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36. Electromagnetic waves. Poynting vector. Another glimpse of relativity

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

PHY204 Lecture 36 [rln36]

Electromagnetic Plane Wave (1)

Maxwell's equations for electric and magnetic fields in free space (no sources):

- Gauss' laws: $\oint \vec{E} \cdot d\vec{A} = 0, \qquad \oint \vec{B} \cdot d\vec{A} = 0.$
- Faraday's and Ampère's laws: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}, \qquad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$

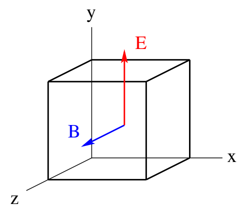
Consider fields of particular directions and dependence on space:

$$\vec{E} = E_y(x,t)\hat{j}, \quad \vec{B} = B_z(x,t)\hat{k}.$$

Gauss' laws are then automatically satisfied.

Use the cubic Gaussian surface to show that

- the net electric flux Φ_E is zero,
- the net magnetic flux Φ_B is zero.



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If electromagnetic waves are electric fields and magnetic fields that change in time and space in a wave-like manner, then it must be possible to cast both fields in the role of wave functions such as described in the previous lecture.

Wave functions are solutions of the wave equations. Electric fields and magnetic fields must satisfy the four Maxwell equations (the two Gauss's laws, Ampère's law, and Faraday's law). Hence, a derivation of the wave equation from Maxwell's equations predicts the existence of electromagnetic waves.

Our starting point are Maxwell's equations in an environment free of sources (charges and conduction currents). Both Gauss's laws now have a zero on the right-hand side. Faraday's and Ampère's laws now look more symmetric.

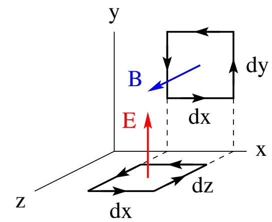
For our demonstration we make the following simplifying assumptions. We consider fields that vary only with coordinate x and time t. Such waves are called plane waves. Then we assume that the electric field only has a y-component and the magnetic field only has a z-component. Such waves are called linearly polarized waves.

These assumptions have the advantage that the two Gauss's laws are automatically satisfied. We demonstrate this for a cubic Gaussian surface as shown. For each field only two faces of the cube produce flux and those two fluxes are equal in magnitude and opposite in sign.

In a region where $\vec{E}$ is pointing up (as shown), the top face yields positive electric flux and the bottom face negative electric flux in equal amounts. In a region where $\vec{B}$ is pointing out (as shown), the front face yields positive magnetic flux and the back face negative magnetic flux in equal amounts.

Electromagnetic Plane Wave (2)

- Faraday's law, $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$,
  applied to loop in (x, y)-plane becomes
  $[E_y(x+dx,t) - E_y(x,t)]dy = -\frac{\partial}{\partial t} B_z(x,t) dx dy$
  $\Rightarrow \frac{\partial}{\partial x} E_y(x,t) = -\frac{\partial}{\partial t} B_z(x,t) \qquad \text{(F)}$
- Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$,
  applied to loop in (x, z)-plane becomes
  $[-B_z(x+dx,t) + B_z(x,t)]dz = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t) dx dz$
  $\Rightarrow -\frac{\partial}{\partial x} B_z(x,t) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t) \qquad \text{(A)}$



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Faraday's law and Ampère's law are not automatically satisfied by the two functions,

$$\vec{E} = E_y(x,t)\hat{\mathbf{j}}, \quad \vec{B} = E_z(x,t)\hat{\mathbf{k}}.$$

The two items on the slide are applications of the two laws to tiny square loops with loop directions selected as shown.

Only two sides of each loop contribute to the line integrals, namely those for which the relevant field is either parallel or antiparallel to $d\vec{l}$. For the remaining two segments the relevant field is perpendicular to $d\vec{l}$, which yields a vanishing dot product.

In each case, one of the infinitesimal elements of length (dy or dz) can be divided out and the other (dx in both instances) can be used to convert the difference quotient into a spatial derivative.

From the application of Faraday's law thus emerges equation (F) and from the application of Ampère's law equation (A). We thus have arrived at two coupled differential equations. Each equation relates the space derivative of one field to the time derivative of the other field.

Our next step will be the decoupling of the two equations into an equation for the electric field and an equation for the magnetic field. This step will be carried out on the next page.

Electromagnetic Plane Wave (3)

Take partial derivatives $\frac{\partial}{\partial x}$(F) and $\frac{\partial}{\partial t}$(A): $\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial t \partial x}, \quad -\frac{\partial^2 B_z}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}.$

$\Rightarrow \frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}$ (E) (wave equation for electric field).

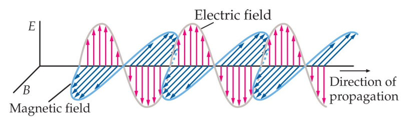
Take partial derivatives $\frac{\partial}{\partial t}$(F) and $\frac{\partial}{\partial x}$(A): $\frac{\partial^2 E_y}{\partial t \partial x} = -\frac{\partial^2 B_z}{\partial t^2}, \quad -\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t \partial x}.$

$\Rightarrow \frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2}$ (B) (wave equation for magnetic field).

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (speed of light).

Sinusoidal solution:

- $E_y(x,t) = E_{max} \sin(kx - \omega t)$
- $B_z(x,t) = B_{max} \sin(kx - \omega t)$



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If we take a space derivative on both sides of equation (F) and a time derivative on both sides of equation (A) they still hold true. One side of each equation thus obtained, namely the side with the mixed derivative, is identical. Therefore the opposite side of each equation must be equal. Bingo! We have arrived at the wave equation for the electric field. The wave speed c is the speed of light as expressed below in terms of the familiar permittivity and permeability constants.

In like manner, we can derive the wave equation for the magnetic field by taking a time derivative on both sides of equation (F) and a space derivative on both sides of equation (A). The wave speed is the same.

We already know that the sinusoidal wave functions as shown at the bottom of the slide are solutions of the wave equation. This solution is known as a linearly polarized electromagnetic wave.

A snapshot of the wave described by these wave functions is shown graphically. We are dealing with a transverse wave. The electric field at any position oscillates in y-direction (up and down) and the magnetic field in z-direction (front and back). The wave propagates in x-direction (to the right).

How many of the four parameters, k, ω, E_{max}, and B_{max} can chosen independently? Why is there no phase shift between the wave functions of the two fields? Look for answers on the next page.

Electromagnetic Plane Wave (4)

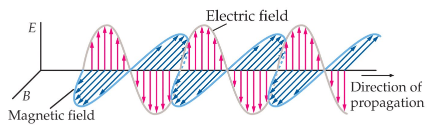
For given wave number k the angular frequency ω is determined, for example by substitution of $E_{max}\sin(kx - \omega t)$ into (E).

For given amplitude E_{max} the amplitude B_{max} is determined, for example, by substituting $E_{max}\sin(kx - \omega t)$ and $B_{max}\sin(kx - \omega t)$ into (A) or (F).

$$\Rightarrow \frac{\omega}{k} = \frac{E_{max}}{B_{max}} = c.$$

The direction of wave propagation is determind by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}.$$



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When we substitute the sinusoidal solution (see slide) into equation (E) or equation (B), the equality of the two sides holds if $\omega/k = c$, which we knew all along. The important point to keep in mind is that ω and k cannot be varied independently. The ratio c is a universal constant.

When we substitute the sinusoidal solution into equation (F) we obtain,

$$kE_{max}\cos(kx - \omega t) = \omega B_{max}\cos(kx - \omega t)$$

From this equation and the previous result, $\omega/k = c$, we conclude that the ratio of field amplitudes is $E_{max}/B_{max} = c$. We also note that the two sides of (F) can only be equal if there is no phase shift between the electric-field and magnetic-field wave functions.

The Poynting vector $\vec{S}$ as stated on the slide captures multiple pieces of information. Its direction, for one, is the direction of wave propagation.

An electromagnetic wave with either $\vec{E}$ or $\vec{B}$ reversed in direction would travel in the opposite direction (to the left). If both fields reverse direction, which they do after every half period, the direction of propagation stays the same. The cross product switches direction if one factor does. It stays the same if both factors switch direction.

Electromagnetic waves with different polarizations are linear superpositions of linearly polarized waves. In a circularly polarized wave, the electric field has fixed magnitude. Its orientation circles around the direction of propagation:

$$\mathbf{E}(x,t) = E_{max}[\sin(kx - \omega t)\hat{\mathbf{j}} + \cos(kx - \omega t)\hat{\mathbf{k}}].$$

The magnetic field does the same.

Energy Transport in Electromagnetic Plane Wave

Fields: $E_y(x,t) = E_{max}\sin(kx - \omega t)$, $B_z(x,t) = B_{max}\sin(kx - \omega t)$.

Energy density: $u(x,t) = \frac{1}{2}\epsilon_0 E_y^2(x,t) + \frac{1}{2\mu_0}B_z^2(x,t)$. [J/m^3]

Use the amplitude relations $\epsilon_0 E_{max}^2 = \epsilon_0 c^2 B_{max}^2 = \frac{1}{\mu_0}B_{max}^2$.

$$u(x,t) = \epsilon_0 E_{max}^2 \sin^2(kx - \omega t) = \frac{1}{\mu_0}B_{max}^2 \sin^2(kx - \omega t) = \frac{E_{max}B_{max}}{c\mu_0}\sin^2(kx - \omega t).$$

Energy transported across area A in time dt: $dU(x,t) = u(x,t)Acdt$. [J]

Power transported per unit area: $\frac{1}{A}\frac{dU}{dt} = u(x,t)c = S(x,t)$. [W/m^2]

Intensity (average power transported per unit area):

$$I = \bar{S} = \frac{E_{max}B_{max}}{2\mu_0} = \frac{\epsilon_0 c}{2}E_{max}^2 = \frac{c}{2\mu_0}B_{max}^2. \quad \text{[W/m}^2\text{]}$$

tsl323

A second piece of information contained in the Poynting vector has to do with the energy transported by an electromagnetic wave. We recall from our discussion of capacitors that electric fields carry energy and from our discussion of inductors that magnetic fields do the same. In an LC oscillator, we have learned, energy is converted back and forth between these two forms.

The expressions for energy density developed in the context of static fields are also valid for time-varying fields such as realized in an electromagnetic wave and stated on the slide. We see that in a linearly polarized wave the energy travels in packages.

Each package has a profile in the shape of a $\sin^2(kx)$ function and extends over half a wavelength. These packages travel at the speed of light in the direction of the Poynting vector.

The magnitude of the Poynting vector is shown to represent the energy per time unit and per unit of cross sectional area (i.e. power per unit area) transported by a plane wave.

The intensity of an electromagnetic wave is defined as the average power transported per unit area.

The intensity of a plane wave remains constant along its path. That is not the case if the wave travels radially away from a source that is point-like. Then energy conservation dictates that the intensity is inversely proportional to the area of the spherical wave front, which means inversely proportional to the square of the distance from the source.

Momentum Transport in Electromagnetic Plane Wave

The momentum transported by an electromagnetic wave is proportional to the energy transported.

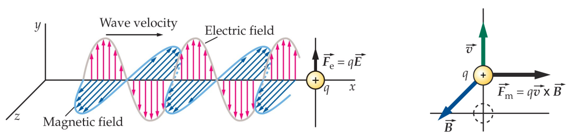
Momentum density: $\frac{\vec{p}}{V} = \frac{\vec{S}}{c^2}$, where $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is the Poynting vector.

When the wave is absorbed by a material surface it exerts an impulse $\vec{F}dt = \Delta\vec{p}$.

The resulting radiation pressure is the average force per unit area:

$$P_{abs} = \frac{\bar{F}}{A} = \frac{p}{Adt} = \frac{p}{Adx}\frac{dx}{dt} = \frac{p}{V}c = \frac{\bar{S}}{c} = \frac{I}{c}.$$

The radiation pressure exerted by a reflected wave is twice as large: $P_{ref} = \frac{2\bar{S}}{c} = \frac{2I}{c}$.



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Electromagnetic waves not only carry energy but also momentum. When a wave is being absorbed at the surface of a massive object, that object recoils. When the surface is a mirror, which reflects the wave, the recoil doubles.

When our skin is exposed to the sun's radiation we feel the energy transfer as heat. The momentum transfer is too small to be perceptible as pressure, but the pressure does exist and is measurable in suitably designed experiments.

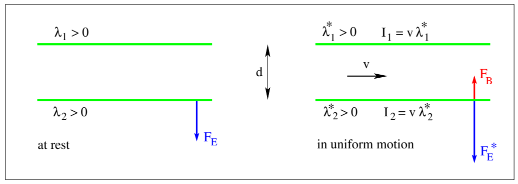
The graphs at the bottom of the slide are meant to provide a plausibility argument for the interaction of a wave with a charged particles to produce a force in forward direction.

The charged particle is being accelerated by the electric field in y-direction. As it picks up velocity in y-direction it is located in a magnetic field in z-direction and thus experiences a magnetic force in x-direction, which is the forward direction.

Note that the Poynting vector is related to the momentum per unit volume of a plane wave as shown on the slide.

Is There Absolute Motion?

Forces between two long, parallel, charged rods



- $\frac{F_E}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1\lambda_2}{d}$ (left), $\frac{F_E^*}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1^*\lambda_2^*}{d}$, $\frac{F_B}{L} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$, (right)
- $\frac{F_E^* - F_B}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1^*\lambda_2^*}{d}\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1\lambda_2}{d}$
- $c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.998 \times 10^8 \mathrm{ms}^{-1}$ (speed of light)
- $\lambda_1^* = \frac{\lambda_1}{\sqrt{1 - v^2/c^2}}$, $\lambda_2^* = \frac{\lambda_2}{\sqrt{1 - v^2/c^2}}$ (due to length contraction)

tsl211

In conclusion of this course, we would like to draw attention to the fact that our understanding of electricity and magnetism not only contributed enormously to a revolution in technology but also to a revolution in our conception of space and time.

We have discussed the slide on this page already in lecture 24. It considers the forces between two long charged rails from the perspective of observer 1, who is at rest relative to them, and from the perspective of observer 2, who is in motion relative to them.

Both observers take Maxwell's equations seriously as laws of nature but come to inconsistent conclusions if they also insist that space and time are absolute, i.e. independent of the motion of the observer or the observed.

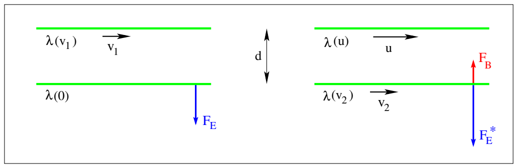
Observer 1 measures a (repulsive) electric force between the two rails. Observer 2 measures, in addition to that electric force, also an (attractive) magnetic force. We have seen that the data of the two measurements can only agree if we allow the rails to contract a bit.

Length contraction of moving objects is one of the consequences of relativistic mechanics. Time dilation (slowing down of moving clocks) and curvature of space in the presence of massive objects are other consequences.

In the design of GPS, all three consequences and others have to be taken into account to make it work.

Catching Up with a Photon? (1)

Forces between two long, parallel, charged rods in relative motion.



- Galilean kinematics predicts $u = v_1 + v_2$.
- Relativistic kinematics requires $v_1 < c,\ v_2 < c,\ u < c$.
- Relativistic dynamics requires $F_E^* - F_B = F_E$.
- Length-contracted charge densities: $\lambda(v) = \dfrac{\lambda(0)}{\sqrt{1 - v^2/c^2}}$.
- Electric currents: $I(v) = \lambda(v)v$.

tsl439

Standing on the platform in a train station, we observe a train moving at velocity v_2 through the station and a passenger inside the train walking at velocity v_1 toward the front. From our perspective, the passenger moves with velocity, $u = v_1 + v_2$, says nonrelativistic mechanics.

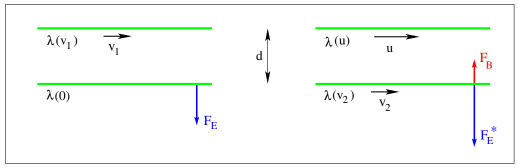
Imagine the train moving with velocity $v_2 = \frac{2}{3}c$ and the passenger walking with velocity $v_1 = \frac{2}{3}c$, where c is the speed of light. From our perspective on the station platform, the passenger would move faster than light, if the above velocity-addition rule were correct.

We mentioned earlier that nothing physical can move faster than the speed of light and be consistent with Maxwell's equations. We can use our setup of charged rails to determine what the modification of the velocity-addition rule has to be.

Observer 1, who sees the lower rail at rest and the upper rail move with velocity v_1 predicts that the two rails only exert a repulsive electric force F_E on each other as calculated on the previous page (see lecture 24 for more details). Observer 2, who sees the lower rail move with velocity v_2 and the upper rail with velocity u, predicts that the two rails exert on each other an attractive magnetic force F_B in addition to the (now modified) repulsive electric force F_E^*. Note that the modified charge density $\lambda(v)$ of a moving rail due to length contraction affects both electric and magnetic forces.

If absolute motion does not exist, as the theory of relativity assumes and asserts, then both observers must predict the same net force. What does that entail for the relation between the velocities v_1, v_2, and u?

Catching Up with a Photon? (2)



- $\frac{F_E}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda(0)\lambda(v_1)}{d}, \qquad \frac{F_E^*}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda(v_2)\lambda(u)}{d}.$
- $\frac{F_B}{L} = \frac{\mu_0}{2\pi}\frac{[\lambda(v_2)v_2][\lambda(u)u]}{d} = \frac{1}{2\pi\epsilon_0}\frac{\lambda(v_2)\lambda(u)}{d}\frac{v_2u}{c^2}.$
- $\frac{F_E^* - F_B}{L} = \frac{F_E}{L} \quad \Rightarrow \quad \frac{1}{2\pi\epsilon_0}\frac{\lambda(v_2)\lambda(u)}{d}\left(1 - \frac{v_2u}{c^2}\right) = \frac{1}{2\pi\epsilon_0}\frac{\lambda(0)\lambda(v_1)}{d}$
- $\Rightarrow \quad \frac{1}{\sqrt{1-v_2^2c^2}}\frac{1}{\sqrt{1-u^2/c^2}}\left(1 - \frac{v_2u}{c^2}\right) = \frac{1}{\sqrt{1-v_1^2/c^2}}$ to be solved for u.
- Relativistic kinematics predicts $u = \frac{v_1+v_2}{1+v_1v_2/c^2} < c$.

tsl440

The first item shows the expressions for the electric force predicted by observers 1 and 2. The second item shows the magnetic force predicted by observer 2 only.

The third item equates the predictions of the two observers. The fourth item substitutes the expression $\lambda(v)$ for charge density from the previous page and simplifies the equation somewhat.

What we see is an algebraic relation between the velocities v_1, v_2, and u, which we can solve into the modified velocity addition rule. The last item shows the result.

If we use $v_1 = v_2 = \frac{2}{3}c$, the values for the imagined train and passenger on the previous page, we, who are on the station platform, see the passenger move not at the impossible superluminary velocity $u = \frac{4}{3}c$ but, as predicted by relativistic kinematics, at the velocity,

$$u = \frac{\frac{2}{3}c + \frac{2}{3}c}{1 + \frac{\left(\frac{2}{3}c\right)\left(\frac{2}{3}c\right)}{c^2}} = \frac{36}{39}c.$$

If we choose either $v_1 = c$ or v_2 or $v_1 = v_2 = c$, the relativistic velocity addition rule predicts $u = c$. There is no way to exceed the speed of light. There is no way to catch up with a photon.

Instead of a quiz, you are invited to share some of your experiences related to PHY204 in the remote instruction mode. This is voluntary, not an obligation in any sense. A parting message will be highly appreciated. It will help us making adjustments in the next round.

We wish you all a continued safe navigation through these challenging times.

With warmest regards
Gerhard Muller and Robert Coyne