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## 32. Single-device ac circuits. RLC series circuit

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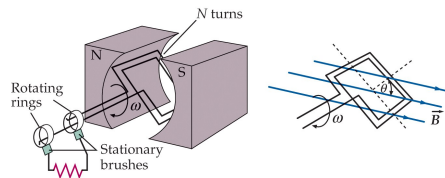
# PHY204 Lecture 32 [r1n32]

## Alternating Current Generator



Coil of  $N$  turns and cross-sectional area  $A$  rotating with angular frequency  $\omega$  in uniform magnetic field  $\vec{B}$ .

- Angle between area vector and magnetic field vector:  $\theta = \omega t$ .
- Flux through coil:  $\Phi_B = NBA \cos(\omega t)$ .
- Induced EMF:  $\mathcal{E} = -\frac{d\Phi_B}{dt} = \mathcal{E}_{max} \sin(\omega t)$  with amplitude  $\mathcal{E}_{max} = NBA\omega$ .
- U.S. household outlet values:
  - $\mathcal{E}_{max} = 120V\sqrt{2} \simeq 170V$
  - $f = 60\text{Hz}$ ,  $\omega = 2\pi f \simeq 377\text{rad/s}$ .



tsl287

This and the next two lectures are about alternating current (ac) circuits. These are circuits driven by EMF sources that deliver a voltage of the form,

$$\mathcal{E}(t) = \mathcal{E}_{max} \sin(\omega t + \phi_0),$$

where  $\mathcal{E}_{max}$  is the amplitude or peak value. The angle  $\phi_0$  depends on the setting of the clock used. The two most frequently used settings are

$$\mathcal{E}(t) = \begin{cases} \mathcal{E}_{max} \sin(\omega t) & : \phi_0 = 0, \\ \mathcal{E}_{max} \cos(\omega t) & : \phi_0 = \pi/2. \end{cases}$$

The slide reviews the process by which alternating EMF voltages are generated, namely by forcing a current loop to rotate in a magnetic field. Here we see Faraday's law at work (see lecture 28).

The slide also states the specs for household outlets in this country. The specs are far from uniform across the globe. Not only do the plugs often not fit, the voltage amplitude is often higher, which may cause harm to appliances and their users when plugged in and turned on without voltage converters.

## Single Device in AC Circuit: Resistor



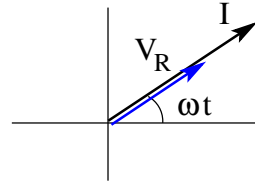
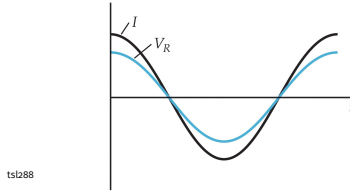
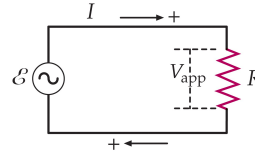
Voltage of ac source :  $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$   
 Current through device:  $I = I_{max} \cos(\omega t - \delta)$

### Resistor

$$V_R = RI = \mathcal{E}_{max} \cos \omega t \Rightarrow I = \frac{\mathcal{E}_{max}}{R} \cos \omega t$$

amplitude:  $I_{max} = \frac{\mathcal{E}_{max}}{R}$ , phase angle:  $\delta = 0$

impedance:  $X_R \equiv \frac{\mathcal{E}_{max}}{I_{max}} = R$  (resistance)



We begin with a discussion of how individual devices respond when connected to an alternating EMF. The responses of a resistor, an inductor, or a capacitor are similar in some aspects, yet different in other aspects.

We first examine the response of a resistor. The ac voltage supplied by the outlet is shown in the top line. Note the particular clock setting. The expression for the current, shown in the next line, is very general and applies to all three devices. What depends on the nature of the device is the amplitude  $I_{max}$  and the phase angle  $\delta$ .

We already know that the current  $I(t)$  through the resistor responds to the voltage  $V_R(t) = \mathcal{E}(t)$  across it by Ohm's law:  $V_R(t) = RI(t)$ . This relationship applied to the general expressions for  $\mathcal{E}(t)$  and  $I(t)$  determines both the current amplitude  $I_{max}$  and the phase angle  $\delta$  (see slide).

The ratio between voltage amplitude and current amplitude in a device or a combination of devices is called impedance. The impedance of a resistor is simply its resistance  $R$ .

Zero phase angle ( $\delta = 0$ ) means that when the voltage has a maximum or a minimum so does the current. Both quantities go through zero at the same time in the same direction (graph at bottom left).

Phasors (graph at bottom right) are a useful representation of oscillating quantities. The length of the phasor represents the amplitude and the projection to the horizontal plane represents the instantaneous values. As a phasor rotates in ccw direction at constant angular velocity, its projection at any instant in time is the value plotted on the diagram (bottom left).



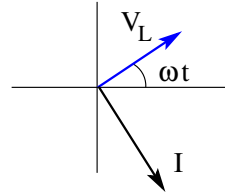
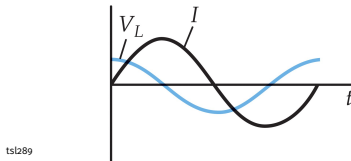
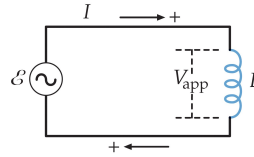
Voltage of ac source :  $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$   
 Current through device:  $I = I_{max} \cos(\omega t - \delta)$

**Inductor**

$$V_L = L \frac{dI}{dt} = \mathcal{E}_{max} \cos \omega t \Rightarrow I = \frac{\mathcal{E}_{max}}{\omega L} \sin(\omega t)$$

amplitude:  $I_{max} = \frac{\mathcal{E}_{max}}{\omega L}$ , phase angle:  $\delta = \frac{\pi}{2}$

impedance:  $X_L \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \omega L$  (inductive reactance)



What changes when we replace the resistor by an inductor? The general expression for voltage and current remain the same.

The response of the current  $I(t)$  to a voltage  $V_L(t) = \mathcal{E}(t)$  is now governed by Faraday's law:  $V_L(t) = L(dI/dt)$ , from which we infer the current amplitude  $I_{max}$  and the phase angle  $\delta$ . Both quantities are now different.

The current phasor now lags behind the voltage phasor as they rotate, which has the consequence that one of the quantities always goes through zero when the other has a maximum or a minimum.

The impedance of the inductor, again defined as the ratio between voltage amplitude and current amplitude, depends on the inductance  $L$  and the angular frequency  $\omega$ . Impedances that do not dissipate energy are called reactances as opposed to resistances that do. Here we have an inductive reactance.

The absence of energy dissipation in inductors has been an earlier topic in the context of current buildup and current shutdown in  $RL$  circuits.

Note the symbol for ac sources in the circuit diagram (top right). We will see this symbol often now.



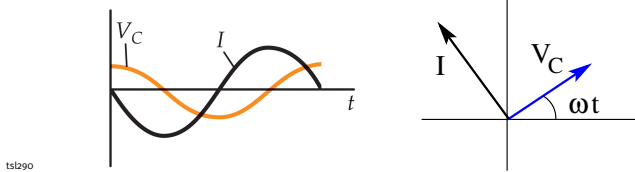
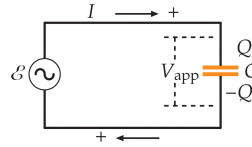
Voltage of ac source :  $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$   
 Current through device:  $I = I_{max} \cos(\omega t - \delta)$

**Capacitor**

$$V_C = \frac{Q}{C} = \mathcal{E}_{max} \cos \omega t \Rightarrow I = \frac{dQ}{dt} = -\omega C \mathcal{E}_{max} \sin(\omega t)$$

amplitude:  $I_{max} = \omega C \mathcal{E}_{max}$ , phase angle:  $\delta = -\frac{\pi}{2}$

$$\text{impedance: } X_C \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \frac{1}{\omega C} \quad (\text{capacitive reactance})$$



We complete this initial single-device analysis with the capacitor. It responds to a voltage across it with a yet different current. Here the current charges or discharges the device.

The voltage  $V_C(t) = \mathcal{E}(t)$  across the capacitor is proportional to the charge  $Q(t)$  on the capacitor,  $V_C(t) = Q(t)/C$ . Given the function  $V_C(t)$  we thus obtain the function  $Q(t)$  directly.

The current  $I(t)$  is the derivative of  $Q(t)$ . Comparison of the results with the general expressions in the top two lines yields the results for the current amplitude and the phase angle.

We see that in this case the current phasor leads the voltage phasor. The two curves (bottom left) are again out of phase by the same amount but with roles interchanged.

When the voltage supplied has a maximum, the current through the capacitor is zero and going negative, whereas the current through the inductor (on the previous page) is zero and going positive.

The impedance of the capacitor depends on the capacitance  $C$  and the angular frequency  $\omega$ . It is a capacitive reactance.

We recall from earlier that there is no energy dissipation in capacitors during charging or discharging. Hence the name reactance for capacitive impedance.

Note that the SI unit of all impedances is Ohm  $[\Omega]$ , irrespective of whether it is a resistance  $R$ , an inductive reactance  $\omega L$ , or a capacitive reactance  $1/\omega C$ .

Note that quantity  $\omega$  has a double role: it is the angular frequency of voltages and currents and it is the angular velocity of the rotational phasor motion.

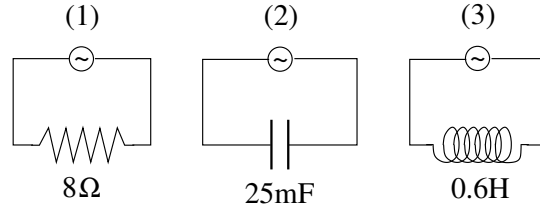
### Single Device in AC Circuit: Application (1)



The ac voltage source  $\mathcal{E} = \mathcal{E}_{max} \sin \omega t$  has an amplitude of  $\mathcal{E}_{max} = 24\text{V}$  and an angular frequency of  $\omega = 10\text{rad/s}$ .

In each of the three circuits, find

- the current amplitude  $I_{max}$ ,
- the current  $I$  at time  $t = 1\text{s}$ .



ts1291

First note the clock setting used: the voltage is zero and increasing at  $t = 0$ . This information is important for part (b) only.

For part (a) we calculate the current amplitude  $I_{max}$  from the voltage amplitude  $\mathcal{E}_{max}$  and the impedance of the device.

$$(1) \quad I_{max} = \frac{\mathcal{E}_{max}}{X_R} = \frac{\mathcal{E}_{max}}{R} = \frac{24\text{V}}{8\Omega} = 3\text{A}.$$

$$(2) \quad I_{max} = \frac{\mathcal{E}_{max}}{X_C} = \frac{\mathcal{E}_{max}}{1/(\omega C)} = \frac{24\text{V}}{4\Omega} = 6\text{A}.$$

$$(3) \quad I_{max} = \frac{\mathcal{E}_{max}}{X_L} = \frac{\mathcal{E}_{max}}{\omega L} = \frac{24\text{V}}{6\Omega} = 4\text{A}.$$

For part (b) we use  $I = I_{max} \sin(\omega t - \delta)$  with  $I_{max}$  from part (a). Your calculator must be set to radians when you evaluate the trigonometric functions.

$$(1) \quad I = I_{max} \sin(10\text{rad}) = (3\text{A})(-0.544) = -1.63\text{A}.$$

$$(2) \quad I = I_{max} \sin(10\text{rad} + \pi/2) = (6\text{A})(-0.839) = -5.03\text{A}.$$

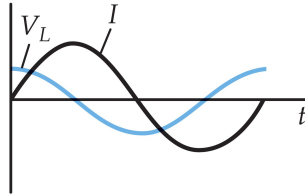
$$(3) \quad I = I_{max} \sin(10\text{rad} - \pi/2) = (4\text{A})(0.839) = 3.36\text{A}.$$

### Single Device in AC Circuit: Application (2)



Consider an ac generator  $\mathcal{E}(t) = \mathcal{E}_{max} \cos(\omega t)$ ,  $\mathcal{E}_{max} = 25\text{V}$ ,  $\omega = 377\text{rad/s}$  connected to an inductor with inductance  $L = 12.7\text{H}$ .

- Find the maximum value of the current.
- Find the current when the emf is zero and decreasing.
- Find the current when the emf is  $-12.5\text{V}$  and decreasing.
- Find the power supplied by the generator at the instant described in (c).



ts1292

Note the different clock setting: the voltage has its maximum value at  $t = 0$ . The general expression for the current in this loop is

$$I(t) = I_{max} \cos(\omega t - \pi/2) = I_{max} \sin(\omega t),$$

where we have used the known phase angle,  $\delta = \pi/2$ , for the inductor.

- (a) We recall that the impedance of an inductor is  $X_L = \omega L$ :

$$I_{max} = \frac{\mathcal{E}_{max}}{X_L} = \frac{\mathcal{E}_{max}}{\omega L} = \frac{25\text{V}}{4788\Omega} = 5.22\text{mA}.$$

(b) The graph, which shows both  $\mathcal{E}(t) = V_C(t)$  and  $I(t)$ , tells us that at the instant in question the current has its maximum value,  $I(t_1) = I_{max}$ . It happens when  $\omega t_1 = \pi/2$ , i.e. at  $t_1 = \pi/2\omega = 4.17\text{ms}$ .

- (c) We use the given voltage function and infer the relevant time  $t_2$  as follows:

$$(25\text{V}) \cos(\omega t_2) = -12.5\text{V} \quad \Rightarrow \quad \cos(\omega t_2) = -\frac{1}{2} \quad \Rightarrow \quad \omega t_2 = \begin{cases} 2.09 \text{ rad} \\ 4.19 \text{ rad} \end{cases}.$$

This angle  $\omega t_2$  is all we need for substitution into the current expression:

$$I(t_2) = I_{max} \sin(\omega t_2) = (5.22\text{mA})(0.866) = 4.52\text{mA}.$$

- (d) Power transfer is the product of voltage and current:

$$P(t_2) = \mathcal{E}(t_2)I(t_2) = (25\text{V}) \underbrace{(\cos(\omega t_2))}_{-0.5} (5.22\text{mA}) \underbrace{(\sin(\omega t_2))}_{0.866} = -56.5\text{mW}.$$

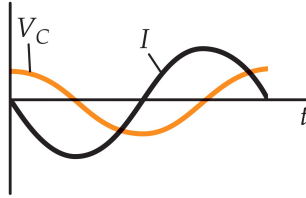
The negative value means that at this instant the inductor releases energy previously stored on it (in the magnetic field).

### Single Device in AC Circuit: Application (3)



Consider an ac generator  $\mathcal{E}(t) = \mathcal{E}_{max} \cos(\omega t)$ ,  $\mathcal{E}_{max} = 25\text{V}$ ,  $\omega = 377\text{rad/s}$  connected to a capacitor with capacitance  $C = 4.15\mu\text{F}$ .

- Find the maximum value of the current.
- Find the current when the emf is zero and decreasing.
- Find the current when the emf is  $-12.5\text{V}$  and increasing.
- Find the power supplied by the generator at the instant described in (c).



ts1293

Here we replace the inductor by a capacitor and answer the same questions again. In the general expression for the current,

$$I(t) = I_{max} \cos(\omega t + \pi/2) = -I_{max} \sin(\omega t),$$

the phase angle is different for the capacitor:  $\delta = -\pi/2$ .

- (a) The impedance of a capacitor is  $X_C = 1/(\omega C)$ :

$$I_{max} = \frac{\mathcal{E}_{max}}{X_C} = \frac{25\text{V}}{639\Omega} = 39.1\text{mA}.$$

(b) The graph, which shows both  $\mathcal{E}(t) = V_L(t)$  and  $I(t)$ , tells us that at the instant in question the current has its minimum value,  $I(t_1) = -I_{max}$ . It happens when  $\omega t_1 = \pi/2$ , i.e. at  $t_1 = \pi/2\omega = 4.17\text{ms}$ .

- (c) We find the relevant time as on the previous page:

$$(25\text{V}) \cos(\omega t_2) = -12.5\text{V} \quad \Rightarrow \quad \cos(\omega t_2) = -\frac{1}{2} \quad \Rightarrow \quad \omega t_2 = \begin{cases} 2.09 \text{ rad} \\ 4.19 \text{ rad} \end{cases}.$$

However, we substitute the result into a different current expression:

$$I(t_2) = -I_{max} \sin(\omega t_2) = -(39.1\text{mA})(-0.866) = 33.9\text{mA}.$$

- (d) Power transfer is the product of voltage and current:

$$P(t_2) = \mathcal{E}(t_2)I(t_2) = (25\text{V}) \underbrace{(\cos(\omega t_2))}_{-0.5} (-39.1\text{mA}) \underbrace{(\sin(\omega t_2))}_{-0.866} = -423\text{mW}.$$

The negative value means that at this instant the capacitor releases energy previously stored on it (in the electric field).





Applied alternating voltage:  $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current:  $I = I_{max} \cos(\omega t - \delta)$

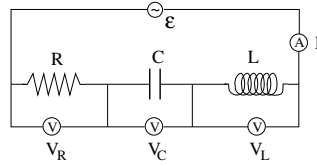
Goals:

- Find  $I_{max}, \delta$  for given  $\mathcal{E}_{max}, \omega$ .
- Find voltages  $V_R, V_L, V_C$  across devices.

Loop rule:  $\mathcal{E} - V_R - V_C - V_L = 0$

Note:

- All voltages are time-dependent.
- In general, all voltages have a different phase.
- $V_R$  has the same phase as  $I$ .



ts1301

Here we begin the analysis of the  $RLC$  series circuit. Given is the ac voltage in the first line. Amplitude  $\mathcal{E}_{max}$  and angular frequency  $\omega$  are known specs.

Also given is the general current expression on the second line. We have used it for single-device circuits, but it holds up for any circuit of resistors, inductors, and capacitors.

It is our task to calculate amplitude  $I_{max}$  and phase angle  $\delta$ . We also wish to calculate the voltages across each device connected to the EMF.

There is only one loop, hence one current. Not counted as a device in the loop is the ammeter, which can always be assumed to have zero impedance (implying zero voltage). The connections to the three voltmeters are not counted as separate loops because they can always be assumed to have infinite impedance (implying zero current).

When we apply the loop rule we must keep in mind that the voltages  $\mathcal{E}(t), V_R(t), V_C(t), V_L(t)$ , across the four devices in the loop have different phases (to be determined).

From the single-device circuits analyzed earlier, we do know the phase relationship between the current  $I(t)$  and each of the voltages  $V_R(t), V_C(t), V_L(t)$ : the current  $I(t)$  is in phase with  $V_R(t)$ , lags behind  $V_L(t)$  by  $\pi/2$  and leads  $V_C(t)$  by  $\pi/2$ .

We also know the ratios of voltage amplitude and current amplitude for the resistor, the capacitor, and the inductor. These ratios are the single-device impedances.

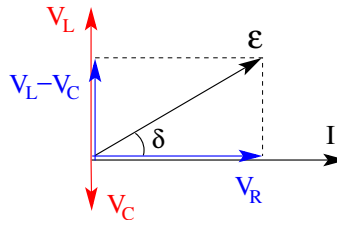
This information enables us to extract the current amplitude  $I_{max}$  and the phase angle  $\delta$  in the general current expression on the second line.



Phasor diagram (for  $\omega t = \delta$ ):

Voltage amplitudes:

- $V_{R,max} = I_{max}X_R = I_{max}R$
- $V_{L,max} = I_{max}X_L = I_{max}\omega L$
- $V_{C,max} = I_{max}X_C = \frac{I_{max}}{\omega C}$



Relation between  $\mathcal{E}_{max}$  and  $I_{max}$  from geometry:

$$\begin{aligned} \mathcal{E}_{max}^2 &= V_{R,max}^2 + (V_{L,max} - V_{C,max})^2 \\ &= I_{max}^2 \left[ R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right] \end{aligned}$$

ts1302

We continue by drawing the phasor diagram as shown for a particular instant in time. At the instant selected, the current assumes its maximum value.

We know the direction of the phasors for  $V_L$ ,  $V_R$ , and  $V_C$  in relation to the phasor for  $I$  (see diagram). We also know the voltage amplitudes in relation to the (yet unknown) current amplitude (see equations to the left). These amplitudes are graphically represented as phasor lengths.

In order that the loop rule is satisfied at all instants in time, we must construct the phasor for the EMF  $\mathcal{E}(t)$  geometrically as a sort of vector sum of voltage phasors as shown. This geometric construction accomplishes what we set as our goal:

- It fixes the ratio between the (known) EMF amplitude and the (unknown) current amplitude.
- It fixes the phase relationship between EMF phasor and current phasor.

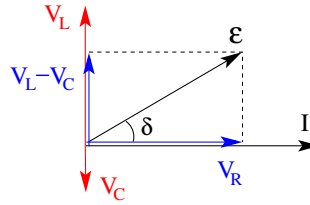
For the first item we use the Pythagorean theorem to calculate the EMF phasor length from the device-voltage phasor lengths (see equation at the bottom of the slide). Each quantity on the right-hand side contains a factor  $I_{max}$ , which we can pull out to arrive at the desired relation. The ratio  $\mathcal{E}_{max}/I_{max}$  is, of course, the impedance of the  $RLC$  series circuit (see next page).



Impedance:  $Z \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \frac{\mathcal{E}_{max}}{Z} = \frac{\mathcal{E}_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$
- $\tan \delta = \frac{V_{L,max} - V_{C,max}}{V_{R,max}} = \frac{\omega L - 1/\omega C}{R}$



Voltages across devices:

- $V_R = RI = RI_{max} \cos(\omega t - \delta) = V_{R,max} \cos(\omega t - \delta)$
- $V_L = L \frac{dI}{dt} = -\omega L I_{max} \sin(\omega t - \delta) = V_{L,max} \cos\left(\omega t - \delta + \frac{\pi}{2}\right)$
- $V_C = \frac{1}{C} \int Idt = \frac{I_{max}}{\omega C} \sin(\omega t - \delta) = V_{C,max} \cos\left(\omega t - \delta - \frac{\pi}{2}\right)$

ts1303

Once we know the impedance  $Z$  of the  $RLC$  series circuit as a function of the circuit specs,  $\omega, R, L, C$ , we can determine the current amplitude  $I_{max}$  from  $Z$  thus calculated and the EMF amplitude  $\mathcal{E}_{max}$ , which also belongs to the circuit specs.

The second item on the slide (repeated from the previous page) gives us the phase angle  $\delta$ , which we need to know for the full current specification. Geometrically, the tangent of  $\delta$  is a ratio of voltage amplitudes as shown. If we pull out the factor  $I_{max}$  in both numerator and denominator, we are left with an expression that depends on the same specs as the impedance.

Mission accomplished! Not entirely, though.

We also wish to calculate the voltages across the resistor, the inductor, and the capacitor as functions of time. This is demonstrated on the last three lines of the slide. Here we use the impedances of single devices as established earlier. The last expression on each line reflects the relative orientation of the voltage phasors shown on the slide.

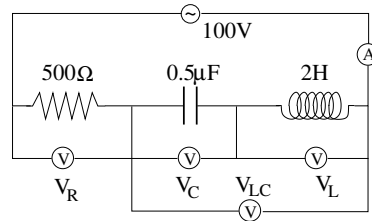
## AC Circuit Application (1)



In this  $RLC$  circuit, the voltage amplitude is  $\mathcal{E}_{max} = 100V$ .

Find the impedance  $Z$ , the current amplitude  $I_{max}$ , and the voltage amplitudes  $V_R, V_C, V_L, V_{LC}$

- (a) for angular frequency is  $\omega = 1000\text{rad/s}$ ,
- (b) for angular frequency is  $\omega = 500\text{rad/s}$ .



ts1304

This is the quiz for lecture 32.

This shows the (straightforward) analysis of a specific  $RLC$  series circuit, run at two different angular frequencies. We work out part (b) and the quiz answers pertain to part (a).

Part (b): It is useful to first determine the single-device impedances:

$$X_R \doteq R = 500\Omega, \quad X_L \doteq \omega L = 1000\Omega, \quad X_C \doteq \frac{1}{\omega C} = 4000\Omega.$$

The impedance  $Z$  and the current amplitude  $I_{max}$  then follow directly:

$$Z = \sqrt{X_R^2 + (X_L - X_C)^2} = 3041\Omega, \quad I_{max} = \frac{\mathcal{E}_{max}}{Z} = 32.9\text{mA}.$$

The voltage amplitudes thus become,

$$V_R^{max} = I_{max}X_R = 16.4V, \quad V_L^{max} = I_{max}X_L = 32.9V,$$

$$V_C^{max} = I_{max}X_C = 131.5V.$$

To answer the last question, we take a look at the phasor diagram on the previous page and recognize that the voltages across inductor and capacitor are opposite in phase. Hence we have

$$V_{LC}^{max} = |V_L^{max} - V_C^{max}| = 98.6V.$$

Part (a): your turn!