PHY 204: Elementary Physics II -- Slides PHY 204: Elementary Physics II (2021)

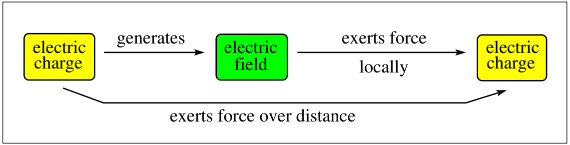
2020

14. Sources of magnetic field: electric currents. Law of Biot and Savart

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

Electric Field of a Point Charge

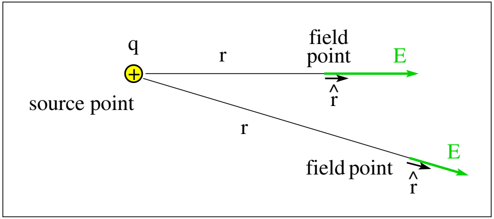


(1) Electric field $\vec{E}$ generated by point charge q: $\vec{E} = k\,\dfrac{q}{r^2}\hat{r}$

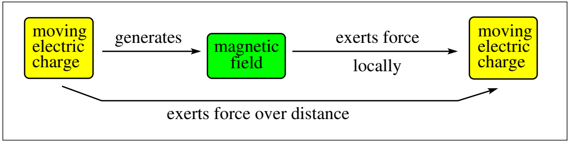
(2) Force $\vec{F}_1$ exerted by field $\vec{E}$ on point charge q_1: $\vec{F}_1 = q_1\vec{E}$

(1+2) Force $\vec{F}_1$ exerted by charge q on charge q_1: $\vec{F}_1 = k\,\dfrac{qq_1}{r^2}\hat{r}$ (static conditions)

- $\epsilon_0 = 8.854 \times 10^{-12}\mathrm{C^2N^{-1}m^{-2}}$
- $k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\mathrm{Nm^2C^{-2}}$
- SI unit of E: [N/C]



Magnetic Field of a Moving Point Charge

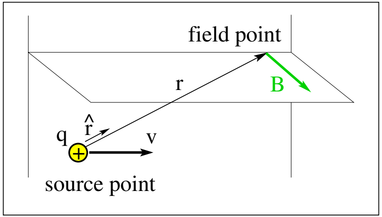


(1) Magnetic field $\vec{B}$ generated by point charge q: $\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$

(2) Force $\vec{F}_1$ exerted by field $\vec{B}$ on point charge q_1: $\vec{F}_1 = q_1 \vec{v}_1 \times \vec{B}$

(1+2) There is a time delay between causally related events over distance.

- Permeability constant
  $\mu_0 = 4\pi \times 10^{-7}$Tm/A



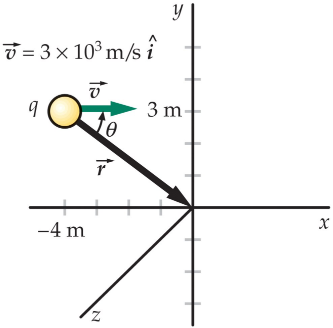
Magnetic Field Application (1)

A particle with charge $q = 4.5\text{nC}$ is moving with velocity $\vec{v} = 3 \times 10^3 \text{m/s}\hat{i}$.

Find the magnetic field generated at the origin of the coordinate system.

- Position of field point relative to particle: $\vec{r} = 4\text{m}\hat{i} - 3\text{m}\hat{j}$
- Distance between Particle and field point: $r = \sqrt{(4\text{m})^2 + (3\text{m})^2} = 5\text{m}$
- Magnetic field:

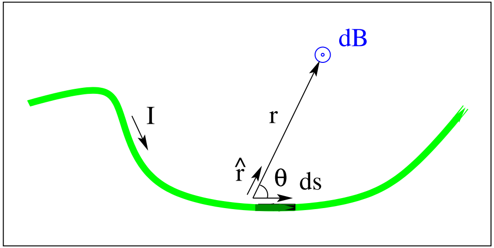
$$\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \vec{r}}{r^3} \\
&= \frac{\mu_0}{4\pi}\frac{q(3 \times 10^3 \text{m/s}\hat{i}) \times (4\text{m}\hat{i} - 3\text{m}\hat{j})}{(5\text{m})^3} \\
&= -\frac{\mu_0}{4\pi}\frac{q(3 \times 10^3 \text{m/s}\hat{i}) \times (3\text{m}\hat{j})}{(5\text{m})^3} \\
&= -3.24 \times 10^{-14}\text{T}\hat{k}.
\end{aligned}$$



Law of Biot and Savart

- Current element: $Id\vec{s} = dq\vec{v}$ [1Am = 1Cm/s]
- Magnetic field of current element: $dB = \dfrac{\mu_0}{4\pi}\dfrac{dqv\sin\theta}{r^2} = \dfrac{\mu_0}{4\pi}\dfrac{Ids\sin\theta}{r^2}$
- Vector relation: $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{Id\vec{s}\times\hat{r}}{r^2}$
- Magnetic field generated by current of arbitrary shape:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{s}\times\hat{r}}{r^2} \quad \text{(Law of Biot and Savart)}$$

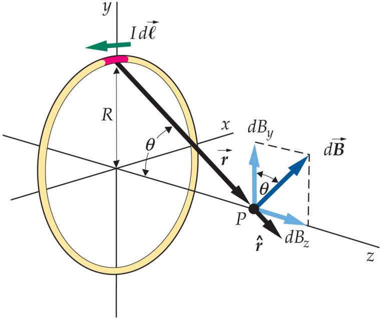


Magnetic Field of Circular Current

- Law of Biot and Savart: $dB = \dfrac{\mu_0}{4\pi}\dfrac{Id\ell}{z^2+R^2}$
- $dB_z = dB\sin\theta = dB\dfrac{R}{\sqrt{z^2+R^2}}$

  $\Rightarrow\ dB_z = \dfrac{\mu_0 I}{4\pi}\dfrac{Rd\ell}{(z^2+R^2)^{3/2}}$
- $B_z = \dfrac{\mu_0 I}{4\pi}\dfrac{R}{(z^2+R^2)^{3/2}}\displaystyle\int_0^{2\pi R} d\ell$

  $\Rightarrow\ B_z = \dfrac{\mu_0 I}{2}\dfrac{R^2}{(z^2+R^2)^{3/2}}$
- Field at center of ring $(z=0)$: $B_z = \dfrac{\mu_0 I}{2R}$
- Magnetic moment: $\mu = I\pi R^2$
- Field at large distance $(z \gg R)$: $B_z \simeq \dfrac{\mu_0}{2\pi}\dfrac{\mu}{z^3}$



Magnetic Field Application (11)

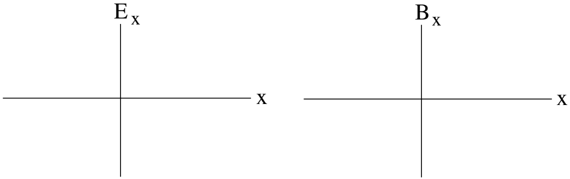
The electric field E_x along the axis of a charged ring and the magnetic field B_x along the axis of a circular current loop are

$$E_x = \frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}}, \qquad B_x = \frac{\mu_0 I}{2}\frac{R^2}{(x^2+R^2)^{3/2}}$$

(a) Simplify both expressions for $x = 0$.

(b) Simplify both expressions for $x \gg R$.

(c) Sketch graphs of $E_x(x)$ and $B_x(x)$.

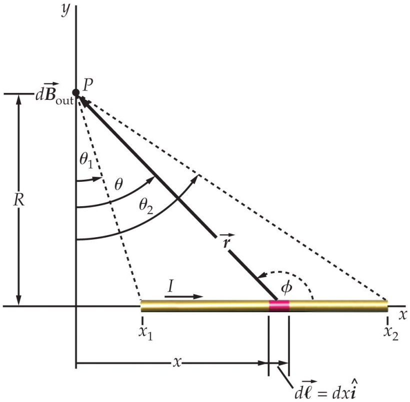


Magnetic Field Generated by Current in Straight Wire (1)

Consider a field point P that is a distance R from the ax

- $dB = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\sin\phi = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\cos\theta$
- $x = R\tan\theta \;\Rightarrow\; \frac{dx}{d\theta} = \frac{R}{\cos^2\theta} = \frac{R}{R^2/r^2} = \frac{r^2}{R}$
- $dB = \frac{\mu_0}{4\pi}\frac{I}{r^2}\frac{r^2 d\theta}{R}\cos\theta = \frac{\mu_0}{4\pi}\frac{I}{R}\cos\theta d\theta$
- $B = \frac{\mu_0}{4\pi}\frac{I}{R}\int_{\theta_1}^{\theta_2}\cos\theta d\theta$

  $= \frac{\mu_0}{4\pi}\frac{I}{R}(\sin\theta_2 - \sin\theta_1)$
- Length of wire: $L = R(\tan\theta_2 - \tan\theta_1)$

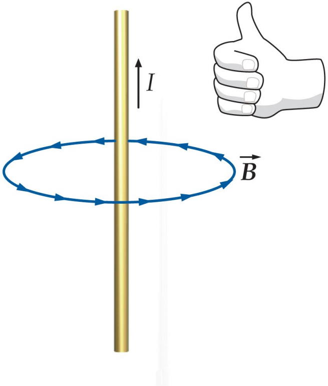


Wire of infinite length: $\theta_1 = -90^\circ,\ \theta_2 = 90^\circ \;\Rightarrow\; B = \frac{\mu_0 I}{2\pi R}$

Magnetic Field Generated by Current in Straight Wire (2)

Consider a current I in a straight wire of infinite length.

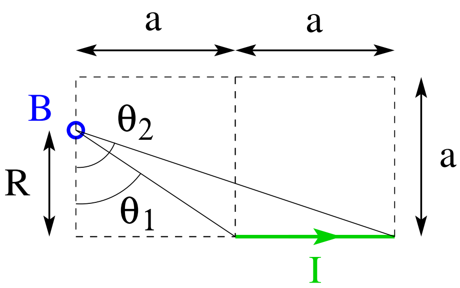
- The magnetic field lines are concentric circles in planes prepedicular to the wire.
- The magnitude of the magnetic field at distance R from the center of the wire is $B = \dfrac{\mu_0 I}{2\pi R}$.
- The magnetic field strength is proportional to the current I and inversely proportional to the distance R from the center of the wire.
- The magnetic field vector is tangential to the circular field lines and directed according to the right-hand rule.



Consider the magnetic field $\vec{B}$ in the limit $R \to 0$.

- $B = \dfrac{\mu_0}{4\pi}\dfrac{I}{R}\left(\sin\theta_2 - \sin\theta_1\right)$
- $\sin\theta_1 = \dfrac{a}{\sqrt{a^2+R^2}} = \dfrac{1}{\sqrt{1+\frac{R^2}{a^2}}} \simeq 1 - \dfrac{1}{2}\dfrac{R^2}{a^2}$
- $\sin\theta_2 = \dfrac{2a}{\sqrt{4a^2+R^2}} = \dfrac{1}{\sqrt{1+\frac{R^2}{4a^2}}} \simeq 1 - \dfrac{1}{2}\dfrac{R^2}{4a^2}$
- $B \simeq \dfrac{\mu_0}{4\pi}\dfrac{I}{R}\left(1 - \dfrac{1}{2}\dfrac{R^2}{4a^2} - 1 + \dfrac{1}{2}\dfrac{R^2}{a^2}\right)$

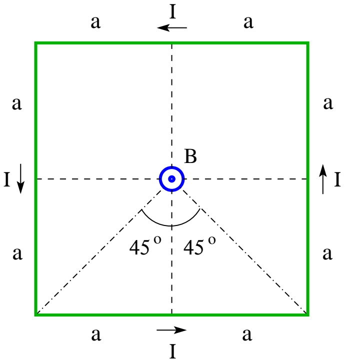
$$= \frac{\mu_0 I}{4\pi}\frac{3R}{8a^2} \xrightarrow{R\to 0} 0$$



Magnetic Field at Center of Square-Shaped Wire

Consider a current-carrying wire bent into the shape of a square with side $2a$.

Find direction and magnitude of the magnetic field generated at the center of the square.



$$B = 4\frac{\mu_0}{4\pi}\frac{I}{a}\left[\sin(45^\circ) - \sin(-45^\circ)\right] = \frac{\sqrt{2}\mu_0 I}{\pi a}.$$

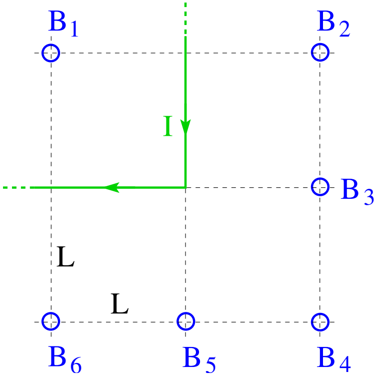
Magnetic Field Application (6)

A current-carrying wire is bent into two semi-infinite straight segments at right angles.

(a) Find the direction $(\odot, \otimes)$ of the magnetic fields $B_1, \ldots, B_6$.

(b) Name the strongest and the weakest fields among them.

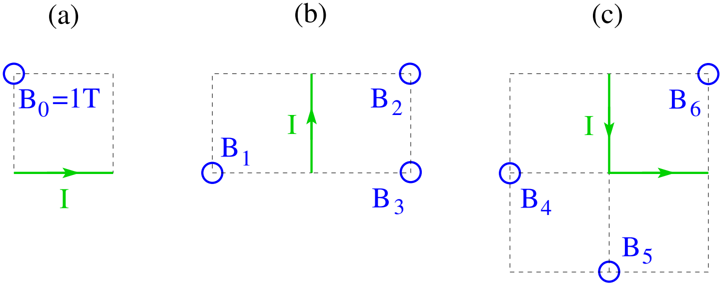
(c) Name all pairs of fields that have equal strength.



Magnetic Field Application (5)

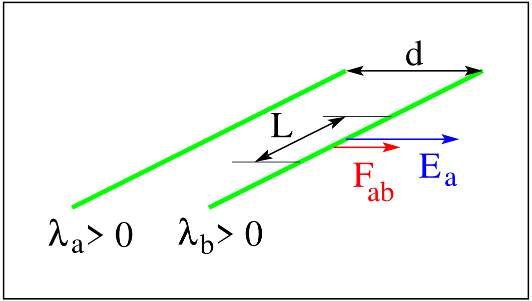
If the current I in (a) generates a magnetic field $B_0 = 1T$ pointing out of the plane

- find magnitude and direction of the fields B_1, B_2, B_3 generated by I in (b),
- find magnitude and direction of the fields B_4, B_5, B_6 generated by I in (c).



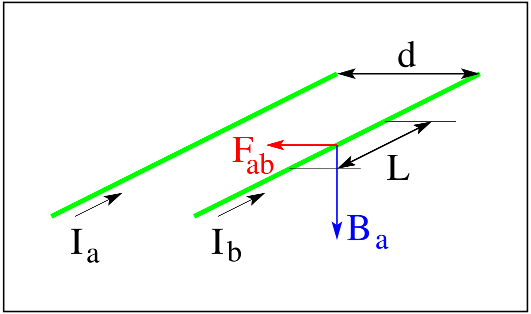
Force Between Parallel Lines of Electric Charge

- Electric charge densities: λ_a, λ_b
- Electric field generated by line a: $E_a = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_a}{d}$
- Electric force on segment of line b: $F_{ab} = \lambda_b L E_a$
- Electric force per unit length (repulsive): $\dfrac{F_{ab}}{L} = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_a\lambda_b}{d}$



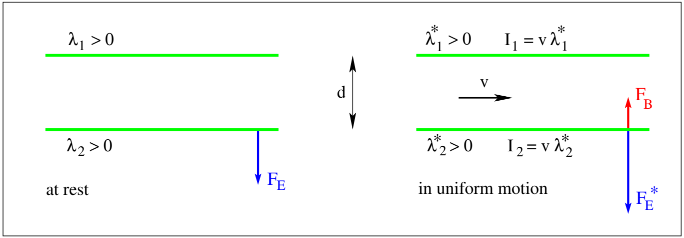
Force Between Parallel Lines of Electric Current

- Electric currents: I_a, I_b
- Magnetic field generated by line a: $B_a = \dfrac{\mu_0}{2\pi}\dfrac{I_a}{d}$
- Magnetic force on segment of line b: $F_{ab} = I_b L B_a$
- Magnetic force per unit length (attractive): $\dfrac{F_{ab}}{L} = \dfrac{\mu_0}{2\pi}\dfrac{I_a I_b}{d}$



Is There Absolute Motion?

Forces between two long, parallel, charged rods

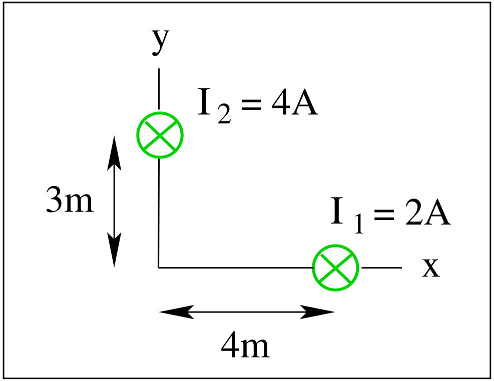


- $\dfrac{F_E}{L} = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_1\lambda_2}{d}$ (left), $\dfrac{F_E^*}{L} = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_1^*\lambda_2^*}{d}$, $\dfrac{F_B}{L} = \dfrac{\mu_0}{2\pi}\dfrac{I_1 I_2}{d}$, (right)
- $\dfrac{F_E^* - F_B}{L} = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_1^*\lambda_2^*}{d}\left(1 - \dfrac{v^2}{c^2}\right) = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda_1\lambda_2}{d}$
- $c = \dfrac{1}{\sqrt{\epsilon_0\mu_0}} = 2.998 \times 10^8 \mathrm{ms}^{-1}$ (speed of light)
- $\lambda_1^* = \dfrac{\lambda_1}{\sqrt{1 - v^2/c^2}}$, $\lambda_2^* = \dfrac{\lambda_2}{\sqrt{1 - v^2/c^2}}$ (due to length contraction)

Magnetic Field Application (12)

Consider two infinitely long straight currents I_1 and I_2 as shown.

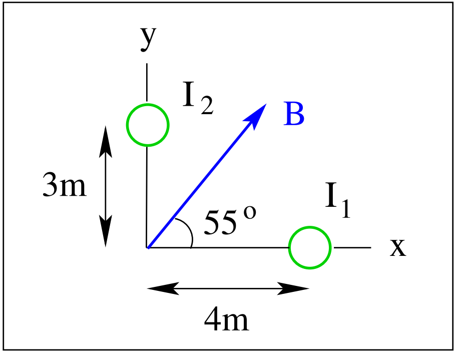
- Find the components B_x and B_y of the magnetic field at the origin of the coordinate system.



Magnetic Field Application (13)

Two straight electric currents I_1 and I_2 of infinite length directed perpendicular to the xy-plane generate a magnetic field of magnitude $B = 6.4 \times 10^{-7}$T in the direction shown.

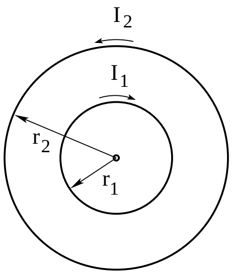
- Find the magnitude and direction $(\odot, \otimes)$ of each current.



Consider two circular currents $I_1 = 3$A at radius $r_1 = 2$m and $I_2 = 5$A at radius $r_2 = 4$m in the directions shown.

(a) Find magnitude B and direction $(\odot, \otimes)$ of the resultant magnetic field at the center.

(b) Find magnitude μ and direction $(\odot, \otimes)$ of the magnetic dipole moment generated by the two currents.



Consider two circular currents $I_1 = 3\text{A}$ at radius $r_1 = 2\text{m}$ and $I_2 = 5\text{A}$ at radius $r_2 = 4\text{m}$ in the directions shown.

(a) Find magnitude B and direction $(\odot, \otimes)$ of the resultant magnetic field at the center.

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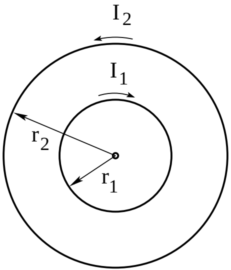


Solution:

(a) $B = \dfrac{\mu_0(3\text{A})}{2(2\text{m})} - \dfrac{\mu_0(5\text{A})}{2(4\text{m})} = (9.42 - 7.85) \times 10^{-7}\text{T}$

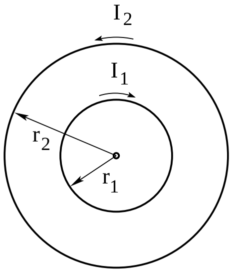
$\Rightarrow B = 1.57 \times 10^{-7}\text{T} \quad \otimes$

Unit Exam III: Problem #1 (Spring '08)

Consider two circular currents $I_1 = 3\text{A}$ at radius $r_1 = 2\text{m}$ and $I_2 = 5\text{A}$ at radius $r_2 = 4\text{m}$ in the directions shown.

(a) Find magnitude B and direction $(\odot, \otimes)$ of the resultant magnetic field at the center.

(b) Find magnitude μ and direction $(\odot, \otimes)$ of the magnetic dipole moment generated by the two currents.



Solution:

(a) $B = \dfrac{\mu_0(3\text{A})}{2(2\text{m})} - \dfrac{\mu_0(5\text{A})}{2(4\text{m})} = (9.42 - 7.85) \times 10^{-7}\text{T}$

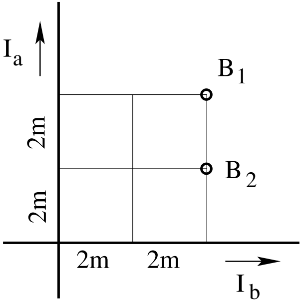
$\Rightarrow B = 1.57 \times 10^{-7}\text{T} \quad \otimes$

(b) $\mu = \pi(4\text{m})^2(5\text{A}) - \pi(2\text{m})^2(3\text{A}) = (251 - 38)\text{Am}^2$

$\Rightarrow \mu = 213\text{Am}^2 \quad \odot$

Consider two infinitely long, straight wires with currents of equal magnitude $I_1 = I_2 = 5\text{A}$ in the directions shown.

Find the direction (in/out) and the magnitude of the magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ at the points marked in the graph.



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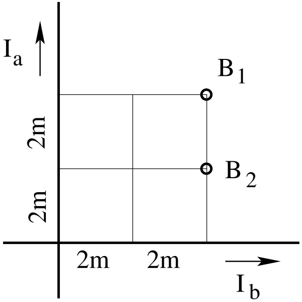
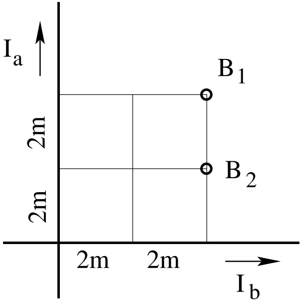


Solution:

- $B_1 = \dfrac{\mu_0}{2\pi}\left(\dfrac{5\text{A}}{4\text{m}} - \dfrac{5\text{A}}{4\text{m}}\right) = 0$ (no direction).

Consider two infinitely long, straight wires with currents of equal magnitude $I_1 = I_2 = 5\mathrm{A}$ in the directions shown.

Find the direction (in/out) and the magnitude of the magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ at the points marked in the graph.



Solution:

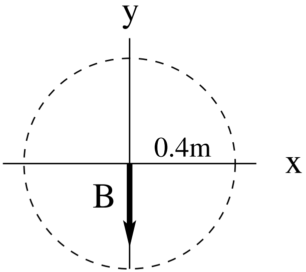
- $B_1 = \frac{\mu_0}{2\pi}\left(\frac{5\mathrm{A}}{4\mathrm{m}} - \frac{5\mathrm{A}}{4\mathrm{m}}\right) = 0$ (no direction).
- $B_2 = \frac{\mu_0}{2\pi}\left(\frac{5\mathrm{A}}{2\mathrm{m}} - \frac{5\mathrm{A}}{4\mathrm{m}}\right) = 0.25\mu\mathrm{T}$ (out of plane).

Intermediate Exam III: Problem #1 (Spring '05)

An infinitely long straight current of magnitude $I = 6$A is directed into the plane ($\otimes$) and located a distance $d = 0.4$m from the coordinate origin (somewhere on the dashed circle). The magnetic field $\vec{B}$ generated by this current is in the negative y-direction as shown.

(a) Find the magnitude B of the magnetic field.

(b) Mark the location of the position of the current $\otimes$ on the dashed circle.



An infinitely long straight current of magnitude $I = 6\mathrm{A}$ is directed into the plane ($\otimes$) and located a distance $d = 0.4\mathrm{m}$ from the coordinate origin (somewhere on the dashed circle). The magnetic field $\vec{B}$ generated by this current is in the negative y-direction as shown.

(a) Find the magnitude B of the magnetic field.

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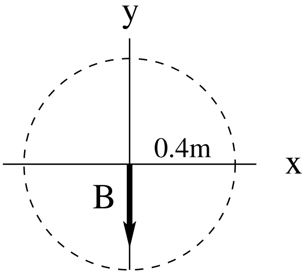

Solution:

(a) $B = \dfrac{\mu_0}{2\pi}\dfrac{I}{d} = 3\mu\mathrm{T}.$

Intermediate Exam III: Problem #1 (Spring '05)

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(a) Find the magnitude B of the magnetic field.

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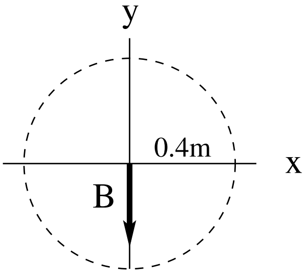
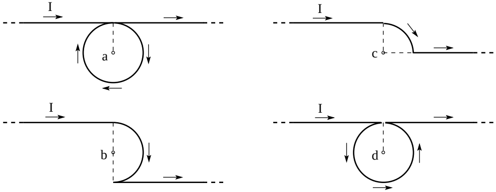


Solution:

(a) $B = \dfrac{\mu_0}{2\pi}\dfrac{I}{d} = 3\mu\text{T}.$

(b) Position of current $\otimes$ is at $y = 0,\ x = -0.4$m.

Two semi-infinite straight wires are connected to a curved wire in the form of a full circle, quarter circle, or half circle of radius $R = 1\text{m}$ in four different configurations. A current $I = 1\text{A}$ flows in the directions shown. Find magnitude B_a, B_b, B_c, B_d and direction $(\odot/\otimes)$ of the magnetic field thus generated at the points a, b, c, d.



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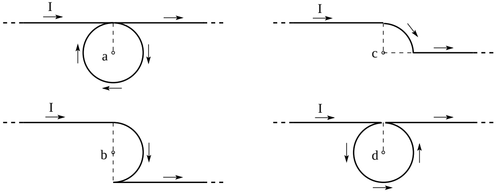
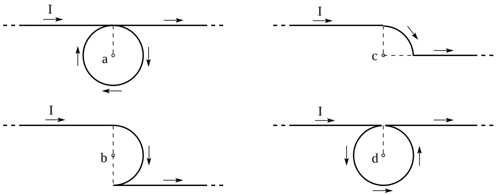


Solution:

$$B_a = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 628\text{nT} + 100\text{nT}| = 828\text{nT} \quad \otimes$$

Two semi-infinite straight wires are connected to a curved wire in the form of a full circle, quarter circle, or half circle of radius $R = 1\text{m}$ in four different configurations. A current $I = 1\text{A}$ flows in the directions shown. Find magnitude B_a, B_b, B_c, B_d and direction $(\odot/\otimes)$ of the magnetic field thus generated at the points a, b, c, d.

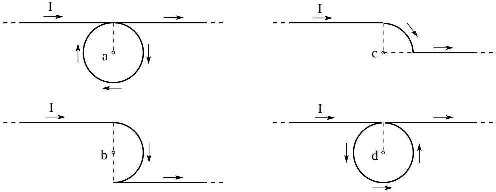


Solution:

$$B_a = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 628\text{nT} + 100\text{nT}| = 828\text{nT} \quad \otimes$$

$$B_b = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4R} - \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 314\text{nT} - 100\text{nT}| = 314\text{nT} \quad \otimes$$

Two semi-infinite straight wires are connected to a curved wire in the form of a full circle, quarter circle, or half circle of radius $R = 1\text{m}$ in four different configurations. A current $I = 1\text{A}$ flows in the directions shown. Find magnitude B_a, B_b, B_c, B_d and direction $(\odot/\otimes)$ of the magnetic field thus generated at the points a, b, c, d.



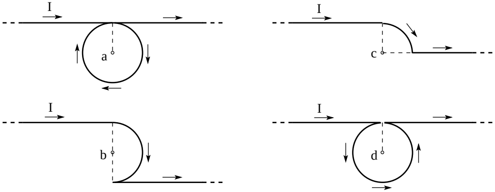
Solution:

$$B_a = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 628\text{nT} + 100\text{nT}| = 828\text{nT} \quad \otimes$$

$$B_b = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4R} - \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 314\text{nT} - 100\text{nT}| = 314\text{nT} \quad \otimes$$

$$B_c = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{8R} + 0 \right| = |100\text{nT} + 157\text{nT}| = 257\text{nT} \quad \otimes$$

Two semi-infinite straight wires are connected to a curved wire in the form of a full circle, quarter circle, or half circle of radius $R = 1\text{m}$ in four different configurations. A current $I = 1\text{A}$ flows in the directions shown. Find magnitude B_a, B_b, B_c, B_d and direction $(\odot/\otimes)$ of the magnetic field thus generated at the points a, b, c, d.



Solution:

$$B_a = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 628\text{nT} + 100\text{nT}| = 828\text{nT} \quad \otimes$$

$$B_b = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4R} - \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} + 314\text{nT} - 100\text{nT}| = 314\text{nT} \quad \otimes$$

$$B_c = \left| \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{8R} + 0 \right| = |100\text{nT} + 157\text{nT}| = 257\text{nT} \quad \otimes$$

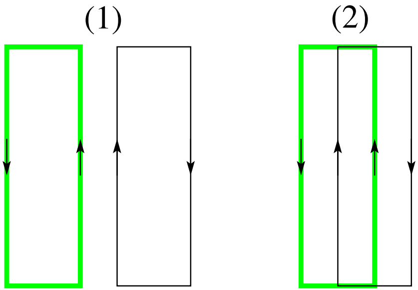
$$B_d = \left| \frac{\mu_0 I}{4\pi R} - \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} \right| = |100\text{nT} - 628\text{nT} + 100\text{nT}| = 428\text{nT} \quad \odot$$

Magnetic Field Application (14)

Consider two pairs of rectangular electric currents flowing in the directions indicated.

(a) What is the direction $(\rightarrow, \leftarrow)$ of the magnetic force experienced by the black rectangle in each case?

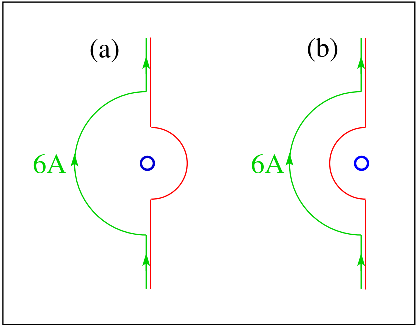
(b) Which black rectangle experiences the stronger magnetic force?



Magnetic Field Application (9)

Two wires of infinite length contain concentric semicircular segments of radii 1m and 2m, respectively.

- If one of the wires carries a 6A current in the direction indicated, what must be the direction $(\uparrow, \downarrow)$ and magnitude of the current in the other wire such that the magnetic field at the center of the semicircles vanishes?



The currents I_1, I_2 in two long straight wires have equal magnitude and generate a magnetic field $\vec{B}$ as shown at three points in space.

- Find the directions $(\odot, \otimes)$ for I_1, I_2 in configurations (a) and (b).