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## Necessary and sufficient conditions for oscillation of delay equations with constant coefficients

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NECESSARY AND SUFFICIENT CONDITIONS  
FOR OSCILLATION OF DELAY EQUATIONS  
WITH CONSTANT COEFFICIENTS

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1. INTRODUCTION

Our aim in this paper is to obtain a necessary and sufficient condition under which all solutions of the delay differential equation (DDE)

$$(1) \quad x'(t) + px(t - \tau) + qx(t - \sigma) = 0,$$

oscillate. Here the coefficients  $p$  and  $q$  are assumed to be real numbers and the delays  $\tau$  and  $\sigma$  are nonnegative real numbers.

For delay differential equations with positive coefficients of the form

$$x'(t) + \sum_{i=1}^n p_i x(t - \tau_i) = 0,$$

necessary and sufficient conditions were obtained by Trnov [3]. See also Ladas, Sficas and Stavroulakis [2].

Our main result is the following theorem.

**Theorem.** *Consider the DDE (1). Assume that the coefficients  $p$  and  $q$  are real numbers and the delays  $\tau$  and  $\sigma$  are nonnegative real numbers. Then the following statements are equivalent:*

- (a) *All solutions of Eq. (1) oscillate.*
- (b) *The characteristic equation*

$$(2) \quad \lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma} = 0$$

*of Eq. (1) has no real roots.*

The importance in the present result is that the coefficients of Eq. (1) are not restricted to be positive.

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large  $t$ .

## 2. PRELIMINARY RESULTS

When  $\tau = \sigma$ , Eq. (1) reduces to an equation with one delay. Also, when  $\sigma = 0$ , the transformation

$$x(t) = y(t) e^{-qt}$$

reduces Eq. (1) to an equation with one delay. Now, for equations with one delay of the form

$$(1') \quad y'(t) + ry(t - \mu) = 0,$$

where  $r$  is a real number and  $\mu$  is a positive real number, it is known that every solution of Eq. (1') oscillates if and only if its characteristic equation

$$\lambda + re^{-\lambda\mu} = 0$$

has no real roots. This result follows from [2] or by observing that

$$\min_{\lambda \in \mathbb{R}} (\lambda + re^{-\lambda\mu}) = \frac{1}{\mu} \ln(r\mu e)$$

and that, as it is known from [1], the condition

$$r\mu e > 1$$

is sufficient for all solutions of Eq. (1') to oscillate.

Hence, in the sequel, without loss of generality, we will assume that the delays  $\tau$  and  $\sigma$  are such that

$$(3) \quad \tau > \sigma > 0.$$

Set

$$F(\lambda) = \lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma}$$

and assume that  $F(\lambda)$  has no real roots. As  $F(+\infty) = +\infty$ , it follows that

$$F(\lambda) > 0 \quad \text{for every } \lambda \in \mathbb{R}.$$

In particular,

$$(4) \quad F(0) = p + q > 0.$$

Also

$$F(-\infty) = +\infty,$$

which implies that

$$(5) \quad p > 0.$$

Finally,

$$(6) \quad m \equiv \min_{\lambda \in \mathbb{R}} (\lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma}) > 0.$$

The following lemma summarizes the above observations.

**Lemma 1.** *Assume that (3) holds. Then the inequalities (4), (5), and (6) are necessary conditions for Eq. (2) to have no real roots.*

Let  $x(t)$  be a solution of Eq. (1) and set

$$(7) \quad z(t) = x(t) - p \int_{t-\tau}^{t-\sigma} x(s) \, ds.$$

Then we have the following result.

**Lemma 2.** *Assume that  $x(t)$  is a solution of Eq. (1). Then  $z(t)$  is also a solution of Eq. (1).*

*Proof.* From (7) and Eq. (1) we find that

$$(8) \quad z'(t) = x'(t) - p[x(t - \sigma) - x(t - \tau)] = -(p + q)x(t - \sigma)$$

and so

$$pz'(s + \sigma) = -p(p + q)x(s).$$

Integrating this equation from  $t - \tau$  to  $t - \sigma$  and using (7) and (8), we obtain

$$p[z(t) - z(t + \sigma - \tau)] = (p + q)[z(t) - x(t)] = (p + q)z(t) + z'(t + \sigma)$$

or equivalently

$$z'(t) + pz(t - \tau) + qz(t - \sigma) = 0.$$

The proof is complete.

The following lemma describes the asymptotic behavior of the function  $z(t)$  as  $t \rightarrow \infty$ .

**Lemma 3.** *Consider the DDE (1) and assume that*

$$(9) \quad \tau > \sigma > 0, \quad p + q > 0, \quad \text{and} \quad p > 0.$$

*Let  $x(t)$  be an eventually positive solution of Eq. (1) and define  $z(t)$  as given by (7). Then the following statements are true:*

(a) *Assume that*

$$p(\tau - \sigma) \leq 1.$$

*Then  $z(t)$  is an eventually positive and decreasing solution of Eq. (1).*

(b) *Assume that*

$$p(\tau - \sigma) > 1.$$

*Set*

$$w(t) = -z(t).$$

*Then  $w(t)$  is an eventually positive and increasing solution of Eq. (1).*

*Proof.* Let  $t_0$  be such that  $x(t) > 0$  for  $t \geq t_0$ . Then:

(a) From Lemma 2 we know that  $z(t)$  is a solution of Eq. (1) and from (8) we see that  $z(t)$  is a decreasing function of  $t$ . To prove that  $z(t)$  is positive, it suffices to show that

$$(10) \quad \lim_{t \rightarrow \infty} z(t) = 0.$$

First, we claim that  $\lim_{t \rightarrow \infty} z(t)$  is a finite number. Otherwise,

$$\lim_{t \rightarrow \infty} z(t) = -\infty$$

which implies that  $z(t)$  is eventually negative and  $x(t)$  is unbounded. Hence, there exists a  $t_1 \geq t_0 + \max\{\tau, \sigma\}$  such that

$$z(t_1) < 0 \quad \text{and} \quad x(t_1) = \max_{t_0 \leq s \leq t_1} x(s)$$

It follows, from (7), that

$$0 > z(t_1) = x(t_1) - p \int_{t_1 - \tau}^{t_1 - \sigma} x(s) ds \geq x(t_1) [1 - p(\tau - \sigma)] \geq 0.$$

This contradiction establishes our claim that

$$l \equiv \lim_{t \rightarrow \infty} z(t)$$

is finite.

Integrating both sides of (8) from  $t_0$  to  $t$  and letting  $t \rightarrow \infty$ , we see that

$$l - z(t_0) = -(p + q) \int_{t_0}^{\infty} x(s - \sigma) ds$$

which shows that  $x \in L^1[t_0, \infty)$ . From Eq. (1), it follows that  $x' \in L^1[t_0, \infty)$ . Hence,  $\lim_{t \rightarrow \infty} x(t)$  exists and it has to be zero (because  $x \in L^1[t_0, \infty)$ ). Thus,  $\lim_{t \rightarrow \infty} x(t) = 0$  and, from (7), we conclude that (10) holds.

(b) From Lemma 2 and the linearity of Eq. (1), it follows that  $w(t)$  is a solution of Eq. (1). From (7), we also see that

$$(11) \quad w'(t) = (p + q)x(t - \sigma) > 0$$

and so  $w(t)$  is an increasing function of  $t$ . To show that  $w(t)$  is eventually positive, it suffices to prove that

$$(12) \quad \lim_{t \rightarrow \infty} w(t) = +\infty.$$

Otherwise,

$$\lim_{t \rightarrow \infty} w(t) \equiv l$$

exists and is finite. Integrating both sides of (11) from  $t_0$  to  $t$  and letting  $t \rightarrow \infty$ , we find

$$l - w(t_0) = (p + q) \int_{t_0}^{\infty} x(s - \sigma) ds$$

which shows that  $x \in L^1[t_0, \infty)$ . As in the proof of part (a), we conclude that  $l = 0$ . Therefore,  $w(t)$  increases to zero as  $t \rightarrow \infty$ , which implies that  $w(t) < 0$ . Thus, there exists a  $t_1 \geq t_0 + \max\{\tau, \sigma\}$  such that

$$w(t_1) < 0 \quad \text{and} \quad x(t_1) = \min_{t_0 \leq s \leq t_1} x(s).$$

It follows, from (7), that

$$0 > w(t_1) = -z(t_1) = -x(t_1) + p \int_{t_1-\tau}^{t_1-\sigma} x(s) ds \geq x(t_1) [-1 + p(\tau - \sigma)] > 0.$$

This contradiction establishes (12) and the proof is complete.

### 3. PROOF OF MAIN RESULT

(a)  $\Rightarrow$  (b). Otherwise, Eq. (2) has a real root  $\lambda_0$ . Then  $x(t) = e^{\lambda_0 t}$  is a nonoscillatory solution of Eq. (1) which contradicts our assumption.

(b)  $\Rightarrow$  (a). We may (and do) assume that (3) holds. Since Eq. (2) has no real roots, the inequalities (4), (5), and (6) are satisfied.

Next, we distinguish the following cases.

Case 1.  $p(\tau - \sigma) \leq 1$ . Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution  $x(t)$ . Setting

$$z(t) = x(t) - p \int_{t-\tau}^{t-\sigma} x(s) ds,$$

we know, by Lemma 3(a), that  $z(t)$  is a positive and decreasing solution of Eq. (1). Furthermore, from (8), we have

$$z'(t) + (p + q)x(t - \sigma) = 0, \quad t \geq t_0.$$

Set

$$z_0(t) = z(t)$$

and

$$(13) \quad z_n(t) = z_{n-1}(t) - p \int_{t-\tau}^{t-\sigma} z_{n-1}(s) ds, \quad n = 1, 2, \dots$$

Then, for each  $n = 1, 2, \dots$ , the function  $z_n(t)$  is also a positive and decreasing solution of Eq. (1) such that

$$(14) \quad z'_n(t) + (p + q)z_{n-1}(t - \sigma) = 0, \quad t \geq t_0.$$

For each  $n = 1, 2, \dots$ , we define the set

$$A(z_n) = \{\lambda > 0: z'_n(t) + \lambda z_n(t) \leq 0\}.$$

The proof will be accomplished by proving that  $A(z_n)$  has the following contradictory properties:

(i) For each  $n = 1, 2, \dots$ , the set  $A(z_n)$  is nonempty and bounded above by a number independent of  $n$ .

(ii)  $\lambda \in A(z_n) \Rightarrow (\lambda + m) \in A(z_{n+1})$ ,  $n = 1, 2, \dots$ , where  $m$  is the positive number defined by (6).

Clearly,

$$z_n(t) \leq z_{n-1}(t) \leq z_{n-1}(t - \sigma)$$



and (14) yields

$$z'_n(t) + (p + q) z_n(t) \leq 0.$$

Therefore,

$$(p + q) \in \Lambda(z_n)$$

which proves that  $\Lambda(z_n) \neq \emptyset$ .

Next, we prove that  $\Lambda(z_n)$  is bounded above by a number independent of  $n$ . Indeed, integrating both sides of (14) from  $t - \sigma$  to  $t$ , we find

$$z_n(t) - z_n(t - \sigma) + (p + q) \int_{t-\sigma}^t z_{n-1}(s - \sigma) ds = 0$$

which implies that

$$-z_n(t - \sigma) + \sigma(p + q) z_{n-1}(t - \sigma) < 0$$

and hence

$$(15) \quad z_{n-1}(t) < \frac{1}{\sigma(p + q)} z_n(t), \quad n = 1, 2, \dots$$

From (7) we find that

$$z_n(t) \leq z_{n-1}(t) - p(\tau - \sigma) z_{n-1}(t - \sigma)$$

and so, using (15), we obtain

$$p(\tau - \sigma) z_{n-1}(t - \sigma) \leq z_{n-1}(t) - z_n(t) < \frac{1}{\sigma(p + q)} z_n(t).$$

Therefore,

$$(16) \quad (p + q) z_{n-1}(t - \sigma) < A z_n(t), \quad n = 1, 2, \dots,$$

where

$$A = \frac{1}{\sigma p(\tau - \sigma)}.$$

Using (16) in (14), we conclude that

$$z'_n(t) + A z_n(t) > 0$$

which proves that

$$A \notin \Lambda(z_n)$$

and so  $\Lambda(z_n)$  is bounded from above by  $A$ .

Next, we will prove (ii). Let  $\lambda \in \Lambda(z_n)$  and set

$$z_n(t) = e^{-\lambda t} \phi_n(t).$$

Then

$$\phi'_n(t) = e^{\lambda t} [z'_n(t) + \lambda z_n(t)] \leq 0$$

and, from (7), we see that

$$(17) \quad z_{n+1}(t) = e^{-\lambda t} \phi_n(t) - p \int_{t-\tau}^{t-\sigma} e^{-\lambda s} \phi_n(s) ds \leq$$

$$\begin{aligned} &\leq e^{-\lambda t} \phi_n(t) - \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) e^{-\lambda t} \phi_n(t - \sigma) \leq \\ &\leq \left[ 1 - \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) \right] e^{-\lambda t} \phi_n(t - \sigma). \end{aligned}$$

Also, from (14), we find

$$(18) \quad z'_{n+1}(t) = -(p + q) e^{\lambda t} e^{-\lambda t} \phi_n(t - \sigma).$$

Hence, from (17) and (18), we have

$$\begin{aligned} &z'_{n+1}(t) + (\lambda + m) z_{n+1}(t) \leq \\ &\leq \left[ -(p + q) e^{\lambda t} + (\lambda + m) - (\lambda + m) \frac{p}{\lambda} (e^{\lambda \tau} - e^{\lambda \sigma}) \right] e^{-\lambda t} \phi_n(t - \sigma) \leq \\ &\leq (p + q) (e^{\lambda \sigma} - e^{\lambda \tau}) e^{-\lambda t} \phi_n(t - \sigma) < 0 \end{aligned}$$

which proves that

$$(\lambda + m) \in \Lambda(z_{n+1})$$

and completes the proof in this case.

Case 2.  $p(\tau - \sigma) > 1$ . Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution  $y(t)$ . Setting

$$w(t) = -z(t) = -x(t) + p \int_{t-\tau}^{t-\sigma} x(s) ds,$$

by Lemma 3(b), we conclude that  $w(t)$  is an eventually positive and increasing solution of Eq. (1) such that

$$w'(t) - (p + q) x(t - \sigma) = 0, \quad t \geq t_0.$$

Set

$$w_0(t) = w(t)$$

and

$$(19) \quad w_n(t) = -w_{n-1}(t) + p \int_{t-\tau}^{t-\sigma} w_{n-1}(s) ds, \quad n = 1, 2, \dots$$

Then, for each  $n = 1, 2, \dots$ , the function  $w_n(t)$  is also an eventually positive and increasing solution of Eq. (1) and such that

$$(20) \quad w'_n(t) - (p + q) w_{n-1}(t - \sigma) = 0, \quad t \geq t_0.$$

For each  $n = 1, 2, \dots$ , we define the set

$$M(w_n) = \{ \mu > 0: w'_n(t) - \mu w_n(t) \geq 0 \}.$$

We also set

$$\mu_0 = \frac{p + q}{p(\tau - \sigma)} \quad \text{and} \quad m_0 = \frac{m\mu_0}{p} e^{\mu_0 \sigma}.$$

The proof will be accomplished by proving that  $M(w_n)$  has the following contradictory properties:

(i) For each  $n = 1, 2, \dots$

$$\mu_0 \in M(w_n) \quad \text{and} \quad -q \notin M(w_n),$$

that is, the set  $M(w_n)$  is nonempty and bounded from above by a number independent of  $n$ .

(ii) For every  $\mu \in M(w_n)$ , with  $\mu > \mu_0$ ,

$$(\mu + m_0) \in M(w_{n+1}), \quad n = 1, 2, \dots$$

First, we will prove (i). From (19), we see that for each  $n = 1, 2, \dots$

$$w_n(t) \leq p(\tau - \sigma) w_{n-1}(t - \sigma)$$

and, using this in (20), we find that

$$w'_n(n) - \frac{p+q}{p(\tau-\sigma)} w_n(t) \geq 0.$$

Hence

$$\mu_0 = \frac{p+q}{p(\tau-\sigma)} \in M(w_n), \quad n = 1, 2, \dots$$

Now, from (20) and the fact that  $w_n(t)$  is a solution of Eq. (1), we have

$$(p+q) w_{n-1}(t - \sigma) = w'_n(t) = -p w_n(t - \tau) - q w_n(t - \sigma)$$

which implies that  $q < 0$  and

$$(p+q) w_{n-1}(t - \sigma) < -q w_n(t - \sigma) < -q w_n(t).$$

Using this in (20), we find that

$$w'_n(t) - (-q) w_n(t) < 0$$

which proves that

$$-q \notin M(w_n), \quad n = 1, 2, \dots$$

Finally, we will prove (ii). Let  $\mu \in M(w_n)$  with  $\mu \geq \mu_0$ , and set

$$w_n(t) = e^{\mu t} \psi_n(t).$$

Then

$$\psi'_n(t) = e^{-\mu t} [w'_n(t) - \mu w_n(t)] \geq 0$$

and, from (19), we find that

$$\begin{aligned} (21) \quad w_{n+1}(t) &= -e^{\mu t} \psi_n(t) + p \int_{t-\tau}^{t-\sigma} e^{\mu s} \psi_n(s) ds \leq \\ &\leq -e^{\mu t} \psi_n(t) + \frac{p}{\mu} (e^{-\mu \sigma} - e^{-\mu \tau}) e^{\mu t} \psi_n(t - \sigma) \leq \\ &\leq e^{\mu t} \psi_n(t - \sigma) \left[ -1 + \frac{p}{\mu} (e^{-\mu \sigma} - e^{-\mu \tau}) \right]. \end{aligned}$$

Also, from (20), we have

$$(22) \quad w'_{n+1}(t) = (p + q) e^{-\mu\sigma} e^{\mu t} \psi_n(t - \sigma).$$

Hence, from (21) and (22), we obtain

$$\begin{aligned} & w'_{n+1}(t) - (\mu + m_0) w_{n+1}(t) \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[ (p + q) e^{-\mu\sigma} + (\mu + m_0) - (\mu + m_0) \frac{p}{\mu} (e^{-\mu\sigma} - e^{-\mu\tau}) \right] \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[ (\mu + p e^{-\mu\tau} + q e^{-\mu\sigma}) + m_0 \left( 1 - \frac{p}{\mu} e^{-\mu\sigma} \right) \right] \geq \\ & \geq e^{\mu t} \psi_n(t - \sigma) \left[ m + m_0 \left( 1 - \frac{p}{\mu_0 e^{\mu_0 \sigma}} \right) \right] = e^{\mu t} \psi_n(t - \sigma) m_0 \geq 0 \end{aligned}$$

which implies that

$$(\mu + m_0) \in M(w_{n+1}), \quad n = 1, 2, \dots$$

The proof of the theorem is complete.

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