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Research Article

Multiple Attractors for a Competitive System of Rational Difference Equations in the Plane

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We investigate global dynamics of the following systems of difference equations $x_{n+1} = \beta_1 x_n / (B_1 x_n + y_n)$, $y_{n+1} = (\alpha_2 + \gamma_2 y_n)/(A_2 + x_n)$, $n = 0, 1, 2, \ldots$, where the parameters β_1, B_1, β_2, α_2, γ_2, A_2 are positive numbers, and initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$. We show that this system has up to three equilibrium points with various dynamics which depends on the part of parametric space. We show that the basins of attractions of different locally asymptotically stable equilibrium points or nonhyperbolic equilibrium points are separated by the global stable manifolds of either saddle points or of nonhyperbolic equilibrium points. We give an example of globally attractive nonhyperbolic equilibrium point and semistable non-hyperbolic equilibrium point.

1. Introduction

In this paper we consider the following rational system of difference equations

$$
\begin{aligned}
x_{n+1} &= \frac{\beta_1 x_n}{B_1 x_n + y_n}, \\
y_{n+1} &= \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n},
\end{aligned}
\qquad n = 0, 1, 2, \ldots,
\tag{1.1}
$$

where the parameters β_1, B_1, β_2, α_2, γ_2, A_2 are positive numbers, and initial conditions x_0 and y_0 are nonnegative numbers such that $x_0 + y_0 > 0$. System (1.1) was mentioned in [1] as one of three systems of open problem 3 which asked for the description of global dynamics of some rational systems of difference equations. In notation used to labels systems of linear fractional

difference equations used in [1] system (1.1) is known as (3.19) and (4.1). In this paper, we provide the precise description of global dynamics of the system (1.1). We show that the system (1.1) may have between zero and three equilibrium points, which may have different local character. If the system (1.1) has one equilibrium point, then this point is either locally asymptotically stable or saddle point or nonhyperbolic equilibrium point. If the system (1.1) has two equilibrium points, then they are either locally asymptotically stable, and nonhyperbolic, or locally asymptotically stable and saddle point. If the system (1.1) has three equilibrium points then two of the equilibrium points are locally asymptotically stable and the third point, which is between these two points in South-East ordering defined below, is a saddle point. The major problem for global dynamics of the system (1.1) is determining the basins of attraction of different equilibrium points. The difficulty in analyzing the behavior of all solutions of the system (1.1) lies in the fact that there are many regions of parameters where this system possesses different equilibrium points with different local character and that in several cases the equilibrium point is nonhyperbolic. However, all these cases can be handled by using recent results in [2]. The dual of this system is the system where x_n and y_n replace their role, and it was labeled as system (4.1) and (3.19) in [1]. Dynamics of this system immediately follows from the results proven here, by simply replacing the roles of x_n and y_n.

System (1.1) is a competitive system, and our results are based on recent results about competitive systems in the plane, see [2, 3]. System (1.1) has a potential to be used as a mathematical model for competition. In fact, the first equation of (1.1) is of Leslie-Gower type, and the second equation can be considered to be of Leslie-Gower type with stocking (or immigration) represented with the term α_2, see [4–7]. Here β_1, γ_2 are the inherent birth rates while B_1 and A_2 are related to the density-dependent effects on newborn recruitment. Finally, α_2 affects stocking for species with state variable y_n.

In Section 2, we present some general results about competitive systems in the plane. In Section 3 contains some basic facts such as the nonexistence of period-two solution of system (1.1). In Section 4 analyzes local stability which is fairly complicated for this system. Finally, in Section 5 gives global dynamics for all values of parameters. This section finishes with an introduction of a new terminology for different type scenarios for competitive systems that can be used to give a simple classification of all possible global behavior for system (1.1). The interesting feature of this paper is that there are five regions of the parameters in which one of the equilibrium points is nonhyperbolic, and yet we are able to describe the global dynamics in all five cases. To achieve this goal, we use new method of proving stability of nonhyperbolic equilibrium points introduced in [2].

2. Preliminaries

Consider a first-order system of difference equations of the form

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n), \\
y_{n+1} &= g(x_n, y_n),
\end{aligned}
\qquad n = 0, 1, 2, \ldots, \qquad (x_{-1}, x_0) \in \mathcal{I} \times \mathcal{I},
\tag{2.1}
$$

where $f, g : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$, $f(x, y)$ is nondecreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and nondecreasing in y. Such system is called *competitive*. One may associate a competitive map T to a competitive system (2.1) by setting $T = (f, g)$ and considering T on $\mathcal{B} = \mathcal{I} \times \mathcal{I}$.

We now present some basic notions about competitive maps in plane. Define a partial order $\preceq$ on $\mathbb{R}^2$ so that the positive cone is the fourth quadrant, that is, $(x^1, y^1) \preceq (x^2, y^2)$ if and only if $x^1 \leq x^2$ and $y^1 \geq y^2$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ the *order interval* $[\![\mathbf{x}, \mathbf{y}]\!]$ is the set of all $\mathbf{z}$ such that $\mathbf{x} \preceq \mathbf{z} \preceq \mathbf{y}$. A set $\mathscr{A}$ is said to be *linearly ordered* if $\preceq$ is a total order on $\mathscr{A}$. If a set $\mathscr{A} \subset \mathbb{R}^2$ is linearly ordered by $\preceq$, then the infimum $\mathbf{i} = \inf \mathscr{A}$ and supremum $\mathbf{s} = \sup \mathscr{A}$ of $\mathscr{A}$ exist in $\overline{\mathbb{R}}^2 = [-\infty, \infty] \times [-\infty, \infty]$. If both $\mathbf{i}$ and $\mathbf{s}$ belong to $\mathbb{R}^2$, then the linearly ordered set $\mathscr{A}$ is bounded, and conversely. We note that the ordering $\preceq$ may be extended to the extended plane $\overline{\mathbb{R}}^2$ in a natural way. For example, $(0, \infty) \preceq (a, b)$ if $a \geq 0$ or $a = \infty$. If $\mathbf{x} \in \mathbb{R}^2$, we denote with $Q_\ell(\mathbf{x})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\overline{\mathbb{R}}^2$ relative to $\mathbf{x}$, that is, $Q_1(x, y) = \{(u, v) \in \overline{\mathbb{R}}^2 : u \geq x, \ v \geq y\}$, $Q_2(x, y) = \{(u, v) \in \overline{\mathbb{R}}^2 : x \geq u, \ v \geq y\}$, and so on.

A *map* T on a set $\mathscr{B} \subset \mathbb{R}^2$ is a continuous function $T : \mathscr{B} \to \mathscr{B}$. The map is *smooth* on $\mathscr{B}$ if the interior of $\mathscr{B}$ is nonempty and if T is continuously differentiable on the interior of $\mathscr{B}$. A set $\mathscr{A} \subset \mathscr{B}$ is *invariant* for the map T if $T(\mathscr{A}) \subset \mathscr{A}$. A point $\mathbf{x} \in \mathscr{B}$ is a *fixed point* of T if $T(\mathbf{x}) = \mathbf{x}$, and a *minimal period-two point* if $T^2(\mathbf{x}) = \mathbf{x}$ and $T(\mathbf{x}) \neq \mathbf{x}$. A *period-two point* is either a fixed point or a minimal period-two point. The *orbit of* $\mathbf{x} \in \mathscr{B}$ is the sequence $\{T^\ell(\mathbf{x})\}_{\ell=0}^{\infty}$. A *minimal period two orbit* is an orbit $\{\mathbf{x}_\ell\}_{\ell=0}^{\infty}$ for which $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 = \mathbf{x}_2$. The *basin of attraction* of a fixed point $\mathbf{x}$ is the set of all $\mathbf{y}$ such that $T^n(\mathbf{y}) \to \mathbf{x}$. A fixed point $\mathbf{x}$ is a *global attractor* on a set $\mathscr{A}$ if $\mathscr{A}$ is a subset of the basin of attraction of $\mathbf{x}$. A fixed point $\mathbf{x}$ is a *saddle point* if T is differentiable at $\mathbf{x}$, and the eigenvalues of the Jacobian matrix of T at $\mathbf{x}$ are such that one of them lies in the interior of the unit circle in $\mathbb{R}^2$, while the other eigenvalue lies in the exterior of the unit circle. If $T = (T_1, T_2)$ is a map on $\mathcal{R} \subset \mathbb{R}^2$, define the sets $\mathcal{R}_T(-, +) := \{(x, y) \in \mathcal{R} : T_1(x, y) \leq x, \ T_2(x, y) \geq y\}$ and $\mathcal{R}_T(+, -) := \{(x, y) \in \mathcal{R} : T_1(x, y) \geq x, \ T_2(x, y) \leq y\}$. For $\mathscr{A} \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the *distance from x to $\mathscr{A}$* as $\mathrm{dist}(x, \mathscr{A}) := \inf\{\|x - y\| : y \in \mathscr{A}\}$.

A map T is *competitive* if $T(\mathbf{x}) \preceq T(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, and T is *strongly competitive* if $\mathbf{x} \preceq \mathbf{y}$ implies that $T(\mathbf{x}) - T(\mathbf{y}) \in \{(u, v) : u > 0, \ v < 0\}$. If T is differentiable, a sufficient condition for T to be strongly competitive is that the Jacobian matrix of T at any $\mathbf{x} \in \mathscr{B}$ has the sign configuration

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}. \tag{2.2}$$

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [8, 9] for competitive maps, and [10, 11] for difference equations.

If $\mathscr{A}$ is any subset of $\mathbb{R}^k$, we shall use the notation $\mathrm{clos}\,(\mathscr{A})$ to denote the closure of $\mathscr{A}$ in $\mathbb{R}^k$, and $\mathscr{A}^\circ$ to denote the interior of $\mathscr{A}$.

The next results are stated for order-preserving maps on $\mathbb{R}^n$ and are known but given here for completeness. See [12] for a more general version valid in ordered Banach spaces.

Theorem 2.1. *For a nonempty set $R \subset \mathbb{R}^n$ and $\preceq$ a partial order on $\mathbb{R}^n$, let $T : R \to R$ be an order-preserving map, and let $a, b \in R$ be such that $a \prec b$ and $[\![a, b]\!] \subset R$. If $a \preceq T(a)$ and $T(b) \preceq b$, then $[\![a, b]\!]$ is invariant and*

(i) *there exists a fixed point of T in $[\![a, b]\!]$,*

(ii) *if T is strongly order preserving, then there exists a fixed point in $[\![a, b]\!]$ which is stable relative to $[\![a, b]\!]$,*

(iii) *if there is only one fixed point in $[\![a,b]\!]$, then it is a global attractor in $[\![a,b]\!]$ and therefore asymptotically stable relative to $[\![a,b]\!]$.*

Corollary 2.2. *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1,u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1,u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

Define a *rectangular region* $\mathcal{R}$ in $\mathbb{R}^2$ to be the cartesian product of two intervals in $\mathbb{R}$.

Remark 2.3. It follows from the Perron-Frobenius theorem and a change of variables [9] that, at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

Theorem 2.4. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \text{int}(Q_1(\overline{x}) \cup Q_3(\overline{x}))$ is nonempty (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

(a) *The map T has a C^1 extension to a neighborhood of $\overline{x}$.*

(b) *The Jacobian matrix $J_T(\overline{x})$ of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axes.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{x}$ that is invariant and a subset of the basin of attraction of $\overline{x}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 2.7 and in the examples in [2] that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 2.5. *For the curve $\mathcal{C}$ of Theorem 2.4 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

(i) *The map T has no fixed points nor periodic points of minimal period two in Δ.*

(ii) *The map T has no fixed points in Δ, $\det J_T(\overline{x}) > 0$, and $T(x) = \overline{x}$ has no solutions $x \in \Delta$.*

(iii) *The map T has no points of minimal period two in Δ, $\det J_T(\overline{x}) < 0$, and $T(x) = \overline{x}$ has no solutions $x \in \Delta$.*

In many cases, one can expect the curve $\mathcal{C}$ to be smooth.

Theorem 2.6. *Under the hypotheses of Theorem 2.4, suppose that there exists a neighborhood $\mathcal{U}$ of $\overline{x}$ in $\mathbb{R}^2$ such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian matrix of T at each $x \in \Delta$ is invertible. Then, the curve $\mathcal{C}$ in the conclusion of Theorem 2.4 is of class C^k.*

In applications, it is common to have rectangular domains $\mathcal{R}$ for competitive maps. If a competitive map has several fixed points, often the domain of the map may be split

into rectangular invariant subsets such that Theorem 2.4 could be applied to the restriction of the map to one or more subsets. For maps that are strongly competitive near the fixed point, hypothesis (b) of Theorem 2.4 reduces just to $|\lambda| < 1$. This follows from a change of variables [9] that allows the Perron-Frobenius theorem to be applied to give that at any point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axes.

Smith performed a systematic study of competitive and cooperative maps in [9, 13, 14] and in particular introduced invariant manifolds techniques in his analysis [13–15] with some results valid for maps on n-dimensional space. Smith restricted attention mostly to competitive maps T that satisfy additional constraints. In particular, T is required to be a diffeomorphism of a neighborhood of $\mathbb{R}^n_+$ that satisfies certain conditions (this is the case if T is orientation preserving or orientation reversing), and that the coordinate semiaxes are invariant under T. For such class of maps (as well as for cooperative maps satisfying similar hypotheses), Smith obtained results on invariant manifolds passing through hyperbolic fixed points and a fairly complete description of the phase-portrait when $n = 2$, especially for those cases having a unique fixed point on each of the open positive semiaxes. In our results, presented here, we removed all these constraints and added the precise analysis of invariant manifolds of nonhyperbolic equilibrium points. The invariance of coordinate semiaxes seems to be serious restriction in the case of competitive models with constant stocking or harvesting, see [16] for stocking.

The next result is useful for determining basins of attraction of fixed points of competitive maps. Compare to Theorem 4.4 in [13], where hyperbolicity of the fixed point is assumed, in addition to other hypotheses.

Theorem 2.7. (A) *Assume the hypotheses of Theorem 2.4, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2.4. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely,*

$$
\begin{aligned}
\mathcal{W}_- &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists \, y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\
\mathcal{W}_+ &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists \, y \in \mathcal{C} \text{ with } y \preceq_{se} x\},
\end{aligned}
\tag{2.3}
$$

such that the following statements are true:

    (i) *$\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.*

    (ii) *$\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.*

(B) *If, in addition to the hypotheses of part (A), $\overline{x}$ is an interior point of $\mathcal{R}$, and T is C^2 and strongly competitive in a neighborhood of $\overline{x}$, then T has no periodic points in the boundary of $Q_1(\overline{x}) \cup Q_3(\overline{x})$ except for $\overline{x}$, and the following statements are true.*

    (iii) *For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_2(\overline{x})$ for $n \geq n_0$.*

    (iv) *For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_4(\overline{x})$ for $n \geq n_0$.*

Basins of attraction of *period-two solutions* or period-two orbits of certain systems or maps can be effectively treated with Theorems 2.4 and 2.7. See [2, 6, 11] for the hyperbolic case; for the nonhyperbolic case, see examples in [2, 17].

If T is a map on a set $\mathcal{R}$ and if $\overline{x}$ is a fixed point of T, the *stable set* $\mathcal{W}^s(\overline{x})$ of $\overline{x}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{x}\}$, and *unstable set* $\mathcal{W}^u(\overline{x})$ of $\overline{x}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1}, \ x_0 = x, \lim_{n \to -\infty} x_n = \overline{x} \right\}. \tag{2.4}$$

When T is noninvertible, the set $\mathcal{W}^s(\overline{x})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{x})$ and $\mathcal{W}^u(\overline{x})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 2.8. *In addition to the hypotheses of part (B) of Theorem 2.7, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axes. If the curve $\mathcal{C}$ of Theorem 2.4 has endpoints in $\partial \mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{x})$ of $\overline{x}$, and the unstable set $\mathcal{W}^u(\overline{x})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{x})$ in $\mathcal{R}$ are fixed points of T.*

The following result gives information on *local* dynamics near a fixed point of a map when there exists a characteristic vector whose coordinates have negative product and such that the associated eigenvalue is hyperbolic. This is a well-known result, valid in much more general setting which we include it here for completeness. A point (x, y) is a *subsolution* if $T(x, y) \preceq_{se} (x, y)$, and (x, y) is a *supersolution* if $(x, y) \preceq_{se} T(x, y)$. An *order interval* $[\![(a, b), (c, d)]\!]$ is the cartesian product of the two compact intervals $[a, c]$ and $[b, d]$.

Theorem 2.9. *Let T be a competitive map on a rectangular set $\mathcal{R} \subset \mathbb{R}^2$ with an isolated fixed point $\overline{x} \in \mathcal{R}$ such that $\mathcal{R} \cap \text{int}(Q_2(\overline{x}) \cup Q_4(\overline{x})) \neq \emptyset$. Suppose that T has a C^1 extension to a neighborhood of $\overline{x}$. Let $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^2$ be an eigenvector of the Jacobian matrix of T at $\overline{x}$, with associated eigenvalue $\mu \in \mathbb{R}$. If $v^{(1)} v^{(2)} < 0$, then there exists an order interval $\mathcal{I}$ which is also a relative neighborhood of $\overline{x}$ such that, for every relative neighborhood $\mathcal{U} \subset \mathcal{I}$ of $\overline{x}$, the following statements are true.*

    (i) *If $\mu > 1$, then $\mathcal{U} \cap \text{int} \, Q_2(\overline{x})$ contains a subsolution, and $\mathcal{U} \cap \text{int} \, Q_4(\overline{x})$ contains a supersolution. In this case, for every $x \in \mathcal{I} \cap \text{int} \, (Q_2(\overline{x}) \cup Q_4(\overline{x}))$, there exists N such that $T^n(x) \notin \mathcal{I}$ for $n \geq N$.*

    (ii) *If $\mu < 1$, then $\mathcal{U} \cap \text{int} \, Q_2(\overline{x})$ contains a supersolution and $\mathcal{U} \cap \text{int} \, Q_4(\overline{x})$ contains a subsolution. In this case, $T^n(x) \to \overline{x}$ for every $x \in \mathcal{I}$.*

In the nonhyperbolic case, we have the following result.

Theorem 2.10. *Assume that the hypotheses of Theorem 2.9 hold, that T is real analytic at $\overline{x}$, and that $\mu = 1$. Let $c_j, d_j, j = 2, 3, \ldots$ be defined by the Taylor series*

$$T(\overline{x} + tv) = \overline{x} + vt + (c_2, d_2)t^2 + \cdots + (c_n, d_n) \, t^n + \cdots . \tag{2.5}$$

Suppose that there exists an index $\ell \geq 2$ such that $(c_\ell, d_\ell) \neq (0, 0)$ and $(c_j, d_j) = (0, 0)$ for $j < l$. If either

    (a) *$c_\ell d_\ell < 0$, or (b)$c_\ell \neq 0$, $T(\overline{x} + tv)^{(2)}$ is affine in t, or (c) $d_\ell \neq 0$, $T(\overline{x} + tv)^{(1)}$ is affine in t, then there exists an order interval $\mathcal{I}$ which is also a relative neighborhood of $\overline{x}$ such that, for every relative neighborhood $\mathcal{U} \subset \mathcal{I}$ of $\overline{x}$, the following statements are true.*

(i) *If ℓ is odd and $(c_\ell, d_\ell) \preceq_{se} (0,0)$, then $\mathcal{U} \cap \operatorname{int} Q_4(\overline{x})$ contains a supersolution, and $\mathcal{U} \cap \operatorname{int} Q_2(\overline{x})$ contains a subsolution. In this case, for every $x \in \mathcal{O} \cap \operatorname{int}(Q_2(\overline{x}) \cup Q_4(\overline{x}))$, there exists N such that $T^n(x) \notin \mathcal{O}$ for $n \geq N$.*

(ii) *If ℓ is odd and $(0,0) \preceq_{se} (c_\ell, d_\ell)$, then $\mathcal{U} \cap \operatorname{int} Q_4(\overline{x})$ contains a subsolution and $\mathcal{U} \cap \operatorname{int} Q_2(\overline{x})$ contains a supersolution. In this case, $T^n(x) \to \overline{x}$ for every $x \in \mathcal{O}$.*

(iii) *If ℓ is even and $(c_\ell, d_\ell) \preceq_{se} (0,0)$, then $\mathcal{U} \cap \operatorname{int} Q_4(\overline{x})$ contains a subsolution and $\mathcal{U} \cap \operatorname{int} Q_2(\overline{x})$ contains a subsolution. In this case, $T^n(x) \to \overline{x}$ for every $x \in \mathcal{O} \cap Q_4(\overline{x})$, and for every $x \in \mathcal{O} \cap \operatorname{int}(Q_2(\overline{x}))$, there exists N such that $T^n(x) \notin \mathcal{O}$ for $n \geq N$.*

(iv) *If ℓ is even and $(0,0) \preceq_{se} (c_\ell, d_\ell)$, then $\mathcal{U} \cap \operatorname{int} Q_2(\overline{x})$ contains a supersolution and $\mathcal{U} \cap \operatorname{int} Q_4(\overline{x})$ contains a supersolution. In this case, $T^n(x) \to \overline{x}$ for every $x \in \mathcal{O} \cap Q_2(\overline{x})$, and, for every $x \in \mathcal{O} \cap \operatorname{int}(Q_4(\overline{x}))$ there exists, N such that $T^n(x) \notin \mathcal{O}$ for $n \geq N$.*

3. Some Basic Facts

In this section, we give some basic facts about the nonexistence of period-two solutions, local injectivity of map T at the equilibrium point.

3.1. Equilibrium Points

The equilibrium points $(\overline{x}, \overline{y})$ of the system (1.1) satisfy

$$
\overline{x} = \frac{\beta_1 \overline{x}}{B_1 \overline{x} + \overline{y}},
$$
$$
\overline{y} = \frac{\alpha_2 + \gamma_2 \overline{y}}{A_2 + \overline{x}}.
\tag{3.1}
$$

Solutions of System (3.1) are

(i) $\overline{x} = 0$, $\overline{y} = \alpha_2 / A_2 - \gamma_2$ when $A_2 > \gamma_2$, that is,

$$
E_1 = \left(0, \frac{\alpha_2}{A_2 - \gamma_2}\right).
\tag{3.2}
$$

(ii) If $\overline{x} \neq 0$, then using System (3.1), we obtain

$$
\overline{y} = \beta_1 - B_1 \overline{x},
$$
$$
0 = \overline{x}^2 B_1 - \overline{x}(B_1(\gamma_2 - A_2) + \beta_1) - (\beta_1(A_2 - \gamma_2) - \alpha_2).
\tag{3.3}
$$

Solutions of System (3.3) are

$$
\overline{x}_{3,2} = \frac{B_1(\gamma_2 - A_2) + \beta_1 \pm \sqrt{D_0}}{2B_1}, \qquad \overline{y}_{2,3} = \frac{B_1(A_2 - \gamma_2) + \beta_1 \pm \sqrt{D_0}}{2},
\tag{3.4}
$$

Table 1

E_1	$A_2 > \gamma_2,\ \alpha_2 > \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1}$ or
	$A_2 > \gamma_2,\ \alpha_2 \leq \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1},\ \beta_1 < \dfrac{\alpha_2}{A_2 - \gamma_2},\ B_1 > \dfrac{\beta_1}{A_2 - \gamma_2}$
$E_1 \equiv E_2 \equiv E_3$	$A_2 > \gamma_2,\ \beta_1 = \dfrac{\alpha_2}{A_2 - \gamma_2},\ B_1 = \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$
$E_1 \equiv E_3$	$A_2 > \gamma_2,\ \beta_1 = \dfrac{\alpha_2}{A_2 - \gamma_2},\ B_1 > \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$
$E_1 \equiv E_2, E_3$	$A_2 > \gamma_2,\ \beta_1 = \dfrac{\alpha_2}{A_2 - \gamma_2},\ B_1 < \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$
E_1, E_2, E_3	$A_2 > \gamma_2,\ B_1(\gamma_2 - A_2) + \sqrt{4B_1\alpha_2} < \beta_1 < \dfrac{\alpha_2}{A_2 - \gamma_2},\ \alpha_2 > B_1(A_2 - \gamma_2)^2$
E_1, E_3	$A_2 > \gamma_2,\ \beta_1 > \dfrac{\alpha_2}{A_2 - \gamma_2}$
$E_1, E_2 = E_3$	$A_2 > \gamma_2,\ (A_2 - \gamma_2)B_1 < \beta_1,\ (B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 = 0$
E_2, E_3	$A_2 < \gamma_2,\ (B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 > 0,\ \beta_1 > B_1(\gamma_2 - A_2)$
$E_2 = E_3$	$A_2 < \gamma_2,\ (B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 = 0,\ \beta_1 > B_1(\gamma_2 - A_2)$
e_1, e_2	$A_2 = \gamma_2,\ \beta_1^2 - 4B_1\alpha_2 \geq 0$
No equilibrium	$\gamma_2 \geq A_2$ and $\alpha_2 \leq \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1},\ \beta_1 < B_1(\gamma_2 - A_2)$ or
	$\gamma_2 \geq A_2$ and $\alpha_2 > \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1}$

where $D_0 = (B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2$, which gives a pair of the equilibrium points $E_2 = (\overline{x}_2, \overline{y}_2)$ and $E_3 = (\overline{x}_3, \overline{y}_3)$.

Geometrically, the equilibrium points are the intersections of two equilibrium curves: $C_1 : x = 0 \cup y = -B_1 x + \beta_1$ and $C_2 : y = \alpha_2/(A_2 - \gamma_2 + x)$. Depending on the values of parameters, C_2 may have between 0 and 3 intersection points with two lines which constitutes C_1.

The algebraic criteria for the existence of the equilibrium points are summarized in Table 1.

Where

$$e_1 = \left(\frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1\alpha_2}}{2B_1}, \frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1\alpha_2}}{2} \right),$$

$$e_2 = \left(\frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1\alpha_2}}{2B_1}, \frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1\alpha_2}}{2} \right). \tag{3.5}$$

Remark 3.1. Observe the following: If the system (1.1) has two or three equilibrium points E_1, E_2, and E_3 then, $E_1 \preceq E_2 \preceq E_3$. Indeed, consider the critical curve $C_2 : y = \alpha_2/(A_2 - \gamma_2 + x)$. Observe that $y(0) = \alpha_2/(A_2 - \gamma_2)$, $y(\overline{x}_2) = \overline{y}_2$, and $y(\overline{x}_3) = \overline{y}_3$. It is obvious that the following

holds $0 \leq \overline{x}_2 \leq \overline{x}_3$. Since, the critical curve C_2 decreases, we have $y(0) \geq y(\overline{x}_2) \geq y(\overline{x}_3)$, that is, $E_1 \leq E_2 \leq E_3$.

Lemma 3.2. *Assume that $x_0 = 0$, $y_0 \in \mathbb{R}^+ = (0, \infty)$. Then the following statements are true for solutions of the system (1.1).*

    (i) *If $A_2 > \gamma_2$, then $x_n = 0$, for all $n \in \mathbb{N}$, and $y_n \to \alpha_2/(A_2 - \gamma_2)$, $n \to \infty$.*

    (ii) *If $A_2 < \gamma_2$, then $x_n = 0$, for all $n \in \mathbb{N}$, and $y_n \to \infty$, $n \to \infty$.*

    (iii) *If $A_2 = \gamma_2$, then $y_n = y_0 + (\alpha_2/\gamma_2)n$, and $x_n = 0$, for all $n \in \mathbb{N}$, $y_n \to \infty$, $n \to \infty$.*

    Assume that $x_0 \neq 0$ and $(x_0, y_0) \in \mathbb{R}_2^+$. Then, the following statements are true for all $n = 1, 2, \ldots$:

    (iv) $x_n \leq \beta_1/B_1$.

    (v) $y_n \leq c(\gamma_2/A_2)^n + (\alpha_2/(A_2 - \gamma_2))$ *and*

        (a) $y_n \geq B_1\alpha_2/(B_1(A_2 - \gamma_2) + \beta_1) + \epsilon_1$, $A_2 > \gamma_2$, *where ϵ_1 is arbitrarily small positive number.*

        (b) $y_n \leq \alpha_2/(A_2 - \gamma_2) + \varepsilon_2$, $\varepsilon > 0$, $A_2 > \gamma_2$, *where ϵ_2 is arbitrarily small positive number.*

Proof. Since (i)–(iv) are immediate consequences of the system (1.1), we will prove only (v).
    Take $x_0 = 0$ and $y_0 \in \mathbb{R}^+$. Then, we have $x_n = 0$ for all $n \in \mathbb{N}$, and

$$y_{n+1} = \frac{\alpha_2}{A_2} + \frac{\gamma_2}{A_2}y_n. \tag{3.6}$$

Solution of (3.6), when $A_2 \neq \gamma_2$ is

$$y_n = c\left(\frac{\gamma_2}{A_2}\right)^n + \frac{\alpha_2}{A_2 - \gamma_2} \tag{3.7}$$

which immediately implies (i) and (ii). Statement (iii) follows from (3.6).
    Equation

$$x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n} \tag{3.8}$$

implies that

$$x_n \leq \frac{\beta_1}{B_1}. \tag{3.9}$$

Using the last inequality, we have

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n} \geq \frac{\alpha_2 + \gamma_2 y_n}{A_2 + (\beta_1/B_1)} = \frac{B_1\alpha_2}{A_2 B_1 + \beta_1} + \frac{\gamma_2 B_1}{B_1 A_2 + \beta_1}y_n, \tag{3.10}$$

which by difference inequality theorem [18] implies the following

$$y_n \geq c \left(\frac{\gamma_2 B_1}{A_2 B_1 + \beta_1} \right)^n + \frac{B_1 \alpha_2}{B_1 (A_2 - \gamma_2) + \beta_1} \geq \frac{B_1 \alpha_2}{B_1 (A_2 - \gamma_2) + \beta_1} + \epsilon_1, \quad n \longrightarrow \infty. \tag{3.11}$$

Furthermore, second equation in (1.1) implies that

$$y_{n+1} \leq \frac{\alpha_2}{A_2} + \frac{\gamma_2}{A_2} y_n \tag{3.12}$$

which, by the difference inequalities argument, see [18], implies that $y_n \leq u_n$, where u_n satisfies (3.6). In view of (3.7) we obtain our conclusion. $\qquad \square$

3.2. Period-Two Solution

In this section, we prove that System (1.1) has no minimal period-two solution which will be essential for application of Theorems 2.5–2.7. The map T associated to System (1.1) is given by

$$T(x, y) = \left(\frac{\beta_1 x}{B_1 x + y}, \frac{\alpha_2 + \gamma_2 y}{A_2 + x} \right). \tag{3.13}$$

Lemma 3.3. *System* (1.1) *has no minimal period-two solution.*

Proof. We have

$$\begin{aligned} T(T(x, y)) &= T \left(\frac{\beta_1 x}{B_1 x + y}, \frac{\alpha_2 + \gamma_2 y}{A_2 + x} \right) \\ &= \left(\frac{\beta_1^2 x (A_2 + x)}{B_1 \beta_1 x (A_2 + x) + (\alpha_2 + \gamma_2 y)(B_1 x + y)}, \frac{(B_1 x + y)(\alpha_2 (A_2 + x) + \gamma_2 (\alpha_2 + \gamma_2 y))}{(A_2 + x)(A_2 (B_1 x + y) + \beta_1 x)} \right). \end{aligned} \tag{3.14}$$

Period-two solution satisfies

$$\frac{\beta_1^2 x (A_2 + x)}{B_1 \beta_1 x (A_2 + x) + (\alpha_2 + \gamma_2 y)(B_1 x + y)} - x = 0, \tag{3.15}$$

$$\frac{(B_1 x + y)(\alpha_2 (A_2 + x) + \gamma_2 (\alpha_2 + \gamma_2 y))}{(A_2 + x)(A_2 (B_1 x + y) + \beta_1 x)} - y = 0. \tag{3.16}$$

We show that this system has no other positive solutions except the equilibrium points.

Equation (3.15) is equivalent to

$$x\left(y\alpha_2 + x^2 B_1\beta_1 - A_2\beta_1^2 + x\left(-\beta_1^2 + B_1(\alpha_2 + A_2\beta_1)\right) + y^2\gamma_2 + xy B_1\gamma_2\right) = 0, \tag{3.17}$$

$$-xy^2 A_2 + x^2 B_1\alpha_2 + x^2 y\left(-A_2 B_1 - \beta_1\right) + y\left(A_2\alpha_2 + \alpha_2\gamma_2\right) + x\left(A_2 B_1\alpha_2 + B_1\alpha_2\gamma_2\right)$$
$$+ y^2\left(-A_2^2 + \gamma_2^2\right) + xy\left(-A_2^2 B_1 + \alpha_2 - A_2\beta_1 + B_1\gamma_2^2\right) = 0. \tag{3.18}$$

If $x = 0$ then we obtain the fixed point E_1. So assume that $x \neq 0$. Then, using (3.17), we have

$$y\alpha_2 + x^2 B_1\beta_1 - A_2\beta_1^2 + x\left(-\beta_1^2 + B_1(\alpha_2 + A_2\beta_1)\right) + y^2\gamma_2 + xy B_1\gamma_2 = 0. \tag{3.19}$$

Equation (3.19) implies that

$$y^2 = \frac{-y\alpha_2 - x^2 B_1\beta_1 + A_2\beta_1^2 - x\left(-\beta_1^2 + B_1(\alpha_2 + A_2\beta_1)\right) - xy B_1\gamma_2}{\gamma_2}. \tag{3.20}$$

Substituting (3.20) into (3.18), we have

$$x + A_2 = 0 \tag{3.21}$$

or

$$A_2\left((y + x B_1)\alpha_2 + (x + A_2)(x B_1 - \beta_1)\beta_1\right)$$
$$+ \left((y + x B_1)\alpha_2 - xy\beta_1\right)\gamma_2 + \beta_1(-x B_1 + \beta_1)\gamma_2^2 = 0. \tag{3.22}$$

Equation (3.21) implies that $x = -A_2$, and (3.23) implies that

$$y = \frac{A_2\left((x + A_2)\beta_1^2 - x B_1(\alpha_2 + (x + A_2)\beta_1)\right) - x B_1\alpha_2\gamma_2 + (x B_1 - \beta_1)\beta_1\gamma_2^2}{A_2\alpha_2 + (\alpha_2 - x\beta_1)\gamma_2}. \tag{3.23}$$

Replacing (3.23) into (3.19), we get

$$-x^2 B_1 - \alpha_2 + \beta_1(A_2 - \gamma_2) + x\left(-A_2 B_1 + \beta_1 + B_1\gamma_2\right) = 0, \tag{3.24}$$

$$-\left(\beta_1(\alpha_2 + \beta_1(A_2 - \gamma_2))(A_2 + \gamma_2)^2\right) + x(\alpha_2 + \beta_1(A_2 - \gamma_2))(A_2 + \gamma_2)(A_2 B_1 - \beta_1) + B_1\gamma_2)$$
$$+ x^2\beta_1\left(A_2^2 B_1 + A_2 B_1\gamma_2 + \beta_1\gamma_2\right) = 0. \tag{3.25}$$

Solutions of (3.24) are the equilibrium points.

Consider (3.25). Discriminant of this equation is given by

$$\Delta = \left(4\beta_1^2\left(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2\right) + (\alpha_2 + \beta_1(A_2 - \gamma_2))(\beta_1 - B_1(A_2\gamma_2))^2\right) \\ \times (\alpha_2 + \beta_1(A_2 - \gamma_2)). \tag{3.26}$$

Now, $\Delta \geq 0$ implies that

$$x_1 = -\frac{(A_2 + \gamma_2)\left(\Delta_1 + \sqrt{\Delta}\right)}{2\beta_1(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2)}, \qquad x_2 = -\frac{(A_2 + \gamma_2)\left(\Delta_1 - \sqrt{\Delta}\right)}{2\beta_1(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2)}, \tag{3.27}$$

where

$$\Delta_1 = (\alpha_2 + \beta_1(A_2 - \gamma_2))(-\beta_1 + B_1(A_2 + \gamma_2)). \tag{3.28}$$

Using (3.23), we obtain

$$y_1 = -\frac{\Delta_2 - \sqrt{\Delta}}{2(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2)}, \qquad y_2 = -\frac{\Delta_2 + \sqrt{\Delta}}{2(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2)}, \tag{3.29}$$

where

$$\Delta_2 = A_2^2 B_1 \beta_1 + \alpha_2(\beta_1 - B_1\gamma_2) + \beta_1\gamma_2(\beta_1 + B_1\gamma_2) + A_2(\beta_1^2 - B_1(\alpha_2 - 2\beta_1\gamma_2)). \tag{3.30}$$

We prove the following claims.

Claim 3.4. For all values of parameters $y_2 < 0$.

Proof. If $\Delta_2 > 0$, then $y_2 < 0$. Now, we assume that $\Delta_2 \leq 0$. Then,

$$y_2 < 0 \Longleftrightarrow \Delta_2 + \sqrt{\Delta} > 0 \Longleftrightarrow \Delta - \Delta_2^2 > 0, \tag{3.31}$$

$$\Delta - \Delta_2^2 = -4\beta_1\left(A_2^2 B_1 + (A_2 B_1 + \beta_1)\gamma_2\right)\left(A_2 B_1(-\alpha_2 + \beta_1\gamma_2) + \gamma_2\left(\beta_1^2 + B_1(-\alpha_2 + \beta_1\gamma_2)\right)\right). \tag{3.32}$$

Equation (3.32) implies that

$$\Delta - \Delta_2^2 > 0 \Longleftrightarrow \alpha_2 > \frac{A_2 B_1 \beta_1 \gamma_2 + \beta_1^2\gamma_2 + B_1\beta_1\gamma_2^2}{B_1(A_2 + \gamma_2)}. \tag{3.33}$$

Since $\Delta_2 \leq 0$ if and only if

$$
B_1 > \frac{\beta_1}{A_2 + \gamma_2}, \qquad \alpha_2 \geq \frac{A_2^2 B_1 \beta_1 + A_2 \beta_1^2 + 2A_2 B - 1\beta_1 \gamma_2 + \beta_1^2 \gamma_2 + B_1 \beta_1 \gamma_2^2}{A_2 B_1 - \beta_1 + B_1 \gamma_2},
$$

$$
\frac{A_2^2 B_1 \beta_1 + A_2 \beta_1^2 + 2A_2 B - 1\beta_1 \gamma_2 + \beta_1^2 \gamma_2 + B_1 \beta_1 \gamma_2^2}{A_2 B_1 - \beta_1 + B_1 \gamma_2} - \frac{A_2 B_1 \beta_1 \gamma_2 + \beta_1^2 \gamma_2 + B_1 \beta_1 \gamma_2^2}{B_1 (A_2 + \gamma_2)}
$$

$$
= \frac{\beta_1 \left(A_2^2 B_1 + (A_2 B_1 + \beta_1) \gamma_2 \right) \left(\beta_1 + B_1 (A_2 + \gamma_2) \right)}{B_1 (A_2 + \gamma_2) \left(-\beta_1 + B_1 (A_2 + \gamma_2) \right)} > 0.
$$

(3.34)

This implies that (3.33) is true. That is $y_2 < 0$. □

Claim 3.5. Assume that $x_1 \geq 0$. Then that $y_1 < 0$.

Proof. Assume that $x_1 > 0$. This is equivalent to

$$
A_2 < \gamma_2, \qquad B_1 + \frac{\beta_1 (A_2 + 3\gamma_2)}{A_2^2 - \gamma_2^2} > 0,
$$

(3.35)

$$
\alpha_2 \leq -\left(\frac{\beta_1 \left(A_2 (A_2 B_1 + \beta_1) + 3\beta_1 \gamma_2 - B_1 \gamma_2^2 \right) \left(\beta_1 + B_1 (A_2 + \gamma_2) \right)}{(\beta_1 - B_1 (A_2 + \gamma_2))} \right).
$$

(3.36)

Now

$$
y_1 < 0 \iff \Delta_2 \geq 0, \qquad \Delta - \Delta_2^2 < 0,
$$

(3.37)

which is equivalent to

$$
\Delta_2 \geq 0 \iff B_1 < \frac{\beta_1}{A_2 + \gamma_2}, \qquad \alpha_2 \leq -\left(\frac{\beta_1 (A_2 + \gamma_2) \left(\beta_1 + B_1 (A_2 + \gamma_2) \right)}{\beta_1 - B_1 (A_2 + \gamma_2)} \right)
$$

(3.38)

or

$$
B_1 \geq \frac{\beta_1}{A_2 + \gamma_2}.
$$

(3.39)

Also,

$$
\Delta - \Delta_2^2 < 0 \iff \alpha_2 < \frac{\beta_1 \gamma_2 \left(\beta_1 + B_1 (A_2 + \gamma_2) \right)}{B_1 (A_2 + \gamma_2)}.
$$

(3.40)

Since,

$$
\frac{\beta_1\left(A_2\left(A_2 B_1+\beta_1\right)+3\beta_1\gamma_2-B_1\gamma_2^2\right)\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)}{\left(\beta_1-B_1\left(A_2+\gamma_2\right)\right)}+\frac{\beta_1\gamma_2\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)}{B_1\left(A_2+\gamma_2\right)}
$$

$$
=\frac{\beta_1\left(A_2^2 B_1+\left(A_2 B_1+\beta_1\right)\gamma_2\right)\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)^2}{B_1\left(A_2+\gamma_2\right)\left(\beta_1-B_1\left(A_2+\gamma_2\right)\right)^2}>0.
$$

(3.41)

This inequality and (3.36) imply (3.40).
Since

$$
\frac{\beta_1\left(A_2\left(A_2 B_1+\beta_1\right)+3\beta_1\gamma_2-B_1\gamma_2{}^2\right)\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)}{\left(\beta_1-B_1\left(A_2+\gamma_2\right)\right)^2}-\left(\frac{\beta_1\left(A_2+\gamma_2\right)\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)}{\beta_1-B_1\left(A_2+\gamma_2\right)}\right)
$$

$$
=\frac{2\beta_1\left(A_2{}^2 B_1+\left(A_2 B_1+\beta_1\right)\gamma_2\right)\left(\beta_1+B_1\left(A_2+\gamma_2\right)\right)}{\left(\beta_1-B_1\left(A_2+\gamma_2\right)\right)^2}>0.
$$

(3.42)

Last inequality, (3.36) and (3.39) imply that $\Delta_2 \geq 0$. So we prove, if $x_1 > 0$, then $y_1 < 0$.
Assume that $x_1 = 0$.
We have

$$
x_1=0 \iff \Delta_1+\sqrt{\Delta}=0.
$$

(3.43)

Since,

$$
\Delta_1^2-\Delta=4\beta_1^2\left(\alpha_2+\beta_1\left(A_2-\gamma_2\right)\right)\left(A_2+\gamma_2\right)^2\left(B_1 A_2^2+B_1\gamma_2 A_2+\beta_1\gamma_2\right),
$$

(3.44)

we have that $x_1 = 0$ if and only if

$$
A_2<\gamma_2, \qquad \alpha_2+A_2\beta_1=\beta_1\gamma_2, \qquad \Delta_1\leq 0,
$$

(3.45)

which is true, because

$$
\Delta_1:=\left(\alpha_2+\beta_1\left(A_2-\gamma_2\right)\right)\left(-\beta_1+B_1\left(A_2+\gamma_2\right)\right).
$$

(3.46)

Replacing α_2 with $\beta_1\left(\gamma_2-A_2\right)$ in the formula for y_1, we obtain that $y_1 = -\beta_1$.   ☐

Hence, there does not exist period-two solution.   ☐

4. Linearized Stability Analysis

The Jacobian matrix of the map T, given by (3.13), has the form

$$J_T = \begin{pmatrix} \dfrac{\beta_1 y}{(B_1 x + y)^2} & -\dfrac{\beta_1 x}{(B_1 x + y)^2} \\[2ex] -\dfrac{\alpha_2 + \gamma_2 y}{(A_2 + x)^2} & \dfrac{\gamma_2}{A_2 + x} \end{pmatrix}. \tag{4.1}$$

The determinant of (4.1) is given by

$$\det J_T(x, y) = \frac{\beta_1 (y A_2 \gamma_2 - x \alpha_2)}{(x + A_2)^2 (y + x B_1)^2}. \tag{4.2}$$

The value of the Jacobian matrix of T at the equilibrium point $E = (\overline{x}, \overline{y})$, $\overline{x} \neq 0$ is

$$J_T(\overline{x}, \overline{y}) = \begin{pmatrix} \dfrac{\overline{y}}{B_1 \overline{x} + \overline{y}} & -\dfrac{\overline{x}}{B_1 \overline{x} + \overline{y}} \\[2ex] -\dfrac{\overline{y}}{A_2 + \overline{x}} & \dfrac{\gamma_2}{A_2 + \overline{x}} \end{pmatrix}. \tag{4.3}$$

The determinant of (4.3) is given by

$$\det J_T(\overline{x}, \overline{y}) = \frac{\overline{y}(\gamma_2 - \overline{x})}{(A_2 + \overline{x})(B_1 \overline{x} + \overline{y})}, \tag{4.4}$$

and the trace of (4.3) is

$$\operatorname{Tr} J_T(\overline{x}, \overline{y}) = \frac{\overline{y}}{B_1 \overline{x} + \overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}}. \tag{4.5}$$

The characteristic equation has the form

$$\lambda^2 - \lambda \left(\frac{\overline{y}}{B_1 \overline{x} + \overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}} \right) + \frac{\overline{y}(\gamma_2 - \overline{x})}{(A_2 + \overline{x})(B_1 \overline{x} + \overline{y})} = 0. \tag{4.6}$$

Theorem 4.1. *Assume that $A_2 > \gamma_2$. Then there exists the equilibrium point E_1 and*

(i) *$E_1(0, \alpha_2/(A_2 - \gamma_2))$ is locally asymptotically stable if $\beta_1 < \alpha_2/(A_2 - \gamma_2)$,*

(ii) *E_1 is a saddle point if $\beta_1 > \alpha_2/(A_2 - \gamma_2)$. The corresponding eigenvalues are*

$$\lambda_1 = \frac{\gamma_2}{A_2} \in (0, 1), \qquad \lambda_2 = \frac{\beta_1 (A_2 - \gamma_2)}{\alpha_2} \in (1, +\infty), \tag{4.7}$$

(iii) E_1 *is nonhyperbolic if* $\beta_1 = \alpha_2/(A_2 - \gamma_2)$. *The corresponding eigenvalues are*

$$\lambda_1 = \frac{\gamma_2}{A_2} \in (0,1), \qquad \lambda_2 = 1, \tag{4.8}$$

and the corresponding eigenvectors are $(0,1)$ *and* $(-1/\alpha_2, 1)$, *respectively.*

Proof. The Jacobian matrix (4.1) at the equilibrium point $E_1(0, \alpha_2/(A_2 - \gamma_2))$,

$$J_T(E_1) = \begin{pmatrix} \dfrac{\beta_1}{\overline{y}} & 0 \\ -\dfrac{\alpha_2 + \gamma\overline{y}}{A_2^2} & \dfrac{\gamma_2}{A_2} \end{pmatrix}. \tag{4.9}$$

Note that the Jacobian matrix (4.9) implies that the map T is not strongly competitive at the equilibrium point E_1.

The determinant of (4.9) is given by

$$\det J_T(\overline{x}, \overline{y}) = \frac{\beta_1}{\overline{y}} \frac{\gamma_2}{A_2} = \frac{\beta_1 \gamma_2 (A_2 - \gamma_2)}{\alpha_2 A_2}. \tag{4.10}$$

Note that, under the hypothesis of Theorem, the determinant is greater than zero.

The trace of (4.9) is

$$\mathrm{Tr} J_T(\overline{x}, \overline{y}) = \frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2} = \frac{\beta_1 (A_2 - \gamma_2)}{\alpha_2} + \frac{\gamma_2}{A_2}. \tag{4.11}$$

An equilibrium point is locally asymptotically stable if the following conditions are satisfied

$$\left| \mathrm{Tr} J_T(\overline{x}, \overline{y}) \right| < 1 + \det J_T(\overline{x}, \overline{y}) < 2. \tag{4.12}$$

Now, these two conditions become

$$\frac{\beta_1 (A_2 - \gamma_2)}{\alpha_2} + \frac{\gamma_2}{A_2} < 1 + \frac{\beta_1 \gamma_2 (A_2 - \gamma_2)}{\alpha_2 A_2} < 2. \tag{4.13}$$

Condition

$$\frac{\beta_1 (A_2 - \gamma_2)}{\alpha_2} + \frac{\gamma_2}{A_2} < 1 + \frac{\beta_1 \gamma_2 (A_2 - \gamma_2)}{\alpha_2 A_2} \tag{4.14}$$

implies that

$$(A_2 - \gamma_2)\beta_1 < \alpha_2. \tag{4.15}$$

If $\beta_1 < \alpha_2/(A_2 - \gamma_2)$, then this condition is satisfied.

Condition $1 + (\beta_1\gamma_2(A_2 - \gamma_2))/\alpha_2 A_2 < 2$ is equivalent to

$$\frac{\beta_1\gamma_2(A_2 - \gamma_2)}{\alpha_2 A_2} < 1. \tag{4.16}$$

It is easy to see that the condition is satisfied if $\beta_1 < \alpha_2/(A_2 - \gamma_2)$.

Next, we prove (ii).

An equilibrium point is a saddle if and only if the following conditions are satisfied

$$\left|\mathrm{Tr} J_T(\overline{x}, \overline{y})\right| > \left|1 + \det J_T(\overline{x}, \overline{y})\right|,$$
$$\mathrm{Tr}^2 J_T(\overline{x}, \overline{y}) - 4 \det J_T(\overline{x}, \overline{y}) > 0. \tag{4.17}$$

The first condition is equivalent to

$$(A_2 - \gamma_2)\beta_1 > \alpha_2, \tag{4.18}$$

which is satisfied if $\beta_1 > \alpha_2/(A_2 - \gamma_2)$. The second condition is equivalent to

$$\left(\frac{\beta_1(A_2 - \gamma_2)}{\alpha_2} - \frac{\gamma_2}{A_2}\right)^2 > 0. \tag{4.19}$$

Finally, we prove (iii).

An equilibrium point is nonhyperbolic if the following conditions are satisfied

$$\left|\mathrm{Tr} J_T(\overline{x}, \overline{y})\right| = \left|1 + \det J_T(\overline{x}, \overline{y})\right|,$$
$$\left(\det J_T(\overline{x}, \overline{y}) = 1 \quad \text{or} \quad \left|Tr J_T(\overline{x}, \overline{y})\right| \le 2\right). \tag{4.20}$$

The first condition is equivalent to

$$(A_2 - \gamma_2)\beta_1 = \alpha_2, \tag{4.21}$$

which is satisfied if $\beta_1 = \alpha_2/A_2 - \gamma_2$.

The second condition becomes

$$\frac{\beta_1(A_2 - \gamma_2)}{\alpha_2} + \frac{\gamma_2}{A_2} = 1 + \frac{\gamma_2}{A_2} < 2, \tag{4.22}$$

establishing part (iii). □

We now perform a similar analysis for the other cases in Table 1.

Theorem 4.2. *Assume that*

$$A_2 > \gamma_2, \qquad B_1(\gamma_2 - A_2) + \sqrt{4B_1\alpha_2} < \beta_1 < \frac{\alpha_2}{A_2 - \gamma_2}, \qquad \alpha_2 > B_1(A_2 - \gamma_2)^2. \qquad (4.23)$$

Then E_1, E_2, and E_3 exist and

(i) *the equilibrium point E_1 is locally asymptotically stable,*

(ii) *the equilibrium point E_2 is a saddle point. Furthermore, if $B_1(\gamma_2 - A_2) + \sqrt{4B_2\alpha_2} < \beta_1 < \alpha_2/(A_2 + B_1\gamma_2)$ and $\alpha_2 > B_1 A_2^2$, then the smaller eigenvalue belongs to the interval $(-1, 0)$, and the larger eigenvalue belongs to $(1, +\infty)$. In all other cases, the smaller eigenvalues is in $(0, 1)$.*

That is $|\lambda_1| < 1$ is given by

$$\lambda_1 = \frac{\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2 - \sqrt{\left(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2\right)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}}{2\beta_1(A_2 + \overline{x}_2)}, \qquad (4.24)$$

and the corresponding eigenvector is

$$v_1 = \Bigg(\beta_1\gamma_2 - (A_2 + \overline{x}_2)\overline{y}_2$$

$$+ \sqrt{\left(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2\right)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}, 2\beta_1\overline{y}_2\Bigg). \qquad (4.25)$$

Eigenvalue λ_2, where $|\lambda_2| > 1$, is given by

$$\lambda_2 = \frac{\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2 + \sqrt{\left(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2\right)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}}{2\beta_1(A_2 + \overline{x}_2)}. \qquad (4.26)$$

(iii) *The equilibrium point E_3 is locally asymptotically stable.*

Proof. By Theorem 4.1, (i) holds.

Evaluating the Jacobian matrix (4.3) at the equilibrium point E_2, we obtain

$$J_T = \begin{pmatrix} \dfrac{\overline{y}}{\beta_1} & -\dfrac{\overline{x}}{\beta_1} \\[2ex] -\dfrac{\overline{y}}{A_2 + \overline{x}} & \dfrac{\gamma_2}{A_2 + \overline{x}} \end{pmatrix}. \qquad (4.27)$$

Note that the Jacobian matrix (4.27) implies that the map T is strongly competitive.

The determinant of (4.27) is given by

$$\det = \frac{\overline{y}\gamma_2}{\beta_1(A_2 + \overline{x})} - \frac{\overline{x}\,\overline{y}}{\beta_1(A_2 + \overline{x})}, \qquad (4.28)$$

and the trace of (4.27) is given by

$$\mathrm{Tr} J_T(\overline{x}, \overline{y}) = \frac{\overline{y}}{\beta_1} + \frac{\gamma_2}{A_2 + \overline{x}}. \tag{4.29}$$

The equilibrium point E_2 is a saddle if and only if (4.17) is satisfied. The first condition is equivalent to

$$\frac{\overline{y}}{\beta_1} + \frac{\gamma_2}{A_2 + \overline{x}} > \left| 1 + \frac{\overline{y}\gamma}{\beta_1(A_2 + \overline{x})} - \frac{\overline{x}\,\overline{y}}{\beta_1(A_2 + \overline{x})} \right| \tag{4.30}$$

which is equivalent to

$$\overline{y}(A_2 + \overline{x}) + \beta_1 \gamma_2 > \beta_1(A_2 + \overline{x}) + \overline{y}(\gamma_2 - \overline{x})$$

$$\implies \overline{y}(A_2 + \overline{x}) + \beta_1(\overline{x} - (A_2 - \gamma_2)) - \overline{y}(\gamma_2 - \overline{x}) > 0$$

$$\implies \overline{y}(A_2 + \overline{x} - \gamma_2) - \beta_1(A_2 - \gamma_2 + \overline{x}) > -\overline{x}\,\overline{y} \tag{4.31}$$

$$\implies (A_2 - \gamma_2 + \overline{x})(\overline{y} - \beta_1) > -\overline{x}\,\overline{y}$$

$$\implies (\beta_1 - \overline{y})(A_2 - \gamma_2 + \overline{x}) < \overline{x}\,\overline{y}.$$

In light of (3.3) $\beta_1 - \overline{y}_2 = B_1 \overline{x}_2$, and by using (3.4), $A_2 - \gamma_2 + \overline{x}_2 = \overline{y}_3/B_1$. Now, we have

$$B_1 \overline{x}_2 \frac{\overline{y}_3}{B_1} < \overline{x}_2 \overline{y}_2. \tag{4.32}$$

This implies that $\overline{y}_3 < \overline{y}_2$ which is true.
  Condition

$$\mathrm{Tr}^2 J_T(\overline{x}, \overline{y}) - 4 \det J_T(\overline{x}, \overline{y}) > 0 \tag{4.33}$$

is equivalent to

$$\left(\frac{\overline{y}_2}{\beta_1} - \frac{\gamma_2}{\beta_1(A_2 + \overline{x}_2)} \right)^2 + \frac{\overline{x}_2 \overline{y}_2}{\beta_2(A_2 + \overline{x}_2)} > 0, \tag{4.34}$$

which is true.
  To prove the second part of the statement (ii), we use the characteristic equation (4.6) of System (1.1) at the equilibrium point. Now, we have

$$\lambda_1 + \lambda_2 = \frac{\overline{y}}{\beta_1} + \frac{\overline{y}\gamma_2}{(A_2 + \overline{x})\beta_1}, \qquad \lambda_1 \lambda_2 = \frac{\overline{y}(\gamma_2 - \overline{x})}{(A_2 + \overline{x})\beta_1}. \tag{4.35}$$

Since the map T is strongly competitive, the Jacobian matrix (4.27) has two real and distinct eigenvalues, the larger one in absolute value being positive.

The first equation implies that either both eigenvalues are positive, or the smaller one is negative. First, we show that, under hypothesis (ii) of theorem, the product of these two eigenvalues is less than zero. In order to prove that, it is enough to prove that $\gamma_2 - \overline{x}_2 < 0$.

We have

$$\begin{aligned}
\gamma_2 - \overline{x}_2 &= \gamma_2 - \frac{B_1(\gamma_2 - A_2) + \beta_1 - \sqrt{(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2}}{2B_1} \\
&= \frac{B_1(A_2 + \gamma_2) - \beta_1 + \sqrt{(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2}}{2B_1}.
\end{aligned}$$

(4.36)

Now $\gamma_2 - \overline{x}_2 < 0$ if and only if

$$B_1(A_2 + \gamma_2) - \beta_1 + \sqrt{(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2} < 0,$$

(4.37)

which holds if

$$\beta_1 > B_1(A_2 + \gamma_2), \qquad (B_1(A_2 + \gamma_2) - \beta_1)^2 - (B_1(\gamma_2 - A_2) - \beta_1)^2 + 4B_1\alpha_2 > 0,$$

$$B_1(A_2 + \gamma_2) < \beta_1 < \frac{\alpha_2}{A_2} + B_1\gamma_2,$$

$$\frac{\alpha_2}{A_2} + B_1\gamma_2 - (A_2B_1 + \gamma_2B_1) = \frac{\alpha_2 - A_2^2 B_1}{A_2} > 0 \iff B_1 < \frac{\alpha_2}{A_2^2}.$$

(4.38)

Also, we have

$$\frac{\alpha_2}{A_2} + B_1\gamma_2 - \left(B_1(\gamma_2 - A_2) + 2\sqrt{B_1\alpha_2}\right) \geq 0.$$

(4.39)

In all other cases $\gamma_2 - \overline{x}_2 > 0$.

This proves that the smaller eigenvalue is negative. Since the equilibrium point is a saddle point, it has to belong to $(-1, 0)$. The larger one belongs to $(1, +\infty)$. The proof of second statement is similar.

Now, we prove that E_3 is locally asymptotically stable.

Notice that

$$\left| \operatorname{Tr} J_T(\overline{x}, \overline{y}) \right| < 1 + \det J_T(\overline{x}, \overline{y}) < 2$$

(4.40)

implies that $\overline{y}_2 > \overline{y}_3$ which is true. $\square$

Theorem 4.3. *Assume that*

$$A_2 > \gamma_2, \qquad \beta_1 > (A_2 - \gamma_2)B_1, \qquad (B_1(A_2 - \gamma_2) + \beta_1)^2 - 4\alpha_2 B_1 = 0. \qquad (4.41)$$

Then $E_1, E_2 = E_3$ exist and:

(i) The equilibrium point E_1 is locally asymptotically stable.

(ii) The equilibrium point $E_2 = E_3 = (B_1(\gamma_2 - A_2) + \beta_1/2B_1, B_1(A_2 - \gamma_2) + \beta_1/2)$ is nonhyperbolic. The eigenvalues of the Jacobian matrix evaluated at E_2 are

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2}{2\beta_1(A_2 B_1 + \beta_1 + B_1\gamma_2)}, \qquad (4.42)$$

and the corresponding eigenvectors, respectively, are

$$\left(-\frac{1}{B_1}, 1\right), \qquad \left(\frac{(-A_2 B_1 + \beta_1 + B_1\gamma_2)(A_2 B_1 + \beta_1 + B_1\gamma_2)}{2B_1\beta_1(A_2 B_1 + \beta_1 - B_1\gamma_2)}, 1\right). \qquad (4.43)$$

Furthermore, if $\beta_1 > B_1(A_2 + \gamma_2)$, then $\lambda_2 \in (-1, 0)$. If $B_1(A_2 - \gamma_2) < \beta_1 < B_1(A_2 + \gamma_2)$, then $\lambda_2 \in (0, 1)$.

Proof. By Theorem 4.1, E_1 is a locally asymptotically stable.

Now, we prove that E_2 is nonhyperbolic.

The Jacobian matrix (4.3) at the equilibrium point $E_2 = E_3 = (B_1(\gamma_2 - A_2) + \beta_1/2B_1, B_1(A_2 - \gamma_2) + \beta_1/2)$ is

$$J_T(E_2) = \begin{pmatrix} \dfrac{\overline{y}}{\beta_1} & -\dfrac{\overline{x}}{\beta_1} \\ -\dfrac{\overline{y}}{A_2 + \overline{x}} & \dfrac{\gamma_2}{A_2 + \overline{x}} \end{pmatrix}. \qquad (4.44)$$

The eigenvalues of (4.44) satisfy

$$\left(\frac{B_1(A_2 - \gamma_2) + \beta_1}{2\beta_1} - \lambda\right)\left(\frac{2B_1\gamma_2}{2B_1 A_2 + B_1(\gamma_2 - A_2) + \beta_1} - \lambda\right)$$
$$- \frac{B_1(\gamma_2 - A_2) + \beta_1}{2\beta_1 B_1} \frac{B_1(B_1(A_2 - \gamma_2) + \beta_1)}{2B_1 A_2 + B_1(\gamma_2 - A_2) + \beta_1} = 0, \qquad (4.45)$$

and are given as

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2}{2\beta_1(A_2 B_1 + \beta_1 + B_1\gamma_2)}. \qquad (4.46)$$

Hence, E_2 is nonhyperbolic.

Notice that $|\lambda_2| < 1$. Now, we show that λ_2 can be in $(-1, 0)$ or $(0, 1)$.
We have

$$\lambda_2 = \frac{A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2}{2\beta_1(A_2 B_1 + \beta_1 + B_1\gamma_2)} = \frac{(B_1(A_2 + \gamma_2) - \beta_1)(B_1(A_2 - \gamma_2) + \beta_1)}{2\beta_1(A_2 B_1 + \beta_1 + B_1\gamma_2)}, \qquad (4.47)$$

which is negative if $\beta_1 > B_1(A_2 + \gamma_2)$, and positive if $\beta_1 < B_1(A_2 + \gamma_2)$. $\square$

Theorem 4.4. *Assume that*

$$A_2 > \gamma_2, \qquad \beta_1 > \frac{\alpha_2}{A_2 - \gamma_2}. \qquad (4.48)$$

Then, E_1 and E_3 exist and

(i) *the equilibrium point E_1 is a saddle point,*

(ii) *the equilibrium point E_3 is locally asymptotically stable.*

Proof. By Theorem 4.1, (i) holds. Observe that the assumption of Theorem 4.4 implies that the x coordinate of the equilibrium point E_2 is less than zero.

The proof that the equilibrium point E_3 is locally asymptotically stable is similar to the corresponding proof of Theorem 4.2. $\square$

Theorem 4.5. *The following statements are true.*

(a) *Assume*

$$A_2 < \gamma_2, \qquad (B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 > 0, \qquad \beta_1 > B_1(\gamma_2 - A_2). \qquad (4.49)$$

Then E_2 and E_3 exist and

(i) *the equilibrium point E_2 is a saddle. If $\beta_1 > B_1(\gamma_2 - A_2)$, then the larger eigenvalue is in $(1, +\infty)$, the smaller eigenvalue is in $(-1, 0)$. If $B_1(\gamma_2 - A_2) < \beta_1 < B_1(\gamma_2 + A_2)$, then the smaller eigenvalue is in $(0, 1)$. That is, $|\lambda_1| < 1$ is given by*

$$\lambda_1 = \frac{\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2 - \sqrt{(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}}{2\beta_1(A_2 + \overline{x}_2)}, \qquad (4.50)$$

and the corresponding eigenvector is

$$v_1 = \Big(\beta_1\gamma_2 - (A_2 + \overline{x}_2)\overline{y}_2$$

$$+ \sqrt{(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}, 2\beta_1\overline{y}_2 \Big). \qquad (4.51)$$

Eigenvalue λ_2, where $|\lambda_2| > 1$, is given by

$$\lambda_2 = \frac{\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2 + \sqrt{(\beta_1\gamma_2 + (A_2 + \overline{x}_2)\overline{y}_2)^2 + 4\beta_1(A_2 + \overline{x}_2)(\overline{x}_2 - \gamma_2)\overline{y}_2}}{2\beta_1(A_2 + \overline{x}_2)}. \tag{4.52}$$

(ii) *The equilibrium point E_3 is locally asymptotically stable.*

(b) *Assume that $A_2 < \gamma_2$, $(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 = 0$, $\beta_1 > B_1(\gamma_2 - A_2)$.Then there exists a unique positive equilibrium $E_2 = E_3 = (B_1(\gamma_2 - A_2) + \beta_1/2B_1, \ B_1(A_2 - \gamma_2) + \beta_1/2)$ which is nonhyperbolic. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2/2\beta_1(A_2B_1 + \beta_1 + B_1\gamma_2)$, and the corresponding eigenvectors, respectively, are*

$$\left(-\frac{1}{B_1}, 1\right), \qquad \left(\frac{(-A_2B_1 + \beta_1 + B_1\gamma_2)(A_2B_1 + \beta_1 + B_1\gamma_2)}{2B_1\beta_1(A_2B_1 + \beta_1 - B_1\gamma_2)}, 1\right). \tag{4.53}$$

If $\beta_1 > B_1(A_2 + \gamma_2)$, then $\lambda_2 \in (-1,0)$. If $B_1(\gamma_2 - A_2) < \beta_1 < B_1(\gamma_2 + A_2)$ then $\lambda_2 \in (0,1)$.

Proof. The proof of statements (a) is similar to the proof of the statements (ii) and (iii) of the Theorem 4.2.

Now, we prove statement (b).

The characteristic equation of the system (1.1) at the equilibrium point $E_2 = E_3$ has the form

$$\left(\frac{B_1(A_2 - \gamma_2) + \beta_1}{2\beta_1} - \lambda\right)\left(\frac{2B_1\gamma_2}{2B_1A_2 + B_1(\gamma_2 - A_2) + \beta_1} - \lambda\right)$$
$$-\frac{B_1(\gamma_2 - A_2) + \beta_1}{2\beta_1 B_1}\frac{B_1(B_1(A_2 - \gamma_2) + \beta_1)}{2B_1A_2 + B_1(\gamma_2 - A_2) + \beta_1} = 0 \tag{4.54}$$

in which solutions are eigenvalues of $J_T(E_2)$

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2}{2\beta_1(A_2B_1 + \beta_1 + B_1\gamma_2)}. \tag{4.55}$$

Hence, $E_2 = E_3$ is nonhyperbolic. Notice that $|\lambda_2| < 1$.

Now, we determine that the sign of λ_2

$$\lambda_2 = \frac{A_2^2 B_1^2 - \beta_1^2 + 2B_1\beta_1\gamma_2 - B_1^2\gamma_2^2}{2\beta_1(A_2B_1 + \beta_1 + B_1\gamma_2)} = \frac{(B_1(A_2 + \gamma_2) - \beta_1)(B_1(A_2 - \gamma_2) + \beta_1)}{2\beta_1(A_2B_1 + \beta_1 + B_1\gamma_2)} \tag{4.56}$$

is negative if $\beta_1 > B_1(A_2 + \gamma_2)$, and positive if $B_1(\gamma_2 - A_2) < \beta_1 < B_1(\gamma_2 + A_2)$. □

Now, we consider the special case of the system (1.1) when $A_2 = \gamma_2$.
In this case, the system (1.1) becomes

$$
\begin{aligned}
x_{n+1} &= \frac{\beta_1 x_n}{B_1 x_n + y_n}, \\
y_{n+1} &= \frac{\alpha_2 + A_2 y_n}{A_2 + x_n},
\end{aligned}
\qquad n = 0, 1, 2, \ldots. \tag{4.57}
$$

If the following condition holds

$$
\beta_1^2 - 4B_1 \alpha_2 \geq 0, \tag{4.58}
$$

then the system (4.57) has two positive equilibrium points

$$
\begin{aligned}
e_1 &= \left(\frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2B_1}, \ \frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2} \right), \\
e_2 &= \left(\frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2B_1}, \ \frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2} \right).
\end{aligned} \tag{4.59}
$$

We prove the following.

Theorem 4.6. *Assume that*

$$
A_2 = \gamma_2. \tag{4.60}
$$

Then the following statements hold.

(i) *If $\beta_1^2 - 4B_1 \alpha_2 = 0$, then the system (4.57) has the unique equilibrium point $e = (\beta_1/2B_1, \beta_1/2)$ which is nonhyperbolic. The following holds:*

    (a) *If $\alpha_2 < B_1 A_2^2$, then $\lambda_1 = 1$, and $\lambda_2 = 2A_2 B_1 - \beta_1/2(2A_2 B_1 + \beta_1) \in (0, 1)$, and the corresponding eigenvectors, respectively, are*

$$
\left(-\frac{1}{B_1}, 1 \right), \qquad \left(\frac{(-2A_2 B_1 - \beta_1)}{2B_1 \beta_1}, 1 \right). \tag{4.61}
$$

    (b) *If $\alpha_2 > B_1 A_2^2$, then $\lambda_1 = 1$ and $\lambda_2 = 2A_2 B_1 - \beta_1/2(2A_2 B_1 + \beta_1) \in (-1, 0)$, and the corresponding eigenvectors, respectively, are*

$$
\left(-\frac{1}{B_1}, 1 \right), \qquad \left(\frac{(-2A_2 B_1 - \beta_1)}{2B_1 \beta_1}, 1 \right). \tag{4.62}
$$

(c) If $\alpha_2 = B_1 A_2^2$, then $\lambda_1 = 1$ and $\lambda_2 = 0$, and the corresponding eigenvectors, respectively, are

$$\left(-\frac{1}{B_1}, 1\right), \qquad \left(\frac{1}{B_1}, 1\right). \tag{4.63}$$

(ii) If $\beta_1^2 - 4B_1\alpha_2 > 0$, then the system (4.57) has two positive equilibrium points: e_1 is locally asymptotically stable and e_2 is a saddle point. The following holds.

(d) If $2A_2 B_1 > \beta_1$ or $2A_2 B_1 \le \beta_1$ and $B_1 A_2^2 + \alpha_2 < A_2\beta_1$, then $\lambda_2 \in (1, +\infty)$ and $\lambda_1 \in (0, 1)$.

(e) If $2A_2 B_1 < \beta_1$ and $B_1 A_2^2 + \alpha_2 > A_2\beta_1$, then $\lambda_2 \in (1, +\infty)$ and $\lambda_1 \in (-1, 0)$, where

$$\lambda_1 = \frac{2\alpha_2 B_1 + 3A_2\beta_1 B_1 + A_2 B_1 \sqrt{\beta_1^2 - 4B_1\alpha_2} - \sqrt{\mathcal{F}}}{2\beta_1\left(2A_2 B_1 + \beta_1 - \sqrt{\beta_1^2 - 4B_1\alpha_2}\right)}, \tag{4.64}$$

where

$$\mathcal{F} = \left(2\alpha_2 B_1 + 3A_2\beta_1 B_1 + A_2\sqrt{\beta_1^2 - 4B_1\alpha_2}B_1\right)^2 \\ - 8B_1\beta_1\left(B_1\beta_1 A_2^2 - \alpha_2\beta_1 + \left(B_1 A_2^2 + \alpha_2\right)\sqrt{\beta_1^2 - 4B_1\alpha_2}\right), \tag{4.65}$$

and $|\lambda_1| < 1$.

The corresponding eigenvector for both cases (c) and (d) is $v_1 = (v_1^{(1)}, v_2^{(1)})$, where

$$v_1^{(1)} = \left(2A_2 B_1 + \beta_1 - \sqrt{\beta_1^2 - 4B_1\alpha_2}\right)\left(-2\alpha_2 B_1 + A_2\beta_1 B_1 - A_2 B_1\sqrt{\beta_1^2 - 4B_1\alpha_2} + \sqrt{\mathcal{D}}\right),$$

$$v_2^{(1)} = 4A_2\beta_1^2 B_1^2 + 8\alpha_2\beta_1 B_1^2 + 4A_2\beta_1 B_1^2\sqrt{\beta_1^2 - 4B_1\alpha_2}, \tag{4.66}$$

where

$$\mathcal{D} = 2B_1\Big(-2A_2^2\alpha_2 B_1^2 + 2\alpha_2^2 B_1 + A_2^2\beta_1^2 B_1 + 6A_2\alpha_2\beta_1 B_1 + 4\alpha_2\beta_1^2 \\ + \left(-B_1\beta_1 A_2^2 + 2B_1\alpha_2 A_2 - 4\alpha_2\beta_1\right)\sqrt{\beta_1^2 - 4B_1\alpha_2}\Big). \tag{4.67}$$

Proof. Assume that $\beta_1^2 - 4B_1\alpha_2 = 0$. Then we have that

$$e_1 = e_2 = e = \left(\frac{\beta_1}{2B_1}, \frac{\beta_1}{2}\right). \tag{4.68}$$

The characteristic equation associated to the system (4.57) at the equilibrium point e is given by

$$\lambda^2 - \lambda \left(\frac{1}{2} - \frac{2B_1 A_2}{2B_1 A_2 + \beta_1} \right) + \frac{B_1 A_2}{2B_1 A_2 + \beta_1} - \frac{\beta_1}{2(2B_1 A_2 + \beta_1)} = 0. \tag{4.69}$$

Solutions of (4.69) are

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{2A_2 B_1 - \beta_1}{2(2A_2 B_1 + \beta_1)}. \tag{4.70}$$

It is easy to see that $|\lambda_2| < 1$.

Now, assume that $\alpha_2 < B_1 A_2^2$. Since $\beta_1 > 0$, from $\beta_1^2 - 4B_1 \alpha_2 = 0$, we have $\beta_1 = 2\sqrt{B_1 \alpha_2}$. The numerator of λ_2 is $2A_2 B_1 - 2\sqrt{B_1 \alpha_2} > 0$. Assume the opposite, that is, $2A_2 B_1 - 2\sqrt{B_1 \alpha_2} < 0$. Then, we have

$$A_2^2 B_1^2 < B_1 \alpha_2 \Longrightarrow \alpha_2 > B_1 A_2^2 \tag{4.71}$$

which is a contradiction. So, we confirmed (a).

Assume that $\alpha_2 > B_1 A_2^2$. Then the numerator of $\lambda_2 < 0$. If $2A_2 B_1 - 2\sqrt{B_1 \alpha_2} > 0$, then we have

$$A_2^2 B_1^2 > B_1 \alpha_2 \Longrightarrow \alpha_2 < B_1 A_2^2 \tag{4.72}$$

which is a contradiction. So, (b) holds.

Assume that $\beta_1^2 - 4B_1 \alpha_2 > 0$. Then there are two positive equilibrium points

$$e_1 = \left(\frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2B_1}, \frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2} \right),$$

$$e_2 = \left(\frac{\beta_1 - \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2B_1}, \frac{\beta_1 + \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2} \right). \tag{4.73}$$

Now, we prove that e_1 is a locally asymptotically stable equilibrium point.

We check the conditions for locally asymptotically stable equilibrium point. We have

$$\frac{\overline{y}}{\beta_1} + \frac{A_2}{A_2 + x} < 1 + \frac{\overline{y} A_2}{\beta_1(A_2 + x)} - \frac{\overline{x}\,\overline{y}}{\beta_1(A_2 + \overline{x})}. \tag{4.74}$$

This implies that

$$y(A_2 + \overline{x}) + \beta_1 A_2 < \beta_1(A_2 + \overline{x}) + \overline{y} A_2 - \overline{x}\,\overline{y}, \tag{4.75}$$

which is equivalent to $\bar{x}(2\bar{y} - \beta_1) < 0$ and

$$2\bar{y} - \beta_1 = \beta_1 - \sqrt{\beta_1^2 - 4B_1\alpha_2} - \beta_1 = -\sqrt{\beta_1^2 - 4B_1\alpha_2} < 0, \tag{4.76}$$

which is true.

Now, we check condition $1 + \det J_T(\bar{x}, \bar{y}) < 2$. We have that

$$\frac{\bar{y}A_2}{\beta_1(A_2 + \bar{x})} - \frac{\bar{x}\,\bar{y}}{\beta_1(A_2 + \bar{x})} < 1. \tag{4.77}$$

This implies that $\bar{y}A_2 - \bar{x}\,\bar{y} < \beta_1 A_2 + \beta_1 \bar{x}$ which is true, since

$$-\frac{A_2\sqrt{B_1^2 - 4B_1\alpha_2}}{2} - \alpha_2 < \frac{\beta_1 A_2}{2} + \frac{\beta_1^2 + \beta_1\sqrt{\beta_1^2 - 4B_1\alpha_2}}{2B_1}. \tag{4.78}$$

Hence, e_1 is a locally asymptotically stable equilibrium point.

Now, we prove that e_2 is a saddle. We check the condition (4.17).

Condition $|\text{Tr } J_T(\bar{x}, \bar{y})| > |1 + \det J_T(\bar{x}, \bar{y})|$ is equivalent to $\bar{x}(2\bar{y} - \beta_1) > 0$. This is true, since

$$2\bar{y} - \beta_1 = \beta_1 + \sqrt{\beta_1^2 - 4B_1\alpha_2} - \beta_1 = -\sqrt{\beta_1^2 - 4B_1\alpha_2} > 0. \tag{4.79}$$

Condition

$$\text{Tr}^2 J_T(\bar{x}, \bar{y}) - 4\det J_T(\bar{x}, \bar{y}) > 0 \tag{4.80}$$

is equivalent to

$$\left(\frac{\bar{y}}{\beta_1} - \frac{A_2}{A_2 + \bar{x}}\right)^2 + \frac{\bar{x}\,\bar{y}}{\beta_1(A_2 + \bar{x})} > 0. \tag{4.81}$$

Hence, e_2 is a saddle.

Now, we prove the statements (c) and (d).

The characteristic equation associated to the system (4.57) at the equilibrium point has the following form

$$\lambda^2 - \lambda\left(\frac{\bar{y}}{B_1\bar{x} + \bar{y}} + \frac{A_2}{A_2 + \bar{x}}\right) + \frac{\bar{y}(A_2 - \bar{x})}{(A_2 + \bar{x})(B_1\bar{x} + \bar{y})} = 0. \tag{4.82}$$

Now, we have that

$$\lambda_1 + \lambda_2 = \frac{\bar{y}}{B_1\bar{x} + \bar{y}} + \frac{A_2}{A_2 + \bar{x}} > 0, \qquad \lambda_1\lambda_2 = \frac{\bar{y}(A_2 - \bar{x})}{(A_2 + \bar{x})(B_1\bar{x} + \bar{y})}. \tag{4.83}$$

Consider $A_2 - \overline{x}$. We have that

$$A_2 - \overline{x} = \frac{2B_1 A_2 - \beta_1 + \sqrt{\beta_1^2 - 4B_1 \alpha_2}}{2B_1},$$

(4.84)

$$\beta_1^2 - 4B_1 \alpha_2 - \left(\beta_1 - 2B_1 A_2\right)^2 = -4B_1 \left(B_1 A_2^2 - \beta_1 A_2 + \alpha_2\right)$$

which implies statements (c) and (d).   □

5. Global Behavior

In this section, we present the results on global behavior of the system (1.1).

Theorem 5.1. *Table 2 describes the global behavior of the system* (1.1)

Proof. Throughout the proof of theorem $\preceq$ will denote $\preceq_{\mathrm{se}}$.

$(\mathcal{R}_1, \mathcal{R}_2)$ In view of **Lemma 3.2**, the map T which corresponds to the system (1.1) has an attractive and invariant box $\mathcal{B} = [0, \beta_1/B_1] \times [L_y, U_y]$, where $L_y = B_1 \alpha_2 / B_1 (A_2 - \gamma_2) + \beta_1$, $U_y = \alpha_2 / (A_2 - \gamma_2)$, which contains a unique fixed point E_1. By **Theorem 2.1**, every solution of the system (1.1) converges to E_1. Clearly, the basin of attraction of the equilibrium point E_1 is given by $[0, +\infty)^2 \setminus \{(0,0)\}$.

$(\mathcal{R}_3)$ By **Lemma 3.2** $x_0 = 0$ implies that $x_n = 0$, for all $n \in \mathbb{N}$, and $y_n \to \alpha_2/(A_2 - \gamma_2)$, $n \to \infty$, which shows that y-axes is a subset of the basin of attraction $\mathcal{B}(E_1)$. Furthermore, every solution of (1.1) enters and stays in the box $\mathcal{B}$, and so we can restrict our attention to solutions that starts in $\mathcal{B}$. Clearly the set $Q_4(E_3) \cap \mathcal{B}$ is an invariant set with a single equilibrium point E_3, and so every solution that starts there is attracted to E_3. In view of **Corollary 2.2**, the interior of rectangle $[\![E_1, E_3]\!]$ is attracted to either E_1 or E_3, and because E_3 is the local attractor, it is attracted to E_3. If $(x, y) \in \mathcal{B} \setminus ([\![E_1, E_3]\!] \cup (Q_4(E_3) \cap \mathcal{B}) \cup \{(0, y) : y \geq 0\})$, then there exist the points $(x_l, y_l) \in [\![E_1, E_3]\!]$ and $(x_u, y_u) \in Q_4(E_3) \cap \mathcal{B}$ such that $(x_l, y_l) \preceq_{\mathrm{se}} (x, y) \preceq_{\mathrm{se}} (x_u, y_u)$. Consequently, $T^n((x_l, y_l)) \preceq_{\mathrm{se}} T^n((x, y)) \preceq_{\mathrm{se}} T^n((x_u, y_u))$ for all $n = 1, 2, \ldots$ and so $T^n((x, y)) \to E_3$ as $n \to \infty$, which completes the proof.

$(\mathcal{R}_4)$ The first part of this Theorem is proven in **Theorem 4.4**.

Now, we prove a global result

$$J_T(E_1) = \begin{pmatrix} \dfrac{\overline{y}}{\beta_1} & 0 \\[2ex] -\dfrac{\overline{y}}{A_2} & \dfrac{\gamma_2}{A_2} \end{pmatrix}.$$

(5.1)

The eigenvalues of $J_T(E_1)$ are given by $\lambda_1 = \beta_1/y$ and $\lambda_2 = \gamma_2/A_2$ and so

$$\beta_1 > \frac{\alpha_2}{A_2 - \gamma_2} \implies \lambda_1 > 1, \quad A_2 > \gamma_2 \implies \lambda_2 < 1.$$

(5.2)

The eigenvector of T at E_1 that corresponds to the eigenvalue $\lambda_2 < 1$ is $(0, y)$.

Table 2: Global behavior of system (1.1). GAS stands for globally asymptotically stable.

Region		Global behavior
$\mathcal{R}_1$	$A_2 > \gamma_2,\ \alpha_2 > \dfrac{(\beta_1 - B_1(A_2 - \gamma_2))^2}{4B_1}$, or $A_2 > \gamma_2,\ \alpha_2 \le \dfrac{(\beta_1 - B_1(A_2 - \gamma_2))^2}{4B_1}$, $\beta_1 < \dfrac{\alpha_2}{A_2 - \gamma_2},\ B_1 > \dfrac{\beta_1}{A_2 - \gamma_2}$	There exists a unique equilibrium $E_1(0, \alpha_2/(A_2 - \gamma_2))$, and it is G.A.S.. The basin of attraction of E_1 is $[0, +\infty)^2 \setminus \{(0,0)\}$
$\mathcal{R}_2$		
$\mathcal{R}_3$	$A_2 > \gamma_2,\ \beta_1 = \dfrac{\alpha_2}{(A_2 - \gamma_2)}$, $B_1 < \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$	There exist two equilibrium points $E = E_1 = E_2 = (0, \alpha_2/(A_2 - \gamma_2)) = (0, \beta_1)$ which is nonhyperbolic, and E_3, which is locally asymptotically stable. Furthermore, the positive part of the y-$axes$ is the basin of attraction $\mathcal{B}(E_1)$ of E_1. The equilibrium point E_3 is globally asymptotically stable with the basin of attraction $\mathcal{B}(E_3) = (0, \infty) \times [0, \infty)$
$\mathcal{R}_4$	$A_2 > \gamma_2,\ \beta_1 = \dfrac{\alpha_2}{A_2 - \gamma_2}$, $B_1 < \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$	There exist two equilibrium points E_1 which are a saddle, and E_3, which is a locally asymptotically stable equilibrium point. Furthermore, the positive part of y-axes is the global stable manifold of $\mathcal{W}^s(E_1)$. The equilibrium point E_3 is G.A.S. with the basin of attraction $\mathcal{B}(E_3) = (0, \infty) \times [0, \infty)$
$\mathcal{R}_5$	$\gamma_2 < A_2$, $B_1(\gamma_2 - A_2) + \sqrt{4B_1\alpha_2} < \beta_1 < \dfrac{\alpha_2}{A_2 - \gamma_2}$, $B_1 < \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$	There exist three equilibrium points E_1 and E_3 which are locally asymptotically stable and E_2 which is a saddle. The global stable manifold $\mathcal{W}^s(E_2)$ separates the positive quadrant so that all orbits which start below this manifold are attracted to the point E_3, and all orbits which start above this manifold are attracted to the equilibrium point E_1. All orbits that start on $\mathcal{W}^s(E_2)$ are attracted to E_2. The global unstable manifold $\mathcal{W}^u(E_2)$ is the graph of a continuous strictly decreasing function, and $\mathcal{W}^u(E_2)$ has endpoints E_a and E_3
$\mathcal{R}_6$	$\gamma_2 < A_2,\ (A_2 - \gamma_2)B_1 < \beta_1$, $(B_1(A_2 - \gamma_2) + \beta_1)^2 - 4\alpha_2 B_1 = 0$	There exist two equilibrium points E_1, which is locally asymptotically stable, and $E_2 \equiv E_3$, which is non-hyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_{E_2}$ which is a part of the basin of attraction of E_2. Every solution that starts above this curve is attracted to the equilibrium point E_1; every solution that starts below this curve converges to E_2
$\mathcal{R}_7$	$\gamma_2 < A_2,\ \beta_1 = \dfrac{\alpha_2}{A_2 - \gamma_2}$, $B_1 \ge \dfrac{\alpha_2}{(A_2 - \gamma_2)^2}$	There exists a unique equilibrium point $E_1 = E_2 = E_3$ which is nonhyperbolic. All orbits are attracted to the equilibrium point E_1
$\mathcal{R}_8$	$A_2 < \gamma_2$, $(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 > 0$, $\beta_1 > B_1(\gamma_2 - A_2)$	There exist two equilibrium points E_2, which is a saddle, and E_3, which is a locally asymptotically stable equilibrium point. Furthermore, there exists the global stable manifold $\mathcal{W}^2(E_2)$ that separates the positive quadrant so that all orbits below this manifold are attracted to the equilibrium point E_3, and all orbits above this manifold are asymptotic to $(0, \infty)$. All orbits that start on $\mathcal{W}^s(E_2)$ are attracted to E_2. The global unstable manifold $\mathcal{W}^u(E_2)$ is the graph of a continuous strictly decreasing function, and $\mathcal{W}^u(E_2)$ has endpoints E_2 and E_3

Table 2: Continued.

Region		Global behavior
$\mathcal{R}_9$	$A_2 < \gamma_2,$ $(B_1(\gamma_2 - A_2) - \beta_1)^2 - 4B_1\alpha_2 > 0,$ $\beta_1 > B_1(\gamma_2 - A_2)$	There exists a unique equilibrium point $E = E_2 = E_3$ which is nonhyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_E$ which is a part of the basin of attraction of E. Every solution that stars below this curve is attracted to E; every solution that starts above this curve is asymptotic to $(0, \infty)$
$\mathcal{R}_{10}$	$A_2 = \gamma_2, \beta_1^2 - 4B_1\alpha_2 = 0,$	There exists a unique equilibrium $e = (\beta_1/2B_1, \beta_1/2)$ which is nonhyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_E$, which is a part of the basin of attraction of E. Every solution that starts below this curve is attracted to the equilibrium point: every solution that starts above this curve is asymptotic to $(0, \infty)$
$\mathcal{R}_{11}$	$A_2 = \gamma_2, \beta_1^2 - 4B_1\alpha_2 > 0$	There exist two equilibrium points e_1, which is locally asymptotically stable, and e_2, which is a saddle equilibrium point. Furthermore, there exists the global stable manifold $W^s(e_2)$ that separates the positive quadrant so that all orbits below this manifold are attracted to the equilibrium point e_1, and allorbits above this manifold are asymptotic to $(0, \infty)$. All orbits that starts on $W^s(e_2)$ are attracted to e_2. The global unstable manifold $W^u(e_2)$ is the graph of a continuous strictly decreasing function, and $W^u(e_2)$ has an endpoint at e_1
$\mathcal{R}_{12}$	$\gamma_2 \geq A_2$ and either $\alpha_2 \leq \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1},$ $\beta_1 < B_1(\gamma_2 - A_2)$ or	The system (1.1) does not possess an equilibrium point. Its global behavior is described as follows
$\mathcal{R}_{13}$	$\alpha_2 > \dfrac{(\beta_1 + B_1(A_2 - \gamma_2))^2}{4B_1}$	$x_n \to 0 \quad y_n \to \infty, \quad n \to \infty$

The rest of the proof is similar to the proof of part $(\mathcal{R}_3)$ and uses some continuity arguments.

$(\mathcal{R}_5)$ The first part of this Theorem is proven in Theorem 4.2. Lemma 3.3 states that the system (1.1) has no minimal period-two solution. Take that $\mathcal{R} = \mathbb{R}_+^2 \setminus \{(0,0)\}$. T is strongly monotone in $\mathcal{R}$ and differentiable in int $\mathcal{R} = \mathcal{R}^\circ$ (interior of $\mathcal{R}$). E_2 is a saddle point and $E_2 \in \mathcal{R}^\circ$. Then, all hypothesis (a)–(d) of Theorem 2.6 are satisfied. In light of Theorems 2.6 and 2.7, there exist the global stable manifold $W^s(E_2)$ and the global unstable manifold $W^u(E_2)$ which are the graphs of a continuous strictly monotonic functions. The global stable manifold $W^s(E_2)$ separates the first quadrant into two invariant regions W_- (below the stable) manifold and W_+ (above the stable manifold) which are connected. Each orbit starting above $W^s(E_2)$ remains above and is asymptotic to E_1. Each orbit starting below $W^s(E_2)$ remains below and is asymptotic to E_3. This implies that E_1 and E_3 are global attractors. Theorem 4.2 implies that they are globally asymptotically stable.

$(\mathcal{R}_6)$ Notice that in this case the eigenvector which corresponds to nonhyperbolic eigenvalue $\lambda_1 = 1$ at E_2 is $\mathbf{v}_2 = (-1/B_1, 1)$, see Theorem 4.3. Thus, the hypotheses of Theorems 2.4 and 2.7 are satisfied at the equilibrium point E_2, and the conclusions

of Theorems 2.4, 2.5, and 2.7 follow. Let $\mathcal{C}$, $\mathcal{W}_-$ and $\mathcal{W}_+$ be the sets given in the conclusion of Theorems 2.4 and 2.7. Let $S := \{(x,y) : 0 \le x \le \beta_1/B_1,\ 0 \le y\}$. Since $\beta_1 x/B_1 x + y \le \beta_1/B_1$ for $x \ge 0$, $y \ge 0$, $x + y > 0$, the map T satisfies $T([0,\infty)^2 \setminus (0,0)) \subset S$. Thus, $T(\mathcal{C} \cup \mathcal{W}_+) \subset (\mathcal{C} \cup \mathcal{W}_+) \cap S$, which implies that $T(\mathcal{C} \cup \mathcal{W}_+)$ is bounded. In view of Theorem 2.7, every solution which starts in $\mathcal{W}_+$ eventually enters $Q_4(E_2)$, and so is in rectangle $S \cap Q_4(E_2)$, which by Theorem 2.1, implies that all such solutions converge to the equilibrium point E_2.

If (x,y) is in $\mathcal{W}_-$, by Theorem 2.7, the orbit of (x,y) eventually enters $Q_2(E_2)$. Assume (without loss of generality) that $(x,y) \in$ int $Q_2(E_2)$.

In view of Corollary 2.2 and the fact that E_1 is a local attractor $[\![E_1, E_2]\!]$ is a subset of the basin of attraction of E_1. Let (x_0, y_0) be any point in $\mathcal{W}_+$. Then there exists $\tilde{y}_0 \ge \max\{y_0, \alpha_2/A_2 - \gamma_2\}$ such that $(0, \tilde{y}_0) \preceq_{se} (x_0, y_0)$. Now $(0, \tilde{y}_0) \preceq_{se} (0, \tilde{y}_0) = (0, \alpha_2 + \gamma_2\tilde{y}_0/A_2)$, which implies that $\{T^n(0,\tilde{y}_0)\} = \{(0,\tilde{y}_n)\}$ is an increasing sequence, and so $\{\tilde{y}_n\}$ is a decreasing sequence and thus is convergent to $\alpha_2/A_2 - \gamma_2$. In view of $(0, \tilde{y}_0) \preceq_{se} (x_0, y_0)$, we conclude that $T^n((0,\tilde{y}_0)) \preceq_{se} T^n((x_0,y_0))$ and so $T^n((x_0,y_0))$ eventually enters $[\![E_1, E_2]\!]$, and so it converges to E_1.

($\mathcal{R}_7$) Let (x_0, y_0) be any point in $[0,\infty)^2 \setminus \{(0,0)\}$. Then there exist points $(\tilde{x}_0, 0)$ and $(0, \tilde{y}_0)$, $\tilde{x}_0 \ge \max\{x_0, \beta_1/B_1\}$, $\tilde{y}_0 \ge \max\{y_0, \alpha_2/(A_2 - \gamma_2)\}$ such that $(0, \tilde{y}_0) \preceq_{se} (x_0, y_0) \le (\tilde{x}_0, 0)$. This gives $T^n((0,\tilde{y}_0)) \preceq_{se} T^n((x_0,y_0)) \le T^n((\tilde{x}_0,0))$ for all $n \ge 1$. Clearly, $T(\tilde{x}_0, 0) = (\beta_1/B_1, \alpha_2/(A_2 + \tilde{x}_0)) \le (\tilde{x}_0, 0)$, which implies that $\{T^n(\tilde{x}_0,0)\} = \{(\tilde{x}_n, \tilde{y}_n)\}$ is a decreasing sequence bounded below by E_1 and so is convergent to E_1. Proof that $\{T^n(0,\tilde{y}_0)\}$ is convergent to E_1 is carried in a same way as in the proof of ($\mathcal{R}_6$). In view of $T^n((0,\tilde{y}_0)) \preceq_{se} T^n((x_0,y_0)) \le T^n((\tilde{x}_0,0))$ we conclude that $\{T^n((x_0,y_0))\}$ converges to E_1.

($\mathcal{R}_8$) Put $T_1(x,y) = \beta_1 x/(B_1 x + y)$, $T_2(x,y) = (\alpha_2 + \gamma_2 y)/(A_2 + x)$. Take $x = (x_0, y_0) \in W_+(E_2) \cap \mathcal{R}(-,+)$, where $\mathcal{R}(-,+) = \{(x,y) \in \mathcal{R} : T_1(x,y) < x,\ T_2(x,y) > y\}$. It is known that $\mathcal{R}(-,+)$ is an invariant set, see [11].

Then we have

$$T_1(x_0, y_0) = \frac{\beta_1 x_0}{B_1 x_0 + y_0} < x_0, \qquad T_2(x_0, y_0) = \frac{\alpha_2 + \gamma_2 y_0}{A_2 + x_0} > y_0. \tag{5.3}$$

This implies

$$(T_1(x_0,y_0), T_2(x_0,y_0)) \preceq_{se} (x_0, y_0) \Longleftrightarrow T(x_0,y_0) \preceq_{se} (x_0,y_0). \tag{5.4}$$

By using monotonicity $T^2(x_0,y_0) \preceq_{se} T(x_0,y_0)$. By using induction $T^{n+1}(x_0,y_0) \preceq_{se} T^n(x_0,y_0)$. This implies that sequence $\{x_n\}$ is non-increasing and $\{y_n\}$ is non-decreasing. By Lemma 3.2, $\{x_n\}$ is bounded, hence it must converges. By using equation for x_{n+1} we see that the limit is zero. Since, $\{y_n\}$ is unbounded and nondecreasing then $y_n \to \infty$, $n \to \infty$.

By Theorems 2.4 and 2.7, all orbits below this manifold are attracted to the equilibrium point E_3.

($\mathcal{R}_9$) Since the hypotheses of Theorems 2.4, and 2.7 are satisfied at the equilibrium point E_2, the conclusions of Theorems 2.4, 2.5, and 2.7 follow. Let $\mathcal{C}$, $\mathcal{W}_-$ and $\mathcal{W}_+$ be the

sets given in the conclusion of Theorems 2.4 and 2.7. Let $S := \{(x, y) : 0 \leq x \leq \beta_1/B_1, \ 0 \leq y\}$. Since $\beta_1 x/B_1 x + y \leq \beta_1/B_1$ for $x \geq 0$, $y \geq 0$, $x + y > 0$, the map T satisfies $T([0, \infty)^2 \setminus (0, 0)) \subset S$. Thus $T(\mathcal{C} \cup \mathcal{W}_+) \subset (\mathcal{C} \cup \mathcal{W}_+) \cap S$, which implies that $T(\mathcal{C} \cup \mathcal{W}_+)$ is bounded. In view of Theorem 2.7 every solution which starts in $\mathcal{W}_+$ eventually enters $Q_4(E_2)$ and so is in rectangle $S \cap Q_4(E_2)$, which by Theorem 2.1, implies that all such solutions converge to the equilibrium point E_2.

If (x, y) is in $\mathcal{W}_-$, by Theorem 2.7 the orbit of (x, y) eventually enters $Q_2(E_2)$. Assume (without loss of generality) that $(x, y) \in \text{int } Q_2(E_2)$.

A calculation gives

$$T(E_2 + t\mathbf{v}_2) = \left(\frac{\beta_1 + B_1(\gamma_2 - A_2) - 2t}{2B_1}, \frac{B_1(2\alpha_2 + \gamma_2(\beta_1 + B_1(A_2 - \gamma_2) + 2t)}{\beta_1 + B_1(A_2 + \gamma_2) - 2t} \right) \tag{5.5}$$

for all t and

$$\frac{d}{dt}T(E_2 + t\mathbf{v}_2) = \left(-\frac{1}{B_1}, \frac{2B_1(\beta_1 + 2\alpha_2 + B_1(A_2 + \gamma_2) + \beta_1\gamma_2 + B_1\gamma_2(A_2 - \gamma_2))}{(\beta_1 + B_1(A_2 + \gamma_2) - 2t)^2} \right),$$

$$\frac{1}{2}\frac{d^2}{dt^2}T(E_2 + t\mathbf{v}_2)|_{t=0} = \left(0, \frac{4B_1(\beta_1 + 2\alpha_2 + B_1(A_2 + \gamma_2) + \beta_1\gamma_2 + B_1\gamma_2(A_2 - \gamma_2))}{(\beta_1 + B_1(A_2 + \gamma_2) - 2t)^3} \right).$$

$$\tag{5.6}$$

Since in expansion (2.5) we have (c_2, d_2) with $d_2 > 0$ and $T(E_2 + t\mathbf{v}_2)^{(1)}$ is affine in t, by Theorem 2.10 in any relative neighborhood of E_2 there exists a subsolution $(w, z) \in Q_2(E_2)$, that is, $T(w, z) \preceq_{se} (w, z)$. Choose one such (w, z) so that $(x, y) \preceq_{se} (w, z)$. Since T is competitive, $T^{n+1}(w, z) \leq T^n(w, z)$ for $n = 0, 1, 2, \ldots$. The monotonically decreasing sequence $\{T^n(w, z)\}$ in $Q_2(E)$ is unbounded below, since if it were not it would converge to the unique fixed point in $Q_2(E)$, namely E, which is not possible. Let $(w_n, z_n) := T^n(w, z)$, $n = 0, 1, \ldots$. Then $(w_n, z_n) \in S$ for $n = 1, 2, \ldots$, hence $\{w_n\}$ is bounded. It follows that $\{z_n\}$ is monotone and unbounded. From (1.1) it follows that $w_n \to 0$. Since $T^n(x, y) \preceq_{se} (w_n, z_n)$, it follows that $T^n(x, y) \to (0, \infty)$.

$(\mathcal{R}_{10})$ The eigenvalues of the map T are $\lambda_1 = 1$ and $\lambda_2 = 2A_2B_1 - \beta_1/2(2A_2B_1 + \beta_1)$. The corresponding eigenvectors are

$$\left(-\frac{1}{B_1}, 1 \right), \qquad \left(\frac{2A_2B_1 + \beta_1}{2B_1\beta_1}, 1 \right). \tag{5.7}$$

The existence of the continuous curve $\mathcal{W}_E$ follows as in the proof of $(\mathcal{R}_9)$ as well as convergence of all points which start in $\mathcal{W}_+$.

If (x, y) is in $\mathcal{W}_-$, by Theorem 2.7 the orbit of (x, y) eventually enters $Q_2(E_2)$. Assume (without loss of generality) that $(x, y) \in \text{int } Q_2(E_2)$.

A straightforward calculation gives

$$T(E_2 + t\mathbf{v}_2) = \left(\frac{\beta_1 - 2t}{2B_1}, \frac{B_1(2\alpha_2 + A_2(\beta_1 + 2t))}{\beta_1 + 2B_1 A_2 - 2t} \right) \tag{5.8}$$

for all t and

$$\frac{d}{dt}T(E_2 + t\mathbf{v}_2) = \left(-\frac{1}{B_1}, \frac{(\beta_1 + 2B_1 A_2)^2}{(\beta_1 + 2B_1 A_2 - 2t)^2} \right),$$

$$\frac{1}{2}\frac{d^2}{dt^2}T(E_2 + t\mathbf{v}_2)|_{t=0} = \left(0, \frac{2}{\beta_1 + 2B_1 A_2} \right). \tag{5.9}$$

Since in expansion (2.5) we have (c_2, d_2) with $d_2 > 0$ and $T(E_2 + t\mathbf{v}_2)^{(1)}$ is affine in t, by Theorem 2.10 in any relative neighborhood of E_2 there exists a subsolution $(w, z) \in Q_2(E_2)$, that is, $T(w, z) \preceq_{se} (w, z)$. The rest of the proof is same as the proof of the case $(\mathcal{R}_{10})$.

$(\mathcal{R}_{11})$ The proof, which is similar to the proof of $(\mathcal{R}_8)$, follows as an immediate application of Lemmas 3.2 and 3.3, and Theorems 2.4, 2.7, and 4.6.

$(\mathcal{R}_{12})$ For every point $(x_0, y_0) \in [0, \infty)^2 \setminus \{(0,0)\}$, there exists $\tilde{x}_0 \geq \beta_1/B_1$ such that $(x_0, y_0) \preceq (\tilde{x}_0, 0)$. Clearly, $T(\tilde{x}_0, 0) = (\beta_1/B_1, \alpha_2/(A_2 + \tilde{x}_0)) \preceq (\tilde{x}_0, 0)$, which implies that $\{T^n(\tilde{x}_0, 0)\} = \{(\tilde{x}_n, \tilde{y}_n)\}$ is a decreasing sequence, and so $\tilde{y}_n$ is an increasing sequence. If $\tilde{y}_n$ would be convergent, then $\{(\tilde{x}_n, \tilde{y}_n)\}$ would converge to an equilibrium of system (1.1) which is impossible. Thus, $\lim_{n \to \infty} \tilde{y}_n = \infty$, which implies that $\lim_{n \to \infty} \tilde{y}_n = 0$. In view of $(x_0, y_0) \preceq (\tilde{x}_0, 0)$, we obtain that $T^n((x_0, y_0)) \preceq T^n((\tilde{x}_0, 0))$ and so

$$x_n \longrightarrow 0, \quad y_n \longrightarrow \infty, \quad n \longrightarrow \infty \tag{5.10}$$

follows.   □

Remark 5.2. We can see from Theorem 5.1 that system (1.1) exhibits variety of behaviors in different ranges of parameters. These behaviors can be classified in a few categories that verbally describe situation. Here, we use some terminology introduced in [19].

A *coexistence attractor* is one in which both species are present. An *exclusion attractor* is one in which one species is absent and the other species is present. By *multiple mixed-type attractors*, we mean a scenario that includes at least one coexistence attractor and at least one exclusion attractor. Park in [20, 21] observed the coexistence case in an experimental treatment that also included cases of competitive exclusion, that is, he observed a case termed to be multiple mixed-type attractors. Competition theory is primarily an equilibrium theory that is exemplified, by its limited number of asymptotic outcomes: a globally attracting coexistence equilibrium; a globally attracting exclusion equilibrium; at least two attracting exclusion equilibria; at least two attracting coexistence equilibria; a continuum of nonhyperbolic equilibria. (In this context, globally attracting means within the positive cone

of state space.) Four of these five asymptotic alternatives are illustrated by the LeslieGower model (the discrete analog of the famous LotkaVolterra differential equation model), see [2, 5, 17]

$$x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n}, \quad y_{n+1} = \frac{b_2 y_n}{1 + c_2 x_n + y_n}, \quad n = 0, 1, \ldots \tag{5.11}$$

and fifth alternative, precisely two attracting coexistence equilibria are illustrated by the LeslieGower model with stockings, see [16]

$$x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n} + h_1, \quad y_{n+1} = \frac{b_2 y_n}{1 + c_2 x_n + y_n} + h_2, \quad n = 0, 1, \ldots, \tag{5.12}$$

where the parameters b_1, b_2, c_1, c_2, h_1, and h_2 are positive numbers, and the initial conditions x_0, y_0 are arbitrary nonnegative numbers. Here, b_1 and b_2 are the inherent birth rates, c_1 and c_2 the density-dependent effects on newborn recruitment, and h_1 and h_2 are constant stockings. With this in mind, we introduce the following terminology. Let E_1, E_2 and E_3 be three equilibrium points of general competitive system (2.1) in south-east ordering $E_1 \preceq_{se} E_2 \preceq_{se} E_3$ and assume that E_1 and E_3 are attractors, and E_2 is a saddle point or nonhyperbolic equilibrium having one characteristic values in $(-1, 1)$. If E_1 and E_3 belong to y-axes and x-axes and E_2 is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1, 1)$) then we say that system (2.1) exhibits *saddle competitive exclusion* (resp., *nonhyperbolic competitive exclusion*). If E_1 and E_3 belong to the interior of first quadrant and E_2 is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1, 1)$) then we say that system (2.1) exhibits *saddle competitive coexistence* (resp., *nonhyperbolic competitive coexistence*). If one of the equilibrium points E_1 and E_3 is on the axes and the other is in the interior of first quadrant and E_2 is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1, 1)$), then we say that system (2.1) exhibits *saddle competitive exclusion to coexistence* (resp., *nonhyperbolic competitive exclusion-to-coexistence*). If there exists a single attractor in the interior of first quadrant which attracts all points where it is defined except eventually the points on the axes, we say that system (2.1) exhibits *global competitive coexistence* or *global competitive exclusion* depending on whether the attractor is on the axes or in the interior of first quadrant. If any of the attractors one the axes is the point $(0, \infty)$ or $(\infty, 0)$, such a situation will be named *singular*.

Using this terminology, we can describe the global behavior of system (1.1) in a concise way as follows. System (1.1) exhibits global competitive exclusion if the parameters belong to the regions $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, or $(\mathcal{R}_7)$ and global competitive coexistence if the parameters belong to the region $(\mathcal{R}_4)$.

System (1.1) exhibits saddle competitive exclusion to coexistence if the parameters belong to the region $(\mathcal{R}_5)$. System (1.1) exhibits nonhyperbolic competitive exclusion to coexistence if the parameters belong to the region $(\mathcal{R}_6)$. System (1.1) exhibits singular saddle competitive exclusion to coexistence if the parameters belong to the regions $(\mathcal{R}_8)$ or $(\mathcal{R}_9)$. System (1.1) exhibits singular nonhyperbolic competitive exclusion to coexistence if the parameters belong to the region $(\mathcal{R}_{10})$ or $(\mathcal{R}_{11})$. Finally, System (1.1) exhibits singular competitive exclusion when the parameters belong to the region $(\mathcal{R}_{12})$. With this terminology the possible scenarios for this system are limited, and transition from one scenario to another could be possible explained by using the bifurcation theory. In particular, transition from

global competitive exclusion to global competitive coexistence was explained in [3], and some related results can be found in [19], where an attempt has been made to explain the transitions from one scenario to another by using evolutionary game theory.

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