2020

20. Alternating current circuits. Impedance. Resonance

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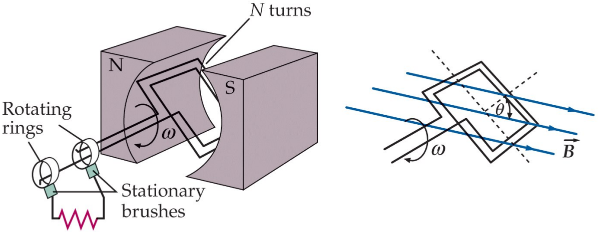
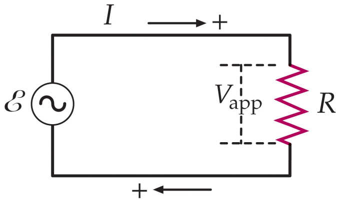
Alternating Current Generator

Coil of N turns and cross-sectional area A rotating with angular frequency ω in uniform magnetic field $\vec{B}$.

- Angle between area vector and magnetic field vector: $\theta = \omega t$.
- Flux through coil: $\Phi_B = NBA\cos(\omega t)$.
- Induced EMF: $\mathcal{E} = -\dfrac{d\Phi_B}{dt} = \mathcal{E}_{max}\sin(\omega t)$ with amplitude $\mathcal{E}_{max} = NBA\omega$.
- U.S. household outlet values:
  - $\mathcal{E}_{max} = 120\text{V}\sqrt{2} \simeq 170\text{V}$
  - $f = 60\text{Hz}, \quad \omega = 2\pi f \simeq 377\text{rad/s}.$



Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

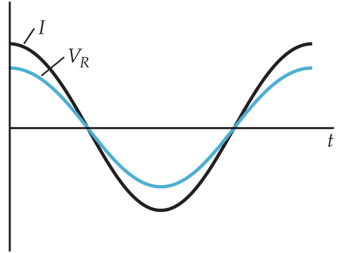


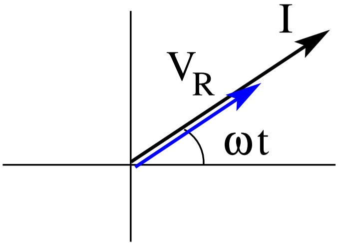
Resistor

$$V_R = RI = \mathcal{E}_{max} \cos \omega t \ \Rightarrow \ I = \frac{\mathcal{E}_{max}}{R} \cos \omega t$$

amplitude: $I_{max} = \dfrac{\mathcal{E}_{max}}{R}$, phase angle: $\delta = 0$

impedance: $X_R \equiv \dfrac{\mathcal{E}_{max}}{I_{max}} = R$ (resistance)

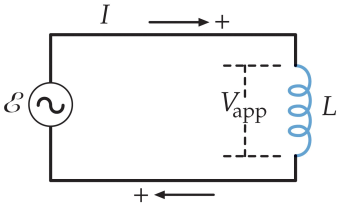




Single Device in AC Circuit: Inductor

Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

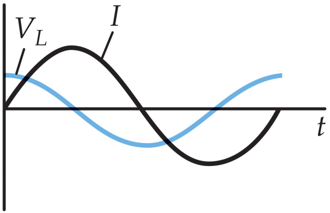
Inductor

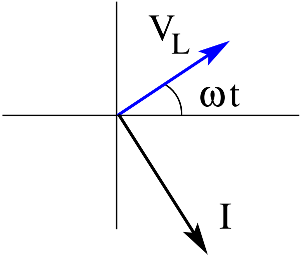


$$V_L = L\frac{dI}{dt} = \mathcal{E}_{max} \cos \omega t \;\Rightarrow\; I = \frac{\mathcal{E}_{max}}{\omega L} \sin(\omega t)$$

amplitude: $I_{max} = \dfrac{\mathcal{E}_{max}}{\omega L}$, phase angle: $\delta = \dfrac{\pi}{2}$

impedance: $X_L \equiv \dfrac{\mathcal{E}_{max}}{I_{max}} = \omega L$ (inductive reactance)





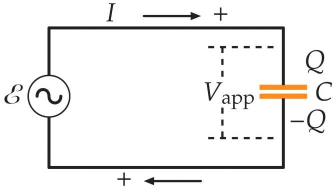
Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

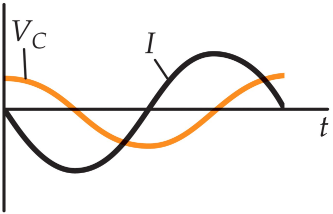
Capacitor

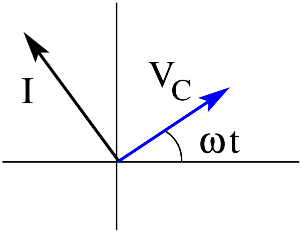
$$V_C = \frac{Q}{C} = \mathcal{E}_{max} \cos \omega t \;\Rightarrow\; I = \frac{dQ}{dt} = -\omega C \mathcal{E}_{max} \sin(\omega t)$$

amplitude: $I_{max} = \omega C \mathcal{E}_{max}$, phase angle: $\delta = -\frac{\pi}{2}$

impedance: $X_C \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \frac{1}{\omega C}$ (capacitive reactance)







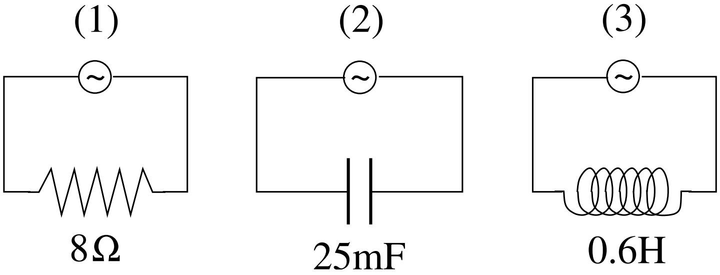
Single Device in AC Circuit: Application (1)

The ac voltage source $\mathcal{E} = \mathcal{E}_{max} \sin \omega t$ has an amplitude of $\mathcal{E}_{max} = 24$V and an angular frequency of $\omega = 10$rad/s.

In each of the three circuits, find

(a) the current amplitude I_{max},

(b) the current I at time $t = 1$s.



Single Device in AC Circuit: Application (2)

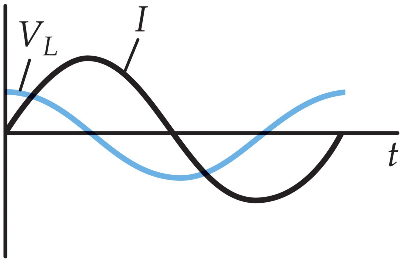
Consider an ac generator $\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$, $\mathcal{E}_{max} = 25\text{V}$, $\omega = 377\text{rad/s}$ connected to an inductor with inductance $L = 12.7\text{H}$.

(a) Find the maximum value of the current.

(b) Find the current when the emf is zero and decreasing.

(c) Find the current when the emf is -12.5V and decreasing.

(d) Find the power supplied by the generator at the instant described in (c).



Single Device in AC Circuit: Application (3)

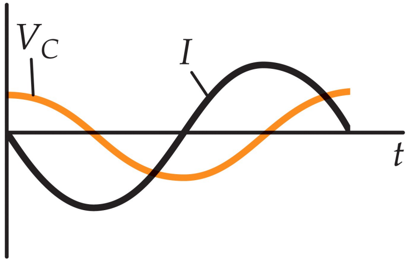
Consider an ac generator $\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$, $\mathcal{E}_{max} = 25\text{V}$, $\omega = 377\text{rad/s}$ connected to a capacitor with capacitance $C = 4.15\mu\text{F}$.

(a) Find the maximum value of the current.

(b) Find the current when the emf is zero and decreasing.

(c) Find the current when the emf is -12.5V and increasing.

(d) Find the power supplied by the generator at the instant described in (c).



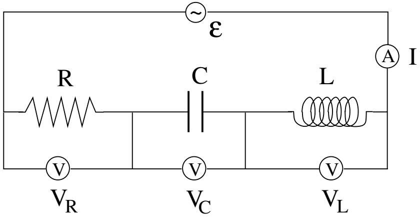
RLC Series Circuit (1)

Applied alternating voltage: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current: $I = I_{max} \cos(\omega t - \delta)$

Goals:

- Find I_{max}, δ for given $\mathcal{E}_{max}, \omega$.
- Find voltages V_R, V_L, V_C across devices.



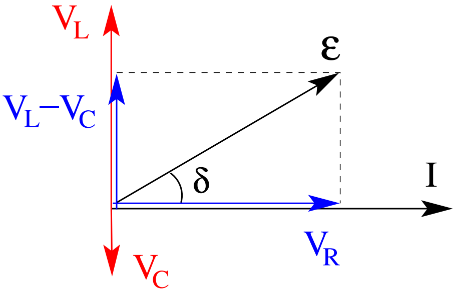
Loop rule: $\mathcal{E} - V_R - V_C - V_L = 0$

Note:

- All voltages are time-dependent.
- In general, all voltages have a different phase.
- V_R has the same phase as I.

RLC Series Circuit (2)

Phasor diagram (for $\omega t = \delta$):



Voltage amplitudes:

- $V_{R,max} = I_{max} X_R = I_{max} R$
- $V_{L,max} = I_{max} X_L = I_{max} \omega L$
- $V_{C,max} = I_{max} X_C = \dfrac{I_{max}}{\omega C}$

Relation between $\mathcal{E}_{max}$ and I_{max} from geometry:

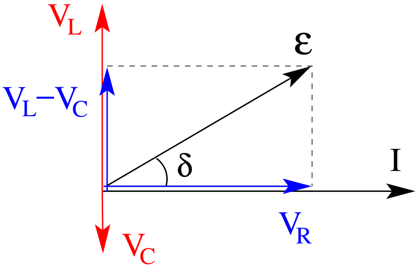
$$\begin{aligned} \mathcal{E}^2_{max} &= V^2_{R,max} + (V_{L,max} - V_{C,max})^2 \\ &= I^2_{max} \left[R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2 \right] \end{aligned}$$

RLC Series Circuit (3)

Impedance: $Z \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \frac{\mathcal{E}_{max}}{Z} = \frac{\mathcal{E}_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$
- $\tan\delta = \frac{V_{L,max} - V_{C,max}}{V_{R,max}} = \frac{\omega L - 1/\omega C}{R}$



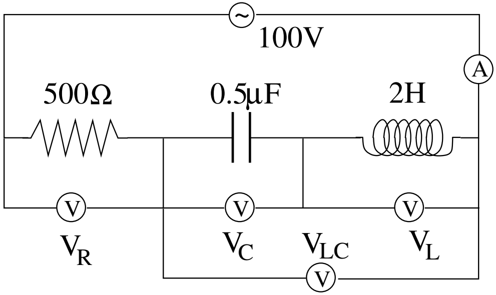
Voltages across devices:

- $V_R = RI = RI_{max}\cos(\omega t - \delta) = V_{R,max}\cos(\omega t - \delta)$
- $V_L = L\frac{dI}{dt} = -\omega L I_{max}\sin(\omega t - \delta) = V_{L,max}\cos\left(\omega t - \delta + \frac{\pi}{2}\right)$
- $V_C = \frac{1}{C}\int I dt = \frac{I_{max}}{\omega C}\sin(\omega t - \delta) = V_{C,max}\cos\left(\omega t - \delta - \frac{\pi}{2}\right)$

In this RLC circuit, the voltage amplitude is $\mathcal{E}_{max} = 100$V.

Find the impedance Z, the current amplitude I_{max},
and the voltage amplitudes V_R, V_C, V_L, V_{LC}

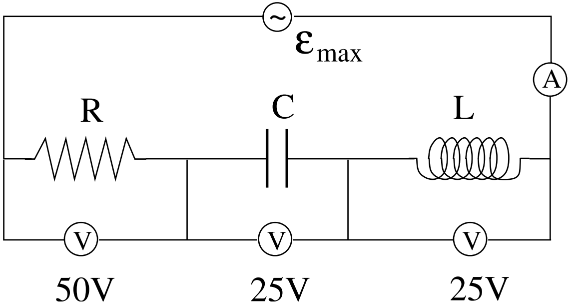
(a) for angular frequency is $\omega = 1000$rad/s,
(b) for angular frequency is $\omega = 500$rad/s.



AC Circuit Application (2)

In this RLC circuit, we know the voltage amplitudes V_R, V_C, V_L across each device, the current amplitude $I_{max} = 5$A, and the angular frequency $\omega = 2$rad/s.

- Find the device properties R, C, L and the voltage amplitude $\mathcal{E}_{max}$ of the ac source.



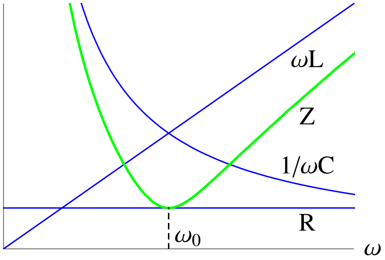
Impedances: RLC in Series (1)

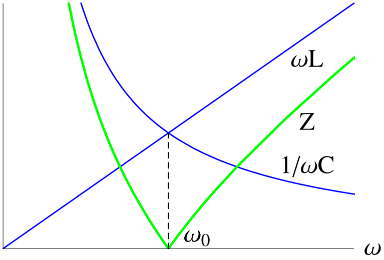
$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

resonance at $\omega_0 = \dfrac{1}{\sqrt{LC}}$

limit $R \to 0$

$$Z = \left|\omega L - \frac{1}{\omega C}\right|$$

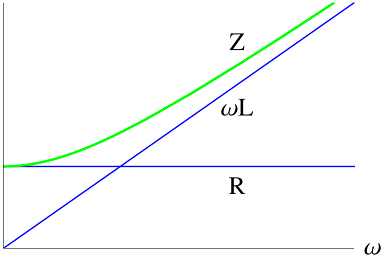




Impedances: RLC in Series (2)

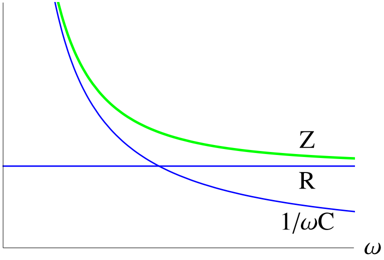
limit $C \to \infty$

$$Z = \sqrt{R^2 + (\omega L)^2}$$

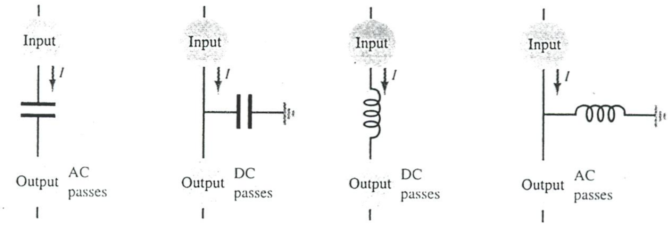


limit $L \to 0$

$$Z = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$$



Filters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Output
AC
passes
Input
I
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DC
passes
Input
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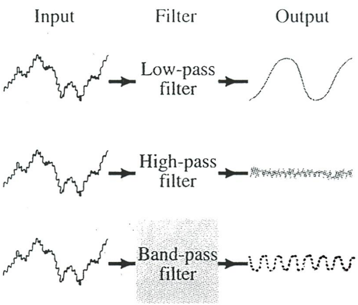
Output
DC
passes
Input
I
Output
AC
passes


Input
Filter
Output
Low-pass filter
High-pass filter
Band-pass filter

RLC Series Resonance (1)

impedance

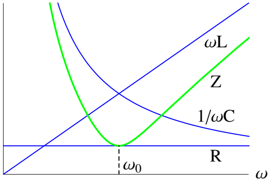
current

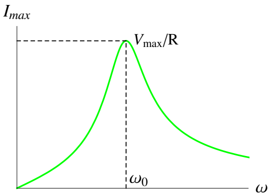
phase angle

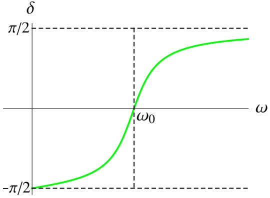
$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$I_{max} = \frac{V_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\delta = \frac{\omega L - 1/\omega C}{R}$$





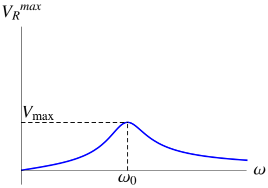


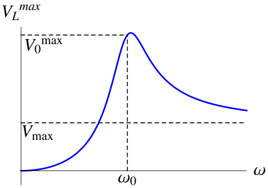
resonance angular frequency:

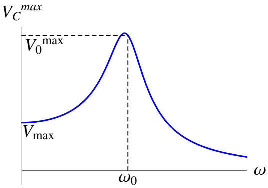
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

RLC Series Resonance (2)

resistor	inductor	capacitor
$V_R^{max} = I_{max} R$	$V_L^{max} = I_{max} \omega L$	$V_C^{max} = I_{max} / \omega C$







- relaxation times: $\tau_{RC} = RC, \quad \tau_{RL} = L/R$
- angular frequencies: $\omega_L = \dfrac{\omega_0}{\sqrt{1 - (\omega_0 \tau_{RC})^2/2}}, \quad \omega_C = \omega_0 \sqrt{1 - (\omega_0 \tau_{RC})^2/2}$
- voltages: $V_0^{max} = V_{max}\, \omega_0 \tau_{RL}, \quad V_L^{max}(\omega_L) = V_C^{max}(\omega_C) = \dfrac{V_0^{max}}{\sqrt{1 - (\omega_0 \tau_{RC})^2/4}}$

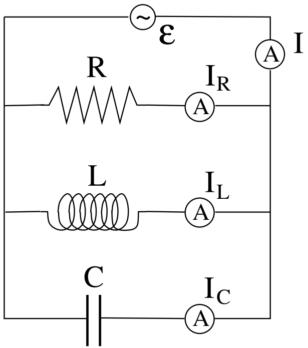
RLC Parallel Circuit (1)

Applied alternating voltage: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current: $I = I_{max} \cos(\omega t - \delta)$

Goals:

- Find I_{max}, δ for given $\mathcal{E}_{max}, \omega$.
- Find currents I_R, I_L, I_C through devices.



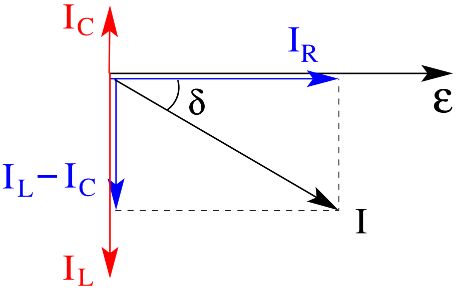
Junction rule: $I = I_R + I_L + I_C$

Note:

- All currents are time-dependent.
- In general, each current has a different phase
- I_R has the same phase as $\mathcal{E}$.

RLC Parallel Circuit (2)

Phasor diagram (for $\omega t = \delta$):



Current amplitudes:

- $I_{R,max} = \dfrac{\mathcal{E}_{max}}{X_R} = \dfrac{\mathcal{E}_{max}}{R}$
- $I_{L,max} = \dfrac{\mathcal{E}_{max}}{X_L} = \dfrac{\mathcal{E}_{max}}{\omega L}$
- $I_{C,max} = \dfrac{\mathcal{E}_{max}}{X_C} = \mathcal{E}_{max}\omega C$

Relation between $\mathcal{E}_{max}$ and I_{max} from geometry:

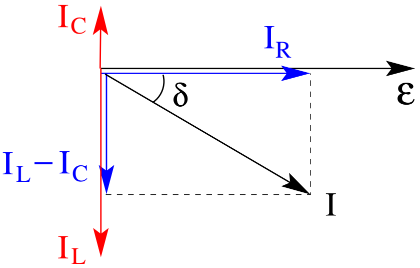
$$\begin{aligned} I_{max}^2 &= I_{R,max}^2 + (I_{L,max} - I_{C,max})^2 \\ &= \mathcal{E}_{max}^2 \left[\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C \right)^2 \right] \end{aligned}$$

RLC Parallel Circuit (3)

Impedance: $\dfrac{1}{Z} \equiv \dfrac{I_{max}}{\mathcal{E}_{max}} = \sqrt{\dfrac{1}{R^2} + \left(\dfrac{1}{\omega L} - \omega C\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \dfrac{\mathcal{E}_{max}}{Z} = \mathcal{E}_{max}\sqrt{\dfrac{1}{R^2} + \left(\dfrac{1}{\omega L} - \omega C\right)^2}$
- $\tan\delta = \dfrac{I_{L,max} - I_{C,max}}{I_{R,max}} = \dfrac{1/\omega L - \omega C}{1/R}$



Currents through devices:

- $I_R = \dfrac{\mathcal{E}}{R} = \dfrac{\mathcal{E}_{max}}{R}\cos(\omega t) = I_{R,max}\cos(\omega t)$
- $I_L = \dfrac{1}{L}\displaystyle\int \mathcal{E} dt = \dfrac{\mathcal{E}_{max}}{\omega L}\sin(\omega t) = I_{L,max}\cos\left(\omega t - \dfrac{\pi}{2}\right)$
- $I_C = C\dfrac{d\mathcal{E}}{dt} = -\omega C\mathcal{E}_{max}\sin(\omega t) = I_{C,max}\cos\left(\omega t + \dfrac{\pi}{2}\right)$

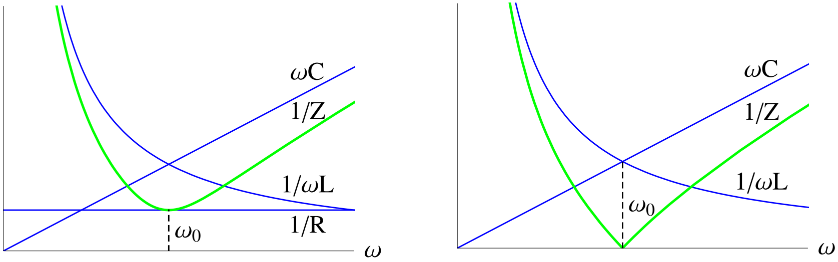
Impedances: RLC in Parallel (1)

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

resonance at $\omega_0 = \frac{1}{\sqrt{LC}}$

limit $R \to \infty$

$$\frac{1}{Z} = \left|\omega C - \frac{1}{\omega L}\right|$$



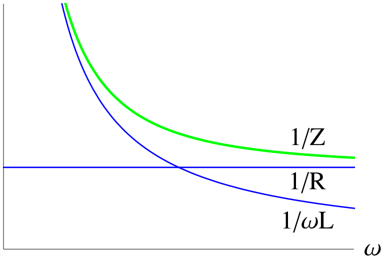
Impedances: RLC in Parallel (2)

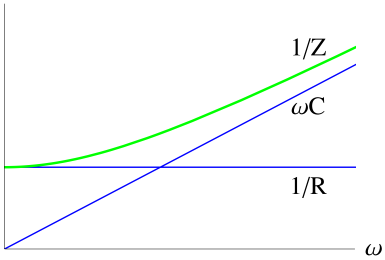
limit $C \to 0$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \frac{1}{(\omega L)^2}}$$

limit $L \to \infty$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + (\omega C)^2}$$

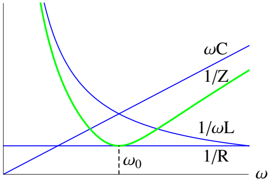




RLC Parallel Resonance (1)

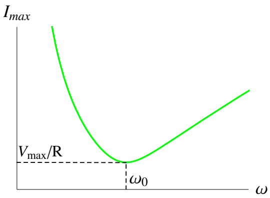
impedance

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$



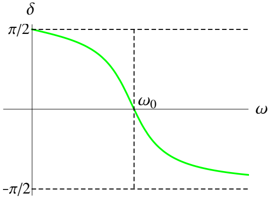
current

$$I_{max} = \frac{V_{max}}{Z}$$



phase angle

$$\delta = \frac{1/\omega L - \omega C}{1/R}$$



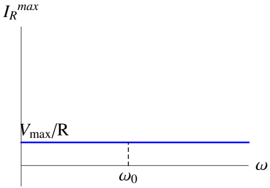
resonance angular frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

RLC Parallel Resonance (2)

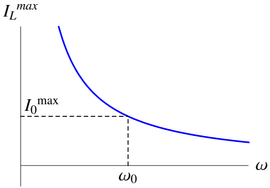
resistor

$$I_R^{max} = V_{max}/R$$



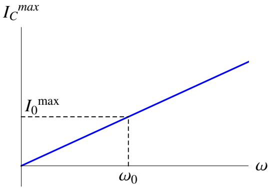
inductor

$$I_L^{max} = V_{max}/\omega L$$



capacitor

$$I_C^{max} = V_{max}\,\omega C$$



currents at resonance:

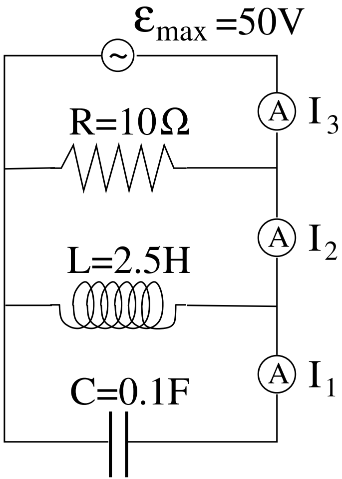
$$I_R^{max} = \frac{V_{max}}{R}, \quad I_L^{max} = I_C^{\max} = I_0^{max} = V_{max}\sqrt{\frac{C}{L}}.$$

AC Circuit Application (3)

Find the current amplitudes I_1, I_2, I_3

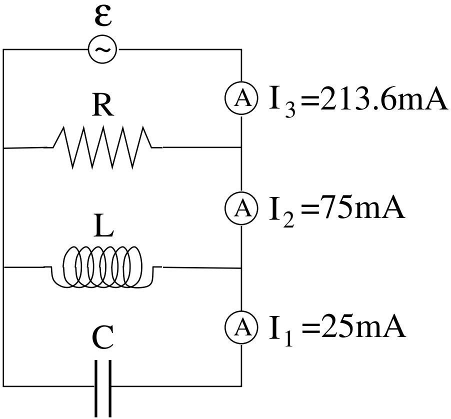
(a) for angular frequency $\omega = 2$rad/s,

(b) for angular frequency $\omega = 4$rad/s.



AC Circuit Application (4)

Given the current amplitudes I_1, I_2, I_3 through the three branches of this RLC circuit, and given the amplitude $\mathcal{E}_{max} = 100$V and angular frequency $\omega = 500$rad/s of the ac source, find the device properties R, L, C.



Power in AC Circuits

Voltage of ac source: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Current through circuit: $I = I_{max} \cos(\omega t - \delta)$

Instantaneous power supplied: $P(t) = \mathcal{E}(t)I(t) = [\mathcal{E}_{max} \cos \omega t][I_{max} \cos(\omega t - \delta)]$

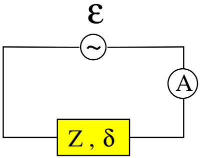
Use $\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$

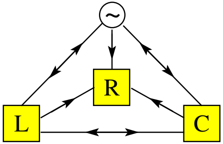
$$\Rightarrow P(t) = \mathcal{E}_{max} I_{max} [\cos^2 \omega t \cos \delta + \cos \omega t \sin \omega t \sin \delta]$$

Time averages: $[\cos^2 \omega t]_{AV} = \frac{1}{2}, \quad [\cos \omega t \sin \omega t]_{AV} = 0$

Average power supplied by source: $P_{AV} = \frac{1}{2} \mathcal{E}_{max} I_{max} \cos \delta = \mathcal{E}_{rms} I_{rms} \cos \delta$

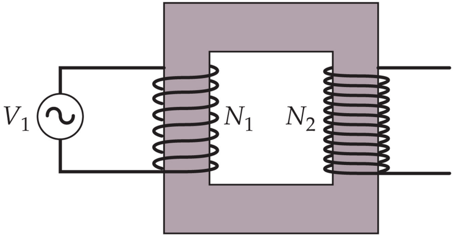
Power factor: $\cos \delta$





Transformer

- Primary winding: N_1 turns
  $$V_1(t) = V_1^{(rms)} \cos(\omega t), \quad I_1(t) = I_1^{(rms)} \cos(\omega t - \delta_1)$$
- Secondary winding: N_2 turns
  $$V_2(t) = V_2^{(rms)} \cos(\omega t), \quad I_2(t) = I_2^{(rms)} \cos(\omega t - \delta_2)$$
- Voltage amplitude ratio: $\dfrac{V_1^{(rms)}}{V_2^{(rms)}} = \dfrac{N_1}{N_2}$
- Power transfer: $V_1^{(rms)} I_1^{(rms)} \cos\delta_1 = V_2^{(rms)} I_2^{(rms)} \cos\delta_2$



AC Circuit Application (6)

Consider an RLC series circuit with inductance $L = 88m$H, capacitance $C = 0.94\mu$F, and unknown resistance R.

The ac generator $\mathcal{E} = \mathcal{E}_{max} \sin(\omega t)$ has amplitude $\mathcal{E}_{max} = 24$V
and frequency $f = 930$Hz. The phase angle is $\delta = 75°$.

(a) Find the resistance R.

(b) Find the current amplitude I_{max}.

(c) Find the maximum energy U_L^{max} stored in the inductor.

(d) Find the maximum energy U_C^{max} stored in the capacitor.

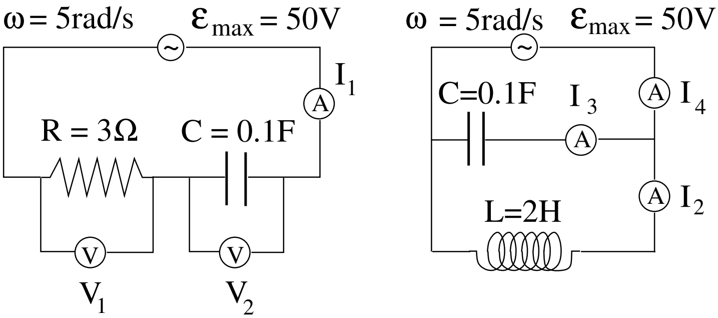
(e) Find the time t_1 at which the current has its maximum value I_{max}.

(f) Find the time t_2 at which the charge on the capacitor has its maximum value Q_{max}.

Consider the two ac circuits shown.

(a) In the circuit on the left, determine the current amplitude I_1 and the voltage amplitudes V_1 and V_2.

(b) In the circuit on the right, determine the current amplitudes I_2, I_3, and I_4.

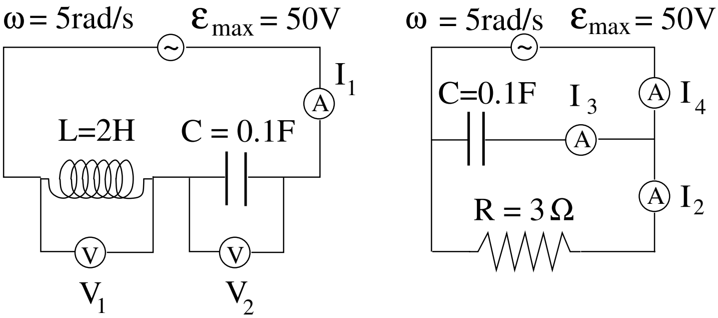


AC Circuit Application (8)

Consider the two ac circuits shown.

(a) In the circuit on the left, determine the maximum value of current I_1 and the maximum value of voltages V_1 and V_2.

(b) In the circuit on the right, determine the maximum value of currents I_2, I_3, and I_4.

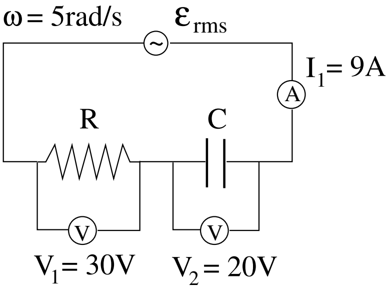


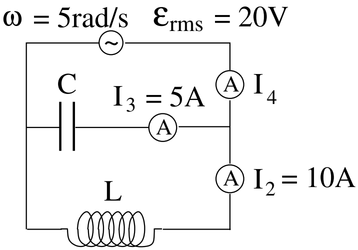
AC Circuit Application (9)

In the two ac circuits shown the ammeter and voltmeter readings are rms values.

(a) In the circuit on the left, find the resistance R of the resistor, the capacitance C of the capacitor, the impedance Z of the two devices combined, and the voltage $\mathcal{E}_{rms}$ of the power source.

(b) In the circuit on the right, find the capacitance C of the capacitor, the inductance L of the inductor, the impedance Z of the two devices combined, and the rms value of the current I_4.



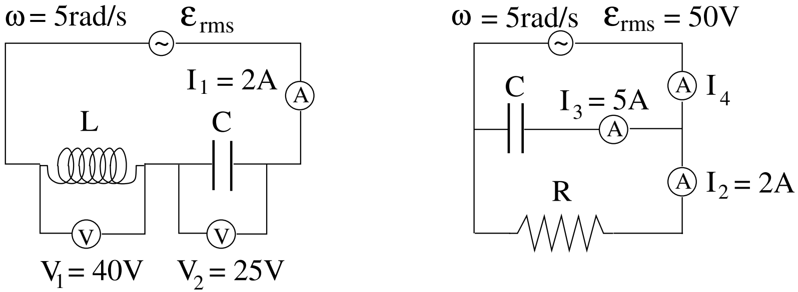


AC Circuit Application (10)

In the two ac circuits shown the ammeter and voltmeter readings are rms values.

(a) In the circuit on the left, find the capacitance C of the capacitor, the inductance L of the inductor, the impedance Z of the two devices combined, and the voltage $\mathcal{E}_{rms}$ of the power source.

(b) In the circuit on the right, find the capacitance C of the capacitor, the resistance R of the resistor, the impedance Z of the two devices combined, and the rms value of the current I_4.



AC Circuit Application (5)

Find the current amplitudes I_1, I_2, I_3, I_4 in the four RLC circuits shown.