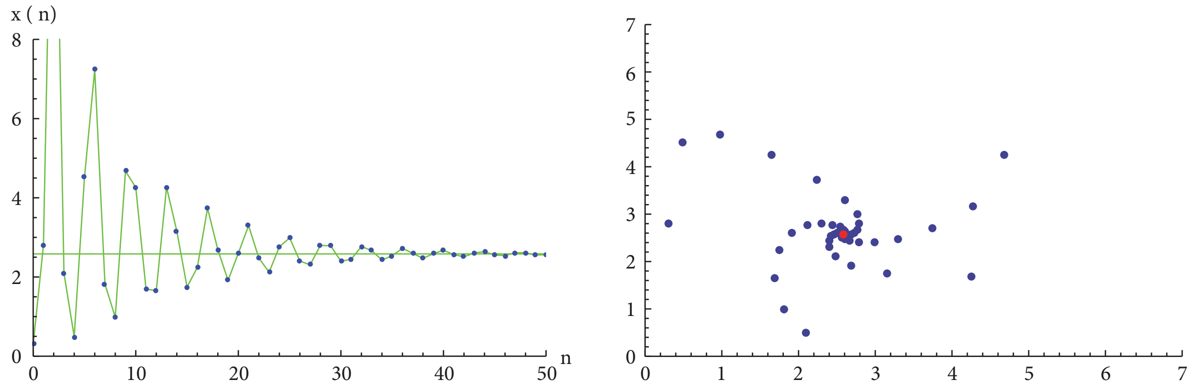
Hence,

$$f\left(\frac{B}{b},\frac{B}{b}\right) = \frac{Fb^2 + B^3}{B^2(b+c)} \in \left[\frac{B}{b},\sqrt{\frac{bF}{Bc}}\right],$$
$$\text{if } c < b \text{ and } F_g = \frac{B^3}{b^3}c < F < \frac{B^3}{bc}. \tag{57}$$

The following calculation will show that $F_d < B^3/bc$. Indeed,

$$\frac{4B^3}{(b+c)^2} < \frac{B^3}{bc} \Longleftrightarrow$$
$$-B^3\frac{(b-c)^2}{bc(b+c)^2} < 0, \tag{58}$$

which is true.



$Fg < F \leq F_d < F_0$

Figure 3: The orbit and the phase portrait for $b = 2$, $c = 1$, $B = 4$, $F = 28$, $F_g = 8$, $F_0 = 128$, and $(x_0, x_{-1}) = (0.3, 2.8)$ generated by Dynamica 4 [16].

Also, since $g(\sqrt{bF/Bc})g(B/b) < 0$, it means that the equilibrium point $\overline{x}$ belongs to the invariant interval $[L, U]$.

Now, by using Lemma 7, Theorems 2 and 4, we get the conclusion that the equilibrium $\overline{x}$ is globally asymptotically stable. □

For some numerical values of parameters we give a visual evidence for Theorem 8. See Figure 3.

Lemma 9. *Assume that $F \neq F_g$.*

(i) *If $x_{n-1} < \sqrt{bF/Bc}$, then $x_{n+1} > B/b$.*

(ii) *If $x_{n-1} > \sqrt{bF/Bc}$, then $x_{n+1} < B/b$.*

(iii) *If $x_{n-1} = \sqrt{bF/Bc}$, then $x_{n+1} = B/b$.*

Proof. Since the map associated with the right-hand side of Equation (1) is always decreasing in the second variable, we have that

$$x_{n-1} \begin{Bmatrix} < \\ > \\ = \end{Bmatrix} \sqrt{\frac{bF}{Bc}} \Longrightarrow$$

$$x_{n+1} = \frac{Bx_n x_{n-1} + F}{bx_n x_{n-1} + cx_{n-1}^2} \begin{Bmatrix} > \\ < \\ = \end{Bmatrix} \frac{Bx_n\sqrt{bF/Bc} + F}{bx_n\sqrt{bF/Bc} + bF/B} = \frac{B}{b}. \quad (59)$$

□

Note that under assumption of Lemma 9, the following inequality holds:

$$\min\left\{\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right\} < \overline{x} < \max\left\{\sqrt{\frac{bF}{Bc}}, \frac{B}{b}\right\}. \quad (60)$$

(c) $F = F_g < F_0$. By substituting parameter $F = F_g = (B/b)^3 c = E^3 c$, where $\overline{x} = E = B/b$, in Equation (1), we obtain

$$\begin{aligned} x_{n+1} &= \frac{Bx_n x_{n-1} + \left(B^3/b^3\right)c}{bx_n x_{n-1} + cx_{n-1}^2} \\ &= \frac{(B/b)\,x_n x_{n-1} + (B/b)^3\,(c/b)}{x_n x_{n-1} + (c/b)\,x_{n-1}^2} \\ &= \frac{Ex_n x_{n-1} + E^3\,(c/b)}{x_n x_{n-1} + (c/b)\,x_{n-1}^2}. \end{aligned} \quad (61)$$

Lemma 10. *(i) Assume that $F = F_g < F_0$, i.e., $c < b$. Then Equation (61) does not possess a minimal period-four solution.*

(ii) Assume that $F = F_g = F_0$, i.e., $c = b$. Then Equation (61) has the minimal period-four solutions of the form

$$\ldots, E, \frac{E^2}{t}, E, t, E, \frac{E^2}{t}, E, t, \ldots \quad (62)$$

where $E = \overline{x} = (B/b)(= B/c)$ and $t > 0$ is an arbitrary number depending on initial conditions x_0 and x_{-1}.

Proof. Suppose that Equation (61) has a minimal period-four solution $\ldots x, y, z, t, x, y, z, t, \ldots$; then it holds

$$\begin{aligned} x &= \frac{Etz + E^3\,(c/b)}{tz + (c/b)\,z^2}, \\ y &= \frac{Ext + E^3\,(c/b)}{xt + (c/b)\,t^2}, \\ z &= \frac{Exy + E^3\,(c/b)}{xy + (c/b)\,x^2}, \\ t &= \frac{Eyz + E^3\,(c/b)}{yz + (c/b)\,y^2}, \end{aligned} \quad (63)$$

where $E = \overline{x}$. By eliminating x and y we obtain

$$\begin{aligned} z\,(z - E)\,W\,(t, z) &= 0, \\ t\,(t - E)\,U\,(t, z) &= 0, \end{aligned} \quad (64)$$

where the functions $W(t,z)$ and $U(t,z)$ can be written in the polynomial form as

$$W(t,z) = \beta_4 z^4 + \beta_3 z^3 + \beta_2 z^2 + \beta_1 z + \beta_0, \quad (65)$$

$$U(t,z) = \alpha_5 z^5 + \alpha_4 z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0, \quad (66)$$

where

$$\begin{aligned}
\beta_4 &= c^3 tE\left(b^2E - c^2t\right),\\
\beta_3 &= c\left(bc^3E^4 - c^4t^2E^2 + b^2c^2t^4 - b^2c^2t^2E^2\right.\\
&\quad \left. + 2b^3ct^2E^2 - 3bc^3t^3E + b^4t^3E\right),\\
\beta_2 &= b^5t^4E - c^5t^2E^3 + b^3c^2t^5 - bc^4t^3E^2 + 2b^2c^3tE^4\\
&\quad - 2b^2c^3t^4E + b^4ct^3E^2 - b^2c^3t^2E^3 + 2b^3c^2t^2E^3\\
&\quad - 2b^3c^2t^3E^2 - bc^4tE^4 + b^4ct^4E,\\
\beta_1 &= bctE^3\left(-c^3E^2 + 2b^3t^2 - c^3t^2 + bc^2E^2 + b^2ct^2\right.\\
&\quad \left. - 3bc^2tE + b^2ctE\right),\\
\beta_0 &= bc^2tE^5\left(b^2t - c^2E\right),\\
\alpha_5 &= bc^4tE > 0,\\
\alpha_4 &= c^2\left(b^2t^2(cE + bt) - c^3E^3 + \left(b^3 - c^3\right)tE^2\right.\\
&\quad \left. + \left(b^2 - c^2\right)ct^2E\right),\\
\alpha_3 &= b\left(bc^3E^4 + b^4t^3E - 2c^4tE^3 - 2c^4t^3E - 2c^4t^2E^2\right.\\
&\quad \left. + b^3ct^4 - bc^3t^2E^2 + b^2c^2t^3E + b^3ct^2E^2\right),\\
\alpha_2 &= bctE\left(-2c^3E^3 + b^2cE^3 + 2b^3tE^2 + b^3t^2E\right.\\
&\quad \left. - bc^2t^2E - bc^2t^3 - bc^2tE^2 - b^2ct^2E\right),\\
\alpha_1 &= b^2c^2tzE^4(bE + 2t(b - c)) > 0,\\
\alpha_0 &= b^2c^3tE^6 > 0.
\end{aligned} \quad (67)$$

Since $z \neq 0$ and $t \neq 0$, from system (64), we obtain the following four cases:

(1) The system

$$\begin{aligned} z - E &= 0,\\ t - E &= 0, \end{aligned} \quad (68)$$

implies $z = t = E$, and by using (63), we get $x = y = E$.

(2) The system

$$\begin{aligned} z - E &= 0,\\ U(t,z) &= 0, \end{aligned} \quad (69)$$

implies $z = E$ and

$$\begin{aligned} U(t,E) &= E^3(b - c)(b + c)\left(ct^2 + tE(b + c) + cE^2\right)\\ &\quad \cdot (cE + bt)^2 > 0 \end{aligned} \quad (70)$$

if $c < b$. If $b = c$, then $U(t,z) = 0$ is satisfied for every $t > 0$, and by using system (63), it follows that the periodic solution of the minimal period four is of the form (62).

(3) The system

$$\begin{aligned} W(t,z) &= 0,\\ t - E &= 0, \end{aligned} \quad (71)$$

implies $t = E$ and

$$\begin{aligned} W(E,z) &= E^3(b - c)(b + c)\left(cz^2 + zE(b + c) + cE^2\right)\\ &\quad \cdot \left(c^2z^2 + bzE(b + c) + bcE^2\right) > 0, \end{aligned} \quad (72)$$

so the conclusion is the same as in the previous case.

(4) The system

$$\begin{aligned} W(t,z) &= 0,\\ U(t,z) &= 0, \end{aligned} \quad (73)$$

demands more detailed analysis.

(a) Assume that $b > c$. Then we can write $b = c + \xi$, $\xi > 0$. Consider the polynomials $W(t,z)$ and $U(t,z)$ as polynomials in one variable t:

$$\begin{aligned} W(t,z) &= a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0,\\ U(t,z) &= b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0, \end{aligned} \quad (74)$$

where

$$\begin{aligned}
a_5 &= c^2z^2(c + \xi)^3,\\
a_4 &= z^2(c + \xi)^2\left(c^3z + \xi^3E + 4c\xi^2E + 5c^2\xi E\right),\\
a_3 &= 2c^5zE\left(E^2 - zE - z^2\right) + c^4z\xi E\left(10E^2 - 3zE\right.\\
&\quad \left. + z^2\right) + 3c^3z\xi^2E\left(5E^2 + 2z^2\right) + c^2z\xi^3E\left(2zE\right.\\
&\quad \left. + 9E^2 + 4z^2\right) + cz\xi^4E\left(zE + 2E^2 + z^2\right),\\
a_2 &= c^5E\left(E^4 - 2zE^3 - z^4\right) + c^4\xi E^2\left(3E^3 - 3zE^2\right.\\
&\quad \left. + 4z^2E + 4z^3\right) + c^3\xi^2E^2\left(3E^3 + 5z^2E + 5z^3\right)\\
&\quad + c^2\xi^3E^2(z + E)\left(E^2 + 2z^2\right),\\
a_1 &= c^3E^2(c + \xi)\left(c\left(z^4 - E^4\right) + z^4\xi + cz^2E^2 + 2z^2\xi E^2\right.\\
&\quad \left. + z\xi E^3\right),\\
a_0 &= c^4z^3E^4(c + \xi),
\end{aligned}$$

$$b_4 = cz^2 (c+\xi)^2 \left(z (c+\xi)^2 - c^2 E\right),$$
$$b_3 = z^2 E (c+\xi)^3 \left(z (c+\xi)^2 - c^2 E\right) + cz^2 \xi E^2 (2c+\xi) \cdot (c+\xi)^2 + c^2 z^3 (c+\xi) \left(\xi^2 E + 2c\xi E + z (c+\xi)^2 - c^2 E\right),$$
$$b_2 = c^5 z^2 E (E-z)^2 + 2c^4 zE\xi \left(E^3 + 3zE^2 + 2z^3\right) + c^3 zE\xi^2 \left(4E^3 + 11zE^2 + 5z^2 E + 2z^3\right) + 2c^2 zE^2 \xi^3 \left(4zE + E^2 + 2z^2\right) + cz^2 E^2 \xi^4 (z + 2E),$$
$$b_1 = c^5 E (E-z)(z+E) \left(E^3 + z \left(E^2 - z^2\right)\right) + c^4 \xi E \left(z^2 E^3 + 2 \left(E^3 - z^3\right) E^2 + 3zE^4 + 3z^4 E + z^5\right) + c^3 \xi^2 E^2 \left(3z^2 E^2 + E^4 + 3zE^3 + 3z^4\right) + c^2 z \xi^3 E^2 \left(E^3 + zE^2 + z^3\right),$$
$$b_0 = c^3 z^3 E^3 \left(E (c+\xi)^2 - c^2 z\right). \tag{75}$$

If $z \in [c^2 E/(c+\xi)^2, E]$, then $b_i \geqslant 0$, for $i = 0, 1, 4$ and $b_2 > 0, b_3 > 0$, so we have that

$$U(t,z) > 0 \quad \text{for } z \in \left[\frac{c^2 E}{(c+\xi)^2}, E\right]. \tag{76}$$

Since $W(t,z)$ and $U(t,z)$ are polynomials of the fifth and fourth degrees, respectively, the resultant of these polynomials is the determinant of the ninth degree:

$$res_t(W,U) = \det \begin{bmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 \\ 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}, \tag{77}$$

from which we obtain

$$res_t(W,U) = (z-E)^4 \Upsilon_1(z) \Upsilon_2(z) \Upsilon_3(z) \Upsilon_4(z) \cdot \Phi(z, E, c+\xi, c), \tag{78}$$

where

$$\Upsilon_1(z) = \xi z^2 + \xi E z - cE^2,$$
$$\Upsilon_2(z) = \left(c^2 z^2 + \xi b E z - (c+\xi) cE^2\right)^2,$$
$$\Upsilon_3(z) = (c+\xi)^9 c^{17} z^9 E^{10} (z+E)^6 \left((2c+\xi) z^2 + c (z+E) E\right) > 0,$$
$$\Upsilon_4(z) = c^2 z^2 + z (2c+\xi) bE + (c+\xi) cE^2 > 0,$$
$$\Phi(z, E, c+\xi, c) = c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0,$$
$$c_4 = 8c^2 \left(2c\xi + \xi^2 + 2c^2\right)(c+\xi)^7,$$
$$c_3 = -2E \left(2c\xi + \xi^2 + 2c^2\right) \left(-55c^2 \xi^4 - 50c^3 \xi^3 + 2c^4 \xi^2 - 4\xi^6 - 24c\xi^5 + 32c^5 \xi + 16c^6\right)(c+\xi)^3,$$
$$c_2 = 96c^{11} E^2 + 512c^{10} \xi E^2 + 1208c^9 \xi^2 E^2 + 1512c^8 \xi^3 E^2 + 768c^7 \xi^4 E^2 - 644c^6 \xi^5 E^2 - 1600c^5 \xi^6 E^2 - 1542c^4 \xi^7 E^2 - 903c^3 \xi^8 E^2 - 338c^2 \xi^9 E^2 - 76c\xi^{10} E^2 - 8\xi^{11} E^2,$$
$$c_1 = -2cE^3 \left(2c\xi + \xi^2 + 2c^2\right) \left(31c^2 \xi^4 + 48c^3 \xi^3 + 62c^4 \xi^2 + 4\xi^6 + 16c\xi^5 + 48c^5 \xi + 16c^6\right)(c+\xi)^2,$$
$$c_0 = 16c^3 E^4 (c+\xi)^8. \tag{79}$$

If the equation $res_t(W,U) = 0$ has solutions for variable z, then they are the common roots of both equations in system (73) for a fixed value of t. One of these positive roots is $z_1 = E$, but for $z = E$ and $t > 0$ system (73) has no solutions since $U(t,E) > 0$, see (76). Therefore, in this case, Equation (61) has no minimal period-four solution.

The positive solution of the equation $\Upsilon_1(z) = 0$ is

$$z_2 = \frac{E\left(-\xi + \sqrt{\xi(\xi + 4c)}\right)}{2\xi}. \tag{80}$$

We will show later that z_2 can not be a component of any positive solutions of system (73).

The positive solution of the equation $\Upsilon_2(z) = 0$ is

$$z_3 = \frac{E\left(-(c+\xi)\xi + \sqrt{(2c+\xi)(c+\xi)\left(\xi^2 - c\xi + 2c^2\right)}\right)}{2c^2}, \tag{81}$$

and $z_3 \in (c^2 E/(c+\xi)^2, E)$. Namely,

$$z_3 < E \Longleftrightarrow 4c^2 \xi^2 > 0, \tag{82}$$

which is true, and

$$\frac{c^2}{(c+\xi)^2} E < z_3 \Longleftrightarrow$$
$$\frac{c^2 E}{(c+\xi)^2}$$

$$-\frac{E\left(-(c+\xi)\xi+\sqrt{(2c+\xi)(c+\xi)(\xi^2-c\xi+2c^2)}\right)}{2c^2} < 0 \Longleftrightarrow -4c^3\xi\left(4c\xi^3+7c^3\xi+7c^2\xi^2+4c^4+\xi^4\right)<0, \quad (83)$$

which is also true. So, system (73) has no solutions since $U(t,E)>0$, see (76).

Now, we prove that the eventually positive roots of the equation $\Phi(z,E,c+\xi,c)=0$ for variable z can belong only to the interval $(c^2E/(c+\xi)^2,E)$ and that every $z\in(c^2E/(c+\xi)^2,E)$ can not be a component of any positive solution of system (73). First, we prove that $\Phi(z,E,c+\xi,c)>0$ for $z>E$. Let $z=E+\gamma$, $\gamma>0$; then we obtain

$$\begin{aligned}\Phi(E+\gamma,E,c+\xi,c) &= 16c^{11}\gamma^4+128c^{10}\gamma^2\xi\left(\gamma^2+E^2+\gamma E\right)+8c^9\xi^2\left(57\gamma^4+16E^4+142\gamma^2E^2+48\gamma E^3+111\gamma^3E\right)\\&+8c^8\xi^3\left(119\gamma^4+76E^4+522\gamma^2E^2+256\gamma E^3+349\gamma^3E\right)+8c^7\xi^4\left(161\gamma^4+162E^4+1101\gamma^2E^2+627\gamma E^3+657\gamma^3E\right)\\&+4c^6\xi^5\left(294\gamma^4+402E^4+3013\gamma^2E^2+1839\gamma E^3+1646\gamma^3E\right)+2c^5\xi^6\left(364\gamma^4+636E^4+5653\gamma^2E^2+3550\gamma E^3+2879\gamma^3E\right)\\&+2c^4\xi^7\left(148\gamma^4+330E^4+3708\gamma^2E^2+2333\gamma E^3+1789\gamma^3E\right)+c^3\xi^8\left(72\gamma^4+221E^4+3375\gamma^2E^2+2082\gamma E^3+1570\gamma^3E\right)\\&+2c^2\xi^9(\gamma+E)\left(4\gamma^3+22E^3+281\gamma E^2+231\gamma^2E\right)+4c\xi^{10}E(\gamma+E)\left(22\gamma^2+E^2+25\gamma E\right)+8\gamma\xi^{11}E(\gamma+E)^2,\end{aligned} \quad (84)$$

i.e., $\Phi(E+\gamma,E,c+\xi,c)>0$. It means that the function $\Phi(z,E,c+\xi,c)$ eventually has the positive roots in the interval $(0,E]$. Since we already considered the case when $z=E$, now we investigate the existence of the positive roots of the equation $\Phi(z,E,b,c)=0$ for $0<z<E$. As we have seen, $U(t,z)>0$ for $c^2E/(c+\xi)^2<z<E$ and $t>0$, so system (73) has no solution in the case when the equation $\Phi(z,E,c+\xi,c)=0$ has the positive roots in the interval $(c^2E/(c+\xi)^2,E)$. This implies that Equation (61) has no minimal period-four solution whenever any root of equation $res_t(W,U)=0$ lies in the interval $(c^2E/(c+\xi)^2,E)$.

Now, we prove that the equation $\Phi(z,E,c+\xi,c)=0$ has no root for variable z if $z\in(0,c^2E/(c+\xi)^2)$ and $\xi>0$. It is easy to see the following:

$$\Phi_0=\Phi(0,E,c+\xi,c)=16c^3E^4(c+\xi)^8>0. \quad (85)$$

For $\Phi_1=\Phi(c^2E/(c+\xi)^2,E,c+\xi,c)$ we have

$$\Phi_1=\frac{c^3\xi^2E^4\left(72c\xi^7+184c^7\xi+278c^2\xi^6+618c^3\xi^5+877c^4\xi^4+820c^5\xi^3+500c^6\xi^2+32c^8+8\xi^8\right)(2c+\xi)^2}{(c+\xi)^4}>0, \quad (86)$$

and

$$\Phi_0-\Phi_1=\frac{c^3E^4\left(2c\xi+2c^2+\xi^2\right)\left(5c\xi^2+4c^2\xi+2c^3+2\xi^3\right)\left(26c\xi^6+36c^6\xi+76c^2\xi^5+135c^3\xi^4+154c^4\xi^3+104c^5\xi^2+4c^7+4\xi^7\right)}{(c+\xi)^4}>0. \quad (87)$$

Now, it is sufficient to prove $d\Phi(z,E,b,c)/dz<0$ in $(0,c^2E/(c+\xi)^2)$, $\xi>0$. Since

$$\frac{d\Phi(z,E,c+\xi,c)}{dz}=2\left((c+\xi)^2+c^2\right)P(z,E,c+\xi,c), \quad (88)$$

where

$$\begin{aligned}P(z,E,c+\xi,c) &= 16c^9(z-E)^3+16c^8\xi(z-E)\left(-8zE+5E^2+7z^2\right)+6c^7\xi^2\left(-29E^3+62zE^2-73z^2E+56z^3\right)\\&+4c^6\xi^3\left(-55E^3+70zE^2-51z^2E+140z^3\right)+c^5\xi^4\left(-189E^3-82zE^2+501z^2E+560z^3\right)\end{aligned}$$

$$+ c^4\xi^5 \left(-126E^3 - 380zE^2 + 1011z^2E + 336z^3\right)$$
$$+ c^3\xi^6 \left(-67E^3 - 379zE^2 + 873z^2E + 112z^3\right)$$
$$+ c^2\xi^7 \left(-24E^3 - 202zE^2 + 417z^2E + 16z^3\right)$$
$$+ 4c\xi^8 E \left(-15zE - E^2 + 27z^2\right)$$
$$+ 4z\xi^9 E (3z - 2E), \quad (89)$$

then for $z = c^2E/k(c+\xi)^2$, $\xi > 0$, $k > 1$ we obtain

$$P\left(\frac{c^2E}{k(c+\xi)^2}, E, c+\xi, c\right) = cE^3 \frac{\Omega(c,\xi,k)}{k^3(c+\xi)^2} < 0, \quad (90)$$

with

$$\Omega(c,\xi,k) = -16c^{10}(k-1)^3 - 16c^9\xi(k-1)\left(6k(k-1) + k^2 + 3\right) - 2c^8\xi^2\left(51k - 186k^2 + 175k^3 - 24\right) - 8c^7\xi^3\left((k-1)\left(20k + 55k^2 + 2\right) + 26k^3\right) - c^6\xi^4 k\left(82k + 488k^2 + 315\left(k^2 - 1\right)\right) - c^5\xi^5 k\left(380k + 487k^2 + 237\left(k^2 - 1\right)\right) - c^4\xi^6 k\left(379k + 424k^2 + 84\left(k^2 - 1\right)\right) - 2c^3\xi^7 k\left(101k + 136k^2 + 6\left(k^2 - 1\right)\right) - c^2\xi^8 k^2(119k + 60) - 8c\xi^9 k^2(4k + 1) - 4\xi^{10}k^3 < 0. \quad (91)$$

Similarly, now we will consider $W(t,z)$ and $U(t,z)$ as polynomials in the variable t (with the coefficients α_i, β_j, $i \in \{0,1,\dots,5\}$, $j \in \{0,1,\dots,4\}$). The resultant of these polynomials is

$$res_z(W,U) = (t-E)^2(t+E)^4((c+\xi)t - cE)^6 \Lambda_1(t) \cdot \Lambda_2(t)\Phi(t,E,c+\xi,c), \quad (92)$$

where

$$\Lambda_1(t) = (c+\xi)^4 c^{18} t^4 E^{15}\left((2c+\xi)t^2 + cE(t+E)\right) \cdot (cE + (c+\xi)t)^4 > 0,$$
$$\Lambda_2(t) = \xi t^2 + \xi Et - cE^2. \quad (93)$$

If the equation $res_z(W,U) = 0$ has solutions for variable t, then they are the common roots of both equations in system (73) for a fixed value of z. One of these positive roots is $t_1 = E$, and for $t = E$ and $z > 0$ system (73) has no solutions since

$$W(E,z,c+\xi,c) = \xi E^3(2c+\xi) \cdot \left(cz^2 + zE(2c+\xi) + cE^2\right) \cdot \left(c^2z^2 + (z(2c+\xi) + cE)(c+\xi)E\right) > 0 \quad (94)$$

for $\xi > 0$. It means that Equation (61) has no minimal period-four solution.

Similarly, for the second root $t_2 = cE/(c+\xi)$, $\xi > 0$ we obtain $\widetilde{U} = U(cE/(c+\xi), z, c+\xi, c)$,

$$\widetilde{U} = \frac{c^3E^2(z+E)^2\left(c^4(z+E)(z-E)^2 + c^3\xi\left(3E^3 + 3z^2E + 2z^3\right) + c^2\xi^2\left(3E^3 + 4zE^2 + 6z^2E + z^3\right) + c\xi^3E(2z+E)^2 + \xi^4 zE(z+E)\right)}{(c+\xi)^2} > 0, \quad (95)$$

where $\xi > 0$. It means that Equation (61) has no minimal period-four solution in this case. As we have already seen, the equation $\Phi(t,E,c+\xi,c) = 0$ has eventually positive roots only in the interval $t \in (c^2E/(c+\xi)^2, E)$, $\xi > 0$. Then,

$$\beta_4 = c^3tE\left((E-t)c^2 + \xi^2E + 2c\xi E\right) > 0,$$
$$\beta_3 = c^5(E-t)\left(E^3 - t^3 + tE^2 + t^2E\right) + c^4\xi\left(4t^2E^2 + E^4 + t^3E + 2t^4\right) + c^3\xi^2t^2(t+E)(t+5E) + 2c^2\xi^3t^2E(2t+E) + ct^3\xi^4E > 0,$$
$$\beta_2 = c^5t(t-E)^2(t+E)^2 + c^4t\xi\left(3E^2\left(E^2 - t^2\right) + 4tE^3 + 5t^3E + 3t^4\right) + c^3t\xi^2\left(2E^4 + 5tE^3 + 14t^3E + 3t^4\right) + c^2t^2\xi^3\left(2E^3 + 2tE^2 + 14t^2E + t^3\right) + ct^3\xi^4E(6t+E) + t^4\xi^5E > 0,$$
$$\beta_1 = ctE^3(c+\xi)\left(c\xi\left((c+\xi)t^2 + ct^2 + cE^2\right) + (c+\xi)t\left(cE\xi + 2\left((c+\xi)^2t - c^2E\right)\right)\right) > 0,$$
$$\beta_0 = c^2tE^5(c+\xi)\left((c+\xi)^2t - c^2E\right) > 0. \quad (96)$$

This implies $W(t,z) > 0$ for $c^2E/(c+\xi)^2 < t < E$ and $z > 0$, so system (73) has no solution in the case when the equation $\Phi(t,E,c+\xi,c) = 0$ has the positive roots in the interval $(c^2E/(c+\xi)^2, E)$, which further means that Equation (61) has no minimal period-four solution.

Also, the positive solution of the equation $\Lambda_2(t) = 0$ is

$$t_3 = \frac{E\left(-\xi + \sqrt{\xi(\xi+4c)}\right)}{2\xi}. \quad (97)$$

Note that $t_3 = z_2$. Now, we prove that (z_2, z_2) can not be solution of system (73). Indeed, suppose the opposite, i.e.,

$$\begin{aligned}
&U(z_2, z_2, c+\xi, c) = 0,\\
&W(z_2, z_2, c+\xi, c) = 0,\\
&\Updownarrow\\
&c^{10} + 2c^9\xi + 8c^8\xi^2 + 26c^7\xi^3 + 45c^6\xi^4 + 43c^5\xi^5\\
&\quad + 13c^4\xi^6 - 27c^3\xi^7 - 35c^2\xi^8 - 15c\xi^9 - 2\xi^{10} = 0, \quad (98)\\
&c^{10} - 2c^9\xi + 27c^8\xi^2 + 51c^7\xi^3 - 30c^6\xi^4 - 165c^5\xi^5\\
&\quad - 214c^4\xi^6 - 148c^3\xi^7 - 58c^2\xi^8 - 12c\xi^9 - \xi^{10} = 0,\\
&\Updownarrow\\
&(c = 0, \xi = 0),
\end{aligned}$$

which is a contradiction with the assumption that $c > 0$ and $\xi > 0$.

Consequently system (73) does not have positive solutions when $b > c$.

(b) Assume that $b = c$. Then, system (73)

$$\begin{aligned}
W(t,z,c,c) &= 0,\\
U(t,z,c,c) &= 0,
\end{aligned} \quad (99)$$

is of the form

$$\begin{aligned}
&(t-E)\left(t^4z^2 + t^3(z+E)z^2\right.\\
&\quad - t^2\left(z^3E + z^2E^2 - 2zE^3\right)\\
&\quad \left.- t\left(z^4E + z^3E^2 + z^2E^3 - E^5\right) - z^3E^3 = 0\right) = 0,\\
&(z-E)\left(t^4z^2 + t^3(z+E)z^2 - t^2\left(z^2E^2 - z^3E\right)\right.\\
&\quad \left.- t\left(-z^4E - z^3E^2 + z^2E^3 + 2zE^4 + E^5\right) - z^3E^3\right)\\
&\quad = 0,
\end{aligned} \quad (100)$$

and combining those equations, we have the following four cases:

(i)

$$\begin{aligned}
t - E &= 0,\\
z - E &= 0,
\end{aligned} \quad (101)$$

and the solution in this case is $t = z = E$,

(ii)

$$\begin{aligned}
&t - E = 0,\\
&t^4z^2 + t^3(z+E)z^2 - t^2\left(z^2E^2 - z^3E\right)\\
&\quad - t\left(-z^4E - z^3E^2 + z^2E^3 + 2zE^4 + E^5\right)\\
&\quad - z^3E^3 = 0,
\end{aligned} \quad (102)$$

and substituting t by E we obtain

$$E^2(z-E)(z+E)^3 = 0 \quad (103)$$

from which we get that the solution is $t = z = E$,

(iii)

$$\begin{aligned}
&t^4z^2 + t^3(z+E)z^2 - t^2\left(z^3E + z^2E^2 - 2zE^3\right)\\
&\quad - t\left(z^4E + z^3E^2 + z^2E^3 - E^5\right) - z^3E^3 = 0,\\
&z - E = 0,
\end{aligned} \quad (104)$$

i.e.,

$$E^2(t-E)(t+E)^3 = 0, \quad (105)$$

and the solution is $t = z = E$,

(iv)

$$\begin{aligned}
&t^4z^2 + t^3(z+E)z^2 - t^2\left(z^3E + z^2E^2 - 2zE^3\right)\\
&\quad - t\left(z^4E + z^3E^2 + z^2E^3 - E^5\right) - z^3E^3 = 0,\\
&t^4z^2 + t^3(z+E)z^2 - t^2\left(z^2E^2 - z^3E\right)\\
&\quad - t\left(-z^4E - z^3E^2 + z^2E^3 + 2zE^4 + E^5\right)\\
&\quad - z^3E^3 = 0.
\end{aligned} \quad (106)$$

By subtracting we get

$$-2tE(z-E)(z+E)\left(zE + E^2 + tz + z^2\right) = 0, \quad (107)$$

i.e.,

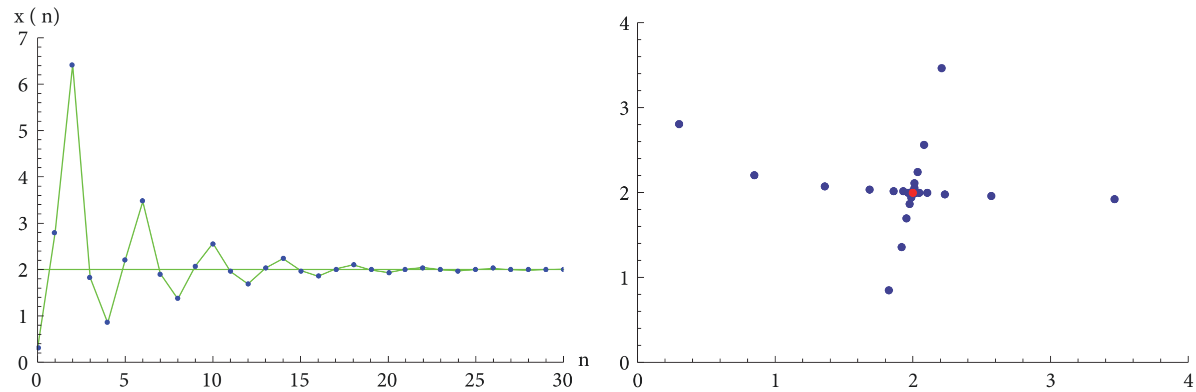
$$z = E. \quad (108)$$

Hence, the solution is $t = z = E$.

This means that (E, E) is a solution of system (73) and that Equation (61) does not possess a minimal period-four solution.

Thus, if $F = F_g < F_0$, then Equation (61) does not possess a minimal period-four solution. Consequently if $F = F_g = F_0$, then Equation (61) has the minimal period-four solutions of the form (62). □

Theorem 11. *Assume that $F = F_g = (B/b)^3c < F_0$. Then, the unique equilibrium point $\overline{x} = B/b$ of Equation (61) is globally asymptotically stable. Also, every solution of Equation*



$F = F_g < F_0$

FIGURE 4: The orbit and the phase portrait for $b = 2$, $c = 1$, $B = 4$, $F = 8$, $F_g = 8$, $F_0 = 128$, and $(x_0, x_{-1}) = (0.3, 2.8)$ generated by Dynamica 4 [16].

(61) oscillates about the equilibrium point $\overline{x}$ with semicycles of length two.

Proof. Notice that

$$x_{n+1} - \frac{B}{b} = \frac{Bc\left(B + bx_{n-1}\right)}{b^2\left(bx_n + cx_{n-1}\right)x_{n-1}}\left(\frac{B}{b} - x_{n-1}\right), \tag{109}$$

i.e., x_{n+1} and x_{n-1} are from the different sides of the equilibrium point (see also Lemma 9, when $\sqrt{bF/Bc} = B/b$). Also, that means x_{n+1} and x_{n+5} are always from the same side of the equilibrium point $\overline{x} = B/b$. Since

$$\begin{aligned} x_{n+4} - x_n &= \frac{Bx_{n+3}x_{n+2} + \left(B^3/b^3\right)c}{bx_{n+3}x_{n+2} + c\left(x_{n+2}\right)^2} - x_n \\ &= \frac{H}{b^3\left(bx_{n+3}x_{n+2} + c\left(x_{n+2}\right)^2\right)}, \end{aligned} \tag{110}$$

where $H = B(b^3x_{n+3}x_{n+2} + B^2c) - b^3(bx_{n+3}x_{n+2} + c(x_{n+2})^2)x_n$ is a linear function in variable x_n, it can be seen that $H = 0 \Longleftrightarrow x_{n+4} = x_n = B/b$ because Equation (61) has no period-two solutions nor period-four solutions (and it holds that $x_n = B/b \Longrightarrow x_{n+2} = B/b \Longrightarrow x_{n+4} = B/b$, see Lemma 9). Also,

$$\begin{aligned} & x_n > \frac{B}{b} \Longrightarrow \\ & H < 0 \Longrightarrow \\ & x_n > x_{n+4} > \frac{B}{b}, \\ & \qquad n \in \mathbb{N}, \\ & x_n < \frac{B}{b} \Longrightarrow \\ & H > 0 \Longrightarrow \\ & x_n < x_{n+4} < \frac{B}{b}, \\ & \qquad n \in \mathbb{N}, \end{aligned} \tag{111}$$

which means that every sequence $\{x_{4k}\}_{k=1}^{\infty}$, $\{x_{4k+1}\}_{k=0}^{\infty}$, $\{x_{4k+2}\}_{k=0}^{\infty}$, $\{x_{4k+3}\}_{k=0}^{\infty}$ is monotone and bounded. That implies that each of the sequences is convergent. Since, by Lemmas 5 and 10, Equation (61) has neither minimal period-two nor period-four solutions, it holds

$$\begin{aligned} \lim_{k\longrightarrow\infty} x_{4k} &= \lim_{k\longrightarrow\infty} x_{4k+1} = \lim_{k\longrightarrow\infty} x_{4k+2} = \lim_{k\longrightarrow\infty} x_{4k+3} \\ &= \overline{x}, \end{aligned} \tag{112}$$

which implies that equilibrium $\overline{x}$ is an attractor and by using Theorem 4, which completes the proof of the theorem. □

For some numerical values of parameters we give a visual evidence for Theorem 11. See Figure 4.

Remark 12. One can see from Theorems 6, 8, and 11 that the equilibrium point $\overline{x}$ is globally asymptotically stable for all values of parameter F such that $0 < F \leq F_d$, where $F_g < F_0$, i.e., $c < b$ (see Figure 5(a)) and for all values of parameter F such that $0 < F < F_g = F_0 = F_d$, i.e., $c = b$ (see Figure 5(b)).

(d) $F = F_g = F_0$. Since $F_g = F_0$ implies $c = b$, Equation (61) is of the form

$$x_{n+1} = \frac{B}{b^3}\frac{b^2x_nx_{n-1} + B^2}{x_n\left(x_n + x_{n-1}\right)}. \tag{113}$$

In this case, by using Lemma 10, we see that Equation (61) has minimal period-four solutions of the form (62). Based on our many numerical simulations and the proof of Theorem 11, we believe that the following conjectures are true.

Conjecture 13. *If $F = F_g = F_0$ (that is $b = c$), then every solution of Equation (61) converges to some period-four solution of the form (62) or to the equilibrium point $\overline{x}$.*

For some numerical values of parameters we give a visual evidence for this case. See Figures 6 and 7.

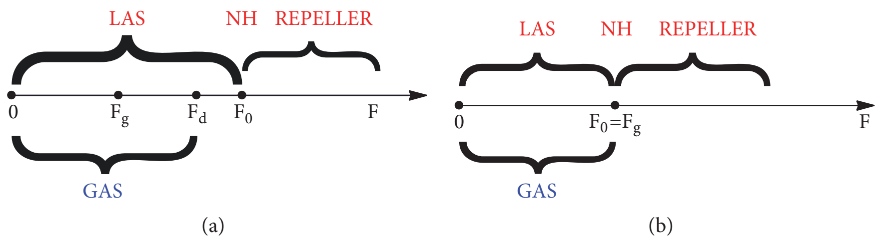


FIGURE 5: Visual representation of local and global asymptotic stability of Equation (1) when (a) $F_g < F_d < F_0$, i.e., $c < b$, and (b) $F_0 = F_g$, i.e., $c = b$.

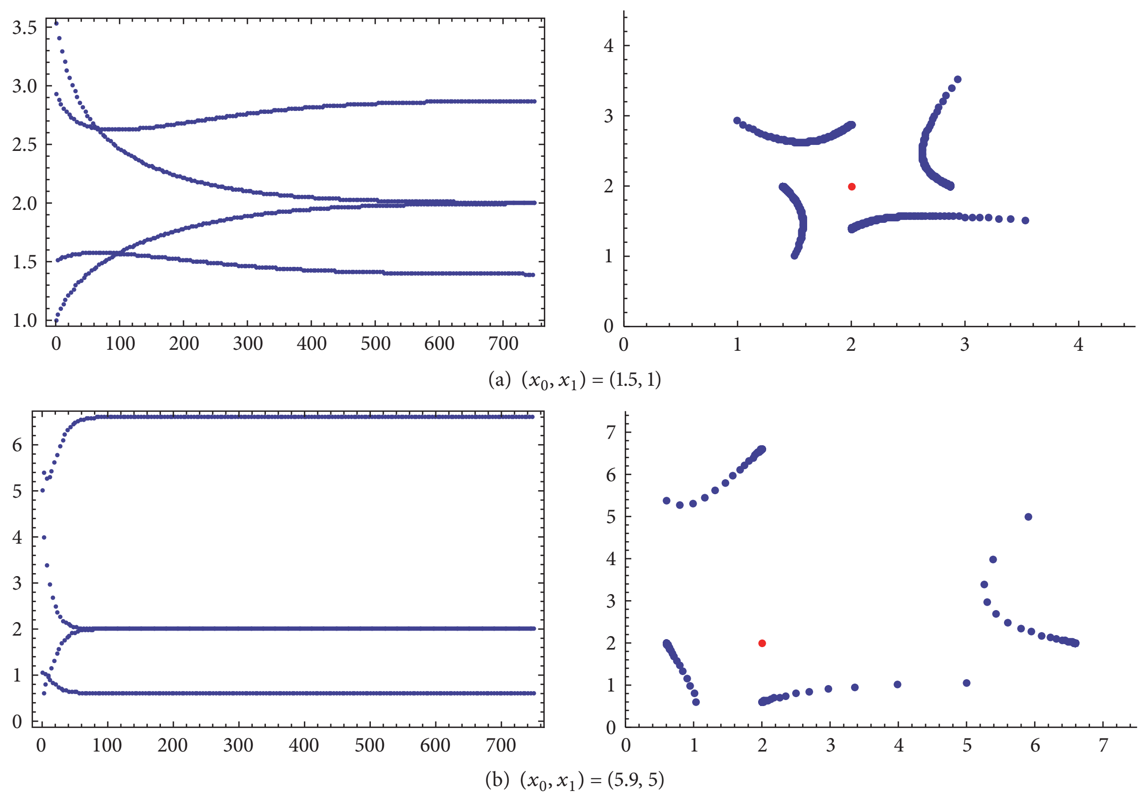


FIGURE 6: The orbit and phase portraits for $b = 1$, $c = 1$, $B = 2$, and $F = F_g = F_0 = 8$ generated by Dynamica 4 [16].

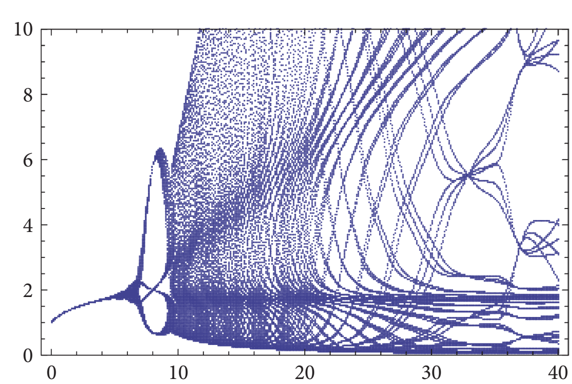


FIGURE 7: Bifurcation diagram in (F, x) plane for $b = 1$, $c = 1$, and $B = 2$, generated by Dynamica 4 [16].

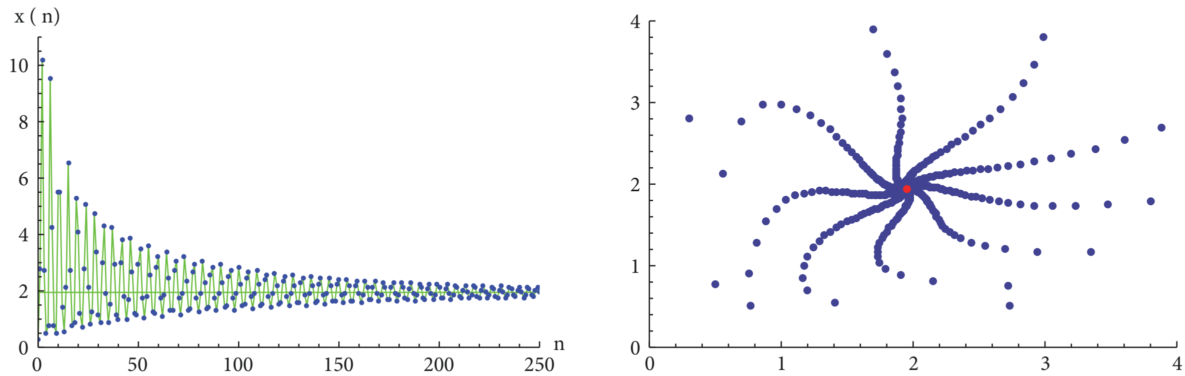


FIGURE 8: The orbit and phase portrait for $c = 2, b = 1, B = 4, F = 7, F_0 = 8, F_g = 128$, and $(x_0, x_{-1}) = (0.3, 2.8)$ generated by *Dynamica* 4 [16].

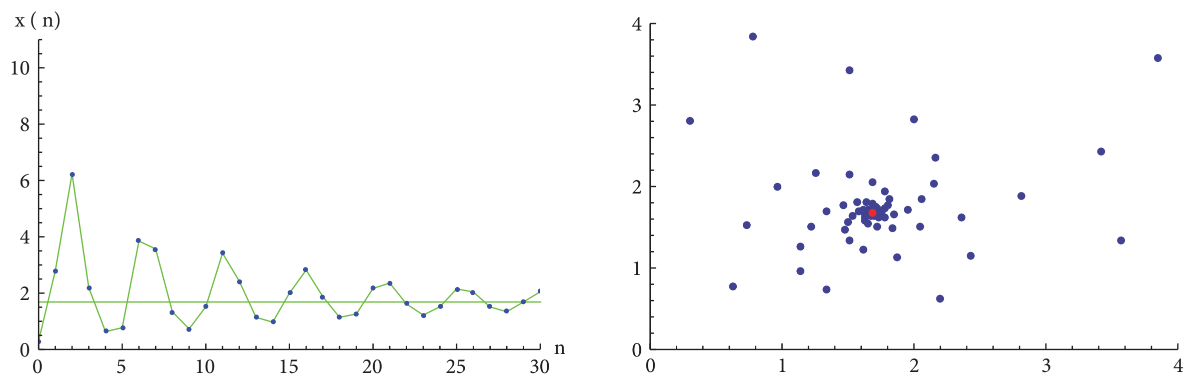


FIGURE 9: The orbit and phase portrait for $c = 2, b = 1, B = 4, F = 3, F_0 = 8, F_g = 128$, and $(x_0, x_{-1}) = (0.3, 2.8)$ generated by *Dynamica* 4 [16].

Case 2 ($F < F_0 < F_g$). We give a visual evidence for some numerical values of parameters which indicates very interesting behaviour and verifies our suspicion that the equilibrium point $\overline{x}$ is globally asymptotically stable in this case also. See Figures 8 and 9.

Conjecture 14. *If $F < F_0 < F_g$ (that is, $b < c$), then the equilibrium point $\overline{x}$ of Equation (1) is globally asymptotically stable.*

4. Naimark-Sacker Bifurcation for $b \neq c$

In this section, we consider bifurcation of a fixed point of map associated with Equation (1) in the case where the eigenvalues are complex conjugates and of unit module. We use the following standard version of the Naimark-Sacker result, see [28, 29]

Theorem 15 (Naimark-Sacker or Poincare-Andronov-Hopf Bifurcation for maps). *Let*

$$F : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \qquad (\lambda, x) \longrightarrow F(\lambda, x) \tag{114}$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) *$F(\lambda, 0) = 0$ for λ near some fixed λ_0;*

(ii) *$DF(\lambda, 0)$ has two nonreal eigenvalues $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) *$(d/d\lambda)|\mu(\lambda)| = d(\lambda_0) \neq 0$ at $\lambda = \lambda_0$;*

(iv) *$\mu^k(\lambda_0) \neq 1$ for $k = 1, 2, 3, 4$.*

Then there is a smooth λ-dependent change of coordinate bringing F into the form

$$F(\lambda, x) = \mathcal{F}(\lambda, x) + O\left(\|x\|^5\right) \tag{115}$$

and there are smooth function $a(\lambda), b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\mathcal{F} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} |\mu(\lambda)| r - a(\lambda) r^3 \\ \theta + \omega(\lambda) + b(\lambda) r^2 \end{pmatrix}. \tag{116}$$

If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant

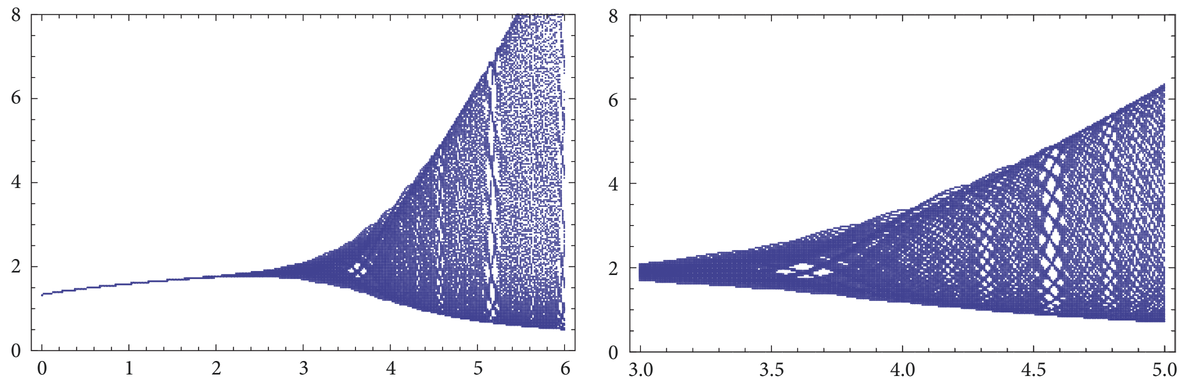


FIGURE 10: Bifurcation diagrams in (F, x) plane for $b = 0.5$, $c = 1$, and $B = 2$, generated by Dynamica 4 [16].

C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\overline{\mu(\lambda)} = \alpha(\lambda) - i\beta(\lambda)$ satisfying $(\alpha(\lambda))^2 + (\beta(\lambda))^2 = 1$ and $\beta(\lambda) \neq 0$. By putting the linear part of such a map into Jordan Canonical form, we may assume F to have the following form near the origin:

$$F(\lambda, x) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}. \tag{117}$$

Then the coefficient $a(\lambda_0)$ of the cubic term in Equation (116) in polar coordinate is equal to

$$a(\lambda_0) = \operatorname{Re}\left(\frac{(1 - 2\mu(\lambda_0)) \overline{\mu(\lambda_0)}^2}{1 - \mu(\lambda_0)} \xi_{11} \xi_{20} \right) + \frac{1}{2} |\xi_{11}|^2 + |\xi_{02}|^2 - \operatorname{Re}\left(\overline{\mu(\lambda_0)} \xi_{21} \right), \tag{118}$$

where

$$\xi_{20} = \frac{1}{8} \left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} - \frac{\partial^2 g_1(0,0)}{\partial x_2^2} + 2 \frac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i \left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} - \frac{\partial^2 g_2(0,0)}{\partial x_2^2} - 2 \frac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2} \right) \right), \tag{119}$$

$$\xi_{11} = \frac{1}{4} \left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} + \frac{\partial^2 g_1(0,0)}{\partial x_2^2} + i \left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} + \frac{\partial^2 g_2(0,0)}{\partial x_2^2} \right) \right), \tag{120}$$

$$\xi_{02} = \frac{1}{8} \left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} - \frac{\partial^2 g_1(0,0)}{\partial x_2^2} - 2 \frac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i \left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} - \frac{\partial^2 g_2(0,0)}{\partial x_2^2} + 2 \frac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2} \right) \right), \tag{121}$$

and

$$\xi_{21} = \frac{1}{16} \left(\frac{\partial^3 g_1}{\partial x_1^3} + \frac{\partial^3 g_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 g_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 g_2}{\partial x_2^3} + i \left(\frac{\partial^3 g_2}{\partial x_1^3} + \frac{\partial^3 g_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 g_1}{\partial x_1^2 \partial x_2} - \frac{\partial^3 g_1}{\partial x_2^3} \right) \right). \tag{122}$$

Theorem 16. *Assume that $b, c, B > 0$, $F_0 = (B/c)^3 b$, and $\overline{x} = B/c$.*

(i) *If $k_2 c < b < c$ or $c < b < k_3 c$, where k_2 and k_3 are positive solutions of the equation $3k^3 - 9k^2 - 3k + 1 = 0$, then there is a neighborhood U of the equilibrium point $\overline{x}$ and $\rho > 0$ such that for $|F - F_0| < \rho$ and $x_0, x_{-1} \in U$ then ω-limit set of solution of Equation (1), with initial condition x_0, x_{-1} is the equilibrium point $\overline{x}$ if $F < F_0$ and belongs to a closed invariant C^1 curve $\Gamma(F_0)$ encircling the equilibrium point $\overline{x}$ if $F > F_0$. Furthermore, $\Gamma(F_0) = 0$.*

(ii) *If $0 < b < k_2 c$ or $k_3 c < b < +\infty$, then there is a neighborhood U of the equilibrium point $\overline{x}$ and a $\rho > 0$ such that for $|F - F_0| < \rho$ and $x_0, x_{-1} \in U$ then α-limit set of x_0, x_{-1} is the equilibrium point $\overline{x}$ if $F > F_0$ and belongs to a closed invariant C^1 curve $\Gamma(F_0)$ encircling the equilibrium point $\overline{x}$ if $F < F_0$. Furthermore, $\Gamma(F_0) = 0$.*

Proof. See Figures 10 and 11 for visual illustration. In order to apply Theorem 15 we make a change of variable

$$\begin{aligned} y_n &= x_n - \overline{x} \Longrightarrow \\ x_n &= y_n + \overline{x}, \\ y_{n+1} &= \frac{B(y_n + \overline{x})(y_{n-1} + \overline{x}) + F}{b(y_n + \overline{x})(y_{n-1} + \overline{x}) + c(y_{n-1} + \overline{x})^2} - \overline{x}. \end{aligned} \tag{123}$$

Set

$$u_n = y_{n-1} \quad \text{and } v_n = y_n \quad \text{for } n = 0, 1, \ldots, \tag{124}$$

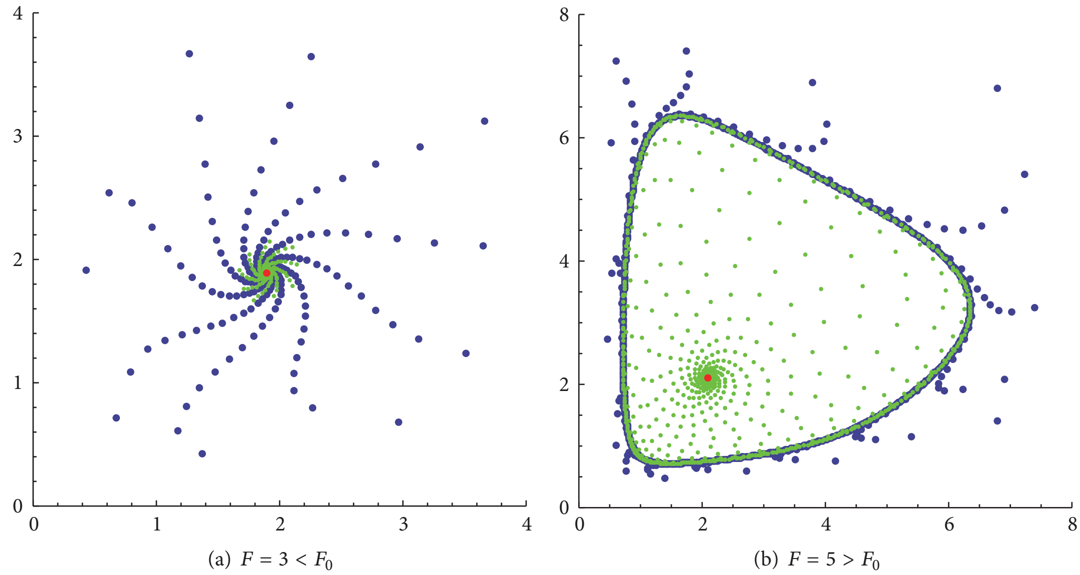


FIGURE 11: Phase portraits when $c = 1, b = 0.5, B = 2, x_{-1} = x_0 = 2.1$ (green), and $x_{-1} = x_0 = 6.8$ (blue), generated by *Dynamica* 4 [16].

then

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= \frac{B(v_n+\overline{x})(u_n+\overline{x})+F}{b(v_n+\overline{x})(u_n+\overline{x})+c(u_n+\overline{x})^2}-\overline{x}. \end{aligned} \quad (125)$$

Let us define the function

$$K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{B(v+\overline{x})(u+\overline{x})+F}{b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2}-\overline{x} \end{pmatrix}. \quad (126)$$

Then $K(u,v)$ has the unique fixed point $(0,0)$. The Jacobian matrix of $K(u,v)$ is given by

$$J_K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{bF(v+\overline{x})+c(u+\overline{x})(2F+B(u+\overline{x})(v+\overline{x}))}{(u+\overline{x})^2(c(u+\overline{x})+b(v+\overline{x}))^2} & \dfrac{-bF+Bc(u+\overline{x})^2}{(u+\overline{x})(c(u+\overline{x})+b(v+\overline{x}))^2} \end{pmatrix} \quad (127)$$

and its value at the zero equilibrium is

$$\begin{aligned} J_0 &= J_K\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\dfrac{bF+2cF+Bc\overline{x}^2}{\overline{x}^3(b+c)^2} & \dfrac{-bF+Bc\overline{x}^2}{\overline{x}^3(b+c)^2} \end{pmatrix}, \end{aligned} \quad (128)$$

i.e.,

$$\begin{aligned} J_0 &= J_K\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\dfrac{c}{b+c}-\dfrac{F}{\overline{x}^3(b+c)} & \dfrac{c}{b+c}-\dfrac{F}{\overline{x}^3(b+c)} \end{pmatrix}. \end{aligned} \quad (129)$$

The eigenvalues $\mu(F), \overline{\mu(F)}$, using (128), are

$$\mu(F) = \frac{-bF+Bc\overline{x}^2 \pm i\sqrt{4(b+c)^2\overline{x}^3\left(bF+2cF+Bc\overline{x}^2\right)-\left(bF-Bc\overline{x}^2\right)^2}}{2(b+c)^2\overline{x}^3} \quad (130)$$

because

$$\left(bF - Bc\overline{x}^2\right)^2 - 4(b+c)^2\overline{x}^3\left(bF + 2cF + Bc\overline{x}^2\right) = \left(bF - Bc\overline{x}^2\right)^2 - 4(b+c)\left(B\overline{x}^2 + F\right)\left(bF + 2cF + Bc\overline{x}^2\right) = -\left(8F^2c + 3F^2b^2 + 3B^2c^2x^4 + 12F^2bc + 4BFb^2x^2 + 12BFc^2x^2 + 4B^2bcx^4 + 18BFbcx^2\right)^2 < 0. \quad (131)$$

Then

$$K\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{c}{b+c} - \frac{F}{\overline{x}^3(b+c)} & \frac{c}{b+c} - \frac{F}{\overline{x}^3(b+c)}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}f_1(F,u,v)\\f_2(F,u,v)\end{pmatrix}, \quad (132)$$

where

$$f_1(F,u,v) = 0,$$
$$f_2(F,u,v) = \frac{B(v+\overline{x})(u+\overline{x}) + F}{b(v+\overline{x})(u+\overline{x}) + c(u+\overline{x})^2} + \frac{c\overline{x}^3 + F}{\overline{x}^3(b+c)}u + \frac{F - c\overline{x}^3}{\overline{x}^3(b+c)}v - \overline{x}. \quad (133)$$

Denote $F_0 = (B/c)^3 b$. For $F = F_0$ we have $\overline{x} = \sqrt[3]{F_0/b} = B/c$. The eigenvalues of J_0 are $\mu(F_0)$ and $\overline{\mu(F_0)}$ where

$$\mu(F_0) = \frac{c - b + i\sqrt{(b+3c)(3b+c)}}{2(b+c)} \quad \text{and} \quad |\mu(F_0)| = 1. \quad (134)$$

The eigenvectors corresponding to $\mu(F_0)$ and $\overline{\mu(F_0)}$ are $v(F_0)$ and $\overline{v(F_0)}$ where

$$v(F_0) = \left(\frac{c - b + i\sqrt{(b+3c)(3b+c)}}{2(b+c)}, 1\right). \quad (135)$$

Further,

$$|\mu(F_0)| = 1,$$
$$\mu^2(F_0) = -\frac{c^2 + 6bc + b^2}{2(b+c)^2} - i\frac{(c-b)\sqrt{(b+3c)(3b+c)}}{2(b+c)^2},$$
$$\mu^3(F_0) = \frac{(b-c)\left(b^2 + 4bc + c^2\right)}{(b+c)^3} + i\frac{2bc\sqrt{(b+3c)(3b+c)}}{(b+c)^3},$$
$$\mu^4(F_0) = -\frac{b^4 - 4bc^3 - 4b^3c - 26b^2c^2 + c^4}{2(b+c)^4} - i\frac{(b-c)\left(b^2 + 6bc + c^2\right)\sqrt{(b+3c)(3b+c)}}{2(b+c)^4}, \quad (136)$$

and $\mu^k(F_0) \neq 1$ for $k = 1,2,3,4$ for $c > 0$, $b > 0$, and $b \neq c$. For $F = F_0$ and $\overline{x} = B/c$

$$K\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\frac{b-c}{b+c}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}h_1(u,v)\\h_2(u,v)\end{pmatrix}, \quad (137)$$

and

$$h_1(u,v) = f_1(F_0,u,v) = 0,$$
$$h_2(u,v) = f_2(F_0,u,v) = \frac{B\left(B^2b + B^2c + Bc^2u + Bc^2v + c^3uv\right)}{c(B+cu)\left(c^2u + Bb + Bc + bcv\right)} + u + \frac{b-c}{b+c}v - \frac{B}{c}. \quad (138)$$

Hence, for $F = F_0$ system (125) is equivalent to

$$\begin{pmatrix}u_{n+1}\\v_{n+1}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\frac{b-c}{b+c}\end{pmatrix}\begin{pmatrix}u_n\\v_n\end{pmatrix} + \begin{pmatrix}h_1(u_n,v_n)\\h_2(u_n,v_n)\end{pmatrix}. \quad (139)$$

Let $\binom{u_n}{v_n} = P\binom{\xi_n}{\eta_n}$, where

$$P = \begin{pmatrix}\frac{c-b}{2(b+c)} & \frac{\sqrt{(b+3c)(3b+c)}}{2(b+c)}\\ 1 & 0\end{pmatrix},$$
$$P^{-1} = \begin{pmatrix}0 & 1\\ \frac{2(b+c)}{\sqrt{(b+3c)(3b+c)}} & -\frac{c-b}{\sqrt{(b+3c)(3b+c)}}\end{pmatrix}. \quad (140)$$

Then system (125) is equivalent to its normal form

$$\begin{pmatrix}\xi_{n+1}\\ \eta_{n+1}\end{pmatrix} = \begin{pmatrix}\frac{c-b}{2(b+c)} & -\frac{\sqrt{(b+3c)(3b+c)}}{2(b+c)}\\ \frac{\sqrt{(b+3c)(3b+c)}}{2(b+c)} & \frac{c-b}{2(b+c)}\end{pmatrix}\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix} + P^{-1}H\left(P\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}\right) \quad (141)$$

where

$$H\begin{pmatrix}u\\ v\end{pmatrix} := \begin{pmatrix}h_1(u,v)\\ h_2(u,v)\end{pmatrix}. \quad (142)$$

Let

$$G\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}g_1(u,v)\\ g_2(u,v)\end{pmatrix} = P^{-1}H\left(P\begin{pmatrix}u\\ v\end{pmatrix}\right). \quad (143)$$

By the straightforward calculation we obtain that

$$\begin{aligned} g_1(u,v) = &-\frac{B}{c} + \frac{b-c}{b+c}u \\ &+ \frac{(c-b)u + \sqrt{(b+3c)(3b+c)}v}{2(b+c)} \\ &+ \Omega(u,v), \end{aligned} \quad (144)$$

where

$$\begin{aligned} \Omega(u,v) &= \frac{2B(b+c)\kappa_1}{c\kappa_2\kappa_3}, \\ \kappa_1 &= 2B^2b^2 + 2B^2c^2 + c^4u^2 + 4B^2bc + 3Bc^3u \\ &\quad - bc^3u^2 + Bc^2v\sqrt{(b+3c)(3b+c)} \\ &\quad + c^3uv\sqrt{(b+3c)(3b+c)} + Bbc^2u, \\ \kappa_2 &= c^2u + 2Bb + 2Bc + cv\sqrt{(b+3c)(3b+c)} \\ &\quad - bcu, \\ \kappa_3 &= 2Bb^2 + 2Bc^2 + c^3u \\ &\quad + c^2v\sqrt{(b+3c)(3b+c)} + bc^2u \\ &\quad + 2b^2cu + 4Bbc, \end{aligned} \quad (145)$$

and

$$g_2(u,v) = \frac{b-c}{\sqrt{(b+3c)(3b+c)}}g_1(u,v). \quad (146)$$

Further,

$$\begin{aligned} \frac{\partial^2 g_1(0,0)}{\partial u^2} &= \frac{c(b-c)\left(3b^2+c^2\right)}{2B(b+c)^3}, \\ \frac{\partial^2 g_1(0,0)}{\partial u\partial v} &= \frac{bc^2\sqrt{(b+3c)(3b+c)}}{B(b+c)^3}, \\ \frac{\partial^2 g_1(0,0)}{\partial v^2} &= \frac{c(3b+c)(b+3c)}{2B(b+c)^2}, \\ \frac{\partial^3 g_1(0,0)}{\partial u^3} &= -\frac{3c^2(b-c)\left(5b^4+4b^3c+6b^2c^2+c^4\right)}{4B^2(b+c)^5}, \\ \frac{\partial^3 g_1(0,0)}{\partial u^2\partial v} &= \frac{c^2\left(-3b^4-8b^3c+2b^2c^2-8bc^3+c^4\right)\sqrt{(b+3c)(3b+c)}}{4B^2(b+c)^5}, \\ \frac{\partial^3 g_1(0,0)}{\partial u\partial v^2} &= -\frac{c^2(3b+c)(b+3c)\left(-b^3+3b^2c+9bc^2+c^3\right)}{4B^2(b+c)^5}, \\ \frac{\partial^3 g_1(0,0)}{\partial v^3} &= -\frac{3c^2(3b+c)(b+3c)\sqrt{(b+3c)(3b+c)}}{4B^2(b+c)^3}. \end{aligned} \quad (147)$$

Now, by using (118), (119), (120), (121), and (122) we obtain

$$\xi_{11} = \frac{c\left(7bc^2+5b^2c+3b^3+c^3\right)}{4B(b+c)^3}\left(1 + i\frac{b-c}{\sqrt{(b+3c)(3b+c)}}\right),$$

$$\xi_{20} = -\frac{c^2}{4B(b+c)^2}\left(3b+c+i\frac{7b^2+2bc-c^2}{\sqrt{(b+3c)(3b+c)}}\right),$$

$$\xi_{02} = \frac{c^2}{4B(b+c)^3}\left(-\left(5b^2+2bc+c^2\right) + i\frac{-b^3+11b^2c+5bc^2+c^3}{\sqrt{(b+3c)(3b+c)}}\right),$$

$$\xi_{21} = \frac{c^2}{8B^2(b+c)^4}\left(-\left(3b^4+2b^3c+8b^2c^2-c^4\right) + i\frac{3b^5+35b^4c+60b^3c^2+74b^2c^3+33bc^4+3c^5}{\sqrt{(b+3c)(3b+c)}}\right),$$

$$\frac{(1-2\mu(F_0))\overline{\mu(F_0)}^2}{1-\mu(F_0)} = -\frac{8bc+b^2+3c^2}{2(b+c)^2} + i\sqrt{10bc+3b^2+3c^2}\frac{5b^2-c^2}{2(3b+c)(b+c)^2},$$
$$\xi_{11}\xi_{20} = -\frac{c^3\left(3b^3+5b^2c+7bc^2+c^3\right)}{16B^2(b+c)^5}\left(\frac{2\left(b^3+19b^2c+11bc^2+c^3\right)}{3b^2+10bc+3c^2} + i\frac{2\left(5b^2-c^2\right)}{\sqrt{3b^2+10bc+3c^2}}\right),$$
$$\operatorname{Re}\left(\frac{(1-2\mu(F_0))\overline{\mu(F_0)}^2}{1-\mu(F_0)}\xi_{11}\xi_{20}\right) = \frac{c^3\left(13b^2+12bc+3c^2\right)\left(3b^3+5b^2c+7bc^2+c^3\right)}{8B^2(b+c)^4(3b+c)(b+3c)},$$
$$\frac{1}{2}\left|\xi_{11}\right|^2 = \frac{c^2\left(3b^3+5b^2c+7bc^2+c^3\right)^2}{8B^2(b+c)^4(3b+c)(b+3c)},$$
$$\left|\xi_{02}\right|^2 = \frac{19b^3c^4+15b^2c^5+5bc^6+c^7}{4B^2(3b+c)(b+3c)(b+c)^3},$$
$$\operatorname{Re}\left(\overline{\mu(F_0)}\xi_{21}\right) = \frac{c^2\left(3b^4+14b^3c+19b^2c^2+14bc^3+2c^4\right)}{8B^2(b+c)^4} \tag{148}$$

and finally,

$$a(F_0) = -\frac{bc^3\left(3b^3-9b^2c-3bc^2+c^3\right)}{8B^2(b+c)^4(3b+c)}. \tag{149}$$

If we substitute b with kc we obtain

$$\begin{aligned}&3b^3-9b^2c-3bc^2+c^3\\&\quad= 3(kc)^3-9(kc)^2c-3(kc)c^2+c^3\\&\quad= c^3\left(3k^3-9k^2-3k+1\right).\end{aligned} \tag{150}$$

So,

$$\begin{gathered}a(F_0)=0 \Longleftrightarrow\\3k^3-9k^2-3k+1=0.\end{gathered} \tag{151}$$

Solutions, determined numerically, are $k_1 \approx -0.48445$, $k_2 \approx 0.21014$, and $k_3 \approx 3.2743$. Since $b > 0$ and $c > 0$ it must be $k > 0$. Now,

$$\begin{aligned}&a(F_0) > 0 \quad \text{for } b = kc,\ k \in (k_2, 1) \cup (1, k_3),\\&a(F_0) < 0 \quad \text{for } b = kc,\ k \in (0, k_2) \cup (k_3, +\infty).\end{aligned} \tag{152}$$

Further,

$$\mu(F) = \frac{-bF+Bc\overline{x}^2 \pm i\sqrt{4(b+c)^2\overline{x}^3\left(bF+2cF+Bc\overline{x}^2\right)-\left(bF-Bc\overline{x}^2\right)^2}}{2(b+c)^2\overline{x}^3} \tag{153}$$

and $\mu(F)\overline{\mu(F)} = (bF+2cF+Bcx^2)/(b+c)^2x^3$, i.e., $|\mu(F)| = \sqrt{(bF+2cF+Bcx^2)/(b+c)^2x^3}$. By differentiating the equilibrium equation

$$(b+c)x^3 - Bx^2 - F = 0 \tag{154}$$

with respect to F and solving for $x'(F)$ we obtain $x'(F) = 1/(3(b+c)x^2-2Bx)$, i.e., $x(F_0) = \sqrt[3]{F_0/b} = B/c$.

$$x'(F_0) = \frac{1}{3(b+c)(B/c)^2-2B(B/c)} = \frac{c^2}{B^2(3b+c)}. \tag{155}$$

Now,

$$\begin{aligned}&\frac{d\left|\mu(F)\right|}{dF}\\&\quad= \frac{1}{2\sqrt{\left(bF+2cF+Bcx^2\right)/(b+c)^2x^3}}\left(\frac{b+2c+2Bcxx'}{(b+c)^2x^3}\right.\\&\qquad\left.-\frac{3\left(bF+2cF+Bcx^2\right)x'}{(b+c)^2x^4}\right).\end{aligned} \tag{156}$$

By substituting $x'(F)$ in the above expression and considering the fact that $|\mu(F_0)| = 1$, we obtain

$$\frac{d\left|\mu(F)\right|}{dF}(F_0) = \frac{1}{2}\left(\frac{b+2c+2Bc(B/c)\left(c^2/\left(B^2(3b+c)\right)\right)}{(b+c)^2(B/c)^3} - \frac{3\left(c^2/\left(B^2(3b+c)\right)\right)}{B/c}\right), \quad (157)$$

i.e.,

$$\frac{d\left|\mu(F)\right|}{dF}(F_0) = \frac{c^4}{2B^3(b+c)(3b+c)} > 0 \quad (158)$$

and that completes the proof of the theorem. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The article is a joint work of all four authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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