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Dynamic Relative Output-Feedback Consensus for Multi-Agent Systems Subject to Actuator Saturation

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ABSTRACT This paper deals with the problem of leaderless consensus control for a class of non-introspective linear multi-agent systems (MASs) subject to input saturation and external disturbances. A novel dynamic relative output-feedback saturated consensus control protocol is proposed, which utilizes not only relative system outputs but also relative controller states from neighboring agents for distributed feedback control. With this new controller structure, the associated control synthesis conditions that guarantee optimal disturbance attenuation performance are fully characterized as linear matrix inequalities (LMIs) using a complete form of Lyapunov function matrix, which can be solved efficiently via convex optimization. The proposed approach unifies the designs for both continuous-time and discrete-time MASs. Two application examples are used to demonstrate effectiveness and usefulness of the proposed results.

INDEX TERMS Multi-agent systems, consensus control, actuator saturation, dynamic relative output feedback, linear matrix inequality.

I. INTRODUCTION

Multi-agent system (MAS), as a typical class of large-scale interconnected systems, has been one of the most thriving research topics in the controls community. It arises in many practical engineering applications, such as swarm robots coordination [1], multiple aircrafts formation flight [2], autonomous vehicles platooning [3], and cooperative regulation of smart power grids [4], etc. Due to its theoretical and practical importance, numerous mathematical frameworks and tools for modeling and control design of MAS have been emerging rapidly over the past two decades (see [5]–[9] and the references cited therein). Among many others, consensus control is the most fundamental framework, which specifies an interesting objective of driving the states/outputs of all agents in the group to reach a common agreement through certain inter-agent coordination strategies [5]. Early studies of consensus control of MASs were focused on relatively simple system settings with single/double integrator agent dynamics (see, e.g., [10], [11]), which have been gradually extended to general higher-order linear/nonlinear MASs (e.g., [12], [13]). Aiming to bridge the gap between theoretical development of MAS control algorithms and their realistic engineering applications, recent research interests have been steering to consideration of various physical constraints, such as uncertain agent dynamics [14], [40], network-induced communication delays [13], switching topologies [15], and actuator saturations [16], [41], etc.

In particular, saturation is a widely encountered and most dangerous nonlinearity in any realistic control systems [42]–[44]. Knowing that ignoring the effects of input saturations in control system designs could lead to performance degradation or even instability of the system, considerable research efforts have been devoted to the associated problem of saturated MAS consensus control. For example, [17] proposed a new type of "bang-bang" control protocol for a class of MASs to achieve finite-time consensus in the presence of saturation constraints; [18] addressed the input saturated consensus control problem by introducing a new class of coordinated saturation functions; and [19] dealt with the saturation issue in MAS control designs by leveraging the low-gain control method from classical saturated control theory. These earlier results, however, only apply to MASs with special agent dynamics of single/double

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integrators. Such a limitation was nicely bypassed in subsequent research by utilizing various novel methodologies. Particularly, [20] managed to further extend the low-gain control method to general linear MASs, and robust semi-global consensus control performance was achieved therein. These results were later on improved by [21] using an adaptive periodic event-triggered control strategy, and by [22] with a fully-distributed event-triggered control strategy. Beyond state consensus, the output consensus control problem was addressed in [12] with guaranteed semi-global stabilization performance in the presence of input saturation and external disturbances. Further progress has been made in [23] and [16] to enable global consensus performance for continuous-time and discrete-time MASs, respectively. Many other important works along this research direction consider mixed effects of saturation and other physical constraints. Representative works include [24] considering saturated consensus control subject to stochastic disturbances; [25] considering input saturation mixed with switching communication topologies and time delays; and [26] studying the effects of both saturation and energy-bounded disturbance on estimation of the domain of consensus attraction. Although tremendous progress has been made, all the above-mentioned results commonly suffer from one critical drawback, that is, they were all developed based on (relative) state feedback, which might not be applicable in many practical situations when only partial states or measurement outputs are available.

The most straightforward idea to overcome the above drawback is to construct an observer for each agent to online estimate their full state information, which has been well realized by many researchers in the field (e.g., [18], [27]–[30], [45]). However, the state-observer-based method has its inherent limitations in saturated MAS consensus control designs. First, it may yield conservative designs as the associated Lyapunov function used in both system analysis and control synthesis typically needs to be constructed in a special block-diagonal form (see, e.g., [28], [29]). Second, it requires each individual agent to have direct access to their own system's measurement outputs for state estimation, which could be problematic under the MAS distributed context. This is because many MASs are by nature non-introspective [31], meaning that each individual agent in the group is not able to measure the absolute state/output information of its own but only able to sense relative information from its neighbors, possibly due to limited localization/sensing capabilities. One important motivation example of considering non-introspective MASs is that: two vehicles in close proximity may be able to measure their relative distance without either of them having knowledge of their absolute position [31]. As an alternative of observer-based methods, dynamic relative output-feedback methods can be used to surmount the above limitations, as suggested by [32]. Nevertheless, one important and challenging issue yet to be addressed along this research line is how to formulate the associated relative output-feedback saturated consensus control synthesis problem in terms of computationally-tractable conditions. Specifically, existing dynamic relative output-feedback consensus control approaches normally lead to non-convex control synthesis conditions that are often formulated as bilinear matrix inequalities (BMIs), even when ignoring the saturation constraint [33]. This owes to the system-level complexity induced by networked interconnected structures. Solving such resulting non-convex BMIs is NP-hard and needs to resort to global optimization techniques that could be rather computationally expensive.

In this paper, we aim to overcome all the above deficiencies suffered by current state-of-the-art consensus control techniques for saturated MASs. To this end, we propose to develop new saturated consensus controller structures under the dynamic relative output-feedback framework. Specifically, the MAS under consideration has general linear dynamics subject to input saturations and energy-bounded external disturbances, and is assumed to be non-introspective and leaderless. The main contributions of this paper can be summarized as follows:

- A novel dynamic relative output-feedback saturated consensus control protocol is proposed, which consists of a linear relative information feedback control loop and a nonlinear deadzone feedback control loop. This new controller structure is compelling in the sense that it not only utilizes the relative measurement outputs from neighboring agent plants, but also shares relative controller state information for distributed feedback control.
- The associated relative output-feedback saturated consensus control synthesis conditions can be fully characterized as computationally-tractable linear matrix inequalities (LMIs) with a complete form of Lyapunov function. As a result, an optimal consensus control solution that guarantees optimal disturbance attenuation performance can be synthesized efficiently via convex LMI-based optimization.
- The proposed approach unifies the designs for both continuous-time and discrete-time MASs.

The rest of the paper is organized as follows. Section II will first present some preliminary results on graph theory and regional analysis for single saturated linear systems. The problem statement will be specified in Section III, followed by the main results including the new dynamic relative output-feedback saturated consensus controller structure and derivation of the associated convex synthesis conditions. Section IV utilizes two application examples to illustrate the design procedure and demonstrate effectiveness of the proposed approaches. Finally, conclusions are drawn in Section V.

II. PRELIMINARIES

A. NOTATION AND GRAPH THEORY

Throughout the paper, $\mathbb{R}$ is used to represent the set of real numbers. $\mathbb{R}_+$ stands for the set of positive real numbers. $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices, and $\mathbb{R}^n$ represents the set of real $n\times 1$ vectors. I_n and 1_n denote the $n\times n$ identity matrix and an n-dimensional column vector with all elements being 1, respectively. $\mathbb{S}^n$ and $\mathbb{S}^n_+$ are used to denote the sets of real symmetric $n \times n$ matrices and positive definite matrices, respectively. A block diagonal matrix with matrices $X_1, X_2, \cdots, X_p$ on its main diagonal is denoted by $diag\{X_1, X_2, \cdots, X_p\}$. The notation $A \otimes B$ represents the Kronecker product of matrices A and B. For a series of column vectors $x_1, \cdots, x_n$, $col\{x_1, \cdots, x_n\}$ stands for a column vector by stacking them together. The symbol $\star$ in LMIs is used to denote entries that follow from symmetry. For two integers $k_1 < k_2$, we denote $I[k_1, k_2] = \{k_1, k_1 + 1, \cdots, k_2\}$. For a matrix $M \in \mathbb{R}^{m\times n}$, M^T denotes its transpose. The hermitian operator $He\{\cdot\}$ is defined as $He\{M\} = M + M^T$ for real matrices. For $x \in \mathbb{R}^n$, its norm is defined as $\|x\| := (x^T x)^{1/2}$. The space of square integrable (summable) functions is denoted by $\mathcal{L}_2$ (ℓ_2), that is, for any $u \in \mathcal{L}_2$ (or ℓ_2), $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{1/2} < \infty$ (or $\|u\|_2 := \left(\sum_{t=0}^\infty u^T(t)u(t)\right)^{1/2} < \infty$). $Co\{S\}$ denotes the convex hull of a set S. For $P \in \mathbb{S}^{n\times n}_+$, we define an ellipsoid $\mathcal{E}(P, s) := \{x \in \mathbb{R}^n : x^T Px \le s^2\}$.

In this paper, we consider MASs whose interconnection structures are described by an undirected graph. Specifically, a graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E}_d, \mathcal{A})$, where the elements of $\mathcal{V} = \{1, 2, \cdots, N_a\}$ are called vertices, the elements of $\mathcal{E}_d$ are pairs (i, j) with $i, j \in \mathcal{V}, i \neq j$, called edges, and the matrix $\mathcal{A}$ is called the adjacency matrix. If $(i, j) \in \mathcal{E}_d$, it means that agent i can receive information from agent j where these two agents are called adjacent. The adjacency matrix is thus defined as $\mathcal{A} = [a_{ij}]_{N_a\times N_a}$, with $a_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}_d$, and $a_{ij} = 0$ otherwise. The graph $\mathcal{G}$ is called undirected if for every $(i, j) \in \mathcal{E}_d$ also $(j, i) \in \mathcal{E}_d$. For a given vertex, say i, its neighboring set $\mathcal{N}_i$ is defined by $\mathcal{N}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}_d\}$. For a given graph, the Laplacian matrix of the graph is defined as $\mathcal{L} = [l_{ij}]$, where $l_{ii} = \sum_{j\neq i} a_{ij}, l_{ij} = -a_{ij}, i \neq j$. If the graph is undirected, then $\mathcal{L}$ is a positive semi-definite matrix, so all eigenvalues of $\mathcal{L}$ are non-negative real. Zero is always an eigenvalue of the Laplacian $\mathcal{L}$, so it has rank at most $N_a - 1$. Furthermore, an undirected graph is called connected if for every pair of distinct vertices i and j there exists a path from i to j, i.e., a finite set of edges (i_k, i_{k+1}) with $k = 1, 2, \cdots, r - 1$ such that $i_1 = i$ and $i_r = j$. An undirected graph is connected if and only if its Laplacian has rank $N_a - 1$. In that case the zero eigenvalue has multiplicity one, and all other eigenvalues are positive real. The remaining $N_a - 1$ eigenvalues are ordered in an increasing order as $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N_a-1}$.

B. REGIONAL ANALYSIS FOR SINGLE SATURATED LINEAR SYSTEMS

For a single/solo linear system subject to symmetric saturation nonlinearity, the associated symmetric saturation function is defined as $sat(u_i) = sgn(u_i)\min\{\overline{u}_i, |u_i|\}$, where $u = \begin{bmatrix}u_1 \; u_2 \cdots u_{n_u}\end{bmatrix}^T$ represents the control input vector, and $\overline{u}_i$ denotes the saturation level of $sat(u_i)$ ($\forall i \in I[1, n_u]$). Note that for symmetric saturations, the absolute values of the negative and positive saturation levels are identical. Here, we have slightly abused the notation by using $sat(\cdot)$ to denote both the scalar-valued and vector-valued saturation functions. Therefore $sat(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is a vector-valued standard saturation function, i.e., $sat(u) = \begin{bmatrix}sat(u_1) \; sat(u_2) \cdots sat(u_{n_u})\end{bmatrix}^T$. The deadzone nonlinearity is closely related to the saturation function by $dz(u) = u - sat(u)$. Using deadzone functions, a linear system subject to actuator saturation can be represented in the following general form [34]

$$\begin{bmatrix}\mathcal{D}x\\ u\\ z\end{bmatrix} = \begin{bmatrix}A & B_0 & B_1\\ C_0 & D_{00} & D_{01}\\ C_1 & D_{10} & D_{11}\end{bmatrix}\begin{bmatrix}x\\ q\\ w\end{bmatrix}, \quad q = dz(u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $q, u \in \mathbb{R}^{n_u}$ represent some system's internal signals, $w \in \mathbb{R}^{n_w}$ denotes some exogenous signals (e.g., disturbance), and $z \in \mathbb{R}^{n_z}$ represents the system's controlled output for quantifying some performance, such as the $\mathcal{L}_2(\ell_2)$ disturbance attenuation performance. The symbol $\mathcal{D}$ denotes a differentiator for continuous-time systems and a difference operator for discrete-time systems.

To facilitate regional analysis and saturation control design, we will introduce a linear subset as

$$\mathcal{R}(H) = \{x \in \mathbb{R}^n : |Hx|_\infty \le \overline{\mathbf{u}}\}, \tag{2}$$

where $H \in \mathbb{R}^{n_u\times n}$ and $\overline{\mathbf{u}} = [\overline{u}_1, \cdots, \overline{u}_{n_u}]^T$. Then, for the system (1), if $x \in \mathcal{R}(H)$, the deadzone function can be written in the following form according to [35]:

$$dz(u) = \Theta(u - Hx) =: \Theta p,$$

for some $\Theta = diag\{\lambda_{\Theta,1}, \lambda_{\Theta,2}, \cdots, \lambda_{\Theta,n_u}\}$ satisfying $0 \le \lambda_{\Theta,i} \le 1, \forall i \in I[1, n_u]$. Applying this relation to the original system (1), we obtain

$$\begin{bmatrix}\mathcal{D}x\\ p\\ z\end{bmatrix} = \begin{bmatrix}A & B_0 & B_1\\ C_0 - H & D_{00} & D_{01}\\ C_1 & D_{10} & D_{11}\end{bmatrix}\begin{bmatrix}x\\ q\\ w\end{bmatrix}, \quad q = \Theta\tilde{u}. \tag{3}$$

Observe that the resulting system (3) is essentially a linear parameter-varying (LPV) system with a linear fractional transformation (LFT) dependency on Θ [36].

For disturbance attenuation problem, we are mainly concerned with a class of energy-bounded disturbances in continuous time

$$\mathcal{W}_s = \left\{w : \mathbb{R}_+ \to \mathbb{R}^{n_w}, \int_{t=0}^\infty w^T(t)w(t)dt \le s^2, w \in \mathcal{L}_2\right\},$$

or in discrete time

$$\mathcal{W}_s = \left\{w : \mathbb{R}_+ \to \mathbb{R}^{n_w}, \sum_{t=0}^\infty w^T(t)w(t) \le s^2, w \in \ell_2\right\},$$

in which s is a positive scalar whose value is given. The level of disturbance attenuation will be measured by the following regional $\mathcal{L}_2$ (ℓ_2) gain performance index

$$\gamma := \sup_{x(0)=0, w\neq 0, w\in\mathcal{W}_s} \frac{\|z\|_2}{\|w\|_2}. \tag{4}$$

Based on the above results, the following two lemmas provide the regional stability and $\mathcal{L}_2$ (ℓ_2) gain analysis conditions for system (1) in continuous time and discrete time, respectively, by slightly modifying the well-known Scaled Bounded Real Lemma [37].

Lemma 1 (Continuous-Time Case): Consider the continuous-time system (1) subject to actuator saturation and energy-bounded disturbance $w \in \mathcal{W}_s$. If there exist a positive definite matrix $P \in \mathbb{S}_+^n$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar $\gamma \in \mathbb{R}_+$ such that

$$\begin{bmatrix} He\{PA\} & \star & \star & \star \\ C_0 - H + \Gamma B_0^T P & He\{-\Gamma + D_{00}\Gamma\} & \star & \star \\ B_1^T P & D_{01}^T & -\gamma I & \star \\ C_1 & D_{10}\Gamma & D_{11} & -\gamma I \end{bmatrix} < 0, \quad (5)$$

$$\begin{bmatrix} \frac{\bar{u}_k^2}{s^2} & \star \\ H^T\delta_k & P \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u] \quad (6)$$

where the notation δ_k is used to denote the kth column vector of the identity matrix. Then, the system (1) has its trajectories contained within $\mathcal{E}(P, s)$ and achieves a regional $\mathcal{L}_2$ gain less than γ, i.e., $\|z\|_2 < \gamma\|w\|_2$, when $x(0) = 0$.

Proof: First of all, condition (6) ensures that $\mathcal{E}(P, s) \subseteq \mathcal{R}(H)$ and hence the system description (3) is valid within $\mathcal{E}(P, s)$. Then, based on the Scaled Bounded Real Lemma [37], to prove the conclusions, it suffices to have the Lyapunov function $V(x) = x^T Px$ satisfying the following condition for system (3) by noticing that $0 \leq \lambda_{\Theta,i} \leq 1, \forall i \in I[1, n_u]$, and matrices Θ and Γ^{-1} are commutable, i.e., $\Gamma^{-1}\Theta = \Theta\Gamma^{-1}$,

$$\dot{V}(x) + 2q^T\Gamma^{-1}(\tilde{u} - q) + \frac{1}{\gamma^2}z^T z - w^T w < 0.$$

Consequently, by using Schur complement, the above condition can be converted to the matrix inequality

$$\begin{bmatrix} He\{PA\} & \star & \star & \star \\ \Gamma^{-1}(C_0 - H) + B_0^T P & He\{-\Gamma^{-1} + \Gamma^{-1}D_{00}\} & \star & \star \\ B_1^T P & D_{01}^T\Gamma^{-1} & -\gamma I & \star \\ C_1 & D_{10} & D_{11} & -\gamma I \end{bmatrix} < 0. \quad (7)$$

Then, performing a congruence transformation on (7) with $diag\{I, \Gamma, I, I\}$, it verifies condition (5). □

Similarly, the ℓ_2-gain analysis conditions for saturated discrete-time systems are stated in the following lemma.

Lemma 2 (Discrete-Time Case): Consider the discrete-time system (1) subject to actuator saturation and energy-bounded disturbance $w \in \mathcal{W}_s$. If there exist a positive definite matrix $P \in \mathbb{S}_+^n$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar $\gamma \in \mathbb{R}_+$ such that

$$\begin{bmatrix} -P & \star & \star & \star & \star \\ C_0 - H & He\{-\Gamma - D_{00}\Gamma\} & \star & \star & \star \\ 0 & D_{01}^T & -\gamma I & \star & \star \\ PA & PB_0\Gamma & PB_1 & -P & \star \\ C_1 & D_{10}\Gamma & D_{11} & 0 & -\gamma I \end{bmatrix} < 0, \quad (8)$$

$$\begin{bmatrix} \frac{\bar{u}_k^2}{s^2} & \star \\ H^T\delta_k & P \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u]. \quad (9)$$

Then, the system (1) has its trajectories contained within $\mathcal{E}(P, s)$ and achieves a regional ℓ_2 gain less than γ, i.e., $\|z\|_2 < \gamma\|w\|_2$, when $x(0) = 0$.

Proof: The proof follows similar ideas of the proof of Lemma 1. First, condition (9) implies $\mathcal{E}(P, s) \subseteq \mathcal{R}(H)$, and hence the system description (3) is valid within $\mathcal{E}(P, s)$. Then, based on the discrete-time Scaled Bounded Real Lemma [37], to prove the conclusions of the Lemma, it suffices to have the Lyapunov function $V(x) = x^T Px$ satisfying the following condition for the discrete-time system (3) by noticing that $0 \leq \lambda_{\Theta,i} \leq 1, \forall i \in I[1, n_u]$, and matrices Θ and Γ^{-1} are commutable, i.e., $\Gamma^{-1}\Theta = \Theta\Gamma^{-1}$,

$$V(x(t+1)) - V(x(t)) + 2q^T(t)\Gamma^{-1}(\tilde{u}(t) - q(t)) + \frac{1}{\gamma^2}z^T(t)z(t) - w^T(t)w(t) < 0.$$

Consequently, by using Schur complement, the above condition can be converted to the matrix inequality

$$\begin{bmatrix} -P & \star & \star & \star & \star \\ \Gamma^{-1}(C_0 - H) & He\{-\Gamma^{-1} - \Gamma^{-1}D_{00}\} & \star & \star & \star \\ 0 & D_{01}^T\Gamma^{-1} & -\gamma I & \star & \star \\ PA & PB_0 & PB_1 & -P & \star \\ C_1 & D_{10} & D_{11} & 0 & -\gamma I \end{bmatrix} < 0. \quad (10)$$

Then, performing a congruence transformation on (10) with $diag\{I, \Gamma, I, I, I\}$, it verifies condition (8). □

III. MAIN RESULTS

A. PROBLEM STATEMENT

In this paper, we consider a MAS consisting of N_a dynamical agents subject to actuator saturations and unknown external disturbances, whose dynamics can be described by

$$\mathcal{P}_i : \begin{bmatrix} \mathcal{D}x_{p,i} \\ y_i \end{bmatrix} = \begin{bmatrix} A_p & B_{p1} & B_{p2} \\ C_p & D_p & 0 \end{bmatrix} \begin{bmatrix} x_{p,i} \\ w_i \\ sat(u_i) \end{bmatrix}, \quad (11)$$

for all $i \in I[1, N_a]$, where $x_{p,i} \in \mathbb{R}^{n_x}$ is the state, $u_i \in \mathbb{R}^{n_u}$ is the control input, $w_i \in \mathbb{R}^{n_w}$ is energy-bounded disturbance, $y_i \in \mathbb{R}^{n_y}$ is the measurement output. $sat(u_i)$ denotes the saturation function with the saturation level of $\bar{u}_k$ ($k \in I[1, n_u]$).

The subscript i denotes the index of the ith agent. All the system matrices A_p, B_{p1}, B_{p2}, C_p and D_p are known constant matrices with appropriate dimensions. We have the following assumptions regarding the agent dynamics and associated communication network graph $\mathcal{G}$ of the MAS (11).

Assumption 1: (A_p, B_{p2}) *and* (A_p, C_p) *are stabilizable and detectable, respectively.*

Assumption 2: $\mathcal{G}$ *is undirected and connected.*

Given the above system setup, our objective in this paper is to design an output-feedback control protocol for the MAS (11) using relative output information, such that (i) the states of the MAS (11) reach consensus, i.e., $\lim_{t\to\infty}(x_{p,i}(t) - x_{p,j}(t)) = 0$ when $w_i = 0$, $\forall i, j \in I[1, N_a]$; and (ii) the overall closed-loop MAS achieves optimal regional $\mathcal{L}_2(\ell_2)$-gain disturbance attenuation performance in the sense of (4). To more specifically quantify the second objective of disturbance attenuation performance, we define the state error of the ith agent relative to the average state of all agents as

$$z_i = x_{p,i} - \frac{1}{N_a}\sum_{j=1}^{N_a} x_{p,j}, \quad \forall i \in I[1, N_a]. \tag{12}$$

Let $z := col\{z_1, z_2, \cdots, z_{N_a}\} \in \mathbb{R}^{N_a n_x}$, then it is clear that $z = 0$ indicates $x_{p,i} = x_{p,j}$ for all $i, j \in I[1, N_a]$, i.e., consensus of the MAS (11). Therefore, $z(t)$ can be used as the controlled (performance) output of the overall MAS to quantify the state disagreement among all agents. Consequently, the considered regional $\mathcal{L}_2(\ell_2)$-gain disturbance attenuation performance can be characterized as the $\mathcal{L}_2$ (ℓ_2) norm between the energy-bounded disturbance $w := col\{w_1, w_2, \cdots, w_{N_a}\} \in \mathbb{R}^{N_a n_w}$ and the controlled output z, i.e., $\|z\|_2 < \gamma\|w\|_2$, $\forall w \in \mathcal{W}_s$. Since smaller $\gamma > 0$ indicates better disturbance attenuation performance, the optimal consensus control design problem seeks to synthesize such an optimal control protocol that would render a minimal $\mathcal{L}_2$ (ℓ_2) gain γ for the overall closed-loop MAS.

B. DYNAMIC RELATIVE OUTPUT-FEEDBACK CONTROLLER STRUCTURE AND SYSTEM TRANSFORMATIONS

To fulfill the above objectives, we propose to construct a dynamic relative output-feedback control protocol for the MAS (11) in the form of

$$\mathcal{C}_i : \begin{cases} \mathcal{D}x_{c,i} = A_{c1}x_{c,i} + A_{c2}\sum_{j=1}^{N_a} a_{ij}(x_{c,i} - x_{c,j}) \\ \qquad + B_{c1}\sum_{j=1}^{N_a} a_{ij}(y_i - y_j) + B_{c2}dz(u_i) \\ u_i = C_{c1}x_{c,i} + C_{c2}\sum_{j=1}^{N_a} a_{ij}(x_{c,i} - x_{c,j}) \\ \qquad + D_{c1}\sum_{j=1}^{N_a} a_{ij}(y_i - y_j) + D_{c2}dz(u_i) \end{cases} \tag{13}$$

for all $i \in I[1, N_a]$, where $x_{c,i} \in \mathbb{R}^{n_c}$ is the controller state with its order n_c to be determined, a_{ij} are the adjacency elements associated with $\mathcal{G}$. $A_{c1}, A_{c2}, B_{c1}, B_{c2}$ and $C_{c1}, C_{c2}, D_{c1}, D_{c2}$ are controller gain matrices to be synthesized. For facilitate subsequent development, we introduce two matrices $H_C \in \mathbb{R}^{n_u\times n_x}$ and $H_D \in \mathbb{R}^{n_u\times n_c}$, and define an auxiliary set

$$\mathcal{R}\left(\begin{bmatrix}H_C & H_D\end{bmatrix}\right) = \Big\{(x_{p,i}, x_{c,i}) \in \mathbb{R}^{n_x} \times R^{n_c} : \left|\delta_k^T(H_C x_{p,i} + H_D x_{c,i})\right| \le \underline{u}_k, \; k \in I[1, n_u], \quad \forall i \in I[1, N_a]\Big\}.$$

To understand the structure of this new dynamic relative output-feedback protocol, it is seen that each agent will need to use their respective local information including the controller state $x_{c,i}$ and the control input u_i through a deadzone loop for feedback control. In addition, each agent will also need to collect relative information from their neighbors, including relative plant outputs $y_i - y_j$ and relative controller states $x_{c,i} - x_{c,j}$. Note that no absolute measurement output information is needed, which is favorable under the distributed control context, especially when the MAS is non-introspective [31]. The interconnection of the open-loop MAS (11) and the saturated control protocol (13) is depicted in Fig. 1 for illustration.

By substituting $sat(u_i) = u_i - dz(u_i)$ for all $i \in I[1, N_a]$, the original agent plant dynamics (11) can be rewritten in the following form:

$$\begin{cases} \mathcal{D}x_{p,i} = A_p x_{p,i} - B_{p2}q_i + B_{p1}w_i + B_{p2}u_i, \\ y_i = C_p x_{p,i} + D_p w_i, \\ q_i = dz(u_i). \end{cases} \tag{14}$$

Combining the MAS plant (14) and the saturated control protocol (13), one can derive the closed-loop MAS dynamics. Specifically, define the following aggregated vectors

$$\begin{aligned} x_p &= col\{x_{p,1}, x_{p,2}, \cdots, x_{p,N_a}\}, \\ x_c &= col\{x_{c,1}, x_{c,2}, \cdots, x_{c,N_a}\}, \\ q &= col\{q_1, q_2, \cdots, q_{N_a}\}, \\ u &= col\{u_1, u_2, \cdots, u_{N_a}\}, \end{aligned}$$

we have

$$\begin{bmatrix}\mathcal{D}x_p \\ \mathcal{D}x_c \\ u \\ z\end{bmatrix} = \begin{bmatrix} I_{N_a}\otimes A_p + \mathcal{L}\otimes B_{p2}D_{c1}C_p & I_{N_a}\otimes B_{p2}C_{c1} + \mathcal{L}\otimes B_{p2}C_{c2} & I_{N_a}\otimes(B_{p2}D_{c2} - B_{p2}) & I_{N_a}\otimes B_{p1} + \mathcal{L}\otimes B_{p2}D_{c1}D_p \\ \mathcal{L}\otimes B_{c1}C_p & I_{N_a}\otimes A_{c1} + \mathcal{L}\otimes A_{c2} & I_{N_a}\otimes B_{c2} & \mathcal{L}\otimes B_{c1}D_p \\ \mathcal{L}\otimes D_{c1}C_p & I_{N_a}\otimes C_{c1} + \mathcal{L}\otimes C_{c2} & I_{N_a}\otimes D_{c2} & \mathcal{L}\otimes D_{c1}D_p \\ \mathcal{L}_c\otimes I_{n_x} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}x_p \\ x_c \\ q \\ w\end{bmatrix},$$

$$q = dz(u), \tag{15}$$

where $\mathcal{L}$ is the Laplacian of the graph $\mathcal{G}$, and the controlled output equation of z is deduced from (12) with matrix $\mathcal{L}_c$

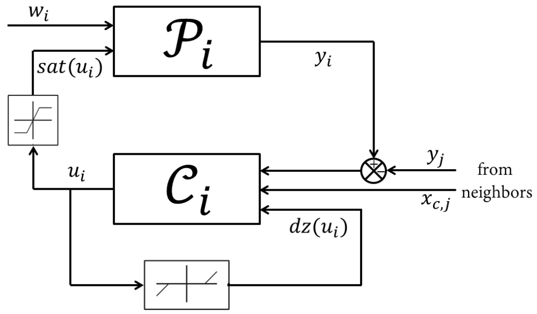


FIGURE 1. Closed-loop structure of the ith agent.

defined as

$$\mathcal{L}_c = \begin{bmatrix} \frac{N_a-1}{N_a} & -\frac{1}{N_a} & \cdots & -\frac{1}{N_a} \\ -\frac{1}{N_a} & \frac{N_a-1}{N_a} & \cdots & -\frac{1}{N_{aa}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N_a} & -\frac{1}{N_a} & \cdots & \frac{N_a-1}{N_a} \end{bmatrix}_{N_a\times N_a} = I_{N_a} - \frac{1}{N_a} 1_{N_a} 1_{N_a}^T.$$

For analysis purpose, we wish to simplify the above closed-loop system formulation through some coordinate transformations. To this end, choose an orthogonal $N_a \times N_a$ matrix with the form of

$$U_1 = \begin{bmatrix} ? & \cdots & ? & \frac{1}{\sqrt{N_a}} \\ ? & \cdots & ? & \frac{1}{\sqrt{N_a}} \\ \vdots & \ddots & \vdots & \frac{1}{\sqrt{N_a}} \\ ? & \cdots & ? & \frac{1}{\sqrt{N_a}} \end{bmatrix}, \tag{16}$$

where "?" denotes elements that we do not care about. It gives $U_1^T 1_{N_a} = [0, \cdots, 0, \sqrt{N_a}]^T$, and the following results can be immediately verified:

$$\begin{aligned} U_1^T \mathcal{L} U_1 &= \begin{bmatrix} \mathcal{L}_1 & 0_{N_a-1} \\ 0_{N_a-1}^T & 0 \end{bmatrix} := \tilde{\mathcal{L}}, \\ U_1^T \mathcal{L}_c U_1 &= U_1^T U_1 - \frac{1}{N_a} U_1^T 1_{N_a} 1_{N_a}^T U_1 \\ &= \begin{bmatrix} I_{N_a-1} & 0_{N_a-1} \\ 0_{N_a-1}^T & 0 \end{bmatrix}, \end{aligned} \tag{17}$$

where the submatrix $\mathcal{L}_1$ is positive definite. Based on this, we perform the following orthogonal transformation to system (15):

$$\begin{aligned} \tilde{x}_p &= (U_1 \otimes I_{n_x}) x_p, & \tilde{x}_c &= (U_1 \otimes I_{n_c}) x_c, \\ \tilde{q} &= (U_1 \otimes I_{n_u}) q, & \tilde{w} &= (U_1 \otimes I_{n_w}) w, \\ \tilde{z} &= (U_1 \otimes I_{n_x}) z, & \tilde{u} &= (U_1 \otimes I_{n_u}) u, \end{aligned}$$

which yields the following transformed model:

$$\begin{bmatrix} \mathcal{D}\tilde{x}_p \\ \mathcal{D}\tilde{x}_c \\ \tilde{u} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} I_{N_a} \otimes A_p + \tilde{\mathcal{L}} \otimes B_{p2} D_{c1} C_p & I_{N_a} \otimes B_{p2} C_{c1} + \tilde{\mathcal{L}} \otimes B_{p2} C_{c2} & I_{N_a} \otimes (B_{p2} D_{c2} - B_{p2}) & I_{N_a} \otimes B_{p1} + \tilde{\mathcal{L}} \otimes B_{p2} D_{c1} D_p \\ \tilde{\mathcal{L}} \otimes B_{c1} C_p & I_{N_a} \otimes A_{c1} + \tilde{\mathcal{L}} \otimes A_{c2} & I_{N_a} \otimes B_{c2} & \tilde{\mathcal{L}} \otimes B_{c1} D_p \\ \tilde{\mathcal{L}} \otimes D_{c1} C_p & I_{N_a} \otimes C_{c1} + \tilde{\mathcal{L}} \otimes C_{c2} & I_{N_a} \otimes D_{c2} & \tilde{\mathcal{L}} \otimes D_{c1} D_p \\ \begin{bmatrix} I_{N_a-1} & 0_{N_a-1} \\ 0_{N_a-1}^T & 0 \end{bmatrix} \otimes I_{n_x} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_p \\ \tilde{x}_c \\ \tilde{q} \\ \tilde{w} \end{bmatrix},$$

$$\tilde{q} = dz(\tilde{u}). \tag{18}$$

In light of the block-diagonal structure of the above system matrices, regarding the $\mathcal{L}_2(\ell_2)$-gain performance from the disturbance $\tilde{w}$ to the controlled output $\tilde{z}$, it is equivalent to consider the following reduced-order system (19) by defining $\tilde{x}_p = col\{\tilde{x}_p^1, \tilde{x}_p^2\}$ with $\tilde{x}_p^2$ being the last n_x elements of $\tilde{x}_p$, likewise defining $\tilde{x}_c^1$, $\tilde{q}^1$, $\tilde{u}^1$, $\tilde{w}^1$, and $\tilde{z}^1$.

$$\begin{bmatrix} \mathcal{D}\tilde{x}_p^1 \\ \mathcal{D}\tilde{x}_c^1 \\ \tilde{u}^1 \\ \tilde{z}^1 \end{bmatrix} = \begin{bmatrix} I_{N_a} \otimes A_p + \mathcal{L}_1 \otimes B_{p2} D_{c1} C_p & I_{N_a} \otimes B_{p2} C_{c1} + \mathcal{L}_1 \otimes B_{p2} C_{c2} & I_{N_a} \otimes (B_{p2} D_{c2} - B_{p2}) & I_{N_a} \otimes B_{p1} + \mathcal{L}_1 \otimes B_{p2} D_{c1} D_p \\ \mathcal{L}_1 \otimes B_{c1} C_p & I_{N_a} \otimes A_{c1} + \mathcal{L}_1 \otimes A_{c2} & I_{N_a} \otimes B_{c2} & \mathcal{L}_1 \otimes B_{c1} D_p \\ \mathcal{L}_1 \otimes D_{c1} C_p & I_{N_a} \otimes C_{c1} + \mathcal{L}_1 \otimes C_{c2} & I_{N_a} \otimes D_{c2} & \mathcal{L}_1 \otimes D_{c1} D_p \\ I_{N_a-1} \otimes I_{n_x} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_p^1 \\ \tilde{x}_c^1 \\ \tilde{q}^1 \\ \tilde{w}^1 \end{bmatrix},$$

$$\tilde{q}^1 = dz(\tilde{u}^1). \tag{19}$$

Remark 1: Through the above system transformations, we have the following observations:

- *First, according to the definition of $\mathcal{L}_2(\ell_2)$-gain performance from disturbance w to controlled output z for system (15), it can be easily verified that*

$$\|z\|_2 < \gamma \|w\|_2 \Leftrightarrow \|\tilde{z}\|_2 < \gamma \|\tilde{w}\|_2 \Leftrightarrow \|\tilde{z}^1\|_2 < \gamma \|\tilde{w}^1\|_2. \tag{20}$$

- *Second, from (18), $z = 0$ is equivalent to $\tilde{x}_p^1 = 0$. Since $z = 0$ implies that the overall MAS achieves consensus, asymptotic stability of system (19) thus ensures asymptotic consensus of system (15).*

In summary, the consensus performance of the closed-loop system (15) is achieved with $\mathcal{L}_2$ (ℓ_2) disturbance attenuation level γ if the system (19) is asymptotically stable and satisfies the $\mathcal{L}_2(\ell_2)$-gain performance with $\|\tilde{z}^1\|_2 < \gamma\|\tilde{w}^1\|_2$. Therefore, the original $\mathcal{L}_2$ (ℓ_2) consensus control problem boils down to $\mathcal{L}_2$ (ℓ_2) stabilization of the transformed system (19).

C. CONVEX SYNTHESIS OF DYNAMIC RELATIVE OUTPUT-FEEDBACK SATURATED CONSENSUS CONTROL

The following lemma is needed to derive the convex synthesis conditions for the proposed dynamic relative output-feedback saturated consensus control protocol.

Lemma 3: Consider the input-saturated MAS (15) with the network graph $\mathcal{G}$. Under Assumptions 1–2, the system achieves the consensus performance with a regional $\mathcal{L}_2$ (ℓ_2) gain less than γ for any disturbance $w_i \in \mathcal{W}_s$ ($\forall i \in I[1, N_a]$), if and only if the following $N_a - 1$ subsystems

$$\begin{bmatrix} \mathcal{D}\bar{x}_{cl,i} \\ \bar{u}_i \\ \bar{z}_i \end{bmatrix} = \begin{bmatrix} A_{cl,i} & B_{cl0} & B_{cl1,i} \\ C_{cl0,i} & D_{cl00} & D_{cl01,i} \\ C_{cl1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{cl,i} \\ \bar{q}_i \\ \bar{w}_i \end{bmatrix}, \quad \bar{q}_i = dz(\bar{u}_i), \quad i \in I[1, N_a - 1] \tag{21}$$

are simultaneously asymptotically stable and achieve the same regional $\mathcal{L}_2$ (ℓ_2) gain less than γ, where $\bar{x}_{cl,i} \in \mathbb{R}^{n_x+n_c}$, $\bar{q}_i \in \mathbb{R}^{n_u}$, $\bar{w}_i \in \mathbb{R}^{n_w}$, $\bar{z}_i \in \mathbb{R}^{n_x}$, and

$$\begin{aligned} A_{cl,i} &= \begin{bmatrix} A_p + \lambda_i B_{p2} D_{c1} C_p & B_{p2} C_{c1} + \lambda_i B_{p2} C_{c2} \\ \lambda_i B_{c1} C_p & A_{c1} + \lambda_i A_{c2} \end{bmatrix}, \\ B_{cl0} &= \begin{bmatrix} B_{p2} D_{c2} - B_{p2} \\ B_{c2} \end{bmatrix}, \quad B_{cl1,i} = \begin{bmatrix} B_{p1} + \lambda_i B_{p2} D_{c1} D_p \\ \lambda_i B_{c1} D_p \end{bmatrix}, \\ C_{cl0,i} &= \begin{bmatrix} \lambda_i D_{c1} C_p & C_{c1} + \lambda_i C_{c2} \end{bmatrix}, \quad C_{cl1} = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix}, \\ D_{cl00} &= D_{c2}, \quad D_{cl01,i} = \lambda_i D_{c1} D_p, \end{aligned} \tag{22}$$

λ_i (for all $i \in I[1, N_a-1]$) are the $N_a - 1$ positive eigenvalues of the Laplacian matrix $\mathcal{L}$.

Proof: With Assumption 2, matrix $\mathcal{L}_1$ of (17) is positive definite. Then there exist an orthogonal matrix $U_2 \in \mathbb{R}^{N_a-1}$ such that

$$U_2^T \mathcal{L}_1 U_2 = diag\{\lambda_1, \lambda_2, \cdots, \lambda_{N_a-1}\} := \Lambda,$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N_a-1}$ are positive eigenvalues of $\mathcal{L}_1$. Based on the observations in Remark 1, proving the $\mathcal{L}_2$ (ℓ_2) consensus performance of system (15) is equivalent to proving $\mathcal{L}_2$ (ℓ_2) stability of system (19). As such, we perform the following orthogonal transformation to system (19): $\bar{x}_p^1 = (U_2 \otimes I_{n_x})\tilde{x}_p^1 := col\{\bar{x}_{p,1}, \bar{x}_{p,2}, \cdots, \bar{x}_{p,N_a-1}\}$, with $\bar{x}_{p,i} \in \mathbb{R}^{n_x}, \forall i \in I[1, N_a - 1]$, $\bar{x}_c^1 = (U_2 \otimes I_{n_c})\tilde{x}_c^1 := col\{\bar{x}_{c,1}, \bar{x}_{c,2}, \cdots, \bar{x}_{c,N_a-1}\}$, with $\bar{x}_{c,i} \in \mathbb{R}^{n_c}, \forall i \in I[1, N_a - 1]$, $\bar{q}^1 = (U_2 \otimes I_{n_u})\tilde{q}^1 := col\{\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_{N_a-1}\}$, with $\bar{q}_i \in \mathbb{R}^{n_u}, \forall i \in I[1, N_a - 1]$, $\bar{w}^1 = (U_2 \otimes I_{n_w})\tilde{w}^1 := col\{\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_{N_a-1}\}$, with $\bar{w}_i \in \mathbb{R}^{n_w}, \forall i \in I[1, N_a - 1]$, $\bar{z}^1 = (U_2 \otimes I_{n_x})\tilde{z}^1 := col\{\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{N_a-1}\}$, with $\bar{z}_i \in \mathbb{R}^{n_x}, \forall i \in I[1, N_a - 1]$, $\bar{u}^1 = (U_2 \otimes I_{n_u})\tilde{u}^1 := col\{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{N_a-1}\}$, with $\bar{u}_i \in \mathbb{R}^{n_u}, \forall i \in I[1, N_a - 1]$. Then, we obtain

$$\begin{bmatrix} \mathcal{D}\bar{x}_p^1 \\ \mathcal{D}\bar{x}_c^1 \\ \bar{u}^1 \\ \bar{z}^1 \end{bmatrix} = \begin{bmatrix} I_{N_a} \otimes A_p + \Lambda \otimes B_{p2} D_{c1} C_p & I_{N_a} \otimes B_{p2} C_{c1} + \Lambda \otimes B_{p2} C_{c2} & I_{N_a} \otimes (B_{p2} D_{c2} - B_{p2}) & I_{N_a} \otimes B_{p1} + \Lambda \otimes B_{p2} D_{c1} D_p \\ \Lambda \otimes B_{c1} C_p & I_{N_a} \otimes A_{c1} + \Lambda \otimes A_{c2} & I_{N_a} \otimes B_{c2} & \Lambda \otimes B_{c1} D_p \\ \Lambda \otimes D_{c1} C_p & I_{N_a} \otimes C_{c1} + \Lambda \otimes C_{c2} & I_{N_a} \otimes D_{c2} & \Lambda \otimes D_{c1} D_p \\ I_{N_a-1} \otimes I_{n_x} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_p^1 \\ \bar{x}_c^1 \\ \bar{q}^1 \\ \bar{w}^1 \end{bmatrix},$$

$$\bar{q}^1 = dz(\bar{u}^1).$$

It is clearly seen that since Λ is diagonal, the above transformed system manages to decouple the system dynamics (19) into $N_a - 1$ independent subsystems as indicated by (21) with $\bar{x}_{cl,i} := col\{\bar{x}_{p,i}, \bar{x}_{c,i}\}$, $\forall i \in I[1, N_a - 1]$. This implies the equivalency between asymptotic stability of system (19) and simultaneously asymptotic stability of all $N_a - 1$ subsystems in (21). Moreover, by the definition of $\mathcal{L}_2(\ell_2)$-gain performance, together with the orthogonal transformation property and the relation of (20), we have

$$\begin{aligned} \|z\|_2 < \gamma\|w\|_2 &\Leftrightarrow \|\tilde{z}\|_2 < \gamma\|\tilde{w}\|_2 \Leftrightarrow \|\tilde{z}^1\|_2 < \gamma\|\tilde{w}^1\|_2 \\ &\Leftrightarrow \|\bar{z}^1\|_2 < \gamma\|\bar{w}^1\|_2 \Leftrightarrow \|\bar{z}_i\|_2 < \gamma\|\bar{w}_i\|_2 \end{aligned}$$

for all $i \in I[1, N_a - 1]$. In summary, system (15) achieves the consensus performance with an $\mathcal{L}_2$ (ℓ_2) gain less than γ if and only if the $N_a - 1$ subsystems (21) all achieve $\mathcal{L}_2$ (ℓ_2) stability with an $\mathcal{L}_2$ (ℓ_2) gain less than γ. This completes the proof. □

Based on Lemma 3, the following two theorems provide the main results for the saturated consensus control synthesis in continuous time and discrete time, respectively.

Theorem 1 (Continuous-Time Case): Consider the continuous-time input-saturated MAS (11). Under Assumptions 1–2, if there exist positive definite matrices $R, S \in \mathbb{S}_+^{n_x}$, rectangular matrices $\hat{A}_{c1}, \hat{A}_{c2} \in \mathbb{R}^{n_x \times n_x}$, $\hat{B}_{c1} \in \mathbb{R}^{n_x \times n_y}$, $\hat{B}_{c2} \in \mathbb{R}^{n_x \times n_u}$, $\hat{C}_{c1}, \hat{C}_{c2} \in \mathbb{R}^{n_u \times n_x}$, $\hat{D}_{c1} \in \mathbb{R}^{n_u \times n_y}$, $\hat{D}_{c2} \in \mathbb{R}^{n_u \times n_u}$, $\begin{bmatrix} \hat{H}_C & \hat{H}_D \end{bmatrix} \in \mathbb{R}^{n_u \times 2n_x}$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar $\gamma \in \mathbb{R}_+$ such that the following conditions hold for all $i \in I[1, N_a - 1]$.

$$\begin{bmatrix} He\left\{A_p R + B_{p2}\hat{C}_{c1} + \lambda_i B_{p2}\hat{C}_{c2}\right\} \\ \hat{A}_{c1} + \lambda_i \hat{A}_{c2} + A_p^T + \lambda_i C_p^T \hat{D}_{c1}^T B_{p2}^T \\ \hat{C}_{c1} - \hat{H}_D + \lambda_i \hat{C}_{c2} + \hat{D}_{c2}^T B_{p2}^T - \Gamma B_{p2}^T \\ \lambda_i D_p^T \hat{D}_{c1}^T B_{p2}^T + B_{p1}^T \\ R \end{matrix}$$

$$\begin{bmatrix} \star & \star & \star & \star \\ He\left\{SA_p + \lambda_i \hat{B}_{c1} C_p\right\} & \star & \star & \star \\ \lambda_i \hat{D}_{c1} C_p - \hat{H}_C + \hat{B}_{c2}^T & He\{-\Gamma + \hat{D}_{c2}\} & \star & \star \\ \lambda_i D_p^T \hat{B}_{c1}^T + B_{p1}^T S & \lambda_i D_p^T \hat{D}_{c1}^T & -\gamma I & \star \\ I & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \frac{\overline{u}_k^2}{s^2} & \star & \star \\ \hat{H}_D^T \delta_k & R & \star \\ \hat{H}_C^T \delta_k & I & S \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u], \tag{24}$$

$$\begin{bmatrix} R & \star \\ I & S \end{bmatrix} > 0. \tag{25}$$

Then the dynamic relative output-feedback saturated control protocol (13) renders the MAS (11) consensus with a regional $\mathcal{L}_2$ gain less than γ for any disturbance $w_i \in \mathcal{W}_s$ ($\forall i \in I[1, N_a]$). Moreover, the controller (13) is of order $n_c = n_x$, and its coefficient matrices can be reconstructed by:

$$\begin{aligned} H_C &= \hat{H}_C, \\ \begin{bmatrix} A_{c1} & A_{c2} & B_{c1} & B_{c2} \\ C_{c1} & C_{c2} & D_{c1} & D_{c2} \\ H_D & 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} N & SB_{p2} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} \hat{A}_{c1} - SA_pR & \hat{A}_{c2} & \hat{B}_{c1} & \hat{B}_{c2} + SB_{p2}\Gamma \\ \hat{C}_{c1} & \hat{C}_{c2} & \hat{D}_{c1} & \hat{D}_{c2} \\ \hat{H}_D - H_C R & 0 & 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} M^T & 0 & 0 & 0 \\ 0 & M^T & 0 & 0 \\ 0 & C_pR & I & 0 \\ 0 & 0 & 0 & \Gamma \end{bmatrix}^{-1}, \end{aligned} \tag{26}$$

where $M, N \in \mathbb{R}^{n_x \times n_x}$ are such that $MN^T = I - RS$.

Proof: From Lemma 3, the regional $\mathcal{L}_2$ consensus performance of the input-saturated MAS (11) is equivalent to the simultaneous regional $\mathcal{L}_2$ stability of $N_a - 1$ number of independent linear systems in the form of (21). As such, combining with Lemma 1, the conclusion of Theorem 1 can be reached if there exist a positive definite matrix $P \in \mathbb{S}_+^{2n_x}$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar γ such that the following inequalities hold for all the $N_a - 1$ number of systems in (21),

$$\begin{bmatrix} He\{PA_{cl,i}\} & \star & \star & \star \\ C_{cl0,i} - H + \Gamma B_{cl0}^T P & He\{-\Gamma + D_{cl00}\Gamma\} & \star & \star \\ B_{cl1,i}^T P & D_{cl01,i}^T & -\gamma I & \star \\ C_{cl1} & 0 & 0 & -\gamma I \end{bmatrix} < 0, \quad \forall i \in I[1, N_a - 1], \tag{27}$$

$$\begin{bmatrix} \frac{\overline{u}_k^2}{s^2} & \star \\ H^T \delta_k & P \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u]. \tag{28}$$

To formulate the above two conditions in convex LMI form, we let $H = \begin{bmatrix} H_C & H_D \end{bmatrix}$ and

$$P = \begin{bmatrix} S & N \\ N^T & \hat{X}^{-1} \end{bmatrix}.$$

Then define

$$Z_1 = \begin{bmatrix} R & I \\ M^T & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I & S \\ 0 & N^T \end{bmatrix},$$

such that $PZ_1 = Z_2$. As a result, we have $\hat{X}^{-1} = -N^T RM^{-T}$. Based on condition (25), it can be verified that

$$Z_1^T PZ_1 = \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0,$$

in turn, $P > 0$. Then, after performing the congruence transformation with matrix $diag\{Z_1, I, I, I\}$ on condition (27), and matrix $diag\{I, Z_1\}$ on condition (28), we have the following results:

$$\begin{aligned} Z_1^T PA_{cl,i} Z_1 &= \begin{bmatrix} A_pR + B_{p2}\hat{C}_{c1} + \lambda_i B_{p2}\hat{C}_{c2} & A_p + \lambda_i B_{p2}\hat{D}_{c1}C_p \\ \hat{A}_{c1} + \lambda_i \hat{A}_{c2} & SA_p + \lambda_i \hat{B}_{c1}C_p \end{bmatrix}, \\ \Gamma B_{cl0}^T PZ_1 &= \begin{bmatrix} \hat{D}_{c2}^T B_{p2}^T - \Gamma B_{p2}^T & \hat{B}_{c2}^T \end{bmatrix}, \\ B_{cl1,i}^T PZ_1 &= \begin{bmatrix} B_{p1}^T + \lambda_i D_p^T \hat{D}_{c1}^T B_{p2}^T & B_{p1}^T S + \lambda_i D_p^T \hat{B}_{c1}^T \end{bmatrix}, \\ (C_{cl0,i} - H)Z_1 &= \begin{bmatrix} \hat{C}_{c1} + \lambda_i \hat{C}_{c2} - \hat{H}_D & \lambda_i \hat{D}_{c1} C_p - \hat{H}_C \end{bmatrix}, \\ C_{cl1} Z_1 &= \begin{bmatrix} R & I \end{bmatrix}, \quad D_{cl00}\Gamma = \hat{D}_{c2}, \\ D_{cl01,i} &= \lambda_i \hat{D}_{c1} D_p, \\ HZ_1 &= \begin{bmatrix} \hat{H}_D & \hat{H}_C \end{bmatrix}. \end{aligned} \tag{29}$$

where

$$\begin{aligned} \hat{A}_{c1} &= SA_pR + SB_{p2}C_{c1}M^T + NA_{c1}M^T, \\ \hat{A}_{c2} &= SB_{p2}D_{c1}C_pR + NB_{c1}C_pR \\ &\quad + SB_{p2}C_{c2}M^T + NA_{c2}M^T, \\ \hat{B}_{c1} &= SB_{p2}D_{c1} + NB_{c1}, \\ \hat{B}_{c2} &= SB_{p2}D_{c2}\Gamma - SB_{p2}\Gamma + NB_{c2}\Gamma, \\ \hat{C}_{c1} &= C_{c1}M^T, \quad \hat{C}_{c2} = D_{c1}C_pR + C_{c2}M^T \\ \hat{D}_{c1} &= D_{c1}, \quad \hat{D}_{c2} = D_{c2}\Gamma, \quad \hat{H}_C = H_C, \\ \hat{H}_D &= H_C R + H_D M^T. \end{aligned} \tag{30}$$

The above congruence transformation results verify equivalency between condition (23) and condition (27), as well as condition (24) and condition (28). The formula (26) for reconstruction of controller gain matrices can also be verified by inverting the relations in (30). This ends the proof. □

Remark 2: It is clearly seen that the saturated consensus control synthesis conditions (23)–(25) in Theorem 1 are formulated in terms of convex LMIs, which can be solved effectively through semi-definite programming [38]. Thus, this *result can be used to pose the following convex LMI-based optimization problem for synthesizing an optimal consensus control protocol that would render the MAS a minimal regional $\mathcal{L}_2$ gain γ:*

$$\min_{R,S,\hat{A}_{c1},\hat{A}_{c2},\hat{B}_{c1},\hat{B}_{c2},\hat{C}_{c1},\hat{C}_{c2},\hat{D}_{c1},\hat{D}_{c2},\hat{H}_C,\hat{H}_D,\Gamma} \quad \gamma \qquad \text{s.t.} \quad (23)\text{–}(25). \tag{31}$$

Thanks to the convexity property of the derived LMI conditions, solving the above LMI-based optimization problem will involve very little computational complexity, as will be illustrated through examples in Section IV.

Parallel to the continuous-time case, the saturated consensus control synthesis condition for discrete-time case is given as follows.

Theorem 2 (Discrete-Time Case): Consider the discrete-time input-saturated MAS (11). Under Assumptions 1–2, if there exist positive definite matrices $R, S \in \mathbb{S}_+^{n_x}$, rectangular matrices $\hat{A}_{c1}, \hat{A}_{c2} \in \mathbb{R}^{n_x \times n_x}$, $\hat{B}_{c1} \in \mathbb{R}^{n_x \times n_y}$, $\hat{B}_{c2} \in \mathbb{R}^{n_x \times n_u}$, $\hat{C}_{c1}, \hat{C}_{c2} \in \mathbb{R}^{n_u \times n_x}$, $\hat{D}_{c1} \in \mathbb{R}^{n_u \times n_y}$, $\hat{D}_{c2} \in \mathbb{R}^{n_u \times n_u}$, $\begin{bmatrix}\hat{H}_C & \hat{H}_D\end{bmatrix} \in \mathbb{R}^{n_u \times 2n_x}$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar $\gamma \in \mathbb{R}_+$ such that condition (32), as shown at the bottom of the next page, and

$$\begin{bmatrix} \frac{\bar{u}_k^2}{s^2} & \star & \star \\ \hat{H}_D^T \delta_k & R & \star \\ \hat{H}_C^T \delta_k & I & S \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u], \tag{33}$$

$$\begin{bmatrix} R & \star \\ I & S \end{bmatrix} > 0, \tag{34}$$

hold for all $i \in I[1, N_a - 1]$. Then the dynamic relative output-feedback saturated control protocol (13) renders the MAS (11) consensus with a regional ℓ_2 gain less than γ for any disturbance $w_i \in \mathcal{W}_s$ ($\forall i \in I[1, N_a]$). Moreover, the controller (13) is of order $n_c = n_x$, and its coefficient matrices can be reconstructed by using (26).

Proof: From Lemma 3, the regional ℓ_2 consensus performance of the input-saturated MAS (11) is equivalent to the simultaneous regional ℓ_2 stability of $N_a - 1$ number of independent linear systems in the form of (21). As such, combining with Lemma 2, the conclusion of Theorem 2 can be reached if there exist a positive definite matrix $P \in \mathbb{S}_+^{2n_x}$, a diagonal positive definite matrix $\Gamma \in \mathbb{S}_+^{n_u}$, and a positive scalar γ such that the following inequalities hold for all the $N_a - 1$ number of systems in (21),

$$\begin{bmatrix} -P & \star & \star & \star & \star \\ C_{cl0,i} - H & He\{-\Gamma - D_{cl00}\Gamma\} & \star & \star & \star \\ 0 & D_{cl01,i}^T & -\gamma I & \star & \star \\ PA_{cl,i} & PB_{cl0}\Gamma & PB_{cl1,i} & -P & \star \\ C_{cl1} & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} \frac{\bar{u}_k^2}{s^2} & \star \\ H^T \delta_k & P \end{bmatrix} \geq 0, \quad \forall k \in I[1, n_u]. \tag{36}$$

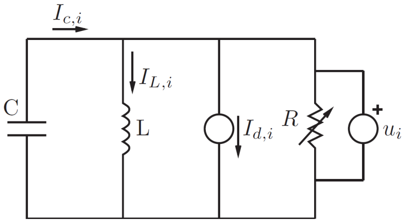


FIGURE 2. LC oscillator circuits for $i \in I[1, 10]$ (Example 1).

To formulate the above two conditions in convex LMI form, we follow a similar idea of the proof of Theorem 1 to let $H = \begin{bmatrix}H_C & H_D\end{bmatrix}$ and

$$P = \begin{bmatrix} S & N \\ N^T & \hat{X}^{-1} \end{bmatrix}, \quad Z_1 = \begin{bmatrix} R & I \\ M^T & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I & S \\ 0 & N^T \end{bmatrix},$$

such that $PZ_1 = Z_2$. As a result, we have $\hat{X}^{-1} = -N^T R M^{-T}$. With condition (34), it can be verified that $Z_1^T P Z_1 = \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0$, in turn, $P > 0$. Then, after performing the congruence transformation with matrix $diag\{Z_1, I, I, Z_1, I\}$ on condition (35), and matrix $diag\{I, Z_1\}$ on condition (36), we obtain the same results as those of (29) and (30). This verifies equivalency between condition (32) and condition (35), as well as condition (33) and condition (36). The controller reconstruction formula (26) can also be verified by inverting the relations in (30). This ends the proof. □

Similar to the continuous-time case, the discrete-time saturated consensus control synthesis conditions (32)–(34) are formulated in terms of convex LMIs. Hence, the associated optimal consensus control protocol that renders the MAS a minimal ℓ_2 gain γ can be synthesized by solving the following convex optimization problem:

$$\min_{R,S,\hat{A}_{c1},\hat{A}_{c2},\hat{B}_{c1},\hat{B}_{c2},\hat{C}_{c1},\hat{C}_{c2},\hat{D}_{c1},\hat{D}_{c2},\hat{H}_C,\hat{H}_D,\Gamma} \quad \gamma \qquad \text{s.t.} \quad (32)\text{–}(34). \tag{37}$$

IV. APPLICATION EXAMPLES

In this section, two examples will be used to demonstrate the effectiveness and real application of the proposed dynamic relative output-feedback saturated consensus control schemes for continuous-time and discrete-time MASs, respectively.

A. EXAMPLE 1: CONTINUOUS-TIME CASE

The first example considers the synchronization problem for multiple electrical circuit systems, which consist of ten perturbed and input-saturated oscillator circuits as shown in Fig. 2.

Specifically, for each circuit system with index $i \in I[1, 10]$, we use $I_{c,i}$ and $I_{L,i}$ to denote the corresponding currents through the capacitor C and the inductor L, respectively, and use $I_{d,i}$ to represent unknown external current perturbations. R is a resistor, and u_i is the control input voltage for the ith circuit system. By applying the Kirchhoff's law, the

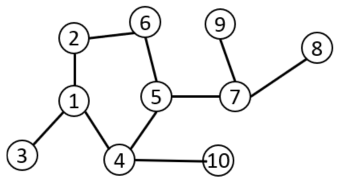


FIGURE 3. Network graph of ten agents.

dynamical behaviour of these LC circuits can be described by

$$LC\ddot{I}_{L,i} + I_{L,i} + \frac{1}{R_i}u_i + I_{d,i} = 0, \quad \forall i \in I[1, 10]. \tag{38}$$

Our goal here is to apply the proposed continuous-time dynamic relative output-feedback consensus control scheme to solve the synchronization problem of the above ten electrical circuits, i.e., driving the current states (including $I_{L,i}$ and $\dot{I}_{L,i}$) of these circuit systems to reach consensus in the presence of unknown external disturbances and control input saturation. To this end, we reformulate the above circuits' dynamics (38) into the state-space form of (11) by defining $x_{p1,i} := I_{L,i}$, $x_{p2,i} := \dot{I}_{L,i}$, $w_i := I_{d,i}$, and $y_i := I_{L,i}$ for all $i \in I[1, 10]$. This yields the associated state-space matrices as

$$\begin{bmatrix} A_p & B_{p1} & B_{p2} \\ C_p & D_p & 0 \end{bmatrix} = \left[\begin{array}{cc:c:c} 0 & 1 & 0 & 0 \\ -\frac{1}{LC} & 0 & -\frac{1}{LC} & -\frac{1}{LCR} \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right].$$

For simulation purpose, we choose the circuit constants $L = 1.5H$, $C = 1F$, $R = 50\Omega$, and assume the saturation level of the voltage inputs u_i as $\underline{u} = 0.2$ for all $i \in I[1, 10]$. The underlying communication network graph $\mathcal{G}$ is given in Fig. 3 with the associated Laplacian matrix as

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

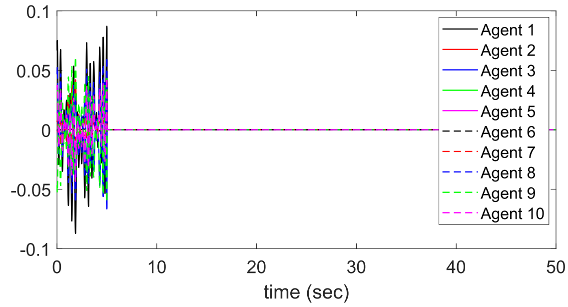


FIGURE 4. Disturbance signals (Example 1).

whose non-zero eigenvalues include $\lambda_1 = 0.2495$, $\lambda_2 = 0.6384$, $\lambda_3 = 0.8471$, $\lambda_4 = 1$, $\lambda_5 = 2$, $\lambda_6 = 2.454$, $\lambda_7 = 3.4902$, $\lambda_8 = 4.3687$, and $\lambda_9 = 4.9521$.

Based on the above system setup, under the proposed consensus control scheme for continuous-time systems, we solve the LMI optimization problem (31) to yield a minimized regional $\mathcal{L}_2$ gain $\gamma = 225.2231$ for the disturbance level $s = 1$. The associated optimal controller matrices can be further obtained with formula (26). Consequently, using the synthesized consensus control protocol in the form of (13), we carry the time-domain simulations under zero initial conditions for all system and controller states. The energy-bounded disturbance signals w_i are assumed to be

$$w(t) = \begin{cases} \big[w_0(t) \quad -0.5\sin(t)w_0(t) \quad -\sin(10t)w_0(t) \\ 0.5\cos(0.1t)w_0(t) \quad 0.2w_0(t) \\ -0.2\sin(t)w_0(t) \quad \cos(t)w_0(t) \\ 0.7\cos(0.1t)w_0(t) \quad -0.7\cos(0.1t)w_0(t)\big[0.5w_0(t)\big]^T, \\ \qquad t \in [0, 5]\ sec \\ 0, \qquad otherwise \end{cases} \tag{40}$$

where $w_0(t) := (3\sin(9t) + 5\sin(\sqrt{13}t) + 7\cos(15t) + 9\cos(19t))/240$ is used to generate random-like noisy signals, according to [39]. Fig. 4 shows the time-domain profile of such energy-bounded random-like disturbance signals. Fig. 5 provides the trajectories of the control input signals $u_i(t)$ for all ten circuit systems. Saturation at magnitude 0.2 and -0.2 can be observed from 0 to 15 sec. The synchronization performance is illustrated in Fig. 6. In particular,

$$\begin{bmatrix} -R & \star & \star & \star & \star & \star & \star \\ -I & -S & \star & \star & \star & \star & \star \\ \hat{C}_{c1} + \lambda_i\hat{C}_{c2} - \hat{H}_D & \lambda_i\hat{D}_{c1}C_p - \hat{H}_C & He\{-\Gamma + \hat{D}_{c2}\} & \star & \star & \star & \star \\ 0 & 0 & \lambda_i D_p^T\hat{D}_{c1}^T & -\gamma I & \star & \star & \star \\ A_pR + B_{p2}\hat{C}_{c1} + \lambda_i B_{p2}\hat{C}_{c2} & A_p + \lambda_i B_{p2}\hat{D}_{c1}C_p & B_{p2}\hat{D}_{c2} - B_{p2}\Gamma & B_{p1} + \lambda_i B_{p2}\hat{D}_{c1}D_p & -R & \star & \star \\ \hat{A}_{c1} + \lambda_i\hat{A}_{c2} & SA_p + \lambda_i\hat{B}_{c1}C_p & \hat{B}_{c2} & SB_{p1} + \lambda_i\hat{B}_{c1}D_p & -I & -S & \star \\ R & I & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{32}$$

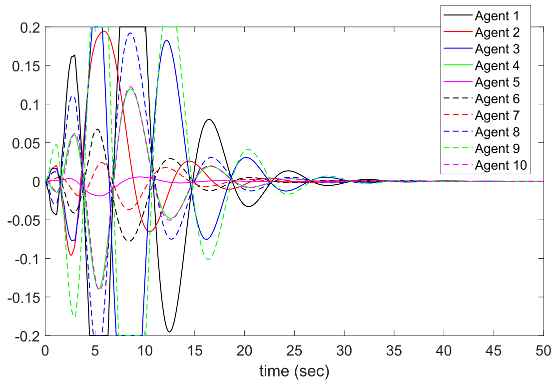


FIGURE 5. **Control inputs (Example 1).**

Figs. 6(a) and 6(b) show that both states of all ten electrical circuits indeed reach consensus with the ultimate consensus trajectories being sinusoidal signals. The consensus performance is more clearly demonstrated in Figs. 6(c) and 6(d), which show the convergence of all consensus error signals (defined in (12)) to zero. Because of the effects of input saturation, some chattering phenomenons appear during the transient process as shown in Figs. 6(b) and (d). The controller state profiles are also provided in Fig. 7 to verify stability of the overall MAS.

B. EXAMPLE 2: DISCRETE-TIME CASE

In the second example, we consider consensus control of a network of ten mass-spring mechanical systems (as shown in Fig. 8), each of which has the following dynamics:

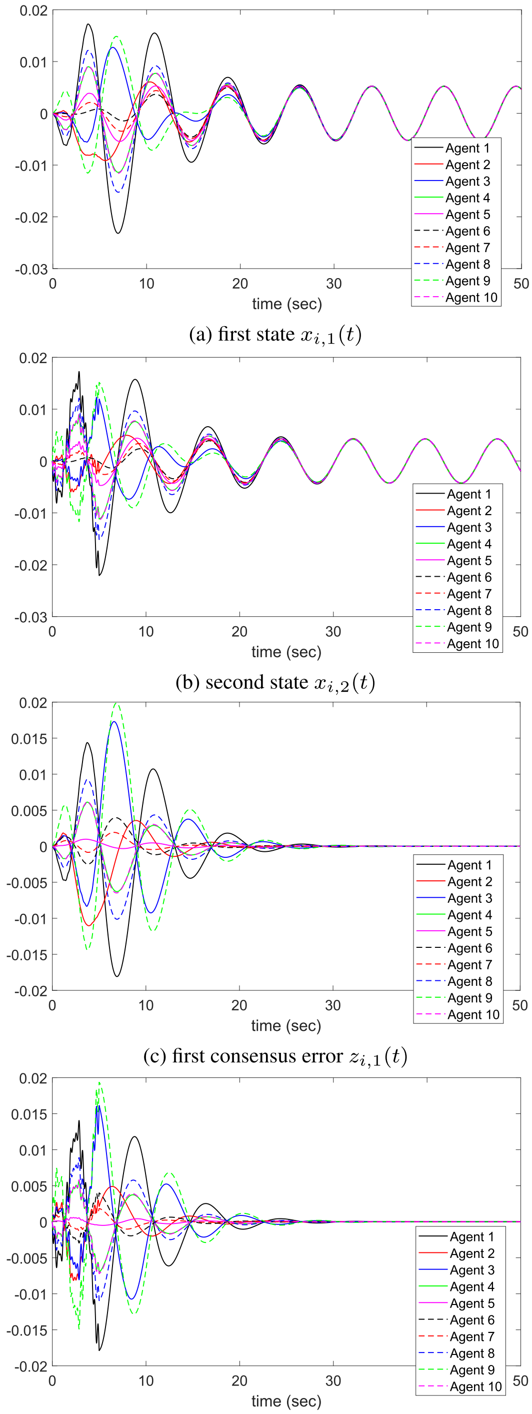
$$m\ddot{x}_i + kx_i = u_i, \quad \forall i \in I[1, 10]$$

where x_i represents displacement of the mass from the equilibrium point, m is the mass, k denotes the spring constant, and u_i is the control input representing external forces applied to the mass. The goal is to drive all these ten mass-spring systems to reach certain consensus behaviors by utilizing the discrete-time version of the proposed dynamic relative output-feedback control protocol. To this end, we rewrite the above system dynamics into the following state-space model by defining $x_{p,i} = [x_i, \dot{x}_i]^T$ and $y_i = x_i$:

$$\dot{x}_{p,i} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x_{p,i} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_i,$$
$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{p,i}, \quad \forall i \in I[1, 10].$$

Then, by discretising the above continuous-time model with a sampling time $T_s = 0.01$ sec, and taking into account the effects of control input saturation and external disturbances/noise w_i, we obtain the following discrete-time model for the multiple mass-spring systems:

$$x_{p,i}(t+1) = \begin{bmatrix} 1 & 0.01 \\ -0.01\frac{k}{m} & 1 \end{bmatrix} x_{p,i}(t) + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} w_i(t)$$



(a) first state $x_{i,1}(t)$

(b) second state $x_{i,2}(t)$

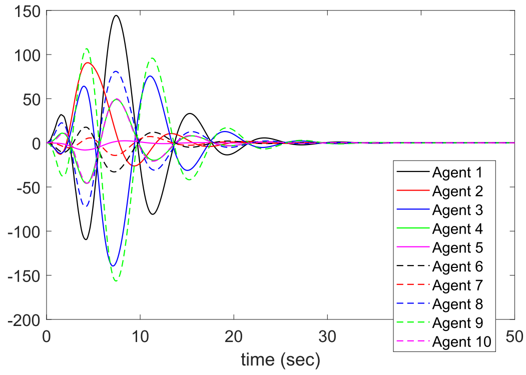
(c) first consensus error $z_{i,1}(t)$

(d) second consensus error $z_{i,2}(t)$

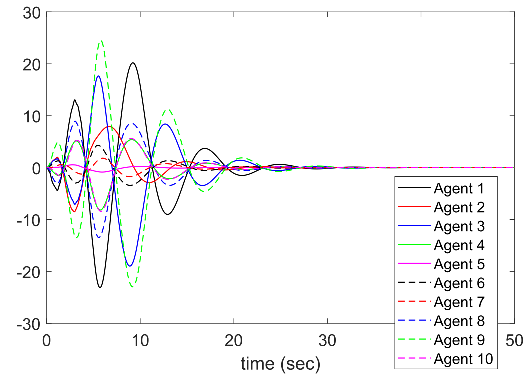
FIGURE 6. **Synchronization of ten oscillator circuits (Example 1).**

$$+ \begin{bmatrix} 0 \\ \frac{0.01}{m} \end{bmatrix} sat(u_i(t)),$$
$$y_i(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{p,i}(t) + w_i(t), \quad \forall i \in I[1, 10].$$

For controller synthesis, we choose the mass $m = 1$ kg, the spring constant $k = 0.1$ N/m, the saturation level of the force control inputs $\overline{u} = 0.5$ for all $i \in I[1, 10]$, and



(a) first state $x_{c,i1}(t)$



(b) second state $x_{c,i2}(t)$

FIGURE 7. **Controller states (Example 1).**

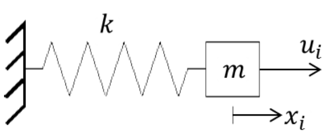


FIGURE 8. **Mass-spring mechanical systems for $i \in I[1, 10]$ (Example 2).**

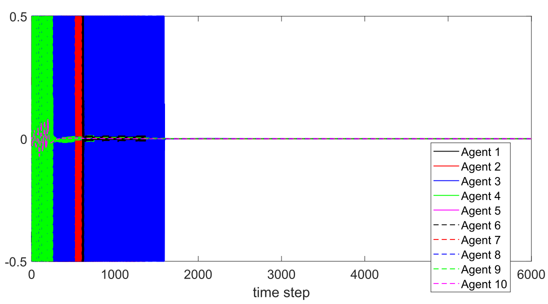
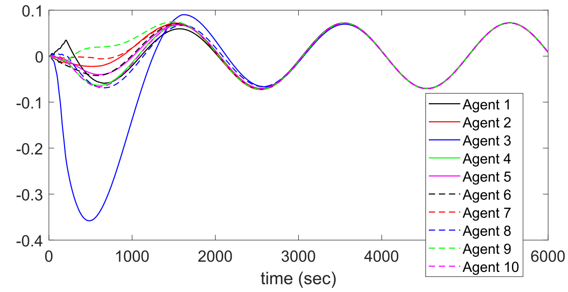
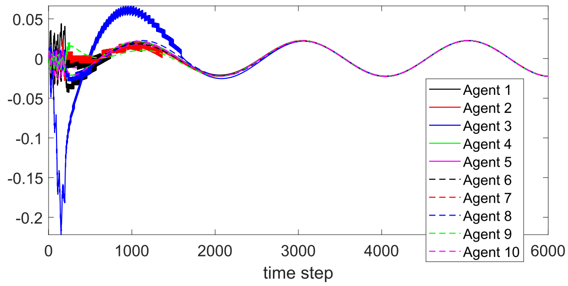


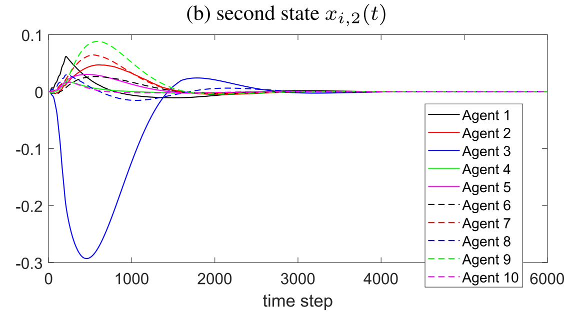
FIGURE 9. **Control inputs (Example 2).**



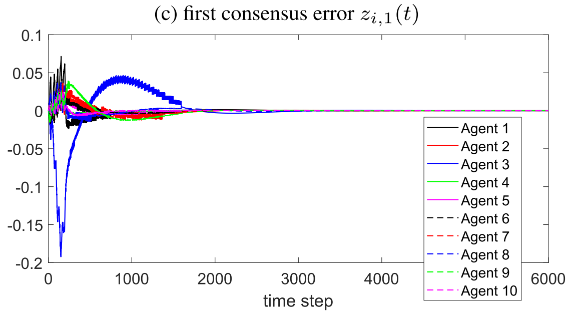
(a) first state $x_{i,1}(t)$



(b) second state $x_{i,2}(t)$



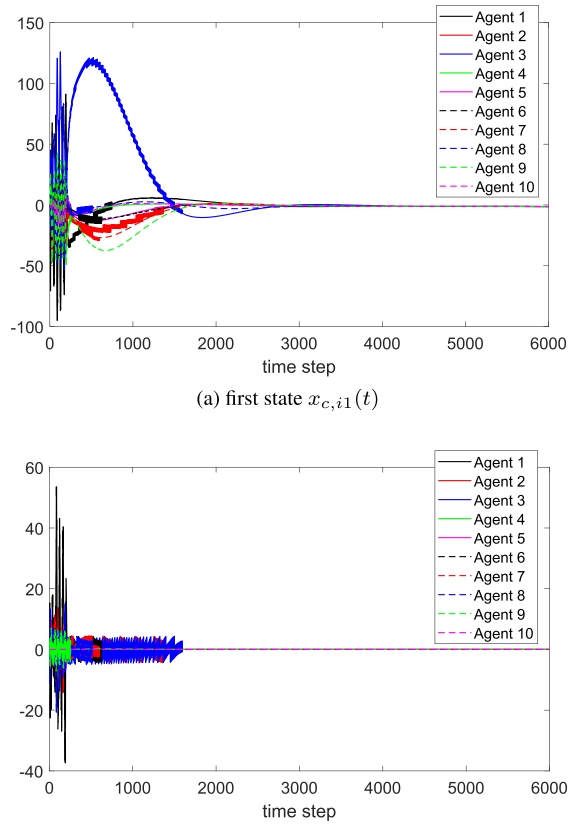
(c) first consensus error $z_{i,1}(t)$



(d) second consensus error $z_{i,2}(t)$

FIGURE 10. **Consensus of ten mass-spring mechanical systems (Example 2).**

assume that the underlying communication network graph $\mathcal{G}$ has the same structure as of Fig. 3, which shares the same Laplacian matrix as given in (39). Based on Theorem 2, we solve the convex LMI optimization problem (37) to achieve the optimal regional ℓ_2 gain $\gamma = 48.4417$ for the disturbance level $s = 150$. The associated optimal controller matrices are also obtained using (26). With an energy-bounded random-like disturbance satisfying a similar profile of (40) but starting from time step 0 to time step 200, we conduct the time-domain simulations with zero initial conditions to all system and controller states. The simulation results are plotted in Fig. 9 through Fig. 11. Fig. 9 shows the trajectories of the control input signals of all ten agents. Saturation can be clearly observed for these control signals. The plant state trajectories and the consensus error trajectories of all ten agents are depicted in Figs. 10(a)–(b) and 10(c)–(d), respectively. Similar to the Example 1, because of the effects of input saturation, some chattering phenomenons appear during the transient process as shown in Figs. 10(b) and (d). It is interesting



(a) first state $x_{c,i1}(t)$

(b) second state $x_{c,i2}(t)$

FIGURE 11. Controller states (Example 2).

to see that ten mass-spring systems will ultimately reach consensus on a persistently oscillating behavior. This demonstrates that the proposed control protocol is indeed capable of stabilizing the overall MAS and meeting the consensus specification with disturbance attenuation when subjected to actuator saturation. Moreover, the controller state trajectories are plotted in Fig. 11 for overall stability demonstration.

V. CONCLUSIONS

In this paper, a novel dynamic relative output-feedback consensus control protocol has been proposed for a class of linear MASs subject to input saturation and energy-bounded external disturbances. With this new control protocol, the associated consensus control synthesis conditions have been formulated in terms of computationally-tractable LMIs, such that an optimal consensus control solution that ensures optimal disturbance attenuation performance can be synthesized via efficient convex optimization. Both continuous-time and discrete-time MASs have been addressed in a unified framework. Effectiveness of the proposed results have been demonstrated through two realistic engineering applications, including synchronization of multiple electrical circuits and consensus oscillation of multiple mass-spring mechanical systems. In the future work, it is promising to further extend the proposed methodologies to leader-following MASs, and MASs with more general settings, such as nonlinear and heterogeneous agent dynamics, as well as switching and directed network topologies, etc.

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