11-23-2020

# 29. Inductance and energy stored in inductors. Self-induction. Mutual induction

Gerhard Müller
*University of Rhode Island*, gmuller@uri.edu

Robert Coyne
*University of Rhode Island*, robcoyne@uri.edu

# PHY204 Lecture 29 [rln29]

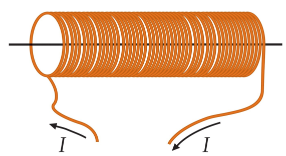
**Inductor and Inductance**

**Inductor** (device):

- A wire that is wound into $N$ turns of some shape and area.
- The current $I$ flowing through the wire generates a magnetic field $\vec{B}$ in its vicinity.
- The magnetic field $\vec{B}$, in turn, produces a magnetic flux $\Phi_B$ through each turn.

**Inductance** (device property):

- Definition: $L = \frac{N\Phi_B}{I}$
- SI unit: 1H = 1Wb/A (one Henry)




tsl265

In this lecture we discuss Faraday's law in action at the level of devices and circuits. The phenomenon of magnetic induction manifests itself in a device called inductor.

Inductors come in a variety of shapes. The solenoid is the prototypical inductor, just as the parallel-plate capacitor is prototypical.

A solenoid is a long wire tightly wound into $N$ turns around a cylinder as shown. Each turn of the wire can be thought of as a loop – a place of action for Faraday's law.

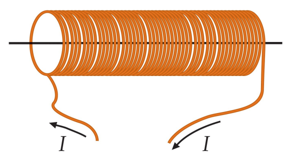
We know from earlier that when a current is sent through the wire, it generates a magnetic field in its vicinity. This magnetic field, in turn, produces a magnetic flux through each loop in the winding.

A time-dependent current $I(t)$ causes a time-dependent magnetic flux $\Phi_B(t)$, which, according th Faraday's law, induces an EMF. We shall examine (a few pages down the line) how the induced EMF depends on the current sent through the inductor.

The relevant device property will turn out to be the inductance $L$ as defined on the slide of this page. Keep in mind that this definition is not limited to solenoids. Take note of the SI unit for inductance: [H]=[Wb/s].

## Inductance of a Solenoid

- $A$: cross-sectional area
- $\ell$: length
- $n$: number of turns per unit length
- $N = n\ell$: total number of turns
- $B = \mu_0 n I$: magnetic field inside solenoid
- $\Phi_B = BA$: magnetic flux through each turn
- $\Rightarrow$ Inductance of solenoid: $L \equiv \frac{N\Phi_B}{I} = \mu_0 n^2 A\ell$




tsl266

Here we calculate the inductance $L$ of a solenoid with given specifications as stated in the first three items. The fourth item combines the previous two for use in the definition of inductance.

The fifth item is the magnetic field $B$ generated inside a long solenoid as calculated (in lecture 25) via Ampère's law and the law of Biot and Savart. The sixth item calculates the magnetic flux through each turn, assuming that the magnetic field inside a long solenoid is uniform. We have justified that assumption earlier.

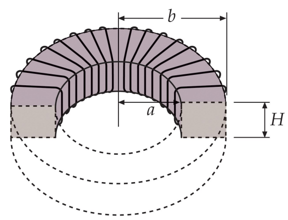
Assembling all these ingredients in the definition of $L$ produces the result shown. For given specifications, this expression produces a number in units of Henry.

The function of an inductor in a circuit is determined its inductance $L$ just as the function of a capacitor is determined by its capacitance $C$ and the function of a resistor by its resistance $R$.

The different functions of the resistor and the capacitor have been highlighted in $RC$ circuits (lecture 18). In the next two lectures, we shall highlight the different functions of resistor, capacitors, and the inductors in the context of $RL$, $LC$, and $RLC$ circuits.

## Inductance of a Toroid

- Total number of turns: $N$
- Magnetic field inside toroid: $B = \frac{\mu_0 I}{2\pi r}$
- Magnetic flux through each turn (loop):
  $\Phi_B = \int_a^b BH\,dr = \frac{\mu_0 INH}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 INH}{2\pi}\ln\frac{b}{a}$
- Inductance: $L \equiv \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 H}{2\pi}\ln\frac{b}{a}$
- Narrow toroid: $s \equiv b - a \ll a$
  $\ln\frac{b}{a} = \ln\left(1+\frac{s}{a}\right) \simeq \frac{s}{a}$
- Inductance: $L = \frac{\mu_0 N^2 (sH)}{2\pi a}$




tsl267

We have discussed earlier that when we bend a solenoid into the shape of a doughnut we get a toroid. In the toroid shown, the turns of wire are rectangular, which simplifies the calculation of $L$ a bit.

The magnetic field $B$ inside a toroid is not uniform (see lecture 25). It depends on the radial distance from the symmetry axis. The calculation of magnetic flux through one (rectangular) turn of the wire thus involves an integration (third item).

Substituting all ingredients into the definition of inductance is easy again and produces the expression shown in the fourth item.

If the width of the rectangular loop is small compared to the radius of the toroid, then the expression for $L$ can be simplified in good approximation as done in the last two items.

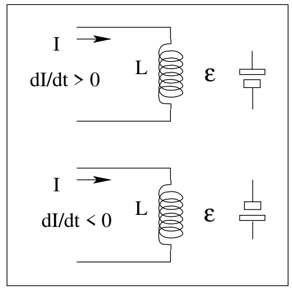
In summary, inductors come in different shapes and sizes. The definition of inductance is common to all. It is a number in units of Henry. For simple designs $L$ can be calculated with good precision. In all cases $L$ can be measured. For that we need to understand what an inductor does to a circuit.

## Self-Induction

The induced electric EMF acts such as to oppose the change in the current that causes it (Lenz's rule).

The presence of an inductance makes the electric current sluggish (resistant to change).

- Faraday's law: $\mathcal{E} = -\frac{d}{dt}(N\Phi_B)$
- Inductance: $L = \frac{N\Phi_B}{I}$
- Self-induced EMF: $\mathcal{E} = -L\frac{dI}{dt}$




tsl268

In an electrical circuit, the inductor is denoted by a coil symbol with a label $L$ as shown on the slide.

What is the relationship between voltage and current in this device? The voltage, in this case, is an induced EMF. Faraday's law is restated in the first item, where $\Phi_B$ is the magnetic flux through one of $N$ turns of the wiring. If we substitute $N\Phi_B$ from the definition of $L$ (second item) into Faraday's law (first item), we end up, in the third item, with the desired relation between voltage and current.

That was easy enough. But watch out. Understanding what self-induced EMF means takes some effort.

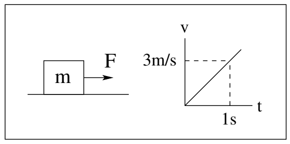
When the current is constant, then its derivative vanishes and so does the induced EMF. In other words, a constant current, no matter what its value or direction is, leaves the inductor invisible as a device.

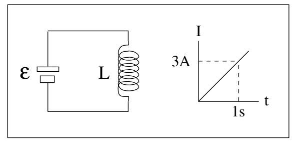
When the current flows in the direction shown and is increasing, then the induced EMF counteracts that increase. That is the meaning of the minus sign. We see the negative feedback of Lenz's rule in action.

Conversely, when the current flows in the same direction but is decreasing, then the induced EMF counteracts the decrease. It pushes the current forward, in a manner of speaking.

The larger the rate of current increase or decrease, the stronger the counteracting induced EMF. Inductors make currents sluggish. In the presence of an inductor, it takes an effort to increase or decrease the current, just it takes an effort to accelerate or decelerate a moving train.

## Inertia: Mechanical vs Electrical




$F = 6\text{N}, \quad m = 2\text{kg}$ $\qquad$ $\mathcal{E} = 6\text{V}, \quad L = 2\text{H}$

Newton's second law

$$F - m\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = \frac{F}{m} = 3\text{m/s}^2$$

Loop rule

$$\mathcal{E} - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L} = 3\text{A/s}$$

tsl520

This page is intended to portray the aspect of inertia associated with a current flowing through an inductor using a mechanical analogy.

On the left you see a block of mass $m$ on a frictionless surface. The block is accelerated from rest by a constant force $F$. Here we have Newton's second law in action. The acceleration, $a = dv/dt$, is inversely proportional to the mass $m$ for a given force $F$. Mass makes the motion sluggish.

On the right we have an inductor connected to a battery. The initial current is zero. The battery forces an increasing current through the inductor. The loop rule, which governs this circuit, has mathematically the same form as Newton's second law. The rate of change of current $dI/dt$ is inversely proportional to the inductance $L$ for a given EMF $\mathcal{E}$. Inductance make the current sluggish.

However, there is more to an inductor than being the cause of inertia for electrical currents.

## Energy Stored in Inductor

Establishing a current in the inductor requires work.
The work done is equal to the potential energy stored in the inductor.

- Current through inductor: $I$ (increasing)
- Voltage induced across inductor: $|\mathcal{E}| = L\dfrac{dI}{dt}$
- Power absorbed by inductor: $P = |\mathcal{E}|I$
- Increment of potential energy: $dU = Pdt = LIdI$
- Potential energy of inductor with current $I$ established:
  $U = L\int_0^I IdI = \frac{1}{2}LI^2$

Q: where is the potential energy stored?
A: in the magnetic field.

tsl269

Earlier in the course, we have learned that the capacitor is a device used for energy storage. When we charge up a capacitor, we add energy in the form of an electric field between the oppositely charged conductors. When the capacitor is discharged, that energy is released to the circuit. The electric field collapses in the process.

Inductors can be used for the same purpose, with the role of the electric field taken over by the magnetic field as we shall see (on the next page).

Establishing a current through an inductor requires work. That work adds potential energy $U$ incrementally. Electric power is the rate at which such work is performed: $P = dU/dt$.

From earlier in the course (resistor cuircuits) we know that the rate at which energy is processed in a device is equal to the product of the current through the device and the voltage across the device: $P = |\mathcal{E}|I$. The slide sets aside minus signs. Power absorbed increases the energy.

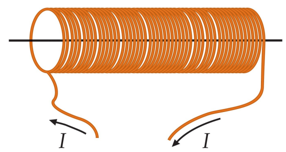
The relevant voltage in this instance is due to self-induction. This brings us to a relation between increments of energy, $dU$, and increments of current, $dI$. That relation is readily integrated as shown.

When we establish a current $I$ through an inductor, the amount $U = \frac{1}{2}LI^2$ of energy is being stored on the inductor, just as when we charge up a capacitor to a voltage $V$, energy $U = \frac{1}{2}CV^2$ is being stored on the capacitor.

## Energy Density Within Solenoid

Energy is stored in the magnetic field inside the solenoid.

- Inductance: $L = \mu_0 n^2 A\ell$
- Magnetic field: $B = \mu_0 nI$
- Potential energy: $U = \frac{1}{2}LI^2 = \frac{1}{2\mu_0}B^2(A\ell)$
- Volume of solenoid interior: $A\ell$
- Energy density of magnetic field: $u_B = \frac{U}{A\ell} = \frac{1}{2\mu_0}B^2$




tsl270

We use the solenoid design for a demonstration that the energy stored on the inductor is located in the magnetic field $B$.

For that purpose we use the expression for inductance derived earlier and the expression for the magnetic field derived even earlier than that.

Upon substitution of $L$ and $I$ from these relations into the expression for energy stored on the inductor derived on the previous page, we arrive at a result for the stored energy $U$ that is a function of the magnetic field $B$.

Since we know that the magnetic field inside a solenoid is uniform, we can infer a magnetic energy density as the ratio of magnetic energy and the volume of the region that contains the magnetic field as done in the last item. This last expression is quite general, not limited to inductors.

| device | Capacitor | Inductor |
|---|---|---|
| device property | $C = \frac{Q}{V}$ [F] | $L = \frac{N\Phi_B}{I}$ [H] |
| energy | $U = \frac{Q^2}{2C}$ [J] | $L = \frac{1}{2}LI^2$ [J] |
| energy density | $u_E = \frac{1}{2}\epsilon_0 \lvert\vec{E}\rvert^2$ [J/m$^2$] | $u_B = \frac{1}{2\mu_0}\lvert\vec{B}\rvert^2$ [J/m$^2$] |

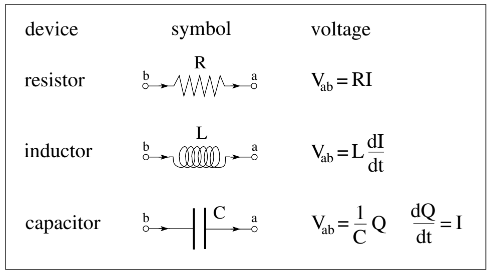
tsl608

This table summarizes important aspects of the capacitor and the inductor, two devices that store and release energies in circuits. They are called reactive devices as opposed to resistive devices, which dissipate energy such as the resistor does.

The last row in the table emphasizes the form of energy in the two devices: electrical energy on the capacitor and magnetic energy on the inductor. Keep in mind that energy is convertible. We shall see how electric energy transforms into magnetic energy and back into electric energy.

Wherever there is a magnetic field $\vec{B}$, there is energy with energy density $u_B$. Likewise, wherever there is an electric field $\vec{E}$, there is energy with energy density $u_E$. Note that $\vec{E}$ and $\vec{B}$ are vector fields. Each component, $E_x, E_y, E_z$ and $B_x, B_y, B_z$ is a function of position $x, y, z$. The energy densities $u_E$ and $u_B$, by contrast, are scalar fields. They have no directions but still are functions of position $x, y, z$.

## Response of Device to Electric Current




Note that that the response of each device to a current $I$ is a voltage $V_{ab} \equiv V_b - V_a$.

- resistor: response proportional to current itself
- inductor: response proportional to derivative of current
- capacitor: response proportional to integral of current

tsl277

Devices are elements of electrical circuits. The three devices discussed here have two terminals each. A current $I$ can flow through each device, meaning that some amount charge enters one terminal and an equal amount exits the other terminal.

This is obvious for the resistor and the inductor. In the case of a capacitor, when some amount of positive charge accumulates on one plate, an equal amount of negative charge accumulates on the opposite plate, implying effectively that an equal amount of positive charge goes off the opposite plate. What actually moves depends on the nature of the conductor.

Each device responds to a current $I$ that flows through it with a voltage $V_{ab}$ across it. That response is qualitatively different in each device.

The response of the resistor is characterized by Ohm's law and that of the inductor by Faraday's law. In the first device, the voltage is proportional to the current, in the second device proportional to the derivative of the current.

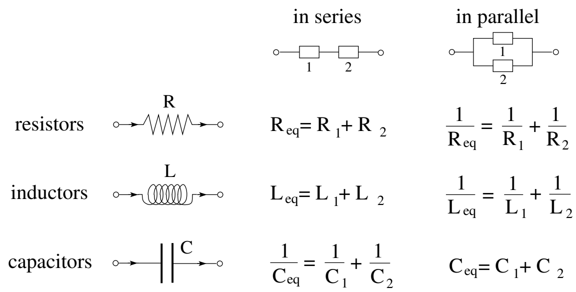
The capacitor responds with a voltage proportional to the charge on it. The instantaneous charge $Q(t)$ is an integral of the current $I(t)$:

$$Q(t) = \int_{t_0}^{t} I(t')dt' \quad \Rightarrow \quad \frac{d}{dt}Q(t) = I(t).$$

Here $t_0$ is a moment in time when there is no charge on the capicitor. When you take the derivative with respect to $t$ in the expression for $Q(t)$, you effectively undo the integral.

A steady current flowing through a resistor produces a steady voltage across it. A steady current flowing through an inductor produces no voltage. No steady current current can flow through a capacitor for more than a short time. The device charges up and block further current in the same direction.

## Devices in Series or in Parallel




tsl521

Earlier in this course, we discussed equivalent resistances and equivalent capacitance of devices connected in series or in parallel. The above slide gives you a summary of the rules including those applicable to inductors.

It may strike you as odd that the rules are the same for resistors and inductors but different for capacitors. The different rules are indeed a source of confusion and mistakes.

The difference is an unfortunate consequence of the way device properties have been named. Yet there are very good reasons for the naming.

If we were to use the quantity $\Gamma = 1/C$ instead of the $C$ to characterize capacitors, the rules for $\Gamma$ would be the same as those for $R$ and $L$. The quantity $\Gamma$ would have to be (awkwardly) named inverse capacitance or something else that triggers the same association. The name "elastance" has been used.

## Mutual Induction

$\Phi_{12}$: magnetic flux through each loop of coil 1 caused by current $I_2$ through coil 2

$\Phi_{21}$: magnetic flux through each loop of coil 2 caused by current $I_1$ through coil 1

$M_{12} = \frac{N_2 \Phi_{12}}{I_2}$ (mutual inductance)

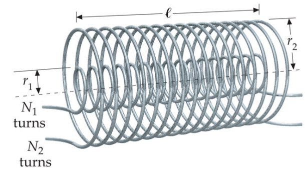
$M_{21} = \frac{N_1 \Phi_{21}}{I_1}$ (mutual inductance)

$\mathcal{E}_1 = -M_{12}\frac{dI_2}{dt}$ (emf induced in coil 2 due to current in coil 1)

$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}$ (emf induced in coil 2 due to current in coil 1)

$M_{12} = M_{21} = M$ (symmetry property)

$M = \mu_0 \frac{N_1}{\ell}\frac{N_2}{\ell}(\ell \pi r_1^2)$ (present configuration)




(a)

tsl414

Mutual inductance, as opposed to self induction, covers an important set of applications of Faraday's law. The slide shows a setup with two coaxial solenoids, for which the effect is easily demonstrated.

If a time-dependent current $I_1(t)$ is being sent through the inner coil, it produces a time-dependent magnetic flux through each turn of both coils. The flux $\Phi_{11}$ would matter for self induction. Here we are interested in the flux $\Phi_{21}$, which matters for mutual induction.

The slide gives you the definition for mutual inductance and the expression for the induced EMF $\mathcal{E}_2$ in coil 2 due to a time-dependent current $I_1(t)$ in coil 1.

It also works the other way around with the same result. The mutual inductances $M_{12}$ and $M_{21}$ are identical. The result for the coaxial-solenoid configuration is shown on the last line.

Mutual induction is at work in induction coils of car engines. Turning on the ignition makes the car battery drive a current through one of the solenoids, which builds up a magnetic field in good time. Interrupting that induction current, causes a steep rate at which the current collapses. The magnetic field collapses at a similar rate, which produces a high voltage.

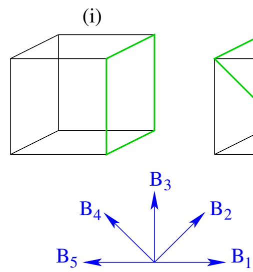
High voltages – much higher than the battery can deliver directly – are needed for some functions of a car engine, e.g. to produce sparks in the cylinders.

## Magnetic Induction: Application (12)

Consider two conducting loops (i) and (ii) (indicated by green lines in cubes of sides $L = 2\text{m}$). Each loop is placed in a region of uniform magnetic field with linearly increasing magnitude, $B(t) = bt, b = 2\text{T/s}$, and one of the five directions indicated.

(a) Find the magnetic flux through each loop as produced by each field.

(b) Find the magnitude and direction of the emf induced by each field in each loop.




tsl263

This is the quiz for lecture 29.

For this quiz, consider only configuration (i).

For part (a) we are only asking you to state if the fluxes $\Phi_B^{(1)}, \ldots, \Phi_B^{(5)}$ are positive, zero, or negative if we assume the convention that the area vector $\vec{A}$ is pointing $\rightarrow$.

For part (b) I we are asking if the direction of the induced currents $I_1, \ldots, I_5$ is cw, zero, or ccw.