2020

15. Ampere's law for the magnetic field with applications

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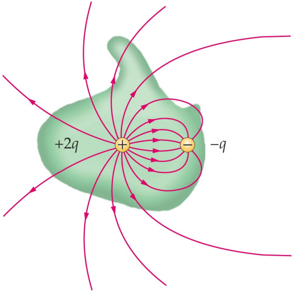
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \qquad \text{i.e.} \qquad \Phi_E = \frac{Q_{in}}{\epsilon_0} \qquad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \mathrm{C}^2 \mathrm{N}^{-1} \mathrm{m}^{-2}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



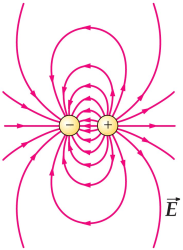
Gauss's Law for Magnetic Field

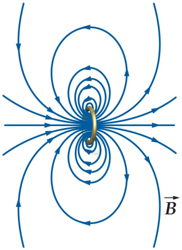
The net magnetic flux Φ_B through any closed surface is equal to zero:

$$\oint \vec{B} \cdot d\vec{A} = 0.$$

There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

The figures below illustrate Gauss's laws for the electric and magnetic fields in the context of an electric dipole (left) and a magnetic dipole (right).





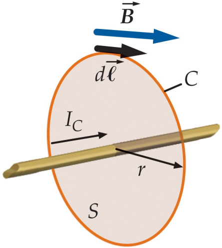
Ampère's Law (Restricted Version)

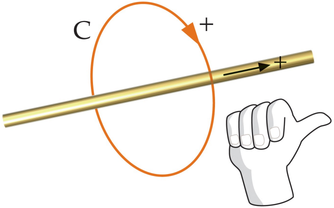
The circulation integral of the magnetic field $\vec{B}$ around any closed curve (loop) C is equal to the net electric current I_C flowing through the loop:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C, \qquad \text{with} \quad \mu_0 = 4\pi \times 10^{-7}\text{Tm/A}$$

The symbol $\oint$ denotes an integral over a closed curve in this context.
Note: Only the component of $\vec{B}$ tangential to the loop contributes to the integral.

The positive current direction through the loop is determined by the right-hand rule.

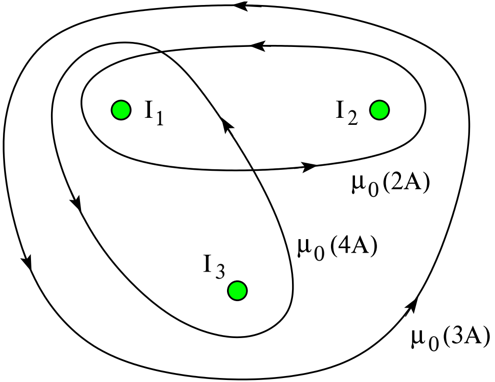




Ampère's Law: Application (1)

The line integrals $\oint \vec{B} \cdot d\vec{s}$ along the three Amperian loops are as indicated.

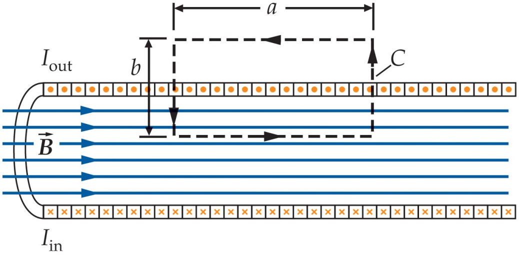
- Find the direction $(\odot, \otimes)$ and the magnitude of the currents I_1, I_2, I_3.



Ampère's Law: Magnetic Field Inside a Long Solenoid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the rectangular Amperian loop shown.

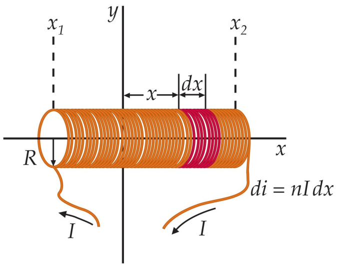
- Magnetic field inside: strong, uniform, directed along axis.
- Magnetic field outside: negligibly weak.
- Number of turns per unit length: n.
- Total current through Amperian loop: $I_C = nIa$ (I is the current in the wire).
- Ampère's law applied to rectangular loop: $Ba = \mu_0 nIa$.
- Magnetic field inside: $B = \mu_0 nI$.

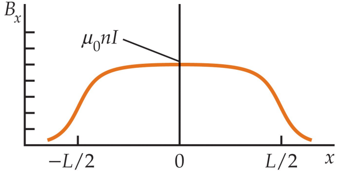


Magnetic Field on the Axis of a Solenoid

- Number of turns per unit length: $n = N/L$
- Current circulating in ring of width dx': $nIdx'$
- Magnetic field on axis of ring: $dB_x = \frac{\mu_0(nIdx')}{2}\frac{R^2}{[(x-x')^2+R^2]^{3/2}}$
- Magnetic field on axis of solenoid:

$$B_x = \frac{\mu_0 nI}{2}R^2\int_{x_1}^{x_2}\frac{dx'}{[(x-x')^2+R^2]^{3/2}} = \frac{\mu_0 nI}{2}\left(\frac{x-x_1}{\sqrt{(x-x_1)^2+R^2}} - \frac{x-x_2}{\sqrt{(x-x_2)^2+R^2}}\right)$$

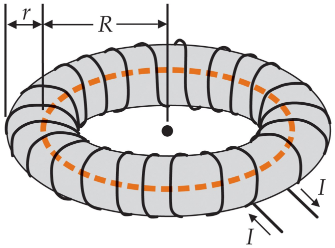




Ampère's Law: Magnetic Field Inside a Toroid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular Amperian loop shown.

- Magnetic field inside: directed tangentially with magnitude depending on R only.
- Magnetic field outside: negligibly weak.
- Number of turns: N.
- Total current through Amperian loop: $I_C = NI$ (I is the current in the wire).
- Ampère's law applied to circular loop: $B(2\pi R) = \mu_0 NI$.
- Magnetic field inside: $B = \dfrac{\mu_0 NI}{2\pi R}$.

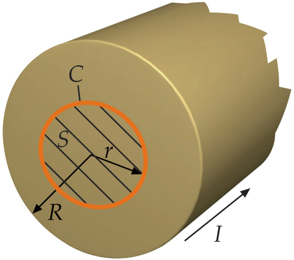


Ampère's Law: Magnetic Field Inside a Wire

Consider a long, straight wire of radius R.
The current I is distributed uniformly over the cross section.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r < R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Line integral: $\oint \vec{B} \cdot d\vec{\ell} = B(2\pi r)$.
- Fraction of current inside loop: $\dfrac{I_C}{I} = \dfrac{\pi r^2}{\pi R^2}$.
- Magnetic field at radius $r < R$: $B = \dfrac{\mu_0 I_C}{2\pi r} = \dfrac{\mu_0 I r}{2\pi R^2}$.
- B increases linearly with r from zero at the center.
- Magnetic field at the perimeter: $B = \dfrac{\mu_0 I}{2\pi R}$.

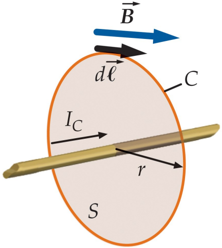


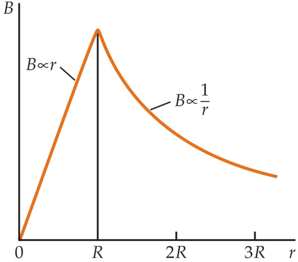
Ampère's Law: Magnetic Field Outside a Wire

Consider a long, straight wire of radius R with current I.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r > R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Current inside loop: $I_C = I$.
- Ampère's law applied: $B(2\pi r) = \mu_0 I$.
- Magnetic field at radius $r > R$: $B = \dfrac{\mu_0 I}{2\pi r}$.

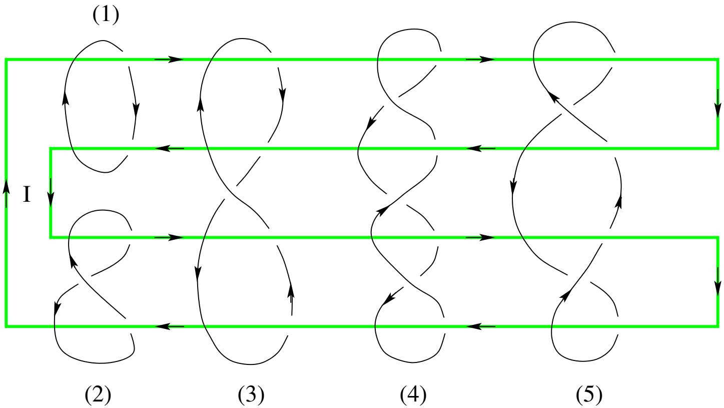




Ampère's Law: Application (2)

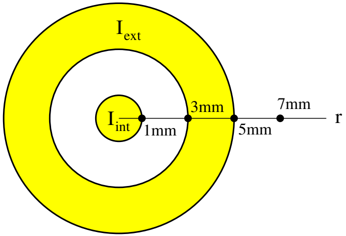
An electric current I flows through the wire in the direction indicated.

- Determine for each of the five Amperian loops whether the line integral $\oint \vec{B} \cdot d\vec{s}$ is positive, negative, or zero.



Unit Exam III: Problem #3 (Spring '12)

The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A} \odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated $(\bullet)$.



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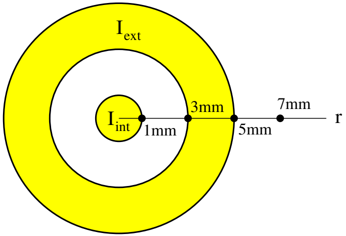

Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow B_1 = 6\mu\text{T} \quad \uparrow$$

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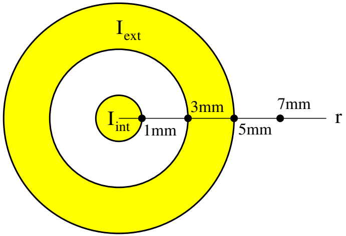

Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow B_1 = 6\mu\text{T} \quad \uparrow$$
$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow B_1 = 2\mu\text{T} \quad \uparrow$$

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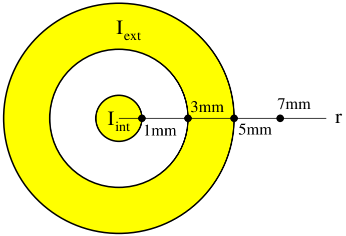

Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \; B_1 = 6\mu\text{T} \;\; \uparrow$$
$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \; B_1 = 2\mu\text{T} \;\; \uparrow$$
$$2\pi(5\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow \; B_1 = 2.4\mu\text{T} \;\; \uparrow$$

Unit Exam III: Problem #3 (Spring '12)

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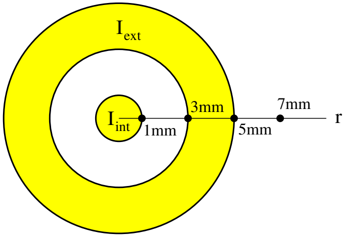


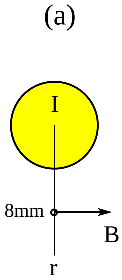
Solution:

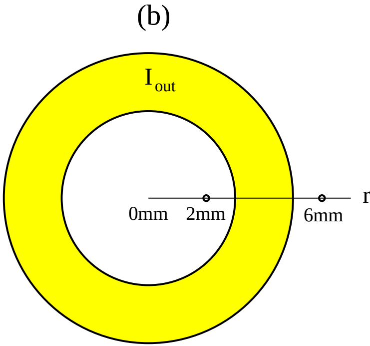
$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow B_1 = 6\mu\text{T} \quad \uparrow$$
$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow B_1 = 2\mu\text{T} \quad \uparrow$$
$$2\pi(5\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow B_1 = 2.4\mu\text{T} \quad \uparrow$$
$$2\pi(7\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow B_1 = 1.71\mu\text{T} \quad \uparrow$$

Unit Exam III: Problem #2 (Spring '08)

(a) Consider a solid wire of radius $R = 3\text{mm}$.
Find magnitude I and direction (in/out) that produces a magnetic field $B = 7\mu\text{T}$ at radius $r = 8\text{mm}$.

(b) Consider a hollow cable with inner radius $R_{int} = 3\text{mm}$ and outer radius $R_{ext} = 5\text{mm}$.
A current $I_{out} = 0.9\text{A}$ is directed out of the plane.
Find direction (up/down) and magnitude B_2, B_6 of the magnetic field at radius $r_2 = 2\text{mm}$ and $r_6 = 6\text{mm}$, respectively.





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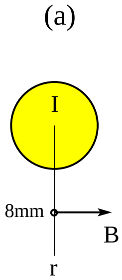

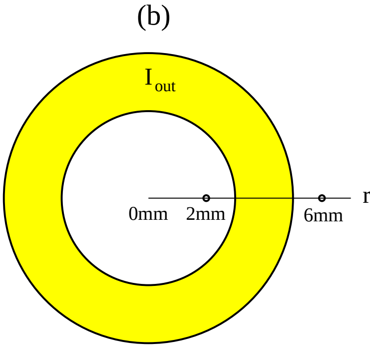

Solution:

(a) $7\mu\text{T} = \dfrac{\mu_0 I}{2\pi(8\text{mm})} \quad \Rightarrow I = 0.28\text{A}$ (out).

(a) Consider a solid wire of radius $R = 3\text{mm}$.
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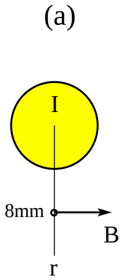
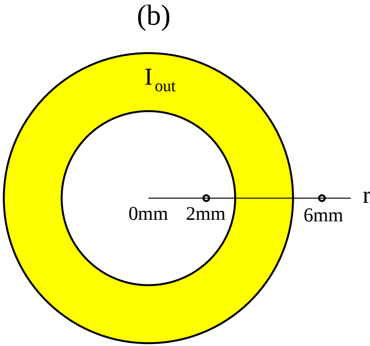
(b) Consider a hollow cable with inner radius $R_{int} = 3\text{mm}$ and outer radius $R_{ext} = 5\text{mm}$.
A current $I_{out} = 0.9\text{A}$ is directed out of the plane.
Find direction (up/down) and magnitude B_2, B_6 of the magnetic field at radius $r_2 = 2\text{mm}$ and $r_6 = 6\text{mm}$, respectively.





Solution:

(a) $7\mu\text{T} = \dfrac{\mu_0 I}{2\pi(8\text{mm})} \quad \Rightarrow \; I = 0.28\text{A}$ (out).

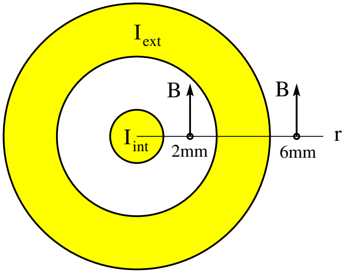
(b) $B_2 = 0, \qquad B_6 = \dfrac{\mu_0(0.9\text{A})}{2\pi(6\text{mm})} = 30\mu\text{T}$ (up).

Unit Exam III: Problem #2 (Spring '11)

The coaxial cable shown has surfaces at radii 1mm, 3mm, and 5mm. The magnetic field is the same at radii 2mm and 6mm, namely $B = 7\mu$T in the direction shown.

(a) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm int}$ flowing through the inner conductor.

(b) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm ext}$ flowing through the outer conductor.



The coaxial cable shown has surfaces at radii 1mm, 3mm, and 5mm. The magnetic field is the same at radii 2mm and 6mm, namely $B = 7\mu$T in the direction shown.

(a) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm int}$ flowing through the inner conductor.

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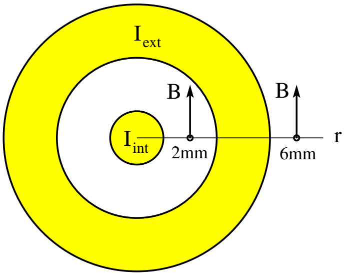

Solution:

(a) $(7\mu\mathrm{T})(2\pi)(0.002\mathrm{m}) = \mu_0 I_{\rm int} \quad \Rightarrow \; I_{\rm int} = 0.07\mathrm{A}$ (out)

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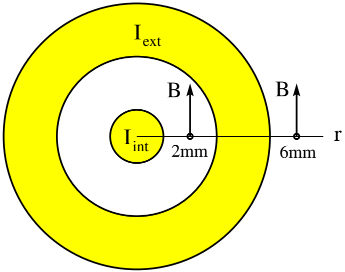


Solution:

(a) $(7\mu\mathrm{T})(2\pi)(0.002\mathrm{m}) = \mu_0 I_{\rm int} \quad \Rightarrow \; I_{\rm int} = 0.07\mathrm{A}$ (out)

(b) $(7\mu\mathrm{T})(2\pi)(0.006\mathrm{m}) = \mu_0 (I_{\rm int} + I_{\rm ext}) \quad \Rightarrow \; I_{\rm int} + I_{\rm ext} = 0.21\mathrm{A}$ (out)

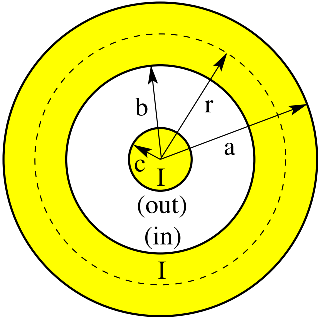
$\Rightarrow \; I_{\rm ext} = 0.14\mathrm{A}$ (out)

Ampère's Law: Coaxial Cable

Consider a long coaxial cable, consisting of two cylindrical conductors separated by an insulator as shown in a cross-sectional view.

There is a current I flowing out of the plane in the inner conductor and a current of equal magnitude I flowing into the plane in the outer conductor.

Calculate the magnetic field B as a function of the radial coordinate r.



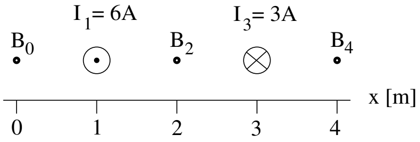
Intermediate Exam III: Problem #2 (Spring '07)

Consider two very long, straight wires with currents $I_1 = 6\text{A}$ at $x = 1\text{m}$ and $I_3 = 3\text{A}$ at $x = 3\text{m}$ in the directions shown. Find magnitude and direction (up/down) of the magnetic field

(a) B_0 at $x = 0$,

(b) B_2 at $x = 2\text{m}$,

(c) B_4 at $x = 4\text{m}$.



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(a) B_0 at $x = 0$,

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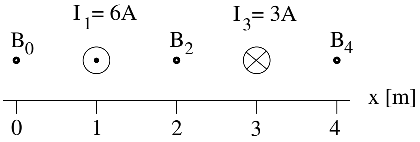

Solution:

(a) $B_0 = -\dfrac{\mu_0(6\text{A})}{2\pi(1\text{m})} + \dfrac{\mu_0(3\text{A})}{2\pi(3\text{m})} = -1.2\mu\text{T} + 0.2\mu\text{T} = -1.0\mu\text{T}$ (down), Consider two very long, straight wires with currents $I_1 = 6\text{A}$ at $x = 1\text{m}$ and $I_3 = 3\text{A}$ at $x = 3\text{m}$ in the directions shown. Find magnitude and direction (up/down) of the magnetic field

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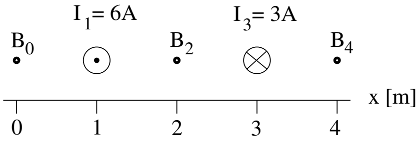


Solution:

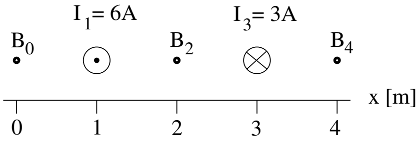
(a) $B_0 = -\dfrac{\mu_0(6\text{A})}{2\pi(1\text{m})} + \dfrac{\mu_0(3\text{A})}{2\pi(3\text{m})} = -1.2\mu\text{T} + 0.2\mu\text{T} = -1.0\mu\text{T}$ (down),

(b) $B_2 = \dfrac{\mu_0(6\text{A})}{2\pi(1\text{m})} + \dfrac{\mu_0(3\text{A})}{2\pi(1\text{m})} = 1.2\mu\text{T} + 0.6\mu\text{T} = 1.8\mu\text{T}$ (up), Consider two very long, straight wires with currents $I_1 = 6\text{A}$ at $x = 1\text{m}$ and $I_3 = 3\text{A}$ at $x = 3\text{m}$ in the directions shown. Find magnitude and direction (up/down) of the magnetic field

(a) B_0 at $x = 0$,

(b) B_2 at $x = 2\text{m}$,

(c) B_4 at $x = 4\text{m}$.



Solution:

(a) $B_0 = -\frac{\mu_0(6\text{A})}{2\pi(1\text{m})} + \frac{\mu_0(3\text{A})}{2\pi(3\text{m})} = -1.2\mu\text{T} + 0.2\mu\text{T} = -1.0\mu\text{T}$ (down),

(b) $B_2 = \frac{\mu_0(6\text{A})}{2\pi(1\text{m})} + \frac{\mu_0(3\text{A})}{2\pi(1\text{m})} = 1.2\mu\text{T} + 0.6\mu\text{T} = 1.8\mu\text{T}$ (up),

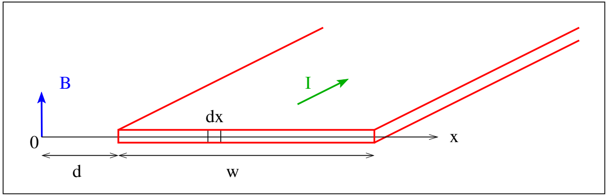
(c) $B_4 = \frac{\mu_0(6\text{A})}{2\pi(3\text{m})} - \frac{\mu_0(3\text{A})}{2\pi(1\text{m})} = 0.4\mu\text{T} - 0.6\mu\text{T} = -0.2\mu\text{T}$ (down).

Magnetic Field Next to Current-Carrying Ribbon

Consider a very long ribbon of width w carrying a current I in the direction shown.

The current density is assumed to be uniform.

Find the magnetic field B generated a distance d from the ribbon as shown.



Divide the ribbon into thin strips of width dx.
Treat each strip as a wire with current $dI = Idx/w$.
Sum up the field contributions from parallel wires.

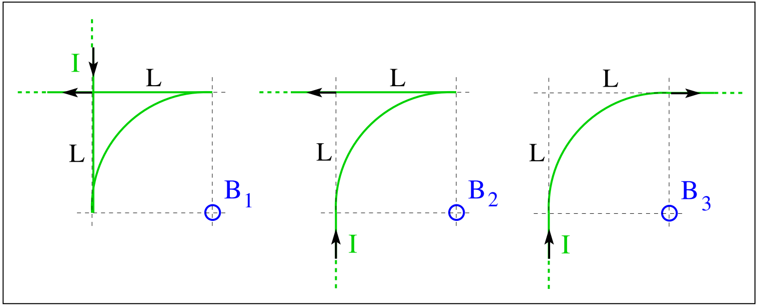
$$dB = \frac{\mu_0}{2\pi}\frac{dI}{x} = \frac{\mu_0 I}{2\pi w}\frac{dx}{x}$$

$$B = \frac{\mu_0 I}{2\pi w}\int_d^{d+w}\frac{dx}{x} = \frac{\mu_0 I}{2\pi w}\ln\left(1+\frac{w}{d}\right)$$

Two semi-infinite straight wires are connected to a segment of circular wire in three different ways. A current I flows in the direction indicated.

(a) Find the direction $(\odot, \otimes)$ of the magnetic fields $\vec{B}_1, \vec{B}_1, \vec{B}_3$.

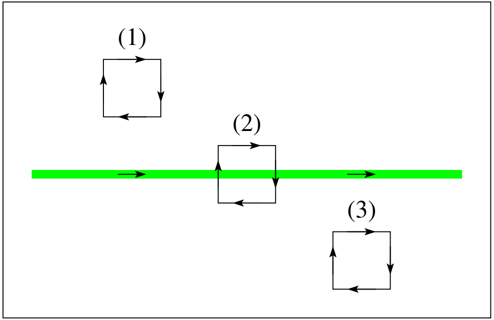
(b) Rank the magnetic fields according to strength.



Magnetic Field Application (8)

Three squares with equal clockwise currents are placed in the magnetic field of a straight wire with a current flowing to the right.

- Find the direction (↑, ↓, zero) of the magnetic force acting on each square.

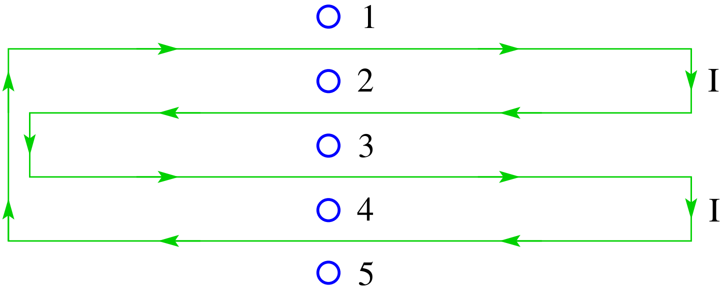


Magnetic Field Application (4)

An electric current I flows through the wire as indicated by arrows.

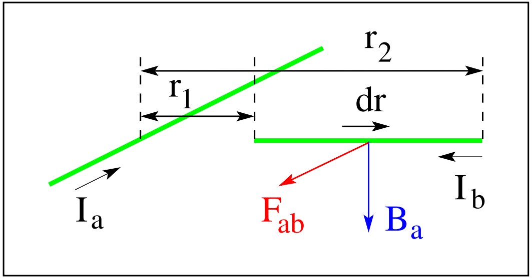
(a) Find the direction $(\odot, \otimes)$ of the magnetic field generated by the current at the points $1, \ldots, 5$.

(b) At which points do we observe the strongest and weakest magnetic fields?



Force Between Perpendicular Lines of Electric Current

- Electric currents: I_a, I_b
- Magnetic field generated by line a: $B_a = \frac{\mu_0}{2\pi}\frac{I_a}{r}$
- Magnetic force on segment dr of line b: $dF_{ab} = I_b B_a dr$
- Magnetic force on line b: $F_{ab} = \frac{\mu_0}{2\pi} I_a I_b \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\mu_0}{2\pi} I_a I_b \ln \frac{r_2}{r_1}$



Magnetic Field Application (10)

Consider two currents of equal magnitude in straight wires flowing perpendicular to the plane.

- In configurations (a) and (b), find the direction $(\rightarrow, \leftarrow, \uparrow, \downarrow)$ of the magnetic field generated by the two currents at points P, Q, R, S