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Research Article

Global Dynamics of a Competitive System of Rational Difference Equations in the Plane

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We investigate global dynamics of the following systems of difference equations $x_{n+1} = (\alpha_1 + \beta_1 x_n)/y_n$, $y_{n+1} = (\alpha_2 + \gamma_2 y_n)/(A_2 + x_n)$, $n = 0, 1, 2, \ldots$, where the parameters α_1, β_1, α_2, γ_2, and A_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $y_0 > 0$. We show that this system has rich dynamics which depend on the part of parametric space. We show that the basins of attractions of different locally asymptotically stable equilibrium points are separated by the global stable manifolds of either saddle points or of nonhyperbolic equilibrium points.

1. Introduction and Preliminaries

In this paper, we study the global dynamics of the following rational system of difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{y_n},$$
$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the parameters α_1, β_1, α_2, γ_2, and A_2 are positive numbers and initial conditions $x_0 \geq 0$ and $y_0 > 0$ are arbitrary numbers. System (1.1) was mentioned in [1] as a part of Open Problem 3 which asked for a description of global dynamics of three specific competitive systems. According to the labeling in [1], system (1.1) is called (21, 29). In this paper, we provide the precise description of global dynamics of system (1.1). We show that system

(1.1) has a variety of dynamics that depend on the value of parameters. We show that system (1.1) may have between zero and two equilibrium points, which may have different local character. If system (1.1) has one equilibrium point, then this point is either locally saddle point or non-hyperbolic. If system (1.1) has two equilibrium points, then the pair of points is the pair of a saddle point and a sink. The major problem is determining the basins of attraction of different equilibrium points. System (1.1) gives an example of semistable non-hyperbolic equilibrium point. The typical results are Theorems 4.1 and 4.5 below.

System (1.1) is a competitive system, and our results are based on recent results developed for competitive systems in the plane; see [2, 3]. In the next section, we present some general results about competitive systems in the plane. The third section deals with some basic facts such as the non-existence of period-two solution of system (1.1). The fourth section analyzes local stability which is fairly complicated for this system. Finally, the fifth section gives global dynamics for all values of parameters.

Let $\mathcal{I}$ and $\mathcal{J}$ be intervals of real numbers. Consider a first-order system of difference equations of the form

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n), \\
y_{n+1} &= g(x_n, y_n),
\end{aligned}
\quad n = 0, 1, 2, \ldots,
\tag{1.2}
$$

where $f : \mathcal{I} \times \mathcal{J} \to \mathcal{I}$, $g : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$, and $(x_0, y_0) \in \mathcal{I} \times \mathcal{J}$.

When the function $f(x, y)$ is increasing in x and decreasing in y and the function $g(x, y)$ is decreasing in x and increasing in y, the system (1.2) is called *competitive*. When the function $f(x, y)$ is increasing in x and increasing in y and the function $g(x, y)$ is increasing in x and increasing in y, the system (1.2) is called *cooperative*. A map T that corresponds to the system (1.2) is defined as $T(x, y) = (f(x, y), g(x, y))$. Competitive and cooperative maps, which are called monotone maps, are defined similarly. *Strongly competitive* systems of difference equations or maps are those for which the functions f and g are coordinate-wise strictly monotone.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $Q_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, that is, $Q_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u, y \geq v\}$, $Q_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u, y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from x to $\mathcal{A}$* as $\mathrm{dist}(\mathrm{x}, \mathcal{A}) := \inf \{\|\mathrm{x} - \mathrm{y}\| : \mathrm{y} \in \mathcal{A}\}$. By $\mathrm{int}\, \mathcal{A}$, we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is competitive if it is nondecreasing with respect to the South-East partial order, that is if the following holds:

$$
\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{se} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \implies F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{se} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.
\tag{1.3}
$$

Competitive systems were studied by many authors; see [4–19], and others. All known results, with the exception of [4, 6, 10], deal with hyperbolic dynamics. The results presented here are results that hold in both the hyperbolic and the non-hyperbolic cases.

We now state three results for competitive maps in the plane. The following definition is from [18].

Definition 1.1. Let R be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : R \rightarrow R$ is said to satisfy condition $(O+)$ if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.

The following theorem was proved by de Mottoni and Schiaffino [20] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [15, 16].

Theorem 1.2. *Let R be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which $(O+)$ holds, then for all $x \in R$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead $(O-)$ holds, then for all $x \in R$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in R, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [18], with the domain of the map specialized to be the Cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 1.3 (Smith [18]). *Let $R \subset \mathbb{R}^2$ be the Cartesian product of two intervals in $\mathbb{R}$. Let $T : R \rightarrow R$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in R$, then T satisfies $(O+)$. If T is injective and $\det J_T(x) < 0$ for all $x \in R$, then T satisfies $(O-)$.*

Theorem 1.4. *Let T be a monotone map on a closed and bounded rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Suppose that T has a unique fixed point $\bar{\mathbf{e}}$ in $\mathcal{R}$. Then $\bar{\mathbf{e}}$ is a global attractor of T on $\mathcal{R}$.*

The following theorems were proved by Kulenović and Merino [3] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. These results are useful for determining basins of attraction of fixed points of competitive maps.

Our first result gives conditions for the existence of a global invariant curve through a fixed point (hyperbolic or not) of a competitive map that is differentiable in a neighborhood of the fixed point, when at least one of two nonzero eigenvalues of the Jacobian matrix of the map at the fixed point has absolute value less than one. A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the Cartesian product of two intervals in $\mathbb{R}$.

Theorem 1.5. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\bar{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \mathrm{int}(Q_1(\bar{x}) \cup Q_3(\bar{x}))$ is nonempty (i.e., $\bar{x}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

(a) *The map T has a C^1 extension to a neighborhood of $\bar{x}$.*

(b) *The Jacobian matrix of T at $\bar{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\bar{x}$ that is invariant and a subset of the basin of attraction of $\bar{x}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\bar{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Corollary 1.6. *If T has no fixed point nor periodic points of minimal period-two in Δ, then the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$.*

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 1.5 reduces just to $|\lambda| < 1$. This follows from a change of variables [18] that allows the Perron-Frobenius Theorem to be applied to give that, at any point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrants, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

The following result gives a description of the global stable and unstable manifolds of a saddle point of a competitive map. The result is the modification of Theorem 1.7 from [12].

Theorem 1.7. *In addition to the hypotheses of Theorem 1.5, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 1.5 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the global stable manifold $\mathcal{W}^s(\overline{x})$ of $\overline{x}$, and the global unstable manifold $\mathcal{W}^u(\overline{x})$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{x})$ in $\mathcal{R}$ are fixed points of T.*

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 1.8. *Assume the hypotheses of Theorem 1.5, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 1.5. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \qquad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \qquad (1.4)$$

such that the following statements are true.

(i) *$\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.*

(ii) *$\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.*

If, in addition, $\overline{x}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{x}$, then T has no periodic points in the boundary of $Q_1(\overline{x}) \cup Q_3(\overline{x})$ except for $\overline{x}$, and the following statements are true.

(iii) *For every $x \in \mathcal{W}^-$, there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_2(\overline{x})$ for $n \geq n_0$.*

(iv) *For every $x \in \mathcal{W}_+$, there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_4(\overline{x})$ for $n \geq n_0$.*

2. Some Basic Facts

In this section we give some basic facts about the nonexistence of period-two solutions, local injectivity of map T at the equilibrium point and $(O+)$ condition.

2.1. Equilibrium Points

The equilibrium points $(\overline{x}, \overline{y})$ of system (1.1) satisfy

$$\overline{x} = \frac{\alpha_1 + \beta_1 \overline{x}}{\overline{y}},$$

$$\overline{y} = \frac{\alpha_2 + \gamma_2 \overline{y}}{A_2 + \overline{x}}. \qquad (2.1)$$

First equation of System (2.1) gives

$$\overline{y} = \frac{\alpha_1}{\overline{x}} + \beta_1. \qquad (2.2)$$

Second equation of System (2.1) gives

$$\overline{y}(A_2 + \overline{x}) = \alpha_2 + \gamma_2 \overline{y}. \qquad (2.3)$$

Now, using (2.2), we obtain

$$\frac{(\alpha_1 + \beta_1 \overline{x})(A_2 + \overline{x})}{\overline{x}} = \alpha_2 + \gamma_2 \frac{\alpha_1 + \beta_1 \overline{x}}{\overline{x}}. \qquad (2.4)$$

This implies

$$(\alpha_1 + \beta_1 \overline{x})(A_2 + \overline{x}) = \alpha_2 \overline{x} + \gamma_2 \alpha_1 + \beta_1 \gamma_2 \overline{x}, \qquad (2.5)$$

which is equivalent to

$$\beta_1 \overline{x}^2 + \overline{x}[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] + \alpha_1(A_2 - \gamma_2) = 0. \qquad (2.6)$$

Solutions of (2.6) are

$$\overline{x}_1 = \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1},$$

$$\overline{x}_2 = \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1}. \qquad (2.7)$$

Table 1

E_1	$A_2 < \gamma_2,$
E_1, E_2	$A_2 > \gamma_2,\ [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) > 0, \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1$
$E_1 \equiv E_2$	$A_2 > \gamma_2,\ [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) = 0, \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1$
E	$A_2 = \gamma_2,\ \alpha_2 > \alpha_1$
No equilibrium	$A_2 > \gamma_2,\ [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) < 0$
No equilibrium	$A_2 > \gamma_2,\ [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) \geq 0,\ \beta_1(A_2 - \gamma_2) > \alpha_2 - \alpha_1$
No equilibrium	$A_2 = \gamma_2,\ \alpha_2 \leq \alpha_1$

Now, (2.2) gives

$$\overline{y}_1 = \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{\left[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)\right]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)},$$

$$\overline{y}_2 = \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) - \sqrt{\left[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)\right]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)}. \tag{2.8}$$

The equilibrium points are:

$$E_1 = (\overline{x}_1, \overline{y}_1),$$
$$E_1 = (\overline{x}_2, \overline{y}_2), \tag{2.9}$$

where $\overline{x}_1, \overline{y}_1, \overline{x}_2, \overline{y}_2$ are given by the above relations.
Note that

$$\overline{xy} \neq \beta_1\gamma_2. \tag{2.10}$$

The discriminant of (2.6) is given by

$$D = \left[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)\right]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2). \tag{2.11}$$

The criteria for the existence of equilibrium points are summarized in Table 1 where

$$E = \left(\frac{\alpha_2 - \alpha_1}{\beta_1}, \frac{\beta_1\alpha_2}{\alpha_2 - \alpha_1}\right). \tag{2.12}$$

2.2. Condition $(O+)$ and Period-Two Solution

In this section we prove three lemmas.

Lemma 2.1. *System (1.1) satisfies either $(O+)$ or $(O-)$. Consequently, the second iterate of every solution is eventually monotone.*

Proof. The map T associated to system (1.1) is given by

$$T(x, y) = \left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\alpha_2 + \gamma_2 y}{A_2 + x} \right). \tag{2.13}$$

Assume

$$T(x_1, y_1) \preceq_{\mathrm{ne}} T(x_2, y_2), \tag{2.14}$$

then we have

$$\frac{\alpha_1 + \beta_1 x_1}{y_1} \leq \frac{\alpha_1 + \beta_1 x_2}{y_2}, \tag{2.15}$$

$$\frac{\alpha_2 + \gamma_2 y_1}{A_2 + x_1} \leq \frac{\alpha_2 + \gamma_2 y_2}{A_2 + x_2}. \tag{2.16}$$

Equations (2.15) and (2.16) are equivalent, respectively, to

$$\alpha_1 (y_2 - y_1) + \beta_1 (x_1 y_2 - x_2 y_1) \leq 0, \tag{2.17}$$

$$\alpha_2 (x_2 - x_1) + A_2 \gamma_2 (y_1 - y_2) + \gamma_2 (x_2 y_1 - x_1 y_2) \leq 0. \tag{2.18}$$

Now, using (2.17) and (2.18), we have the following:

$$\text{If } y_2 \geq y_1 \implies x_1 y_2 < x_2 y_1 \implies x_1 < x_2 \implies (x_1, y_1) \preceq_{\mathrm{ne}} (x_2, y_2),$$
$$\text{If } y_2 < y_1 \implies x_2 < x_1 \text{ and/or } x_2 y_1 - x_1 y_2 < 0 \implies x_2 < x_1 \implies (x_2, y_2) \preceq_{\mathrm{ne}} (x_1, y_1). \tag{2.19}$$

$\square$

Lemma 2.2. *System* (1.1) *has no minimal period-two solution.*

Proof. Set

$$T(x, y) = \left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\alpha_2 + \gamma_2 y}{A_2 + x} \right). \tag{2.20}$$

Then

$$T(T(x, y)) = T\left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\alpha_2 + \gamma_2 y}{A_2 + x} \right)$$
$$= \left(\frac{(x + A_2)(\alpha_1 + (\beta_1(\alpha_1 + x\beta_1))/y)}{\alpha_2 + y\gamma_2}, \frac{y(\alpha_2 + (\gamma_2(\alpha_2 + y\gamma_2))/x + A_2)}{y A_2 + \alpha_1 + x\beta_1} \right). \tag{2.21}$$

Period-two solution satisfies

$$\frac{(x + A_2)(\alpha_1 + (\beta_1(\alpha_1 + x\beta_1))/y)}{\alpha_2 + y\gamma_2} - x = 0, \tag{2.22}$$

$$\frac{y(\alpha_2 + (\gamma_2(\alpha_2 + y\gamma_2))/(x + A_2))}{yA_2 + \alpha_1 + x\beta_1} - y = 0. \tag{2.23}$$

We show that this system has no other positive solutions except equilibrium points. Equations (2.22) and (2.23) are equivalent, respectively, to

$$\frac{xy\alpha_1 + yA_2\alpha_1 - xy\alpha_2 + x\alpha_1\beta_1 + A_2\alpha_1\beta_1 + x^2\beta_1^2 + xA_2\beta_1^2 - xy^2\gamma_2}{y(\alpha_2 + y\gamma_2)} = 0, \tag{2.24}$$

$$\frac{-xy^2A_2 - y^2A_2^2 - xy\alpha_1 - yA_2\alpha_1 + xy\alpha_2 + yA_2\alpha_2 - x^2y\beta_1 - xyA_2\beta_1 + y\alpha_2\gamma_2 + y^2\gamma_2^2}{(x + A_2)(yA_2 + \alpha_1 + x\beta_1)} = 0. \tag{2.25}$$

Equation (2.24) implies

$$yA_2\alpha_1 + xy(\alpha_1 - \alpha_2) + A_2\alpha_1\beta_1 + x^2\beta_1^2 + x\beta_1(\alpha_1 + A_2\beta_1) - xy^2\gamma_2 = 0. \tag{2.26}$$

Equation (2.25) implies

$$-xy^2A_2 - x^2y\beta_1 + xy(-\alpha_1 + \alpha_2 - A_2\beta_1) + y(A_2(-\alpha_1 + \alpha_2) + \alpha_2\gamma_2) + y^2\left(-A_2^2 + \gamma_2^2\right) = 0. \tag{2.27}$$

Using (2.26), we have

$$x^2 = \frac{-yA_2\alpha_1 - xy(\alpha_1 - \alpha_2) - A_2\alpha_1\beta_1 - x\beta_1(\alpha_1 + A_2\beta_1) + xy^2\gamma_2}{\beta_1^2}. \tag{2.28}$$

Putting (2.28) into (2.27), we have

$$y\left(y(x + A_2)\alpha_1 + y\left(-A_2(x + A_2)\beta_1 - xy\gamma_2 + \beta_1\gamma_2^2\right) + \alpha_2(-xy + \beta_1(x + A_2 + \gamma_2))\right) = 0. \tag{2.29}$$

This is equivalent to

$$x = \frac{-yA_2^2\beta_1 + A_2(y\alpha_1 + \alpha_2\beta_1) + \beta_1\gamma_2(\alpha_2 + y\gamma_2)}{-y\alpha_1 + \alpha_2(y - \beta_1) + y(A_2\beta_1 + y\gamma_2)}. \tag{2.30}$$

Putting (2.30) into (2.24), we obtain

$$\beta_1(\alpha_2 + y\gamma_2)\Big(\alpha_2\beta_1 + y^2(A_2 - \gamma_2) + y(\alpha_1 - \alpha_2 + \beta_1(-A_2 + \gamma_2))\Big) = 0 \tag{2.31}$$

or

$$y^2\,\gamma_2\,(A_2 + \gamma_2) + \beta_1\,\gamma_2\,(-\alpha_1 + \beta_1\,(A_2 + \gamma_2)) + y\,(A_2 + \gamma_2)\,(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 + \gamma_2)) = 0. \tag{2.32}$$

From (2.31), we obtain fixed points. In the sequel, we consider (2.32).
Discriminant of (2.32) is given by

$$\Delta := (A_2 + \gamma_2)\Big(-4\beta_1\gamma_2{}^2(-\alpha_1 + \beta_1(A_2 + \gamma_2)) + (A_2 + \gamma_2)(-\alpha_1 + \alpha_2 + \beta_1(A_2 + \gamma_2))^2\Big). \tag{2.33}$$

Real solutions of (2.32) exist if and only if $\Delta \geq 0$. The solutions are given by

$$y_1 = \frac{(-A_2 - \gamma_2)\,(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 + \gamma_2)) - \sqrt{\Delta}}{2\gamma_2\,(A_2 + \gamma_2)},$$

$$y_2 = \frac{(-A_2 - \gamma_2)\,(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 + \gamma_2)) + \sqrt{\Delta}}{2\gamma_2\,(A_2 + \gamma_2)}. \tag{2.34}$$

Using (2.30), we have

$$x_1 = \frac{(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 - \gamma_2))\,(A_2 + \gamma_2) + \sqrt{\Delta}}{2\beta_1\gamma_2},$$

$$x_2 = \frac{(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 - \gamma_2))\,(A_2 + \gamma_2) - \sqrt{\Delta}}{2\beta_1\gamma_2}. \tag{2.35}$$

$\square$

Claim. Assume $\Delta \geq 0$. Then

    (i) for all values of parameters, $y_1 < 0$;

    (ii) for all values of parameters, $x_2 < 0$.

Proof. (1) Assume $(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 + \gamma_2)) > 0$. Then it is obvious that the claim $y_1 < 0$ is true. Now, assume $(-\alpha_1 + \alpha_2 + \beta_1\,(A_2 + \gamma_2)) \leq 0$. Then $y_1 < 0$ if and only if

$$\Delta - (A_2 + \gamma_2)^2(-\alpha_1 + \alpha_2 + \beta_1(A_2 + \gamma_2))^2 > 0, \tag{2.36}$$

which is equivalent to

$$-4\beta_1\gamma_2{}^2(A_2 + \gamma_2)(-\alpha_1 + \beta_1(A_2 + \gamma_2)) > 0. \tag{2.37}$$

This is true since

$$(-\alpha_1 + \beta_1(A_2 + \gamma_2)) < (-\alpha_1 + \alpha_2 + \beta_1(A_2 + \gamma_2)) \leq 0. \tag{2.38}$$

(2) Assume $(-\alpha_1 + \alpha_2 + \beta_1\ (A_2 - \gamma_2)) < 0$. Then it is obvious that $x_2 < 0$. Now, assume

$$(-\alpha_1 + \alpha_2 + \beta_1(A_2 - \gamma_2)) \geq 0. \tag{2.39}$$

Then $x_2 < 0$ if and only if

$$\Delta - (-\alpha_1 + \alpha_2 + \beta_1(A_2 - \gamma_2))^2 (A_2 + \gamma_2)^2 > 0. \tag{2.40}$$

This is equivalent to

$$4\beta_1\gamma_2(A_2 + \gamma_2)\left(A_2(-\alpha_1 + \alpha_2 + A_2\beta_1) + \alpha_2\gamma_2 - \beta_1{\gamma_2}^2\right) > 0. \tag{2.41}$$

Using (2.39), we have

$$\left(A_2(-\alpha_1 + \alpha_2 + A_2\beta_1) + \alpha_2\gamma_2 - \beta_1{\gamma_2}^2\right) \geq A_2\gamma_2\beta_1 + \alpha_2\gamma_2 - \beta_1{\gamma_2}^2 \geq \alpha_1\gamma_2 > 0, \tag{2.42}$$

which implies that the inequality (2.41) is true. $\qquad\qquad\square$

Now, the proof of the Lemma 2.2 follows from the Claim 2.2.

Lemma 2.3. *The map T associated to System* (1.1) *satisfies the following:*

$$T(x, y) = (\overline{x}, \overline{y}) \quad only\ for\ (x, y) = (\overline{x}, \overline{y}). \tag{2.43}$$

Proof. By using (2.1), we have

$$T(x, y) = (\overline{x}, \overline{y}),$$
$$\Longleftrightarrow \frac{\alpha_1 + \beta_1 x}{y} = \frac{\alpha_1 + \beta_1\overline{x}}{\overline{y}}, \tag{2.44}$$
$$\frac{\alpha_2 + \gamma_2 y}{A_2 + x} = \frac{\alpha_2 + \gamma_2\overline{y}}{A_2 + \overline{x}}.$$

First equation implies

$$\alpha_1(\overline{y} - y) + \beta_1(x\overline{y} - \overline{x}y) = 0. \tag{2.45}$$

Second equation implies

$$\alpha_2(\overline{x} - x) + \gamma_2 A_2(y - \overline{y}) = \gamma_2(x\overline{y} - y\overline{x}). \tag{2.46}$$

Note the following

$$x\overline{y} - \overline{x}y = (x - \overline{x})\overline{y} + \overline{x}(\overline{y} - y). \tag{2.47}$$

Using (2.47), Equations (2.45) and (2.46), respectively, become

$$\beta_1 \overline{y}(x - \overline{x}) + (\alpha_1 + \beta_1 \overline{x})(\overline{y} - y) = 0,$$
$$(\alpha_2 + \gamma_2 \overline{y})(x - \overline{x}) + \gamma_2(A_2 + \overline{x})(\overline{y} - y) = 0. \tag{2.48}$$

Note that System (2.48) is linear homogeneous system in $x - \overline{x}$ and $\overline{y} - y$. The determinant of System (2.48) is given by

$$\begin{vmatrix} \beta_1 \overline{y} & \alpha_1 + \beta_1 \overline{y} \\ \alpha_2 + \gamma_2 \overline{y} & \gamma_2(A_2 + \overline{x}) \end{vmatrix}. \tag{2.49}$$

Using (2.1), the determinant of System (2.48) becomes

$$\begin{vmatrix} \beta_1 \overline{y} & \overline{xy} \\ \overline{y}(A_2 + \overline{x}) & \gamma_2(A_2 + \overline{x}) \end{vmatrix} = \overline{y}(A_2 + \overline{x})(\beta_1 \gamma_2 - \overline{xy}) \neq 0. \tag{2.50}$$

This implies that System (2.48) has only trivial solution, that is

$$x = \overline{x}, \qquad y = \overline{y}. \tag{2.51}$$

$\square$

3. Linearized Stability Analysis

The Jacobian matrix of the map T has the following form:

$$J_T = \begin{pmatrix} \dfrac{\beta_1}{y} & -\dfrac{\alpha_1 + \beta_1 x}{y^2} \\ -\dfrac{\alpha_2 + \gamma_2 y}{(A_2 + x)^2} & \dfrac{\gamma_2}{A_2 + x} \end{pmatrix}. \tag{3.1}$$

The value of the Jacobian matrix of T at the equilibrium point is

$$J_T(\overline{x}, \overline{y}) = \begin{pmatrix} \dfrac{\beta_1}{\overline{y}} & -\dfrac{\overline{x}}{\overline{y}} \\ -\dfrac{\overline{y}}{A_2 + \overline{x}} & \dfrac{\gamma_2}{A_2 + \overline{x}} \end{pmatrix}. \tag{3.2}$$

The determinant of (3.2) is given by

$$\det J_T(\overline{x}, \overline{y}) = \frac{\beta_1 \gamma_2 - \overline{xy}}{\overline{y}(A_2 + \overline{x})}.$$ (3.3)

The trace of (3.2) is

$$\text{Tr} J_T(\overline{x}, \overline{y}) = \frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}}.$$ (3.4)

The characteristic equation has the form

$$\lambda^2 - \lambda \left(\frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}} \right) + \frac{\beta_1 \gamma_2 - \overline{xy}}{\overline{y}(A_2 + \overline{x})} = 0.$$ (3.5)

Theorem 3.1. *Assume that $A_2 < \gamma_2$. Then there exists a unique positive equilibrium E_1 which is a saddle point, and the following statements hold.*

(a) *If $\beta_1(\gamma_2 + A_2) < \alpha_1 + \alpha_2$, then $\lambda_1 \in (1, +\infty)$ and $\lambda_2 \in (-1, 0)$.*

(b) *If $\beta_1(\gamma_2 + A_2) > \alpha_1 + \alpha_2$, $\beta_1 A_2 < \alpha_2 < \beta_1 \gamma_2$ and $\alpha_1(\beta_1 \gamma_2 - \alpha_2) < \beta_1 \gamma_2(\alpha_2 - \beta_1 A_2)$, then $\lambda_1 \in (1, +\infty)$ and $\lambda_2 \in (0, 1)$.*

(c) *If $\beta_1(\gamma_2 + A_2) > \alpha_1 + \alpha_2$, $\alpha_2 > \beta_1 \gamma_2$ and $\alpha_1(\alpha_2 - \beta_1 \gamma_2) > \beta_1 \gamma_2(\alpha_2 - \beta_1 A_2)$, then $\lambda_1 \in (1, +\infty)$ and $\lambda_2 \in (0, 1)$.*

(d) *If $\beta_1(\gamma_2 + A_2) > \alpha_1 + \alpha_2$, $\alpha_2 > \beta_1 \gamma_2$ and $\alpha_1(\alpha_2 - \beta_1 \gamma_2) < \beta_1 \gamma_2(\alpha_2 - \beta_1 A_2)$, then $\lambda_1 \in (1, +\infty)$ and $\lambda_2 \in (-1, 0)$.*

Proof. The equilibrium is a saddle point if and only if the following conditions are satisfied:

$$\left| \text{Tr} J_T(\overline{x}, \overline{y}) \right| > \left| 1 + \det J_T(\overline{x}, \overline{y}) \right|, \qquad \text{Tr}^2 J_T(\overline{x}, \overline{y}) - 4 \det J_T(\overline{x}, \overline{y}) > 0.$$ (3.6)

The first condition is equivalent to

$$\frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}} > 1 + \frac{\beta_1 \gamma_2 - \overline{xy}}{\overline{y}(A_2 + \overline{x})}.$$ (3.7)

This implies the following:

$$\begin{aligned}
&\beta_1(A_2 + \overline{x}) + \gamma_2 \overline{y} > \overline{y}(A_2 + \overline{x}) + \beta_1 \gamma_2 - \overline{xy} \\
&\Longleftrightarrow (A_2 + \overline{x})(\beta_1 - \overline{y}) + \gamma_2(\overline{y} - \beta_1) > -\overline{xy} \\
&\Longleftrightarrow (\overline{y} - \beta_1)(A_2 - \gamma_2 + \overline{x}) < \overline{xy}.
\end{aligned}$$ (3.8)

Notice the following:

$$\overline{y}_1 - \beta_1 = \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)} - \beta_1$$

$$= \frac{(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2) + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)}$$

$$= \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1} \frac{2\beta_1}{2(A_2 - \gamma_2)}$$

$$= \overline{x}_2 \frac{\beta_1}{A_2 - \gamma_2}.$$

$$(3.9)$$

That is,

$$\overline{y}_1 - \beta_1 = \overline{x}_2 \frac{\beta_1}{A_2 - \gamma_2}. \tag{3.10}$$

Similarly,

$$A_2 - \gamma_2 + \overline{x}_1 = A_2 - \gamma_2$$

$$+ \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1}$$

$$= \frac{\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2) + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1}$$

$$= \frac{-\beta_1(A_2 - \gamma_2) + (\alpha_1 - \alpha_2) - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2\beta_1}$$

$$= \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)} \frac{A_2 - \gamma_2}{\beta_1}$$

$$= \overline{y}_2 \frac{A_2 - \gamma_2}{\beta_1}.$$

$$(3.11)$$

Now, we have

$$(\overline{y}_1 - \beta_1)(A_2 - \gamma_2 + \overline{x}_1) < \overline{x}_1\overline{y}_1 \Longleftrightarrow \overline{x}_2 \frac{\beta_1}{A_2 - \gamma_2} \overline{y}_2 \frac{A_2 - \gamma_2}{\beta_1} < \overline{x}_1\overline{y}_1. \tag{3.12}$$

This is equivalent to

$$\overline{x}_2\overline{y}_2 < \overline{x}_1\overline{y}_1. \tag{3.13}$$

The last condition is equivalent to

$$\overline{x}_2\overline{y}_2 - \overline{x}_1\overline{y}_1 < 0 \iff -\left(\overline{y}_2(\overline{x}_1 - \overline{x}_2) + \overline{x}_1(\overline{y}_1 - \overline{y}_2)\right) < 0 \tag{3.14}$$

which is true since $\overline{x}_1 > \overline{x}_2$ and $\overline{y}_1 > \overline{y}_2$.

The second condition is equivalent to

$$\left(\frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2 + \overline{x}}\right)^2 - 4\frac{\beta_1\gamma_2}{\overline{y}(A_2 + \overline{x})} + 4\frac{\overline{x}\overline{y}}{\overline{y}(A_2 + \overline{x})} > 0. \tag{3.15}$$

This is equivalent to

$$\left(\frac{\beta_1}{\overline{y}} - \frac{\gamma_2}{A_2 + \overline{x}}\right)^2 + 4\frac{\overline{x}\overline{y}}{\overline{y}(A_2 + \overline{x})} > 0, \tag{3.16}$$

establishing the proof of Theorem 3.1.

Since the map T is strongly competitive, the Jacobian matrix (3.2) has two real and distinct eigenvalues, with the larger one in absolute value being positive.

From (3.5) at E_1, we have

$$\begin{aligned}
\lambda_1 + \lambda_2 &= \frac{\beta_1}{\overline{y}_1} + \frac{\gamma_2}{A_2 + \overline{x}_1}, \\
\lambda_1\lambda_2 &= \frac{\beta_1\gamma_2 - \overline{x}_1\overline{y}_1}{\overline{y}_1(A_2 + \overline{x}_1)}.
\end{aligned} \tag{3.17}$$

The first equation implies that either both eigenvalues are positive or the smaller one is negative.

Consider the numerator of the right-hand side of the second equation. We have

$$\begin{aligned}
\beta_1\gamma_2 - \overline{x}_1\overline{y}_1 &= \beta_1\gamma_2 - \frac{-(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{D}}{2\beta_1} \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{D}}{-2(A_2 - \gamma_2)} \\
&= \beta_1\gamma_2 - \frac{\sqrt{D} + (\alpha_1 + \alpha_2) - \beta_1(A_2 - \gamma_2)}{2} \\
&= \frac{\beta_1(\gamma_2 + A_2) - (\alpha_1 + \alpha_2) - \sqrt{D}}{2},
\end{aligned} \tag{3.18}$$

where $D = [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)$.

(a) If $\beta_1(\gamma_2 + A_2) < \alpha_1 + \alpha_2$, then the smaller root is negative, that is, $\lambda_2 \in (-1, 0)$.

If $\beta_1(\gamma_2 + A_2) > (\alpha_1 + \alpha_2)$, then

$$
\begin{aligned}
&\beta_1(\gamma_2 + A_2) - (\alpha_1 + \alpha_2) > \sqrt{D} \\
&\Longleftrightarrow \beta_1^2(\gamma_2 + A_2)^2 - 2\beta_1(\gamma_2 + A_2)(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2)^2 \\
&> (\alpha_1 - \alpha_2)^2 + 2\beta_1(\alpha_1 - \alpha_2)(A_2 - \gamma_2) + \beta_1^2(A_2 - \gamma_2)^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) \\
&\Longleftrightarrow \beta_1\gamma_2(\beta_1 A_2 - \alpha_2) + \alpha_1(\alpha_2 - \beta_1\gamma_2) > 0.
\end{aligned}
\tag{3.19}
$$

From the last inequality statements $(b), (c)$ and (d) follow. $\qquad \square$

We now perform a similar analysis for the other cases in Table 1.

Theorem 3.2. *Assume*

$$
A_2 > \gamma_2, \quad \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1, \quad [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) > 0. \tag{3.20}
$$

Then E_1, E_2 exist. E_1 is a saddle point; E_2 is a sink. For the eigenvalues of $E_1, \lambda_1(E_1) \in (1, +\infty)$ the following holds.

(a) *If $\beta_1\gamma_2 < \alpha_2 < \beta_1 A_2$, then $\lambda_2 \in (0, 1)$.*

(b) *If $\alpha_2 > \beta_1 A_2$ and $\alpha_1(\alpha_2 - \beta_1\gamma_2) < \beta_1\gamma_2(\alpha_2 - \beta_1 A_2)$, then $\lambda_2 \in (-1, 0)$.*

(c) *If $\alpha_2 < \beta_1\gamma_2$ and $\beta_1\gamma_2(\beta_1 A_2 - \alpha_2) > \alpha_1(\beta_1\gamma_2 - \alpha_2)$, then $\lambda_2 \in (0, 1)$.*

Proof. Note that if $\beta_1 A_2 - \alpha_2 < 0$ and $\alpha_2 - \beta_1\gamma_2 < 0$, then $\alpha_2 > \beta_1 A_2$ and $\alpha_2 < \beta_1\gamma_2$, which implies $A_2 < \gamma_2$, which is a contradiction.

The equilibrium is a sink if the following condition is satisfied:

$$
\left| \operatorname{Tr} J_T(\overline{x}, \overline{y}) \right| < \left| 1 + \det J_T(\overline{x}, \overline{y}) \right| < 2. \tag{3.21}
$$

The condition $|\operatorname{Tr} J_T(\overline{x}, \overline{y})| < |1 + \det J_T(\overline{x}, \overline{y})|$ is equivalent to

$$
\frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{A_2 + \overline{y}} < 1 + \frac{\beta_1\gamma_2}{\overline{y}(A_2 + \overline{x})} - \frac{\overline{x}\,\overline{y}}{\overline{y}(A_2 + \overline{x})}. \tag{3.22}
$$

This implies

$$
\begin{aligned}
&\beta_1(A_2 + \overline{x}) + \gamma_2\overline{y} < \overline{y}(A_2 + \overline{x}) + \beta_1\gamma_2 - \overline{x}\,\overline{y} \\
&\Longleftrightarrow (A_2 + \overline{x})(\beta_1 - \overline{y}) + \gamma_2(\overline{y} - \beta_1) < -\overline{x}\,\overline{y} \\
&\Longleftrightarrow (\overline{y} - \beta_1)(A_2 - \gamma_2 + \overline{x}) > \overline{x}\,\overline{y}.
\end{aligned}
\tag{3.23}
$$

Now, we prove that E_2 is a sink.

We have to prove that

$$(\overline{y}_2 - \beta_1)(A_2 - \gamma_2 + \overline{x}_2) > \overline{x}_2\overline{y}_2. \tag{3.24}$$

Notice the following:

$$\overline{y}_2 - \beta_1 = \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)} - \beta_1$$

$$= \frac{(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2) - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)}$$

$$= \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2(A_2 - \gamma_2)}$$

$$= \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2(A_2 - \gamma_2)} \frac{\beta_1}{A_2 - \gamma_2}$$

$$= \overline{x}_1 \frac{\beta_1}{A_2 - \gamma_2}. \tag{3.25}$$

Similarly,

$$A_2 - \gamma_2 + \overline{x}_2 = A_2 - \gamma_2$$

$$+ \frac{-[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)] - \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{2\beta_1}$$

$$= \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2\beta_1}$$

$$= \frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2)}}{-2(A_2 - \gamma_2)} \frac{A_2 - \gamma_2}{\beta_1}$$

$$= \overline{y}_1 \frac{A_2 - \gamma_2}{\beta_1}. \tag{3.26}$$

Now, condition

$$(\overline{y}_2 - \beta_1)(A_2 - \gamma_2 + \overline{x}_2) > \overline{x}_2\overline{y}_2 \tag{3.27}$$

becomes

$$\overline{x}_1 \frac{\beta_1}{A_2 - \gamma_2} \overline{y}_1 \frac{A_2 - \gamma_2}{\beta_1} > \overline{x}_2 \overline{y}_2, \tag{3.28}$$

that is,

$$\overline{x}_1 \overline{y}_1 > \overline{x}_2 \overline{y}_2, \tag{3.29}$$

which is true. (see Theorem 3.1.)

Condition

$$\left| 1 + \det J_T(\overline{x}, \overline{y}) \right| < 2 \tag{3.30}$$

is equivalent to

$$\frac{\beta_1 \gamma_2}{\overline{y}(A_2 + \overline{x})} - \frac{\overline{xy}}{\overline{y}(A_2 + \overline{x})} < 1. \tag{3.31}$$

This implies

$$\beta_1 \gamma_2 - \overline{xy} < \overline{y}(A_2 + \overline{x}) \iff \beta_1 \gamma_1 - \overline{y} A_2 < 2\overline{xy}. \tag{3.32}$$

We have to prove that

$$\beta_1 \gamma_2 - \overline{y}_2 A_2 < 2\overline{x}_2 \overline{y}_2. \tag{3.33}$$

Using (2.2), we have

$$\beta_1 \gamma_2 - \left(\frac{\alpha_1}{\overline{x}_2} + \beta_1 \right) A_2 < 2\overline{x}_2 \left(\frac{\alpha_1}{\overline{x}_2} + \beta_1 \right). \tag{3.34}$$

This is equivalent to

$$\beta_1 (\gamma_2 - A_2) < 2\alpha_1 + 2\overline{x}_2 \beta_1 + \frac{\alpha_1 A_2}{\overline{x}_2}, \tag{3.35}$$

which is always true since $A_2 > \gamma_2$ and the left side is always negative, while the right side is always positive.

Notice that conditions

$$\overline{x}_1\overline{y}_1 > \overline{x}_2\overline{y}_2,$$

$$\left(\frac{\beta_1}{\overline{y}} - \frac{\gamma_2}{A_2 + \overline{x}}\right)^2 + 4\frac{\overline{xy}}{\overline{y}(A_2 + \overline{x})} > 0,$$

(3.36)

imply that E_1 is a saddle point.

From (3.5) at E_1, we have

$$\lambda_1 + \lambda_2 = \frac{\beta_1}{\overline{y}_1} + \frac{\gamma_2}{A_2 + \overline{x}_1},$$

$$\lambda_1\lambda_2 = \frac{\beta_1\gamma_2 - \overline{x}_1\overline{y}_1}{\overline{y}_1(A_2 + \overline{x}_1)}.$$

(3.37)

The first equation implies that either both eigenvalues are positive or the smaller one is negative.

Consider the numerator of the right-hand side of the second equation. We have

$$\beta_1\gamma_2 - \overline{x}_1\overline{y}_1 = \beta_1\gamma_2 - \frac{-(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{D}}{2\beta_1}\frac{(\alpha_1 - \alpha_2) - \beta_1(A_2 - \gamma_2) + \sqrt{D}}{-2(A_2 - \gamma_2)}$$

$$= \beta_1\gamma_2 - \frac{\sqrt{D} + (\alpha_1 + \alpha_2) - \beta_1(A_2 - \gamma_2)}{2}$$

$$= \frac{\beta_1(\gamma_2 + A_2) - (\alpha_1 + \alpha_2) - \sqrt{D}}{2}.$$

(3.38)

We have

$$\beta_1\gamma_2 - \overline{xy} > 0 \iff \frac{\beta_1(\gamma_2 + A_2) - (\alpha_1 + \alpha_2) - \sqrt{D}}{2} > 0.$$

(3.39)

Inequality

$$\beta_1(\gamma_2 + A_2) - (\alpha_1 + \alpha_2) - \sqrt{D} > 0$$

(3.40)

is equivalent to

$$\beta_1\gamma_2(\beta_1 A_2 - \alpha_2) + \alpha_1(\alpha_2 - \beta_1\gamma_2) > 0,$$

(3.41)

which is obvious if $\beta_1\gamma_2 < \alpha_2 < \beta_1 A_2$. Then inequality (3.41) holds. This confirms (a). The other cases follow from (3.41). $\qquad\square$

Theorem 3.3. *Assume*

$$A_2 > \gamma_2, \quad \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1, \quad [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) = 0. \quad (3.42)$$

Then there exists a unique positive equilibrium point

$$E_1 = E_2 = \left(\frac{(\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2)}{2\beta_1}, \frac{\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2)}{2(A_2 - \gamma_2)} \right), \quad (3.43)$$

which is non-hyperbolic. The following holds.

(a) *If $\beta_1(A_2 + \gamma_2) > \alpha_1 + \alpha_2$, then $\lambda_1 = 1$ and $\lambda_2 \in (0, 1)$.*
(b) *If $\beta_1(A_2 + \gamma_2) < \alpha_1 + \alpha_2$, then $\lambda_1 = 1$ and $\lambda_2 \in (-1, 0)$.*

Proof. Evaluating the Jacobian matrix (3.2) at equilibrium $E_1 = E_2 = (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1, (\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2))$, we have

$$J_T(\overline{x}, \overline{y})$$

$$= \begin{pmatrix} \dfrac{\beta_1}{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)} & -\dfrac{((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1}{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)} \\[3mm] -\dfrac{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)}{A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)} & \dfrac{\gamma_2}{A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)} \end{pmatrix}$$

$$(3.44)$$

The characteristic equation of $J_T(\overline{x}, \overline{y})$ is

$$\det(J_T(\overline{x}, \overline{y}) - \lambda I)$$

$$= \begin{pmatrix} \dfrac{\beta_1}{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)} - \lambda & -\dfrac{((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1}{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)} \\[3mm] -\dfrac{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)}{A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)} & \dfrac{\gamma_2}{A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)} - \lambda \end{pmatrix}$$

$$= 0,$$

$$(3.45)$$

which is simplified to

$$\lambda^2 - \lambda \left(\frac{\beta_1}{(\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2)} + \frac{\gamma_2}{A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)} \right)$$

$$+ \frac{\beta_1\gamma_2 - (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1)((\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2))}{((\beta_1(A_2 - \gamma_2) - (\alpha_1 - \alpha_2))/2(A_2 - \gamma_2))(A_2 + (((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1))} = 0.$$

$$(3.46)$$

Solutions of (3.46) are

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{-\alpha_1^2 + 2\alpha_1\alpha_2 - \alpha_2^2 + A_2^2\beta_1^2 + 2A_2\beta_1^2\gamma_2 - 3\beta_1^2\gamma_2^2}{\alpha_1^2 - 2\alpha_1\alpha_2 + \alpha_2^2 - 2A_2\alpha_1\beta_1 + 2A_2\alpha_2\beta_1 + A_2^2\beta_1^2 - \beta_1^2\gamma_2^2}. \qquad (3.47)$$

Note that λ_2 can be written in the following form:

$$\lambda_2 = \frac{-(\alpha_1 - \alpha_2)^2 + \beta_1^2\left(A_2^2 - \gamma_2^2\right) + 2\beta_1^2\gamma_2\left(A_2 - \gamma_2\right)}{(\alpha_1 - \alpha_2)^2 + 2A_2\beta_1(\alpha_2 - \alpha_1) + \beta_1^2\left(A_2^2 - \gamma_2^2\right)}. \qquad (3.48)$$

Note that $|\lambda_2| < 1$.
The corresponding eigenvectors, respectively, are

$$\left(-\frac{A_2 - \gamma_2}{\beta_1}, 1\right), \qquad \left(-\frac{(A_2 - \gamma_2)(\alpha_1 - \alpha_2 + A_2\beta_1 - \beta_1\gamma_2)(-\alpha_1 + \alpha_2 + A_2\beta_1 + \beta_1\gamma_2)}{\beta_1(-\alpha_1 + \alpha_2 + A_2\beta_1 - \beta_1\gamma_2)^2}, 1\right). \qquad (3.49)$$

Note that the denominator of (3.48) is always positive.
Consider numerator of (3.48)

$$-(\alpha_1 - \alpha_2)^2 + \beta_1^2\left(A_2^2 - \gamma_2^2\right) + 2\beta_1^2\gamma_2\left(A_2 - \gamma_2\right). \qquad (3.50)$$

From

$$\left[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)\right]^2 - 4\alpha_1\beta_1\left(A_2 - \gamma_2\right) = 0, \qquad (3.51)$$

we have

$$(\alpha_1 - \alpha_2)^2 = 4\alpha_1\beta_1\left(A_2 - \gamma_2\right) - 2\beta_1(\alpha_1 - \alpha_2)(A_2 - \gamma_2) - \beta_1^2\left(A_2 - \gamma_2\right)^2. \qquad (3.52)$$

Substituting $(\alpha_1 - \alpha_2)^2$ from (3.52) in (3.50), we obtain

$$\begin{aligned}
&-4\alpha_1\beta_1\left(A_2 - \gamma_2\right) + 2\beta_1(\alpha_1 - \alpha_2)(A_2 - \gamma_2) + \beta_2^2\left(A_2^2 - \gamma_2^2\right) + 2\beta_1^2\gamma_2\left(A_2 - \gamma_2\right) \\
&= 2\beta_1\left(A_2 - \gamma_2\right)(\beta_1(A_2 + \gamma_2) - (\alpha_1 + \alpha_2)).
\end{aligned} \qquad (3.53)$$

Now, (3.48) becomes

$$\lambda_2 = \frac{2\beta_1\left(A_2 - \gamma_2\right)(\beta_1(A_2 + \alpha_2) - (\alpha_1 + \alpha_2))}{(\alpha_1 - \alpha_2)^2 + 2A_2\beta_1(\alpha_2 - \alpha_1) + \beta_1^2\left(A_2^2 - \gamma_2^2\right)}, \qquad (3.54)$$

establishing the proof of the theorem.   □

Now, we consider the special case of System (1.1) when $A_2 = \gamma_2$.
In this case system (1.1) becomes

$$
\begin{aligned}
x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n}{y_n}, \\
y_{n+1} &= \frac{\alpha_2 + A_2 x_n}{A_2 + y_n},
\end{aligned}
\qquad n = 0, 1, 2, \dots. \tag{3.55}
$$

Equilibrium points are solutions of the following system:

$$
\begin{aligned}
\overline{x} &= \frac{\alpha_1 + \beta_1 \overline{x}}{\overline{y}}, \\
\overline{y} &= \frac{\alpha_2 + A_2 \overline{y}}{A_2 + \overline{x}}.
\end{aligned} \tag{3.56}
$$

The second equation implies

$$
\overline{x} = \frac{\alpha_2 - \alpha_1}{\beta_1}, \qquad \alpha_2 > \alpha_1. \tag{3.57}
$$

Now, the first equation implies

$$
\overline{y} = \frac{\beta_1 \alpha_2}{\alpha_2 - \alpha_1}, \qquad \alpha_2 > \alpha_1. \tag{3.58}
$$

The map T associated to System (3.55) is given by

$$
T(x, y) = \left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\alpha_2 + A_2 y}{A_2 + x} \right). \tag{3.59}
$$

The Jacobian matrix of the map T has the following form:

$$
J_T = \begin{pmatrix} \dfrac{\beta_1}{y} & -\dfrac{\alpha_1 + \beta_1 x}{y^2} \\[2ex] -\dfrac{\alpha_2 + A_2 y}{(A_2 + x)^2} & \dfrac{\gamma_2}{A_2 + x} \end{pmatrix}. \tag{3.60}
$$

The value of the Jacobian matrix of T at the equilibrium point is

$$
J_T(\overline{x}, \overline{y}) = \begin{pmatrix} \dfrac{\beta_1}{\overline{y}} & -\dfrac{\overline{x}}{\overline{y}} \\[2ex] -\dfrac{\overline{y}}{A_2 + \overline{x}} & \dfrac{A_2}{A_2 + \overline{x}} \end{pmatrix}. \tag{3.61}
$$

The determinant of (3.61) is given by

$$\det J_T(\overline{x}, \overline{y}) = \frac{\beta_1 A_2 - \overline{x}\overline{y}}{\overline{y}(A_2 + \overline{x})}.$$ (3.62)

The trace of (3.61) is

$$\operatorname{Tr} J_T(\overline{x}, \overline{y}) = \frac{\beta_1}{\overline{y}} + \frac{A_2}{A_2 + \overline{x}}.$$ (3.63)

Theorem 3.4. *Assume*

$$A_2 = \gamma_2, \quad \alpha_2 > \alpha_1.$$ (3.64)

Then there exists a unique positive equilibrium point

$$E = \left(\frac{\alpha_2 - \alpha_1}{\beta_1}, \frac{\beta_1 \alpha_2}{\alpha_2 - \alpha_1} \right)$$ (3.65)

of system (1.1), *which is a saddle point. The following statements hold.*

(a) *If* $\beta_1 A_2 - \alpha_2 > 0$, *then* $\lambda_1 \in (1, +\infty)$ *and* $\lambda_2 \in (0, 1)$.

(b) *If* $\beta_1 A_2 - \alpha_2 < 0$, *then* $\lambda_1 \in (1, +\infty)$ *and* $\lambda_2 \in (-1, 0)$.

Proof. We prove that E is a saddle point.
We check the conditions

$$\left| \operatorname{Tr} J_T(\overline{x}, \overline{y}) \right| > 1 + \det J_T(\overline{x}, \overline{y}), \qquad \operatorname{Tr}^2 J_T(\overline{x}, \overline{y}) - 4 \det J_T(\overline{x}, \overline{y}) > 0.$$ (3.66)

Condition $\left| \operatorname{Tr} J_T(\overline{x}, \overline{y}) \right| > 1 + \det J_T(\overline{x}, \overline{y})$ is equivalent to

$$\frac{\beta_1}{\overline{y}} + \frac{A_2}{A_2 + \overline{x}} > 1 + \frac{\beta_1 A_2}{(A_2 + \overline{x})\overline{y}} - \frac{\overline{x}}{A_2 + \overline{x}}.$$ (3.67)

This implies

$$\beta_1(A_2 + \overline{x}) + A_2 \overline{y} > \overline{y} A_2 + \overline{x}\overline{y} + \beta_1 A_2 - \overline{x}\overline{y} \Longleftrightarrow \beta_1 \overline{x} > 0.$$ (3.68)

Condition

$$\operatorname{Tr}^2 J_T(\overline{x}, \overline{y}) - 4 \det J_T(\overline{x}, \overline{y}) > 0$$ (3.69)

is equivalent to

$$\left(\frac{\beta_1}{\overline{y}} - \frac{\gamma_2}{A_2 + \overline{x}}\right)^2 + 4\frac{\overline{xy}}{\overline{y}(A_2 + \overline{x})} > 0. \tag{3.70}$$

Hence E is a saddle point.

Now,

$$\lambda_1 + \lambda_2 = \frac{\beta_1}{\overline{y}},$$

$$\lambda_1\lambda_2 = \frac{\beta_1 A_2 - \alpha_2}{\overline{y}(A_2 + \overline{x})}. \tag{3.71}$$

The first equation implies that either both eigenvalues are positive or the smaller one is less then zero. The second equation implies that

$$\text{If } \beta_1 A_2 > \alpha_2, \text{ then } \lambda_1 \in (1, +\infty), \ \lambda_2 \in (0, -1);$$

$$\text{If } \beta_1 A_2 < \alpha_2, \text{ then } \lambda_1 \in (1, +\infty), \ \lambda_2 \in (-1, 0), \tag{3.72}$$

establishing the proof of theorem. $\qquad\square$

4. Global Behavior

Theorem 4.1. *Assume*

$$A_2 < \gamma_2. \tag{4.1}$$

Then system (1.1) has a unique equilibrium point E_1 which is a saddle point. Furthermore, there exists the global stable manifold $W^s(E_1)$ that separates the positive quadrant so that all orbits below this manifold are asymptotic to $(\infty, 0)$, and all orbits above this manifold are asymptotic to $(0, \infty)$. All orbits that start on $W^s(E_1)$ are attracted to E_1. The global unstable manifold $W^u(E_1)$ is the graph of a continuous, unbounded, strictly decreasing function.

Proof. The existence of the global stable manifold $W^s(E_1)$ with the stated properties follows from Theorems 1.5, 1.7, and 1.8 and Lemmas 2.1 and 2.2. $\qquad\square$

Theorem 4.2. *Assume*

$$A_2 > \gamma_2, \quad \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1, \quad \left[(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)\right]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) > 0. \tag{4.2}$$

Then system (1.1) has two equilibrium points: E_1 which is a saddle point and E_2 which is a sink. Furthermore, there exists the global stable manifold $W^s(E_1)$ that separates the positive quadrant so that all orbits below this manifold are asymptotic to $(\infty, 0)$, and all orbits above this manifold are attracted to equilibrium E_2. All orbits that start on $W^s(E_1)$ are attracted to E_1. The global unstable

manifold $W^u(E_1)$ is the graph of a continuous, unbounded, strictly decreasing function with end point E_2.

Proof. The existence of the global stable manifold $W^s(E_2)$ with the stated properties follows from Theorems 1.5, 1.7, and 1.8 and Lemmas 2.1 and 2.2. □

Theorem 4.3. *Assume*

$$A_2 > \gamma_2, \quad \beta_1(A_2 - \gamma_2) < \alpha_2 - \alpha_1, \quad [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) = 0. \quad (4.3)$$

Then system (1.1) has a unique equilibrium $E_2 = E_3$ which is non-hyperbolic. The sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$, and $\{y_{2n+1}\}$ are eventually monotonic. Every solution that starts in $Q_4(E_2)$ is asymptotic to $(\infty, 0)$, and every solution that starts in $Q_2(E_2)$ is asymptotic to the equilibrium E_2. Furthermore, there exists the global stable manifold $W^s(E_2)$ that separates the positive quadrant into three invariant regions, so that all orbits below this manifold are asymptotic to $(\infty, 0)$, and all orbits that start above this manifold are attracted to the equilibrium E_2. All orbits that start on $W^s(E_2)$ are attracted to E_2.

Proof. The existence of the global stable manifold $W^s(E_2)$ with the stated properties follows from Theorems 1.5, 1.7, and 1.8 and Lemmas 2.1 and 2.2.

First we prove that for all points $M(x) = (x, (\alpha_1/x) + \beta_1), x \neq 0$, the following holds:

$$M(x) \preceq_{se} T(M(x)). \quad (4.4)$$

Observe that $M(x)$ is actually an arbitrary point on the curve $x = (\alpha_1 + \beta_1 x)/y$, which represents one of two equilibrium curves for system (1.1).

Indeed,

$$T(x, (\alpha_1/x) + \beta_1) = \left(x, \frac{\alpha_2 + \gamma_2((\alpha_1/x) + \beta_1)}{A_2 + x} \right)$$

$$= \left(x, \frac{\alpha_2 x + \gamma_2(\alpha_1 + \beta_1 x)}{(A_2 + x)x} \right). \quad (4.5)$$

Now we have

$$\left(x, \frac{\alpha_1}{x} + \beta_1 \right) \preceq_{se} \left(x, \frac{\alpha_2 x + \gamma_2(\alpha_1 + \beta_1 x)}{x(A_2 + x)} \right)$$

$$\Longleftrightarrow x \leq x, \quad \frac{\alpha_1 + \beta_1 x}{x} \geq \frac{\alpha_2 x + \gamma_2(\alpha_1 + \beta_1 x)}{x(A_2 + x)}. \quad (4.6)$$

The last inequality is equivalent to

$$(\alpha_1 + \beta_1 x)(A_2 + x) \geq \alpha_2 x + \gamma_2 \alpha_1 + \beta_1 \gamma_2 x. \quad (4.7)$$

This is equivalent to

$$\beta_1 x^2 + x(\alpha_1 - \alpha_2 + \beta_1(A_2 - \gamma_2)) + \alpha_1(A_2 - \gamma_2) \geq 0, \tag{4.8}$$

which always holds since the discriminant of the quadratic polynomial on the left-hand side is zero.

Note that $M(x) \preceq_{se} E_2$, and $M(x) = E_2$ for $x = ((\alpha_2 - \alpha_1) - \beta_1(A_2 - \gamma_2))/2\beta_1$.
Monotonicity of the map T implies

$$T^n(M(x)) \preceq_{se} T^{n+1}(M(x)). \tag{4.9}$$

Set $T^n(M(x)) = \{(x_n, y_n)\}$. Then the sequence $\{x_n\}$ is increasing and bounded by x-coordinate of the equilibrium, and the sequence $\{y_n\}$ is decreasing and bounded by y-coordinate of the equilibrium. This implies that $\{(x_n, y_n)\}$ converges to the equilibrium as $n \to \infty$.

Now, take any point $A(x, y) \in Q_2(E_2)$. Then there exists point $M(x^*)$ such that $M(x^*) \preceq_{se} A(x, y) \preceq_{se} E_2$. By using monotonicity of the map T, we obtain

$$T^n(M(x^*)) \preceq_{se} T^n(A(x, y)) \preceq_{se} E_2. \tag{4.10}$$

Letting $n \to \infty$ in (4.10), we have

$$\lim_{n \to \infty} T^n(A(x, y)) = E_2. \tag{4.11}$$

Now, we consider $Q_4(E_2)$. By choosing $M(x)$ such that $E_2 \preceq_{se} M(x)$, we note that

$$E_2 \preceq_{se} M(x) \preceq_{se} T(M(x)). \tag{4.12}$$

By using monotonicity of the map T, we have

$$T^n(M(x)) \preceq_{se} T^{n+1}(M(x)). \tag{4.13}$$

Set $T^n(M(x)) = \{(x_n, y_n)\}$. Then the sequence $\{x_n\}$ is increasing, and the sequence $\{y_n\}$ is decreasing and bounded by y-coordinate of equilibrium and has to converge. If $\{x_n\}$ converges, then $\{(x_n, y_n)\}$ has to converge to the equilibrium, which is impossible. This implies that $x_n \to \infty$, $n \to \infty$. Since $y_{n+1} = (\alpha_2 + \gamma_2 y_n)/(A_2 + x_n)$, then $\lim_{n \to \infty} y_n = 0$.

Now, take any point $B(x, y)$ in $Q_4(E_2)$. Then there is point $M(x^{**})$ such that $E_2 \preceq_{se} M(x^{**}) \preceq_{se} B(x, y)$. Using monotonicity of the map T, we have

$$E_2 \preceq_{se} T^n(M(x^{**})) \preceq_{se} T^n(B(x, y)). \tag{4.14}$$

Since, $T^n(M(x^{**}))$ is asymptotic to $(\infty, 0)$, then $\lim_{n \to \infty} T^n(B(x, y)) = (\infty, 0)$. □

Theorem 4.4. *Assume*

$$A_2 = \gamma_2, \qquad \alpha_2 > \alpha_1. \tag{4.15}$$

Then system (1.1) *has a unique equilibrium E which is a saddle point. Furthermore, there exists the global stable manifold $W^s(E)$ that separates the positive quadrant so that all orbits below this manifold are asymptotic to $(\infty, 0)$, and all orbits above this manifold are asymptotic to $(0, \infty)$. All orbits that start on $W^s(E)$ are attracted to E. The global stable manifold $W^u(E)$ is the graph of a continuous, unbounded, strictly increasing function.*

Proof. The existence of the global stable manifold $W^s(E)$ with the stated properties follows from Theorems 1.5, 1.7, and 1.8 and Lemmas 2.1 and 2.2. $\qquad\square$

Theorem 4.5. *Assume*

$$A_2 > \gamma_2, \quad [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) < 0, \tag{4.16}$$

$$A_2 > \gamma_2, \quad [(\alpha_1 - \alpha_2) + \beta_1(A_2 - \gamma_2)]^2 - 4\alpha_1\beta_1(A_2 - \gamma_2) \geq 0, \quad \beta_1(A_2 - \gamma_2) > \alpha_2 - \alpha_1 \tag{4.17}$$

or

$$A_2 = \gamma_2, \qquad \alpha_2 \leq \alpha_1. \tag{4.18}$$

Then system (1.1) *does not possess an equilibrium point. Its global behavior is described as follows:*

$$x_n \longrightarrow \infty, \quad y_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.19}$$

Proof. If the conditions of this theorem are satisfied, then (2.6) implies that there is no real (if the first condition of this theorem is satisfied) or positive equilibrium points (if the second condition of this theorem is satisfied).

Consider the second equation of system (1.1). That is,

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n}. \tag{4.20}$$

Note the following

$$y_{n+1} \leq \frac{\alpha_2}{A_2} + \frac{\gamma_2}{A_2} y_n. \tag{4.21}$$

Now, consider equation

$$u_{n+1} - \frac{\gamma_2}{A_2} u_n = \frac{\alpha_2}{A_2}. \tag{4.22}$$

Its solution is given by

$$u_n = c\left(\frac{\gamma_2}{A_2}\right)^n + \frac{\alpha_2}{A_2 - \gamma_2}. \tag{4.23}$$

Since $A_2 > \gamma_2$, then letting $n \to \infty$ we obtain that $u_n \to 0$. Now, (4.21) implies

$$y_n \le \frac{\alpha_2}{A_2 - \gamma_2} + \varepsilon, \quad n \longrightarrow \infty. \tag{4.24}$$

This means that sequence $\{y_n\}$ is bounded for $A_2 > \gamma_2$.

In order to prove the global behavior in this case, we decompose System (1.1) into the system of even-indexed and odd-indexed terms as

$$\begin{aligned} x_{2n+1} &= \frac{\alpha_1 + \beta_1 x_{2n}}{y_{2n}}, \\ x_{2n} &= \frac{\alpha_1 + \beta_1 x_{2n-1}}{y_{2n-1}}, \\ y_{2n+1} &= \frac{\alpha_2 + \gamma_2 y_{2n}}{A_2 + x_{2n}}, \\ y_{2n} &= \frac{\alpha_2 + \gamma_2 y_{2n-1}}{A_2 + x_{2n-1}}, \end{aligned} \tag{4.25}$$

for $n = 1, 2, \dots$.

Lemma 2.1 implies that subsequences $\{x_{2n+1}\}$, $\{x_{2n}\}$, $\{y_{2n+1}\}$, and $\{y_{2n}\}$ are eventually monotone.

Since sequence $\{y_n\}$ is bounded, then the subsequences $\{y_{2n+1}\}$ and $\{y_{2n}\}$ must converge. If the sequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ would converge to finite numbers, then the solution of (1.1) would converge to the period-two solution, which is impossible by Lemma 2.2. Thus at least one of the subsequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ tends to $+\infty, n \to \infty$. Assume that $x_{2n} \to \infty$ as $n \to \infty$. In view of third equation of (4.25), $y_{2n+1} \to 0$, and in view of first equation of (4.25), $x_{2n+1} \to \infty$ which by fourth equation of (4.25) implies that $y_{2n+1} \to 0$ as $n \to \infty$.

Now, we prove the case when $A_2 = \gamma_2$ and $\alpha_2 = \alpha_1$.

In this case System (1.1) becomes

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n}{y_n}, \\ y_{n+1} &= \frac{\alpha_1 + A_2 y_n}{A_2 + x_n}. \end{aligned} \tag{4.26}$$

The map T associated to System (4.26) is given by

$$T(x, y) = \left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\alpha_1 + A_2 y}{A_2 + x}\right). \tag{4.27}$$

Equilibrium curves C_1 and C_2 can be given explicitly as the following functions of x:

$$C_1 : y_1 = \frac{\alpha_1}{x} + \beta_1,$$
$$C_2 : y_2 = \frac{\alpha_1}{x}. \tag{4.28}$$

It is obvious that these two curves do not intersect, which means that System (4.26) does not possess an equilibrium point.

Similarly, as in the proof of Theorem 4.3, for all points $C_1(x) = (x, (\alpha_1/x) + \beta_1), x \neq 0$ the following holds:

$$C_1(x) \preceq T(C_1(x)). \tag{4.29}$$

Indeed,

$$T\left(x, \frac{\alpha_1}{x} + \beta_1\right) = \left(x, \frac{\alpha_1 + A_2\left(\frac{\alpha_1}{x} + \beta_1\right)}{A_2 + x}\right)$$
$$= \left(x, \frac{\alpha_1 x + A_2(\alpha_1 + \beta_1 x)}{(A_2 + x)x}\right). \tag{4.30}$$

Now, we have

$$\left(x, \frac{\alpha_1}{x} + \beta_1\right) \preceq_{se} \left(x, \frac{\alpha_1 x + A_2(\alpha_1 + \beta_1 x)}{(A_2 + x)x}\right)$$
$$\Longleftrightarrow x \leq x \text{ and } \frac{\alpha_1}{x} + \beta_1 \geq \frac{\alpha_1 x + A_2(\alpha_1 + \beta_1 x)}{(A_2 + x)x}. \tag{4.31}$$

The last inequality is equivalent to

$$(\alpha_1 + \beta_1 x)(A_2 + x) \geq \alpha_1 x + A_2 \alpha_1 + \beta_1 A_2 x, \tag{4.32}$$

which always holds.

Monotonicity of T implies

$$T^n(C_1(x)) \preceq_{se} T^{n+1}(C_1(x)). \tag{4.33}$$

Set $T^n(C_1(x)) = \{(x'_n, y'_n)\}$. Then the sequence $\{x'_n\}$ is increasing and the sequence $\{y'_n\}$ is decreasing. Since $\{y'_n\}$ is decreasing and $y'_n > 0$, $n = 1, 2, \ldots$, then it has to converge. If $\{x'_n\}$ converges, then $\{(x'_n, y'_n)\}$ has to converge to the equilibrium, which is impossible. This implies that $x'_n \to \infty$, $n \to \infty$. The second equation of System (4.26) implies that $y'_n \to 0$, $n \to \infty$.

Now, take any point $(x, y) \in \mathbb{R}_+^2$. Then there exists point $C_1(x^\star) = (x^\star, \alpha_1/x^\star)$ such that

$$C_1(x^\star) \preceq_{\text{se}} (x, y). \tag{4.34}$$

Monotonicity of T implies

$$T^n(C_1(x^\star)) \preceq_{\text{se}} T^n(x, y). \tag{4.35}$$

Set

$$T^n(x, y) = (x_n, y_n), \quad T^n(C_1(x^\star)) = (x_n'', y_n''). \tag{4.36}$$

Then, we have

$$\begin{aligned} x_n'' &\leq x_n, \\ y_n'' &\geq y_n. \end{aligned} \tag{4.37}$$

Since

$$x_n'' \longrightarrow \infty, \quad y_n'' \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.38}$$

we conclude, using the inequalities (4.37), that

$$x_n \longrightarrow \infty, \quad y_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.39}$$

Similarly, we can prove the case $A_2 = \gamma_2, \ \alpha_2 < \alpha_1$. $\qquad\square$

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