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Myron K. Grammatikopoulos

G. Ladas

University of Rhode Island

Yiannis G. Sficas

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Myron K. Grammatikopoulos; Gerasimos Ladas; Yiannis G. Sficas
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NECESSARY AND SUFFICIENT CONDITIONS
FOR OSCILLATION OF DELAY EQUATIONS
WITH CONSTANT COEFFICIENTS

M. K. GRAMMATIKOPOULOS, Ioannina, G. LADAS, Kingston and Y. G. SFICAS, Ioannina

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1. INTRODUCTION

Our aim in this paper is to obtain a necessary and sufficient condition under which all solutions of the delay differential equation (DDE)

$$(1) \quad x'(t) + px(t - \tau) + qx(t - \sigma) = 0,$$

oscillate. Here the coefficients p and q are assumed to be real numbers and the delays τ and σ are nonnegative real numbers.

For delay differential equations with positive coefficients of the form

$$x'(t) + \sum_{i=1}^n p_i x(t - \tau_i) = 0,$$

necessary and sufficient conditions were obtained by Trnov [3]. See also Ladas, Sficas and Stavroulakis [2].

Our main result is the following theorem.

Theorem. *Consider the DDE (1). Assume that the coefficients p and q are real numbers and the delays τ and σ are nonnegative real numbers. Then the following statements are equivalent:*

- (a) *All solutions of Eq. (1) oscillate.*
- (b) *The characteristic equation*

$$(2) \quad \lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma} = 0$$

of Eq. (1) has no real roots.

The importance in the present result is that the coefficients of Eq. (1) are not restricted to be positive.

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large t .

2. PRELIMINARY RESULTS

When $\tau = \sigma$, Eq. (1) reduces to an equation with one delay. Also, when $\sigma = 0$, the transformation

$$x(t) = y(t) e^{-qt}$$

reduces Eq. (1) to an equation with one delay. Now, for equations with one delay of the form

$$(1') \quad y'(t) + ry(t - \mu) = 0,$$

where r is a real number and μ is a positive real number, it is known that every solution of Eq. (1') oscillates if and only if its characteristic equation

$$\lambda + re^{-\lambda\mu} = 0$$

has no real roots. This result follows from [2] or by observing that

$$\min_{\lambda \in \mathbb{R}} (\lambda + re^{-\lambda\mu}) = \frac{1}{\mu} \ln(r\mu e)$$

and that, as it is known from [1], the condition

$$r\mu e > 1$$

is sufficient for all solutions of Eq. (1') to oscillate.

Hence, in the sequel, without loss of generality, we will assume that the delays τ and σ are such that

$$(3) \quad \tau > \sigma > 0.$$

Set

$$F(\lambda) = \lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma}$$

and assume that $F(\lambda)$ has no real roots. As $F(+\infty) = +\infty$, it follows that

$$F(\lambda) > 0 \quad \text{for every } \lambda \in \mathbb{R}.$$

In particular,

$$(4) \quad F(0) = p + q > 0.$$

Also

$$F(-\infty) = +\infty,$$

which implies that

$$(5) \quad p > 0.$$

Finally,

$$(6) \quad m \equiv \min_{\lambda \in \mathbb{R}} (\lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma}) > 0.$$

The following lemma summarizes the above observations.

Lemma 1. *Assume that (3) holds. Then the inequalities (4), (5), and (6) are necessary conditions for Eq. (2) to have no real roots.*

Let $x(t)$ be a solution of Eq. (1) and set

$$(7) \quad z(t) = x(t) - p \int_{t-\tau}^{t-\sigma} x(s) ds.$$

Then we have the following result.

Lemma 2. *Assume that $x(t)$ is a solution of Eq. (1). Then $z(t)$ is also a solution of Eq. (1).*

Proof. From (7) and Eq. (1) we find that

$$(8) \quad z'(t) = x'(t) - p[x(t - \sigma) - x(t - \tau)] = -(p + q)x(t - \sigma)$$

and so

$$pz'(s + \sigma) = -p(p + q)x(s).$$

Integrating this equation from $t - \tau$ to $t - \sigma$ and using (7) and (8), we obtain

$$p[z(t) - z(t + \sigma - \tau)] = (p + q)[z(t) - x(t)] = (p + q)z(t) + z'(t + \sigma)$$

or equivalently

$$z'(t) + pz(t - \tau) + qz(t - \sigma) = 0.$$

The proof is complete.

The following lemma describes the asymptotic behavior of the function $z(t)$ as $t \rightarrow \infty$.

Lemma 3. *Consider the DDE (1) and assume that*

$$(9) \quad \tau > \sigma > 0, \quad p + q > 0, \quad \text{and} \quad p > 0.$$

Let $x(t)$ be an eventually positive solution of Eq. (1) and define $z(t)$ as given by (7). Then the following statements are true:

(a) *Assume that*

$$p(\tau - \sigma) \leq 1.$$

Then $z(t)$ is an eventually positive and decreasing solution of Eq. (1).

(b) *Assume that*

$$p(\tau - \sigma) > 1.$$

Set

$$w(t) = -z(t).$$

Then $w(t)$ is an eventually positive and increasing solution of Eq. (1).

Proof. Let t_0 be such that $x(t) > 0$ for $t \geq t_0$. Then:

(a) From Lemma 2 we know that $z(t)$ is a solution of Eq. (1) and from (8) we see that $z(t)$ is a decreasing function of t . To prove that $z(t)$ is positive, it suffices to show that

$$(10) \quad \lim_{t \rightarrow \infty} z(t) = 0.$$

First, we claim that $\lim_{t \rightarrow \infty} z(t)$ is a finite number. Otherwise,

$$\lim_{t \rightarrow \infty} z(t) = -\infty$$

which implies that $z(t)$ is eventually negative and $x(t)$ is unbounded. Hence, there exists a $t_1 \geq t_0 + \max\{\tau, \sigma\}$ such that

$$z(t_1) < 0 \quad \text{and} \quad x(t_1) = \max_{t_0 \leq s \leq t_1} x(s)$$

It follows, from (7), that

$$0 > z(t_1) = x(t_1) - p \int_{t_1 - \tau}^{t_1 - \sigma} x(s) ds \geq x(t_1) [1 - p(\tau - \sigma)] \geq 0.$$

This contradiction establishes our claim that

$$l \equiv \lim_{t \rightarrow \infty} z(t)$$

is finite.

Integrating both sides of (8) from t_0 to t and letting $t \rightarrow \infty$, we see that

$$l - z(t_0) = -(p + q) \int_{t_0}^{\infty} x(s - \sigma) ds$$

which shows that $x \in L^1[t_0, \infty)$. From Eq. (1), it follows that $x' \in L^1[t_0, \infty)$. Hence, $\lim_{t \rightarrow \infty} x(t)$ exists and it has to be zero (because $x \in L^1[t_0, \infty)$). Thus, $\lim_{t \rightarrow \infty} x(t) = 0$ and, from (7), we conclude that (10) holds.

(b) From Lemma 2 and the linearity of Eq. (1), it follows that $w(t)$ is a solution of Eq. (1). From (7), we also see that

$$(11) \quad w'(t) = (p + q)x(t - \sigma) > 0$$

and so $w(t)$ is an increasing function of t . To show that $w(t)$ is eventually positive, it suffices to prove that

$$(12) \quad \lim_{t \rightarrow \infty} w(t) = +\infty.$$

Otherwise,

$$\lim_{t \rightarrow \infty} w(t) \equiv l$$

exists and is finite. Integrating both sides of (11) from t_0 to t and letting $t \rightarrow \infty$, we find

$$l - w(t_0) = (p + q) \int_{t_0}^{\infty} x(s - \sigma) ds$$

which shows that $x \in L^1[t_0, \infty)$. As in the proof of part (a), we conclude that $l = 0$. Therefore, $w(t)$ increases to zero as $t \rightarrow \infty$, which implies that $w(t) < 0$. Thus, there exists a $t_1 \geq t_0 + \max\{\tau, \sigma\}$ such that

$$w(t_1) < 0 \quad \text{and} \quad x(t_1) = \min_{t_0 \leq s \leq t_1} x(s).$$

It follows, from (7), that

$$0 > w(t_1) = -z(t_1) = -x(t_1) + p \int_{t_1-\tau}^{t_1-\sigma} x(s) ds \geq x(t_1) [-1 + p(\tau - \sigma)] > 0.$$

This contradiction establishes (12) and the proof is complete.

3. PROOF OF MAIN RESULT

(a) \Rightarrow (b). Otherwise, Eq. (2) has a real root λ_0 . Then $x(t) = e^{\lambda_0 t}$ is a nonoscillatory solution of Eq. (1) which contradicts our assumption.

(b) \Rightarrow (a). We may (and do) assume that (3) holds. Since Eq. (2) has no real roots, the inequalities (4), (5), and (6) are satisfied.

Next, we distinguish the following cases.

Case 1. $p(\tau - \sigma) \leq 1$. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $x(t)$. Setting

$$z(t) = x(t) - p \int_{t-\tau}^{t-\sigma} x(s) ds,$$

we know, by Lemma 3(a), that $z(t)$ is a positive and decreasing solution of Eq. (1). Furthermore, from (8), we have

$$z'(t) + (p + q)x(t - \sigma) = 0, \quad t \geq t_0.$$

Set

$$z_0(t) = z(t)$$

and

$$(13) \quad z_n(t) = z_{n-1}(t) - p \int_{t-\tau}^{t-\sigma} z_{n-1}(s) ds, \quad n = 1, 2, \dots$$

Then, for each $n = 1, 2, \dots$, the function $z_n(t)$ is also a positive and decreasing solution of Eq. (1) such that

$$(14) \quad z'_n(t) + (p + q)z_{n-1}(t - \sigma) = 0, \quad t \geq t_0.$$

For each $n = 1, 2, \dots$, we define the set

$$A(z_n) = \{\lambda > 0: z'_n(t) + \lambda z_n(t) \leq 0\}.$$

The proof will be accomplished by proving that $A(z_n)$ has the following contradictory properties:

(i) For each $n = 1, 2, \dots$, the set $A(z_n)$ is nonempty and bounded above by a number independent of n .

(ii) $\lambda \in A(z_n) \Rightarrow (\lambda + m) \in A(z_{n+1})$, $n = 1, 2, \dots$, where m is the positive number defined by (6).

Clearly,

$$z_n(t) \leq z_{n-1}(t) \leq z_{n-1}(t - \sigma)$$

and (14) yields

$$z'_n(t) + (p + q) z_n(t) \leq 0.$$

Therefore,

$$(p + q) \in \Lambda(z_n)$$

which proves that $\Lambda(z_n) \neq \emptyset$.

Next, we prove that $\Lambda(z_n)$ is bounded above by a number independent of n . Indeed, integrating both sides of (14) from $t - \sigma$ to t , we find

$$z_n(t) - z_n(t - \sigma) + (p + q) \int_{t-\sigma}^t z_{n-1}(s - \sigma) ds = 0$$

which implies that

$$-z_n(t - \sigma) + \sigma(p + q) z_{n-1}(t - \sigma) < 0$$

and hence

$$(15) \quad z_{n-1}(t) < \frac{1}{\sigma(p + q)} z_n(t), \quad n = 1, 2, \dots$$

From (7) we find that

$$z_n(t) \leq z_{n-1}(t) - p(\tau - \sigma) z_{n-1}(t - \sigma)$$

and so, using (15), we obtain

$$p(\tau - \sigma) z_{n-1}(t - \sigma) \leq z_{n-1}(t) - z_n(t) < \frac{1}{\sigma(p + q)} z_n(t).$$

Therefore,

$$(16) \quad (p + q) z_{n-1}(t - \sigma) < A z_n(t), \quad n = 1, 2, \dots,$$

where

$$A = \frac{1}{\sigma p(\tau - \sigma)}.$$

Using (16) in (14), we conclude that

$$z'_n(t) + A z_n(t) > 0$$

which proves that

$$A \notin \Lambda(z_n)$$

and so $\Lambda(z_n)$ is bounded from above by A .

Next, we will prove (ii). Let $\lambda \in \Lambda(z_n)$ and set

$$z_n(t) = e^{-\lambda t} \phi_n(t).$$

Then

$$\phi'_n(t) = e^{\lambda t} [z'_n(t) + \lambda z_n(t)] \leq 0$$

and, from (7), we see that

$$(17) \quad z_{n+1}(t) = e^{-\lambda t} \phi_n(t) - p \int_{t-\tau}^{t-\sigma} e^{-\lambda s} \phi_n(s) ds \leq$$

$$\begin{aligned} &\leq e^{-\lambda t} \phi_n(t) - \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) e^{-\lambda t} \phi_n(t - \sigma) \leq \\ &\leq \left[1 - \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) \right] e^{-\lambda t} \phi_n(t - \sigma). \end{aligned}$$

Also, from (14), we find

$$(18) \quad z'_{n+1}(t) = -(p + q) e^{\lambda t} e^{-\lambda t} \phi_n(t - \sigma).$$

Hence, from (17) and (18), we have

$$\begin{aligned} &z'_{n+1}(t) + (\lambda + m) z_{n+1}(t) \leq \\ &\leq \left[-(p + q) e^{\lambda t} + (\lambda + m) - (\lambda + m) \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) \right] e^{-\lambda t} \phi_n(t - \sigma) \leq \\ &\leq (p + q) (e^{\lambda \sigma} - e^{\lambda \tau}) e^{-\lambda t} \phi_n(t - \sigma) < 0 \end{aligned}$$

which proves that

$$(\lambda + m) \in \Lambda(z_{n+1})$$

and completes the proof in this case.

Case 2. $p(\tau - \sigma) > 1$. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $y(t)$. Setting

$$w(t) = -z(t) = -x(t) + p \int_{t-\tau}^{t-\sigma} x(s) ds,$$

by Lemma 3(b), we conclude that $w(t)$ is an eventually positive and increasing solution of Eq. (1) such that

$$w'(t) - (p + q) x(t - \sigma) = 0, \quad t \geq t_0.$$

Set

$$w_0(t) = w(t)$$

and

$$(19) \quad w_n(t) = -w_{n-1}(t) + p \int_{t-\tau}^{t-\sigma} w_{n-1}(s) ds, \quad n = 1, 2, \dots$$

Then, for each $n = 1, 2, \dots$, the function $w_n(t)$ is also an eventually positive and increasing solution of Eq. (1) and such that

$$(20) \quad w'_n(t) - (p + q) w_{n-1}(t - \sigma) = 0, \quad t \geq t_0.$$

For each $n = 1, 2, \dots$, we define the set

$$M(w_n) = \{ \mu > 0: w'_n(t) - \mu w_n(t) \geq 0 \}.$$

We also set

$$\mu_0 = \frac{p + q}{p(\tau - \sigma)} \quad \text{and} \quad m_0 = \frac{m\mu_0}{p} e^{\mu_0 \sigma}.$$

The proof will be accomplished by proving that $M(w_n)$ has the following contradictory properties:

(i) For each $n = 1, 2, \dots$

$$\mu_0 \in M(w_n) \quad \text{and} \quad -q \notin M(w_n),$$

that is, the set $M(w_n)$ is nonempty and bounded from above by a number independent of n .

(ii) For every $\mu \in M(w_n)$, with $\mu > \mu_0$,

$$(\mu + m_0) \in M(w_{n+1}), \quad n = 1, 2, \dots$$

First, we will prove (i). From (19), we see that for each $n = 1, 2, \dots$

$$w_n(t) \leq p(\tau - \sigma) w_{n-1}(t - \sigma)$$

and, using this in (20), we find that

$$w'_n(n) - \frac{p+q}{p(\tau-\sigma)} w_n(t) \geq 0.$$

Hence

$$\mu_0 = \frac{p+q}{p(\tau-\sigma)} \in M(w_n), \quad n = 1, 2, \dots$$

Now, from (20) and the fact that $w_n(t)$ is a solution of Eq. (1), we have

$$(p+q) w_{n-1}(t - \sigma) = w'_n(t) = -p w_n(t - \tau) - q w_n(t - \sigma)$$

which implies that $q < 0$ and

$$(p+q) w_{n-1}(t - \sigma) < -q w_n(t - \sigma) < -q w_n(t).$$

Using this in (20), we find that

$$w'_n(t) - (-q) w_n(t) < 0$$

which proves that

$$-q \notin M(w_n), \quad n = 1, 2, \dots$$

Finally, we will prove (ii). Let $\mu \in M(w_n)$ with $\mu \geq \mu_0$, and set

$$w_n(t) = e^{\mu t} \psi_n(t).$$

Then

$$\psi'_n(t) = e^{-\mu t} [w'_n(t) - \mu w_n(t)] \geq 0$$

and, from (19), we find that

$$\begin{aligned} (21) \quad w_{n+1}(t) &= -e^{\mu t} \psi_n(t) + p \int_{t-\tau}^{t-\sigma} e^{\mu s} \psi_n(s) ds \leq \\ &\leq -e^{\mu t} \psi_n(t) + \frac{p}{\mu} (e^{-\mu \sigma} - e^{-\mu \tau}) e^{\mu t} \psi_n(t - \sigma) \leq \\ &\leq e^{\mu t} \psi_n(t - \sigma) \left[-1 + \frac{p}{\mu} (e^{-\mu \sigma} - e^{-\mu \tau}) \right]. \end{aligned}$$

Also, from (20), we have

$$(22) \quad w'_{n+1}(t) = (p + q) e^{-\mu\sigma} e^{\mu t} \psi_n(t - \sigma).$$

Hence, from (21) and (22), we obtain

$$\begin{aligned} & w'_{n+1}(t) - (\mu + m_0) w_{n+1}(t) \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[(p + q) e^{-\mu\sigma} + (\mu + m_0) - (\mu + m_0) \frac{p}{\mu} (e^{-\mu\sigma} - e^{-\mu\tau}) \right] \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[(\mu + p e^{-\mu\tau} + q e^{-\mu\sigma}) + m_0 \left(1 - \frac{p}{\mu} e^{-\mu\sigma} \right) \right] \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[m + m_0 \left(1 - \frac{p}{\mu_0 e^{\mu_0 \sigma}} \right) \right] = e^{\mu t} \psi_n(t - \sigma) m_0 \geq 0 \end{aligned}$$

which implies that

$$(\mu + m_0) \in M(w_{n+1}), \quad n = 1, 2, \dots$$

The proof of the theorem is complete.

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Authors' addresses: M. K. Grammatikopoulos, G. Ladas, Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881, USA; M. K. Grammatikopoulos, Y. G. Sficas, Department of Mathematics, University of Ioannina, Ioannina 45332, Greece.