2002

On the stability of solutions of certain systems of differential equations with piecewise constant argument

S. R. Grace

Mustafa Kulenović
University of Rhode Island, mkulenovic@uri.edu

H. El-Metwally
University of Rhode Island

Citation/Publisher Attribution
Grace, S. R., Kulenović, M. R. S., & El-Metwally, H. Czechoslovak Mathematical Journal (2002) 52:449.
http://doi.org/10.1023/A:1021707225207
Available at: http://doi.org/10.1023/A:1021707225207

Czechoslovak Mathematical Journal, 52 (127) (2002), 449–461

ON THE STABILITY OF SOLUTIONS OF CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

S. R. Grace, Orman, M. R. S. Kulenović, and H. El-Metwally*, Kingston

(Received March 22, 1999)

Abstract. We obtain some sufficient conditions for the existence of the solutions and the asymptotic behavior of both linear and nonlinear system of differential equations with continuous coefficients and piecewise constant argument.

Keywords: asymptotic behavior, delay, stability

MSC 2000: 34K20, 34K25, 39A10

1. Introduction

In this paper we study the nonlinear differential equation

$$x'(t) + A(t)x(t) = B(t)x\Big(\Big[t+\frac{1}{2}\Big]\Big) + f\Big(t, x(t), x\Big(\Big[t+\frac{1}{2}\Big]\Big)\Big) \tag{1}$$

where $A(t)$ and $B(t)$ are $r \times r$ continuous matrices, $f\colon \mathbb{R} \times \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ is a continuous function on $\mathbb{R} \times \mathbb{R}^{2r}$, x is a r-vector, and $[.]$ is the integer part of a number.

Also we will investigate the equation

$$x'(t) + A(t)x(t) = B(t)x\Big(\Big[t+\frac{1}{2}\Big]\Big) \tag{2}$$

with continuous matrices A and B.

* On leave from Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt.

Differential equations with piecewise constant argument (EPCA) represent a hybrid of continuous and discrete dynamical systems and therefore combine the properties of both the differential and difference equations. These equations have many applications in the control theory and certain biomedical models [3].

The research of first order differential equations with piecewise constant arguments of delay and advanced type was initiated by Cooke and Wiener [4] and Shah and Wiener [7]. A lot of results concerning the scalar versions of equation (1) have been studied in [1, 5, 8] and the references cited therein.

Our goal in this article is to derive the sufficient conditions for the existence of solutions of equations (1) and (2) and the global asymptotic stability of equations (1) and (2). An important tool in achieving this goal will be the variant of the variation of constant formula for equation (2).

We note that the results related to Theorems 1 and 2 of this paper for equations of the form (2) with constant matrices A and B are included in Theorem 1 [9]. Also results concerning the existence and uniqueness are extensions of Theorem 1.1 [2].

2. Main Results

By a solution of equation (1) on $(-\infty, \infty)$ we mean a function $x(t)$ satisfying the conditions

(i) $x(t)$ is continuous on $(-\infty, \infty)$;

(ii) the derivative $x'(t)$ exists everywhere, with the possible exception of the half-integer points $n + \frac{1}{2}$, where one-sided derivatives exist;

(iii) Eq. (1) is satisfied on each interval $\left[n - \frac{1}{2}, n + \frac{1}{2}\right)$, with integer n.

In what follows we will use the norm of a $p \times q$ matrix A as

$$|A| = \max\{|a_{ij}|\colon\ i = 1, 2, \ldots, p \quad \text{and} \quad j = 1, 2, \ldots, q \quad \text{for} \quad p \leqslant q\}.$$

The trivial solution of equation (1) is said to be globally asymptotically stable if every solution $x(t)$ of equation (1) tends to zero as $t \to \infty$.

I.

In this part, we prove our main results for equation (2). In Theorem 1 and Theorem 2 below we will establish sufficient conditions for the existence and uniqueness of the solutions of Eq. (2). These results follow from the representation of the solutions of Eq. (2).

Theorem 2.1. *Let the matrix*

$$C_n = I - \int_{n+\frac{1}{2}}^{n+1} B(u) \exp\Big\{-\int_u^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u, \qquad n = 0, 1, \ldots$$

be non-singular, then Eq. (2) *has a unique solution on* $[0,\infty)$ *given by*

$$x(t) = \Big[F(t,n) + \int_n^t B(s)F(t,s)\,\mathrm{d}s\Big]a_n, \tag{3}$$

where a_n *is a solution of the difference equation*

$$a_{n+1} = C_n^{-1}D_n a_n, \qquad n = 0,1,\dots, \tag{4}$$

C_n *is defined above,*

$$D_n = F(n+1,n) + \int_n^{n+\frac{1}{2}} B(s)F(n+1,s)\,\mathrm{d}s$$

and

$$F(u,v) = \exp\big\{-\textstyle\int_v^u A(s)\,\mathrm{d}s\big\}.$$

P r o o f. For $t \in \big[n-\frac{1}{2}, n+\frac{1}{2}\big)$, Eq. (2) reduces to

$$x'(t) + A(t)x(t) = B(t)a_n, \qquad a_n = x(n). \tag{5}$$

Integrating Eq. (5) from n to $t < n+\frac{1}{2}$ gives

$$x(t) - \exp\big\{-\textstyle\int_n^t A(s)\,\mathrm{d}s\big\}a_n = \exp\big\{-\textstyle\int_n^t A(s)\,\mathrm{d}s\big\}\displaystyle\int_n^t B(u)\exp\big\{\textstyle\int_n^u A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\,a_n.$$

Hence we obtain

$$x(t) = \Big[\exp\big\{-\textstyle\int_n^t A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^t B(u)\exp\big\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_n,$$

which is equivalent to

$$x(t) = \Big[F(t,n) + \int_n^t B(u)F(t,u)\,\mathrm{d}u\Big]a_n,$$

where

$$F(u,v) = \exp\big\{-\textstyle\int_v^u A(s)\,\mathrm{d}s\big\}.$$

Letting $t \to n+\frac{1}{2}$, we have

$$x\Big(n+\frac{1}{2}\Big) = \Big[\exp\big\{-\textstyle\int_n^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^{n+\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_n. \tag{6}$$

Again, for $t \in \left[n+\frac{1}{2}, n+\frac{3}{2}\right)$, integrating Eq. (2) from $n+1$ to t we obtain

$$x(t) = \left[\exp\Big\{-\int_{n+1}^{t} A(s)\,\mathrm{d}s\Big\} + \int_{n+1}^{t} B(u)\exp\Big\{-\int_{u}^{t} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n+1}.$$

Then as $t \to n+\frac{1}{2}$, the continuity of $x(t)$ yields

(7)
$$x\Big(n+\frac{1}{2}\Big) = \left[\exp\Big\{-\int_{n+1}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\} + \int_{n+1}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n+1}.$$

Equations (6) and (7) imply

$$\begin{aligned}
&\left[\exp\Big\{-\int_{n+1}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\} + \int_{n+1}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n+1} \\
&= \left[\exp\Big\{-\int_{n}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\} + \int_{n}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n}. \qquad (8)
\end{aligned}$$

Multiplying both sides of (8) by $\exp\big\{\int_{n+1}^{n+\frac{1}{2}} A(s)\,\mathrm{d}s\big\}$ yields

$$\begin{aligned}
&\left[I + \int_{n+1}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n+1} \\
&= \left[\exp\Big\{-\int_{n}^{n+1} A(s)\,\mathrm{d}s\Big\} + \int_{n}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left[I - \int_{n+\frac{1}{2}}^{n+1} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n+1} \\
&= \left[\exp\Big\{-\int_{n}^{n+1} A(s)\,\mathrm{d}s\Big\} + \int_{n}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n}.
\end{aligned}$$

Since the matrix C_n is non-singular, we see that

$$\begin{aligned}
a_{n+1} = &\left[I - \int_{n+\frac{1}{2}}^{n+1} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right]^{-1} \\
&\times \left[\exp\Big\{-\int_{n}^{n+1} A(s)\,\mathrm{d}s\Big\} + \int_{n}^{n+\frac{1}{2}} B(u)\exp\Big\{-\int_{u}^{n+1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\right] a_{n},
\end{aligned}$$

or in the simple form

$$a_{n+1} = C_n^{-1} D_n a_n, \qquad n = 0, 1, \ldots$$

where

$$C_n = I - \int_{n+\frac{1}{2}}^{n+1} B(u)F(n+1,u)\,\mathrm{d}u, \qquad F(u,v) = \exp\Big\{-\int_v^u A(s)\,\mathrm{d}s\Big\}$$

and

$$D_n = F(n+1,n) + \int_n^{n+\frac{1}{2}} B(u)F(n+1,u)\,\mathrm{d}u,$$

which proves the Theorem. □

Remark 2.1. Taking $A(t) = -A$ and $B(t) = B$ in Theorem 2.1 above, we get Theorem 1 in [9]. In this case, Eq. (3) takes the form

$$x(t) = \Big[\mathrm{e}^{A(t-n)} + \int_n^t B\,\mathrm{e}^{A(t-u)}\,\mathrm{d}u\Big]a_n,$$

and

$$x(t) = \big[\mathrm{e}^{A}(t-n) - \big(I - \mathrm{e}^{A(t-n)}\big)A^{-1}B\big]a_n.$$

Therefore

$$x(t) = M(t-n)a_n, \qquad a_n = x(n),$$

where

$$M(t) = \mathrm{e}^{At} + \big(\mathrm{e}^{At} - I\big)A^{-1}B$$

and we obtain the result in [9].

Remark 2.2. If A and B are constant matrices and $\left[t+\frac{1}{2}\right]$ is replaced by $[t]$, we get Theorem 1.1 in [2].

Theorem 2.2. *The solution $x(t)$ of Eq. (2) has a unique backward continuation on $(-\infty, 0)$ given by (3) and*

$$a_{n-1} = W_n^{-1}Z_n a_n, \qquad n = 0, -1, -2, \ldots$$

where the matrices W_n and Z_n are given by

$$\text{(9)} \qquad \begin{cases} W_n = I + \displaystyle\int_{n-1}^{n-\frac{1}{2}} B(u)\exp\Big\{-\int_u^{n-1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u, \\ Z_n = \exp\Big\{-\displaystyle\int_n^{n-1} A(s)\,\mathrm{d}s\Big\} + \displaystyle\int_n^{n-\frac{1}{2}} B(u)\exp\Big\{-\int_u^{n-1} A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u \end{cases}$$

provided the matrix W_n is non-singular.

P r o o f. As in the proof of Theorem 2.1, for $t \in [n-\frac{1}{2}, n+\frac{1}{2})$ we have Eq. (3) and as $t \to n-\frac{1}{2}$, we get

$$x\Big(n-\frac{1}{2}\Big) = \Big[\exp\big\{-\textstyle\int_n^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_n. \tag{10}$$

Also, for $t \in [n-\frac{3}{2}, n-\frac{1}{2}\big)$, integrating Eq. (2) from $n-1$ to t and letting $t \to n-\frac{1}{2}$ we obtain

$$x\Big(n-\frac{1}{2}\Big) = \Big[\exp\big\{-\textstyle\int_{n-1}^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_{n-1}^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_{n-1}. \tag{11}$$

From Equations (10) and (11) we find

$$\begin{aligned}&\Big[\exp\big\{-\textstyle\int_{n-1}^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_{n-1}^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_{n-1}\\ &= \Big[\exp\big\{-\textstyle\int_n^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-\frac{1}{2}} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_n,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\Big[I + \displaystyle\int_{n-1}^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-1} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_{n-1}\\ &= \Big[\exp\big\{-\textstyle\int_n^{n-1} A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^{n-\frac{1}{2}} B(u)\exp\big\{-\textstyle\int_u^{n-1} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u\Big]a_n.\end{aligned}$$

Hence we have

$$a_{n-1} = W_n^{-1} Z_n a_n, \qquad n = 0, -1, \ldots$$

where W_n and Z_n are defined by (9). The proof is completed. □

Theorem 2.3. *Assume that*

(i) $\quad |A(t)| \geqslant \delta$ *for some* $\delta > 0$ *and* $|B(t)| \leqslant \gamma$ *where* $0 < \gamma < \delta$.

Then the trivial solution of Eq. (2) *is globally asymptotically stable.*

P r o o f. Let $x(t)$ be a solution of Eq. (2). From Eq. (3) we have

$$|x(t)| = |Z(t)|\,|a_n|, \tag{12}$$

where

$$Z(t) = \exp\big\{-\textstyle\int_n^t A(s)\,\mathrm{d}s\big\} + \displaystyle\int_n^t B(u)\exp\big\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u.$$

It follows from Condition (i) that

$$|Z(t)| \leqslant \mathrm{e}^{-\delta(t-n)} + \gamma \int_n^t \mathrm{e}^{-\delta(t-u)}\,\mathrm{d}u \leqslant 1 + \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta(t-n)}].$$

Then we get

$$|Z(t)| \leqslant 1 + \frac{\gamma}{\delta}. \tag{13}$$

Now Eq. (4) implies

$$a_{n+1} = T_n a_n, \qquad n = 0, 1, \ldots$$

where $T_n = C_n^{-1} D_n$. Hence we obtain

$$a_n = T_{n-1} T_{n-2} \ldots T_0 a_0, \qquad a_0 = x(0), \qquad n = 0, 1, \ldots$$

Thus

$$|a_n| = |T_{n-1}|\,|T_{n-2}| \ldots |T_0|\,|a_0|. \tag{14}$$

Now, Condition (i) gives

$$|D_n| \leqslant \mathrm{e}^{-\delta} + \gamma \int_n^{n+\frac{1}{2}} \mathrm{e}^{-\delta(n+1-u)}\,\mathrm{d}u, \qquad n = 0, 1, \ldots$$

and so

$$|D_n| \leqslant \mathrm{e}^{-\delta} + \frac{\gamma}{\delta}\big[\mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta}\big]. \tag{15}$$

Let $H = \int_{n+\frac{1}{2}}^{n+1} B(u) \exp\big\{-\int_u^{n+1} A(s)\,\mathrm{d}s\big\}\,\mathrm{d}u$. Then by using Condition (i), we have

$$|H| \leqslant \gamma \int_{n+\frac{1}{2}}^{n+1} \mathrm{e}^{-\delta(n+1-u)}\,\mathrm{d}u \leqslant \frac{\gamma}{\delta}\big[1 - \mathrm{e}^{-\delta/2}\big].$$

Therefore, from the fact that $|A| - |B| \leqslant \big||A| - |B|\big| \leqslant |A - B|$ for the $p \times q$ matrices A and B, we find

$$|C_n| = |I - H| \geqslant |I| - |H| \geqslant 1 - \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta/2}]. \tag{16}$$

Thus Inequalities (15) and (16) give

$$|T_n| = |C_n^{-1}|\,|D_n| = |C_n|^{-1}|D_n| \leqslant \frac{\mathrm{e}^{-\delta} + \frac{\gamma}{\delta}[\mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta}]}{1 - \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta/2}]}. \tag{17}$$

Now we obtain $\frac{\gamma}{\delta} < 1$ from (i). Consequently

$$\frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta}] < 1 - \mathrm{e}^{-\delta}.$$

Thus

$$\frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta} + \mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta/2}] < 1 - \mathrm{e}^{-\delta}.$$

Hence we have

$$\frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta/2}] + \frac{\gamma}{\delta}[\mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta}] < 1 - \mathrm{e}^{-\delta}$$

or

$$\mathrm{e}^{-\delta} + \frac{\gamma}{\delta}[\mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta}] < 1 - \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta/2}].$$

Therefore we obtain

$$\frac{\mathrm{e}^{-\delta} + \frac{\gamma}{\delta}[\mathrm{e}^{-\delta/2} - \mathrm{e}^{-\delta}]}{1 - \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\delta/2}]} < 1.$$

Now from (17) we get

$$|T_n| \leqslant \lambda < 1, \qquad n = 0, 1, \ldots \tag{18}$$

where

$$\lambda = \frac{\mathrm{e}^{-\delta} + \frac{\gamma}{\delta}[\mathrm{e}^{-\frac{\delta}{2}} - \mathrm{e}^{-\delta}]}{1 - \frac{\gamma}{\delta}[1 - \mathrm{e}^{-\frac{\delta}{2}}]}.$$

Thus Equations (14) and (18) give

$$|a_n| \leqslant \lambda^n |a_0|,$$

which implies

$$\lim_{n\to\infty} a_n = 0. \tag{19}$$

Thus it follows from (12), (13), and (19) that

$$\lim_{t\to\infty} x(t) = 0$$

and the proof is completed. □

Example. Consider the differential equation

$$x'(t) = A(t)x(t) + B(t)x\Big(\Big[t + \frac{1}{2}\Big]\Big), \qquad t \geqslant 0 \tag{20}$$

where $x(t) \in \mathbb{R}^2$ and $A(t)$ and $B(t)$ are 2×2 matrices given by

$$A(t) = \begin{bmatrix} \sin t - 1 & \cos t \\ \frac{1-\mathrm{e}^{-t^2}}{\mathrm{e}} & \frac{-\lambda}{1+\mathrm{e}^{-t}} \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} -\mathrm{e}^{-\beta}\sin t & -\mathrm{e}^{-\beta}\cos t \\ \frac{\mathrm{e}^{-t^2}-2}{\mathrm{e}^{\beta+1}} & \frac{\lambda \mathrm{e}^{-\beta}}{1+\mathrm{e}^{-t}} \end{bmatrix}$$

for some constants $\lambda > 2$ and $\beta > 0$. By Theorem 2.3, the trivial solution of Eq. (20) is globally asymptotically stable. In fact, the vector $x(t) = \big(\mathrm{e}^{-t}, \mathrm{e}^{-(t+1)}\big)$ is a solution of Eq. (20).

II.

In this section we study Eq. (1).

Theorem 2.4. *Assume that the matrix C_n defined as in Theorem* 2.1 *is also non-singular. Then Eq.* (1) *has a unique solution on $[0, \infty)$ given by*

$$\begin{aligned} x(t) = {} & \Big[\exp\Big\{-\int_n^t A(s)\,\mathrm{d}s\Big\} + \int_n^t B(u)\exp\Big\{-\int_u^t A(s)\,\mathrm{d}s\Big\}\,\mathrm{d}u\Big]a_n \\ & + \int_n^t \exp\Big\{-\int_u^t A(s)\,\mathrm{d}s\Big\} f\big(u, x(u), a_n\big)\,\mathrm{d}u, \end{aligned} \tag{21}$$

where a_n is a solution of the difference equation

$$\begin{aligned} a_{n+1} &= T_n a_n + W_n, \qquad n = 0, 1, \ldots, \\ T_n &= C_n^{-1} D_n, \qquad W_n = C_n^{-1}(E_n + F_n), \end{aligned} \tag{22}$$

the matrices C_n and D_n are defined as in Theorem 2.1 *and the matrices E_n and F_n are defined by*

$$\begin{cases} E_n = \displaystyle\int_{n+\frac{1}{2}}^{n+1} \exp\Big\{-\int_u^{n+1} A(s)\,\mathrm{d}s\Big\} f\big(u, x(u), a_{n+1}\big)\,\mathrm{d}u, \\ F_n = \displaystyle\int_n^{n+\frac{1}{2}} \exp\Big\{-\int_u^{n+1} A(s)\,\mathrm{d}s\Big\} f\big(u, x(u), a_n\big)\,\mathrm{d}u. \end{cases} \tag{23}$$

P r o o f. The proof is similar to that of Theorem 2.1 and therefore it is omitted. □

Theorem 2.5. *Assume that Condition* (i) *in Theorem* 2.3 *holds. Let* $f(t,x,y), t \in \mathbb{R}$, $x, y \in \mathbb{R}^r$ *satisfy*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leqslant N(t)|x_1 - x_2| + K(t)|y_1 - y_2| \tag{ii}$$

where $N(t)$ *and* $K(t)$ *are non-negative scalar functions defined on* (t_0,∞) *such that*

$$\int_{t_0}^{\infty} N(s)\,\mathrm{d}s = \alpha < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} K(s)\,\mathrm{d}s = \beta < \infty, \quad \alpha,\beta \in \mathbb{R}^+. \tag{iii}$$

Then the trivial solution of Eq. (1) *is globally asymptotically stable.*

P r o o f. From Eq. (21) we find

$$|x(t)| \leqslant |z(t)|\,|a_n| + \left|\int_n^t \exp\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\}f\big(u,x(u),a_n\big)\,\mathrm{d}u\right|, \tag{24}$$

where $z(t)$ is defined as in Theorem 2.3. Now, it follows from Conditions (i) and (ii) that

$$\begin{aligned}
&\left|\int_n^t \exp\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\}f\big(u,x(u),a_n\big)\,\mathrm{d}u\right| \\
&\leqslant \int_n^t \exp\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\}[N(u)|x(u)| + K(u)|a_n|]\,\mathrm{d}u \\
&\leqslant \int_n^t N(u)|x(u)|\,\mathrm{d}u + \int_n^t K(u)|a_n|\,\mathrm{d}u.
\end{aligned}$$

By Condition (iii) we obtain

$$\left|\int_n^t \exp\{-\textstyle\int_u^t A(s)\,\mathrm{d}s\}f\big(u,x(u),a_n\big)\,\mathrm{d}u\right| \leqslant \int_n^t N(u)|x(u)|\,\mathrm{d}u + \beta|a_n|. \tag{25}$$

By substituting from (25) in (24), we get

$$|x(t)| \leqslant (|z(t)| + \beta)|a_n| + \int_n^t N(u)|x(u)|\,\mathrm{d}u. \tag{26}$$

Eq. (22) implies

$$a_n = C_{n-1}^{-1}D_{n-1}a_{n-1} + W_{n-1}, \qquad n = 1,2,\ldots$$

Therefore from Theorem 2.1 in [6, p. 13] we have

$$a_n = T_{n-1}T_{n-2}\ldots T_0 a_0 + \sum_{r=0}^{n-1} T_{n-1}T_{n-2}\ldots T_r W_r.$$

Thus

$$|a_n| \leqslant |T_{n-1}|\,|T_{n-2}|\dots|T_0|\,|a_0| + \sum_{r=0}^{n-1} |T_{n-1}|\,|T_{n-2}|\dots|T_r|\,|W_r|.$$

Also as in Theorem 2.3, from (18) we find

$$|a_n| \leqslant \lambda^n|a_0| + \sum_{r=0}^{n-1} \lambda^{n-1-r}|W_r|. \tag{27}$$

Now

$$|W_n| = |C_n^{-1}|\bigg| \int_{n+\frac{1}{2}}^{n+1} \exp\big\{-\textstyle\int_u^{n+1} A(s)\,\mathrm{d}s\big\} f\big(u, x(u), a_{n+1}\big)\,\mathrm{d}u + \int_n^{n+\frac{1}{2}} \exp\big\{-\textstyle\int_u^{n+1} A(s)\,\mathrm{d}s\big\} f\big(u, x(u), a_{n+1}\big)\,\mathrm{d}u\bigg|.$$

Assume that

$$\overline{a_n} = \max_{n\leqslant t\leqslant n+1}\{|x(t)|\}.$$

Then by Condition (i) we have

$$|W_n| \leqslant |C_n^{-1}| \int_n^{n+1} 2\mathrm{e}^{-\delta(n+1-u)}[N(u)+K(u)]\,|\overline{a_n}|\,\mathrm{d}u.$$

So, from Inequality (16) and Condition (iii), we see that

$$|W_n| \leqslant \frac{2(\alpha+\beta)}{1-\frac{\gamma}{\delta}(1-\mathrm{e}^{-\delta})}|\overline{a_n}| = L|\overline{a_n}|, \tag{28}$$

where $L = \frac{2\delta(\alpha+\beta)}{\delta-\gamma(1-\mathrm{e}^{-\delta})}$. Then it follows from (27) and (28) that

$$|\overline{a_n}| \leqslant \lambda^n|a_0| + \sum_{r=0}^{n-1} L\lambda^{n-1-r}|\overline{a_r}|. \tag{29}$$

Applying the discrete Gronwall's inequality [6, p. 21] to Eq. (20), we obtain

$$|\overline{a_n}| \leqslant \lambda^n|a_0| \exp\Big\{L\sum_{r=0}^{n-1}\lambda^{n-1-r}\Big\} = \lambda^n|a_0|\mathrm{e}^{L(\lambda^{n-1}+\lambda^{n-2}+\dots+1)}.$$

Therefore, we have

$$|a_n| \leqslant \lambda^n|a_0|\mathrm{e}^{L\lambda^n/(1-\lambda)}. \tag{30}$$

In view of (26) and (30), we obtain

$$|x(t)| \leqslant (|z(t)| + \beta)\lambda^n |a_0| \mathrm{e}^{L\lambda^n/(1-\lambda)} + \int_n^t N(u)|x(u)|\,\mathrm{d}u.$$

Thus Inequality (13) implies

$$|x(t)| \leqslant \Big(1 + \frac{\gamma}{\delta} + \beta\Big)\lambda^n |a_0|\, \mathrm{e}^{L\lambda^n/(1-\lambda)} + \int_n^t N(u)|x(u)|\,\mathrm{d}u.$$

Again applying the continuous Gronwall's inequality we obtain

$$|x(t)| \leqslant \Big(1 + \frac{\gamma}{\delta} + \beta\Big)\lambda^n |a_0|\, \mathrm{e}^{L\lambda^n/(1-\lambda)} \exp\Big\{\int_n^t N(u)\,\mathrm{d}u\Big\},$$

which by Condition (ii) gives

$$|x(t)| \leqslant \Big(1 + \frac{\gamma}{\delta} + \beta\Big)\lambda^n |a_0|\, \mathrm{e}^{\alpha} \mathrm{e}^{L\lambda^n/(1-\lambda)}.$$

Since $\frac{L}{1-\lambda} > 0$, we obtain

$$|x(t)| \leqslant \Big(1 + \frac{\gamma}{\delta} + \beta\Big)|a_0|\, \mathrm{e}^{\alpha} \lambda^t \mathrm{e}^{L\lambda^t/(1-\lambda)}.$$

Thus

$$\lim_{t\to\infty} x(t) = 0.$$

The proof is completed. □

References

[1] *A. R. Aftabizadeh and J. Wiener*: Oscillatory and periodic solutions of an equation alternately of retarded and advanced type. Appl. Anal. *23* (1986), 219–231.

[2] *A. R. Aftabizadeh and J. Wiener*: Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument. Appl. Anal. *26* (1988), 327–333.

[3] *S. Busenderg and K. L. Cooke*: Models of vertically transmitted diseases with sequential-continuous dynamics. In: Nonlinear Phenomena in Mathematical Science (V. Lakshmikantham, ed.). Academic Press, New York, 1982, pp. 179–187.

[4] *K. L. Cooke and J. Wiener*: Retarded differential equation with piecewise constant delays. J. Math. Anal. Appl. *99* (1984), 265–297.

[5] *K. L. Cooke and J. Wiener*: An equation alternately of retarded and advanced type. Proc. Amer. Math. Soc. *99* (1987), 726–732.

[6] *K. Miller*: Linear Difference Equations. Benjamin, New York, 1968.

[7] *S. M. Shah and J. Wiener*: Advanced differential equations with piecewise constant argument deviations. Internat. J. Math. Sci. *6* (1983), 671–703.

[8] *J. Wiener and A. R. Aftabizadeh*: Differential equations alternately of retarded and advanced type. J. Math. Anal. Appl. *129* (1988), 243–255.
[9] *J. Wiener and K. L. Cooke*: Oscillations in systems of differential equations with piecewise constant argument. J. Math. Anal. Appl. *137* (1989), 221–239.

Authors' addresses: S. R. Grace, Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12211, Egypt, e-mail: `srgrace@alpha1-eng.cairo.eun.eg`; M. R. S. Kulenović and H. El-Metwally, Department of Mathematics, University of Rhode Island, Kingston, RI 02881, USA, e-mail: `kulenm@math.uri.edu`.