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03. Electric field of continuous charge distributions. Electric flux

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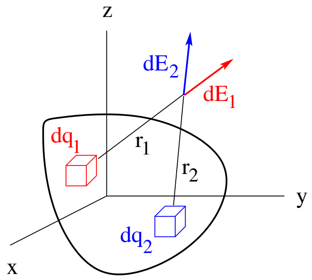
Electric Field of Continuous Charge Distribution

- Divide the charge distribution into infinitesimal blocks.
  - For 3D applications use charge per unit volume: $\rho = \Delta Q/\Delta V$.
  - For 2D applications use charge per unit area: $\sigma = \Delta Q/\Delta A$.
  - For 1D applications use charge per unit length: $\lambda = \Delta Q/\Delta L$.
- Use Coulomb's law to calculate the electric field generated by each block.
- Use the superposition principle to calculate the resultant field from all blocks.
- Use symmetries whenever possible.

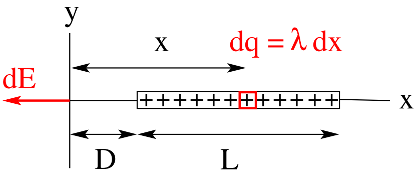
$$d\vec{E}_i = k\frac{dq_i}{r_i^2}\,\hat{r}_i$$

$$\vec{E} = \sum_i d\vec{E}_i \to k\int\frac{dq}{r^2}\,\hat{r}$$



Electric Field of Charged Rod (1)

- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx: $dq = \lambda dx$

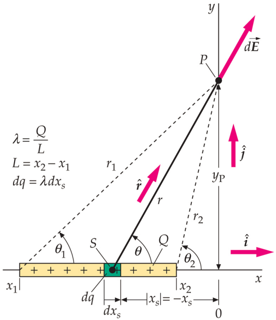


- Electric field generated by slice dx: $dE = \dfrac{kdq}{x^2} = \dfrac{k\lambda dx}{x^2}$
- Electric field generated by charged rod:

$$E = k\lambda \int_D^{D+L} \frac{dx}{x^2} = k\lambda \left[-\frac{1}{x} \right]_D^{D+L} = k\lambda \left[\frac{1}{D} - \frac{1}{D+L} \right] = \frac{kQ}{D(D+L)}$$

- Limiting case of very short rod $(L \ll D)$: $E \simeq \dfrac{kQ}{D^2}$
- Limiting case of very long rod $(L \gg D)$: $E \simeq \dfrac{k\lambda}{D}$

Electric Field of Charged Rod (2)



- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx_s: $dq = \lambda dx_s$
- Trigonometric relations:

$$y_p = r\sin\theta, \quad -x_s = r\cos\theta$$

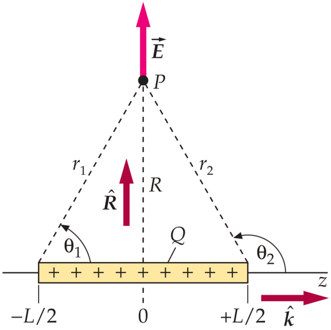
$$x_s = -y_p\cot\theta, \quad dx_s = \frac{y_p d\theta}{\sin^2\theta}$$

- $dE = \dfrac{k\lambda dx_s}{r^2} = \dfrac{k\lambda dx_s}{y_p^2}\sin^2\theta = \dfrac{k\lambda d\theta}{y_p}$
- $dE_y = dE\sin\theta = \dfrac{k\lambda}{y_p}\sin\theta d\theta \Rightarrow E_y = \dfrac{k\lambda}{y_p}\displaystyle\int_{\theta_1}^{\theta_2}\sin\theta d\theta = -\dfrac{k\lambda}{y_p}(\cos\theta_2 - \cos\theta_1)$
- $dE_x = dE\cos\theta = \dfrac{k\lambda}{y_p}\cos\theta d\theta \Rightarrow E_x = \dfrac{k\lambda}{y_p}\displaystyle\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \dfrac{k\lambda}{y_p}(\sin\theta_2 - \sin\theta_1)$

Electric Field of Charged Rod (3)

Symmetry dictates that the resulting electric field is directed radially.

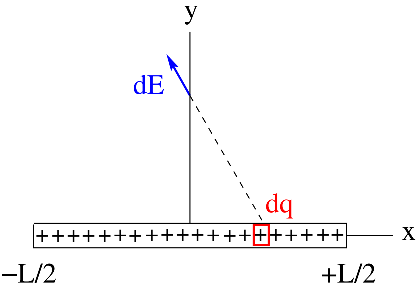
- $\theta_2 = \pi - \theta_1, \quad \Rightarrow \quad \sin\theta_2 = \sin\theta_1, \quad \cos\theta_2 = -\cos\theta_1.$
- $\cos\theta_1 = \dfrac{L/2}{\sqrt{L^2/4 + R^2}}.$
- $E_R = -\dfrac{k\lambda}{R}\left(\cos\theta_2 - \cos\theta_1\right) = \dfrac{k\lambda}{R}\dfrac{L}{\sqrt{L^2/4 + R^2}}.$
- $E_z = \dfrac{k\lambda}{R}\left(\sin\theta_2 - \sin\theta_1\right) = 0.$
- Large distance $(R \gg L)$: $E_R \simeq \dfrac{kQ}{R^2}.$
- Small distances $(R \ll L)$: $E_R \simeq \dfrac{2k\lambda}{R}$
- Rod of infinite length: $\vec{E} = \dfrac{2k\lambda}{R}\hat{R}.$



Electric Field of Charged Rod (4)

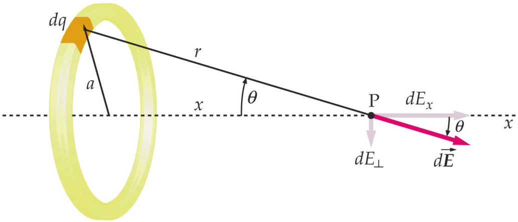
Symmetry dictates that the resulting electric field is directed radially (alternative derivation).

- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx: $dq = \lambda dx$
- $dE = \dfrac{kdq}{r^2} = \dfrac{k\lambda dx}{x^2 + y^2}$
- $dE_y = dE \cos\theta = \dfrac{dEy}{\sqrt{x^2 + y^2}} = \dfrac{k\lambda y dx}{(x^2 + y^2)^{3/2}}$
- $E_y = \displaystyle\int_{-L/2}^{+L/2} \frac{k\lambda y dx}{(x^2 + y^2)^{3/2}} = \left[\frac{k\lambda y x}{y^2\sqrt{x^2 + y^2}}\right]_{-L/2}^{+L/2}$
- $E_y = \dfrac{k\lambda L}{y\sqrt{(L/2)^2 + y^2}} = \dfrac{kQ}{y\sqrt{(L/2)^2 + y^2}}$
- Large distance $(y \gg L)$: $E_y \simeq \dfrac{kQ}{y^2}$
- Small distances $(y \ll L)$: $E_y \simeq \dfrac{2k\lambda}{y}$

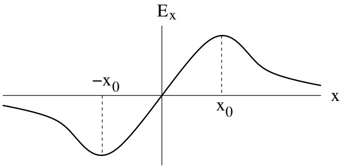


Electric Field of Charged Ring

- Total charge on ring: Q
- Charge per unit length: $\lambda = Q/2\pi a$
- Charge on arc: dq



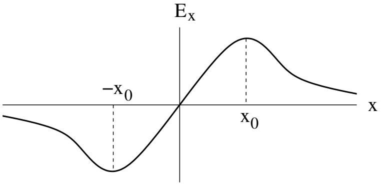
- $dE = \dfrac{kdq}{r^2} = \dfrac{kdq}{x^2 + a^2}$
- $dE_x = dE\cos\theta = dE\,\dfrac{x}{\sqrt{x^2+a^2}} = \dfrac{kxdq}{(x^2+a^2)^{3/2}}$
- $E_x = \dfrac{kx}{(x^2+a^2)^{3/2}}\displaystyle\int dq \;\Rightarrow\; E_x = \dfrac{kQx}{(x^2+a^2)^{3/2}}$
- $|x| \ll a:\; E_x \simeq \dfrac{kQx}{a^3}, \qquad x \gg a:\; E_x \simeq \dfrac{kQ}{x^2}$
- $(dE_x/dx)_{x=x_0} = 0 \quad \Rightarrow\; x_0 = \pm a/\sqrt{2}$



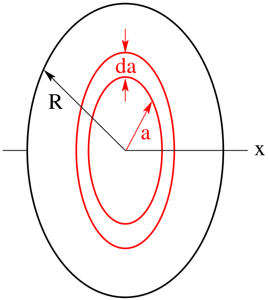
Charged Bead Moving Along Axis of Charged Ring

Consider a negatively charged bead (mass m, charge $-q$) constrained to move without friction along the axis of a positively charged ring.

- Place bead on x-axis near center of ring: $|x| \ll a : \; E_x \simeq \dfrac{kQx}{a^3}$
- Restoring force: $F = -qE_x = -k_s x$ with $k_s = \dfrac{kQq}{a^3}$
- Acceleration: $a = \dfrac{F}{m} = -\dfrac{k_s}{m}\, x$
- Equation of motion: $\dfrac{d^2x}{dt^2} = -\dfrac{k_s}{m}\, x$
- Harmonic oscillation: $x(t) = A\cos(\omega t + \phi)$
- Angular frequency: $\omega = \sqrt{\dfrac{k_s}{m}} = \sqrt{\dfrac{kQq}{ma^3}}$



Electric Field of Charged Disk



- Charge per unit area: $\sigma = \dfrac{Q}{\pi R^2}$
- Area of ring: $dA = 2\pi a da$
- Charge on ring: $dq = 2\pi\sigma a da$

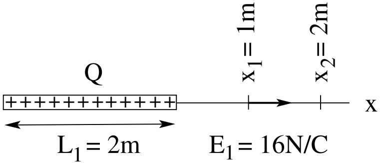
- $dE_x = \dfrac{kxdq}{(x^2+a^2)^{3/2}} = \dfrac{2\pi\sigma kxada}{(x^2+a^2)^{3/2}}$
- $E_x = 2\pi\sigma kx \displaystyle\int_0^R \frac{ada}{(x^2+a^2)^{3/2}} = 2\pi\sigma kx \left[\frac{-1}{\sqrt{x^2+a^2}}\right]_0^R$
- $E_x = 2\pi\sigma k \left[1 - \dfrac{x}{\sqrt{x^2+R^2}}\right]$ for $x > 0$
- $x \ll R: \; E_x \simeq 2\pi\sigma k = \dfrac{\sigma}{2\epsilon_0}$
- Infinite sheet of charge produces uniform electric field perpendicular to plane.

Electric Field of Charged Rubber Band

The electric field at position x along the line of a charged rubber band is

$$E = \frac{kQ}{x(x+L)}$$

The value of E at $x_1 = 1\text{m}$ is $E_1 = 16\text{N/C}$.

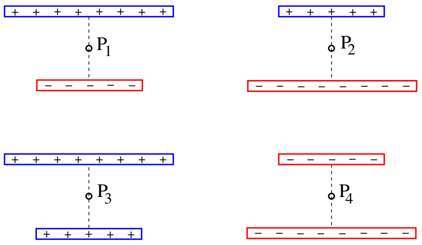


(a) What is the electric field E_2 at a distance $x_2 = 2\text{m}$ from the edge of the band?

(b) To what length L_2 must the band be stretched (toward the left) such that it generates the field $E_2 = 8\text{N/C}$ at $x_1 = 1\text{m}$?

Electric Field Between Charged Rods

Consider four configurations of two charged rods with equal amounts of charge per unit length $|\lambda|$ on them.



(a) Determine the direction of the electric field at points P_1, P_2, P_3, P_4.

(b) Rank the electric field at the four points according to strength.

Electric Field of Charged Semicircle

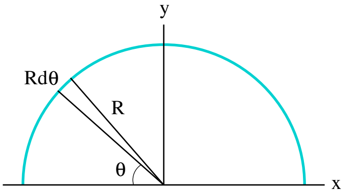
Consider a uniformly charged thin rod bent into a semicircle of radius R.

Find the electric field generated at the origin of the coordinate system.

- Charge per unit length: $\lambda = Q/\pi R$
- Charge on slice: $dq = \lambda R d\theta$ (assumed positive)
- Electric field generated by slice: $dE = k\dfrac{|dq|}{R^2} = \dfrac{k|\lambda|}{R}d\theta$ directed radially (inward for $\lambda > 0$)
- Components of $d\vec{E}$: $dE_x = dE\cos\theta, \quad dE_y = -dE\sin\theta$
- Electric field from all slices added up:

$$E_x = \frac{k\lambda}{R}\int_0^{\pi}\cos\theta\,d\theta = \frac{k\lambda}{R}\Big[\sin\theta\Big]_0^{\pi} = 0$$

$$E_y = -\frac{k\lambda}{R}\int_0^{\pi}\sin\theta\,d\theta = \frac{k\lambda}{R}\Big[\cos\theta\Big]_0^{\pi} = -\frac{2k\lambda}{R}$$

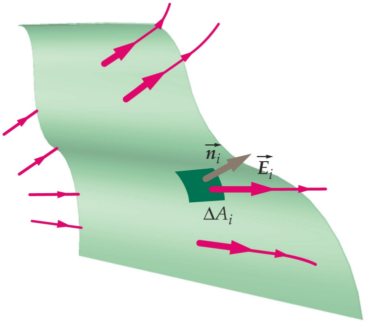


Electric Flux: Definition

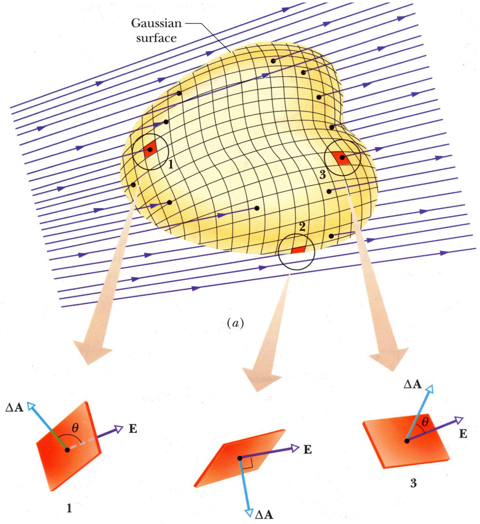
Consider a surface S of arbitrary shape in the presence of an electric field $\vec{E}$.

Prescription for the calculation of the electric flux through S:

- Divide S into small tiles of area ΔA_i.
- Introduce vector $\Delta\vec{A}_i = \hat{n}_i \Delta A_i$ perpendicular to tile.
  - If S is open choose consistently one of two possible directions for $\Delta\vec{A}_i$.
  - If S is closed choose $\Delta\vec{A}_i$ to be directed outward.
- Electric field at position of tile i: $\vec{E}_i$.
- Electric flux through tile i:
  $\Delta\Phi_i^{(E)} = \vec{E}_i \cdot \Delta\vec{A}_i = E_i \Delta A_i \cos\theta_i$.
- Electric flux through S: $\Phi_E = \sum_i \vec{E}_i \cdot \Delta\vec{A}_i$.
- Limit of infinitesimal tiles: $\Phi_E = \int \vec{E} \cdot d\vec{A}$.
- Electric flux is a scalar.
- The SI unit of electric flux is Nm^2/C.

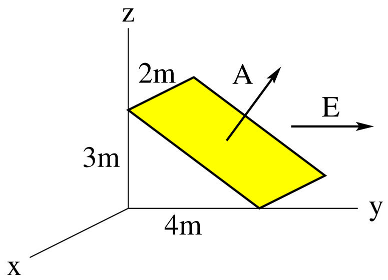


Electric Flux: Illustration



Electric Flux: Application (1)

Consider a rectangular sheet oriented perpendicular to the yz plane as shown and positioned in a uniform electric field $\vec{E} = (2\hat{j})$N/C.



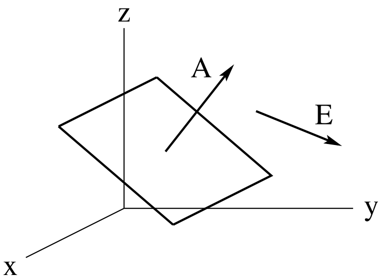
(a) Find the area A of the sheet.

(b) Find the angle between $\vec{A}$ and $\vec{E}$.

(c) Find the electric flux Φ_E through the sheet.

Electric Flux: Application (2)

Consider a plane sheet of paper whose orientation in space is described by the area vector $\vec{A} = (3\hat{j} + 4\hat{k})\text{m}^2$ positioned in a region of uniform electric field $\vec{E} = (1\hat{i} + 5\hat{j} - 2\hat{k})$N/C.



(a) Find the area A of the sheet.

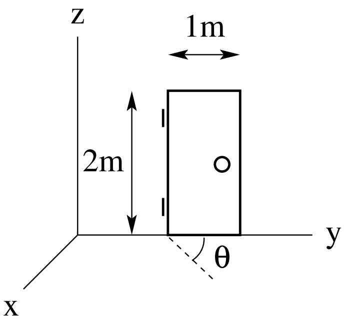
(b) Find the magnitude E of the electric field $\vec{E}$.

(c) Find the electric flux Φ_E through the sheet.

(d) Find the angle θ between vectors $\vec{A}$ and $\vec{E}$.

Electric Flux: Application (3)

The room shown below is positioned in an electric field $\vec{E} = (3\hat{i} + 2\hat{j} + 5\hat{k})$N/C.



(a) What is the electric flux Φ_E through the closed door?

(b) What is the electric flux Φ_E through the door opened at $\theta = 90^\circ$?

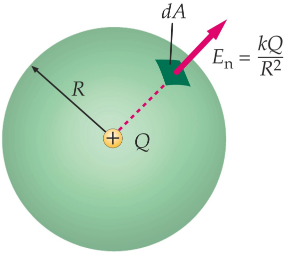
(c) At what angle θ_1 is the electric flux through the door zero?

(d) At what angle θ_2 is the electric flux through the door a maximum?

Electric Flux: Application (4)

Consider a positive point charge Q at the center of a spherical surface of radius R. Calculate the electric flux through the surface.

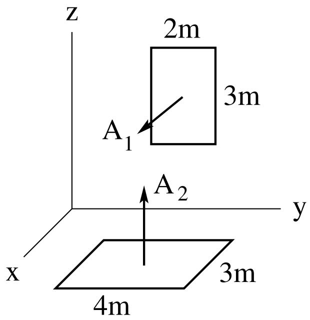
- $\vec{E}$ is directed radially outward. Hence $\vec{E}$ is parallel to $d\vec{A}$ everywhere on the surface.
- $\vec{E}$ has the same magnitude, $E = kQ/R^2$, everywhere on the surface.
- The area of the spherical surface is $A = 4\pi R^2$.
- Hence the electric flux is $\Phi_E \doteq \oint \vec{E} \cdot d\vec{A} = EA = 4\pi kQ$.
- Note that Φ_E is independent of R.



Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive x-direction) and $\vec{A}_2$ (pointing in positive z-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

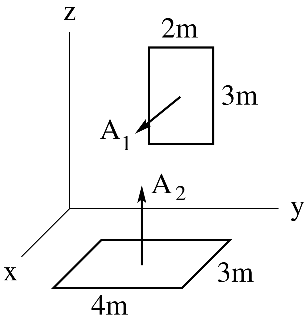
(b) Find the electric flux $\Phi_E^{(2)}$ through area A_2.



Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive x-direction) and $\vec{A}_2$ (pointing in positive z-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

(b) Find the electric flux $\Phi_E^{(2)}$ through area A_2.



Solution:

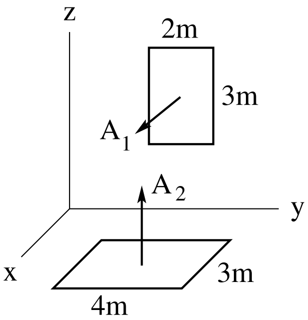
(a) $\vec{A}_1 = 6\hat{i}\,\mathrm{m}^2$,
$\Phi_E^{(1)} = \vec{E} \cdot \vec{A}_1 = (2\mathrm{N/C})(6\mathrm{m}^2) = 12\mathrm{Nm}^2/\mathrm{C}$.

Intermediate Exam I: Problem #3 (Spring '05)

Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive x-direction) and $\vec{A}_2$ (pointing in positive z-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{\imath} + 7\hat{\jmath} - 3\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

(b) Find the electric flux $\Phi_E^{(2)}$ through area A_2.



Solution:

(a) $\vec{A}_1 = 6\hat{\imath}\,\mathrm{m}^2$,
$\Phi_E^{(1)} = \vec{E} \cdot \vec{A}_1 = (2\mathrm{N/C})(6\mathrm{m}^2) = 12\mathrm{Nm}^2/\mathrm{C}$.

(b) $\vec{A}_2 = 12\hat{k}\,\mathrm{m}^2$,
$\Phi_E^{(2)} = \vec{E} \cdot \vec{A}_2 = (-3\mathrm{N/C})(12\mathrm{m}^2) = -36\mathrm{Nm}^2/\mathrm{C}$.