03. Electric field of continuous charge distributions. Electric flux

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

Follow this and additional works at: https://digitalcommons.uri.edu/phy204-slides

Recommended Citation
https://digitalcommons.uri.edu/phy204-slides/28

This Course Material is brought to you for free and open access by the PHY 204: Elementary Physics II (2021) at DigitalCommons@URI. It has been accepted for inclusion in PHY 204: Elementary Physics II -- Slides by an authorized administrator of DigitalCommons@URI. For more information, please contact digitalcommons@etal.uri.edu.
Electric Field of Continuous Charge Distribution

- Divide the charge distribution into infinitesimal blocks.
  - For 3D applications use charge per unit volume: \( \rho = \Delta Q / \Delta V \).
  - For 2D applications use charge per unit area: \( \sigma = \Delta Q / \Delta A \).
  - For 1D applications use charge per unit length: \( \lambda = \Delta Q / \Delta L \).

- Use Coulomb’s law to calculate the electric field generated by each block.
- Use the superposition principle to calculate the resultant field from all blocks.
- Use symmetries whenever possible.

\[
d\vec{E}_i = k \frac{dq_i}{r_i^2} \hat{r}_i \\
\vec{E} = \sum_i d\vec{E}_i \rightarrow k \int \frac{dq}{r^2} \hat{r}
\]
Electric Field of Charged Rod (1)

- Charge per unit length: \( \lambda = \frac{Q}{L} \)
- Charge on slice \( dx \): \( dq = \lambda \, dx \)

- Electric field generated by slice \( dx \): \( \mathbf{dE} = \frac{k \mathbf{dq}}{x^2} = \frac{k \lambda \, dx}{x^2} \)

- Electric field generated by charged rod:

\[
E = k \lambda \int_{D}^{D+L} \frac{dx}{x^2} = k \lambda \left[ -\frac{1}{x} \right]_{D}^{D+L} = k \lambda \left[ \frac{1}{D} - \frac{1}{D+L} \right] = \frac{kQ}{D(D+L)}
\]

- Limiting case of very short rod (\( L \ll D \)): \( E \approx \frac{kQ}{D^2} \)
- Limiting case of very long rod (\( L \gg D \)): \( E \approx \frac{k\lambda}{D} \)
Electric Field of Charged Rod (2)

- Charge per unit length: \( \lambda = Q/L \)
- Charge on slice \( dx_s \): \( dq = \lambda dx_s \)
- Trigonometric relations:
  \[
  y_p = r \sin \theta, \quad -x_s = r \cos \theta
  \]
  \[
  x_s = -y_p \cot \theta, \quad dx_s = \frac{y_p d\theta}{\sin^2 \theta}
  \]

- \( dE = \frac{k\lambda dx_s}{r^2} = \frac{k\lambda dx_s}{y_p^2} \sin^2 \theta = \frac{k\lambda d\theta}{y_p} \)

- \( dE_y = dE \sin \theta = \frac{k\lambda}{y_p} \sin \theta d\theta \Rightarrow E_y = \frac{k\lambda}{y_p} \int_{\theta_1}^{\theta_2} \sin \theta d\theta = -\frac{k\lambda}{y_p} (\cos \theta_2 - \cos \theta_1) \)

- \( dE_x = dE \cos \theta = \frac{k\lambda}{y_p} \cos \theta d\theta \Rightarrow E_x = \frac{k\lambda}{y_p} \int_{\theta_1}^{\theta_2} \cos \theta d\theta = \frac{k\lambda}{y_p} (\sin \theta_2 - \sin \theta_1) \)
Symmetry dictates that the resulting electric field is directed radially.

\[ \theta_2 = \pi - \theta_1, \quad \Rightarrow \quad \sin \theta_2 = \sin \theta_1, \quad \cos \theta_2 = -\cos \theta_1. \]

\[ \cos \theta_1 = \frac{L/2}{\sqrt{L^2/4 + R^2}}. \]

\[ E_R = -\frac{k\lambda}{R} (\cos \theta_2 - \cos \theta_1) = \frac{k\lambda}{R} \left( \frac{L}{\sqrt{L^2/4 + R^2}} \right). \]

\[ E_z = \frac{k\lambda}{R} (\sin \theta_2 - \sin \theta_1) = 0. \]

- Large distance \((R \gg L)\): \(E_R \approx \frac{kQ}{R^2}\).
- Small distances \((R \ll L)\): \(E_R \approx \frac{2k\lambda}{R}\).
- Rod of infinite length: \(\vec{E} = \frac{2k\lambda}{R} \hat{R}\).
Symmetry dictates that the resulting electric field is directed radially (alternative derivation).

- Charge per unit length: $\lambda = Q/L$
- Charge on slice $dx$: $dq = \lambda dx$
- $dE = \frac{kdq}{r^2} = \frac{k\lambda dx}{x^2 + y^2}$
- $dE_y = dE \cos \theta = \frac{dE_y}{\sqrt{x^2 + y^2}} = \frac{k\lambda y dx}{(x^2 + y^2)^{3/2}}$
- $E_y = \int_{-L/2}^{+L/2} \frac{k\lambda y dx}{(x^2 + y^2)^{3/2}} = \left[ \frac{k\lambda y x}{y^2 \sqrt{x^2 + y^2}} \right]_{-L/2}^{+L/2}$
- $E_y = \frac{k\lambda L}{y \sqrt{(L/2)^2 + y^2}} = \frac{kQ}{y \sqrt{(L/2)^2 + y^2}}$
- Large distance ($y \gg L$): $E_y \approx \frac{kQ}{y^2}$
- Small distances ($y \ll L$): $E_y \approx \frac{2k\lambda}{y}$
Electric Field of Charged Ring

- Total charge on ring: \( Q \)
- Charge per unit length: \( \lambda = Q / 2\pi a \)
- Charge on arc: \( dq \)

\[
dE = \frac{kdq}{r^2} = \frac{kdq}{x^2 + a^2}
\]

\[
dE_x = dE \cos \theta = dE \frac{x}{\sqrt{x^2 + a^2}} = \frac{kxdq}{(x^2 + a^2)^{3/2}}
\]

\[
E_x = \frac{kx}{(x^2 + a^2)^{3/2}} \int dq \Rightarrow E_x = \frac{kQx}{(x^2 + a^2)^{3/2}}
\]

- \( |x| \ll a : E_x \simeq \frac{kQx}{a^3}, \quad x \gg a : E_x \simeq \frac{kQ}{x^2} \)
- \( (dE_x/dx)_{x=x_0} = 0 \Rightarrow x_0 = \pm a / \sqrt{2} \)
Consider a negatively charged bead (mass \( m \), charge \(-q\)) constrained to move without friction along the axis of a positively charged ring.

- Place bead on \( x\)-axis near center of ring: \(|x| \ll a\) : \( E_x \simeq \frac{kQx}{a^3} \)
- Restoring force: \( F = -qE_x = -k_s x \) with \( k_s = \frac{kQq}{a^3} \)
- Acceleration: \( a = \frac{F}{m} = -\frac{k_s}{m} x \)
- Equation of motion: \( \frac{d^2x}{dt^2} = -\frac{k_s}{m} x \)
- Harmonic oscillation: \( x(t) = A \cos(\omega t + \phi) \)
- Angular frequency: \( \omega = \sqrt{\frac{k_s}{m}} = \sqrt{\frac{kQq}{ma^3}} \)
Electric Field of Charged Disk

- Charge per unit area: \( \sigma = \frac{Q}{\pi R^2} \)
- Area of ring: \( dA = 2\pi a \, da \)
- Charge on ring: \( dq = 2\pi \sigma \, da \)

\[ dE_x = \frac{kx \, dq}{(x^2 + a^2)^{3/2}} = \frac{2\pi \sigma kx \, da}{(x^2 + a^2)^{3/2}} \]

\[ E_x = 2\pi \sigma k \int_0^R \frac{ada}{(x^2 + a^2)^{3/2}} = 2\pi \sigma k \left[ \frac{-1}{\sqrt{x^2 + a^2}} \right]_0^R \]

\[ E_x = 2\pi \sigma k \left[ 1 - \frac{x}{\sqrt{x^2 + R^2}} \right] \text{ for } x > 0 \]

- \( x \ll R : E_x \simeq 2\pi \sigma k = \frac{\sigma}{2\varepsilon_0} \)

- Infinite sheet of charge produces uniform electric field perpendicular to plane.
The electric field at position $x$ along the line of a charged rubber band is

$$E = \frac{kQ}{x(x+L)}$$

The value of $E$ at $x_1 = 1$ m is $E_1 = 16 \text{N/C}$.

(a) What is the electric field $E_2$ at a distance $x_2 = 2$ m from the edge of the band?

(b) To what length $L_2$ must the band be stretched (toward the left) such that it generates the field $E_2 = 8 \text{N/C}$ at $x_1 = 1$ m?
Consider four configurations of two charged rods with equal amounts of charge per unit length $|\lambda|$ on them.

(a) Determine the direction of the electric field at points $P_1, P_2, P_3, P_4$.

(b) Rank the electric field at the four points according to strength.
Consider a uniformly charged thin rod bent into a semicircle of radius $R$.

Find the electric field generated at the origin of the coordinate system.

- Charge per unit length: $\lambda = \frac{Q}{\pi R}$
- Charge on slice: $dq = \lambda R d\theta$ (assumed positive)
- Electric field generated by slice: $dE = k\frac{|dq|}{R^2} = \frac{k|\lambda|}{R} d\theta$
  directed radially (inward for $\lambda > 0$)
- Components of $d\vec{E}$: $dE_x = dE \cos \theta$, $dE_y = -dE \sin \theta$
- Electric field from all slices added up:

  $$E_x = \frac{k\lambda}{R} \int_0^\pi \cos \theta d\theta = \frac{k\lambda}{R} \left[ \sin \theta \right]_0^\pi = 0$$

  $$E_y = -\frac{k\lambda}{R} \int_0^\pi \sin \theta d\theta = \frac{k\lambda}{R} \left[ \cos \theta \right]_0^\pi = -\frac{2k\lambda}{R}$$
Electric Flux: Definition

Consider a surface $S$ of arbitrary shape in the presence of an electric field $\vec{E}$.

Prescription for the calculation of the electric flux through $S$:

- Divide $S$ into small tiles of area $\Delta A_i$.
- Introduce vector $\Delta \vec{A}_i = \hat{n}_i \Delta A_i$ perpendicular to tile.
  - If $S$ is open choose consistently one of two possible directions for $\Delta \vec{A}_i$.
  - If $S$ is closed choose $\Delta \vec{A}_i$ to be directed outward.
- Electric field at position of tile $i$: $\vec{E}_i$.
- Electric flux through tile $i$: $\Delta \Phi_i^{(E)} = \vec{E}_i \cdot \Delta \vec{A}_i = E_i \Delta A_i \cos \theta_i$.
- Electric flux through $S$: $\Phi_E = \sum_i \vec{E}_i \cdot \Delta \vec{A}_i$.
- Limit of infinitesimal tiles: $\Phi_E = \int \vec{E} \cdot d\vec{A}$.
- Electric flux is a scalar.
- The SI unit of electric flux is $\text{Nm}^2/\text{C}$.
Electric Flux: Illustration
Electric Flux: Application (1)

Consider a rectangular sheet oriented perpendicular to the $yz$ plane as shown and positioned in a uniform electric field $\vec{E} = (2\hat{j})\text{N/C}$.

(a) Find the area $A$ of the sheet.
(b) Find the angle between $\vec{A}$ and $\vec{E}$.
(c) Find the electric flux $\Phi_E$ through the sheet.
Consider a plane sheet of paper whose orientation in space is described by the area vector \( \vec{A} = (3\hat{j} + 4\hat{k}) \text{m}^2 \) positioned in a region of uniform electric field \( \vec{E} = (1\hat{i} + 5\hat{j} - 2\hat{k}) \text{N/C} \).

(a) Find the area \( A \) of the sheet.

(b) Find the magnitude \( E \) of the electric field \( \vec{E} \).

(c) Find the electric flux \( \Phi_E \) through the sheet.

(d) Find the angle \( \theta \) between vectors \( \vec{A} \) and \( \vec{E} \).
The room shown below is positioned in an electric field \( \vec{E} = (3\hat{i} + 2\hat{j} + 5\hat{k}) \text{N/C.} \)

(a) What is the electric flux \( \Phi_E \) through the closed door?
(b) What is the electric flux \( \Phi_E \) through the door opened at \( \theta = 90^\circ \)?
(c) At what angle \( \theta_1 \) is the electric flux through the door zero?
(d) At what angle \( \theta_2 \) is the electric flux through the door a maximum?
Consider a positive point charge $Q$ at the center of a spherical surface of radius $R$. Calculate the electric flux through the surface.

- $\vec{E}$ is directed radially outward. Hence $\vec{E}$ is parallel to $d\vec{A}$ everywhere on the surface.
- $\vec{E}$ has the same magnitude, $E = kQ/R^2$, everywhere on the surface.
- The area of the spherical surface is $A = 4\pi R^2$.
- Hence the electric flux is $\Phi_E = \oint \vec{E} \cdot d\vec{A} = EA = 4\pi kQ$.
- Note that $\Phi_E$ is independent of $R$. 
Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive $x$-direction) and $\vec{A}_2$ (pointing in positive $z$-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$ N/C.

(a) Find the electric flux $\Phi_{E}^{(1)}$ through area $A_1$.

(b) Find the electric flux $\Phi_{E}^{(2)}$ through area $A_2$. 
Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive $x$-direction) and $\vec{A}_2$ (pointing in positive $z$-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k}) \text{N/C}$.

(a) Find the electric flux $\Phi_E^{(1)}$ through area $A_1$.

(b) Find the electric flux $\Phi_E^{(2)}$ through area $A_2$.

Solution:

(a) $\vec{A}_1 = 6\hat{i} \text{m}^2$,

$\Phi_E^{(1)} = \vec{E} \cdot \vec{A}_1 = (2 \text{N/C})(6 \text{m}^2) = 12 \text{Nm}^2/\text{C}$. 

(b) $\vec{A}_2 = 12\hat{k} \text{m}^2$,

$\Phi_E^{(2)} = \vec{E} \cdot \vec{A}_2 = (-3 \text{N/C})(12 \text{m}^2) = -36 \text{Nm}^2/\text{C}$. 

Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive $x$-direction) and $\vec{A}_2$ (pointing in positive $z$-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$ N/C.

(a) Find the electric flux $\Phi_{E}^{(1)}$ through area $A_1$.

(b) Find the electric flux $\Phi_{E}^{(2)}$ through area $A_2$.

Solution:

(a) $\vec{A}_1 = 6\hat{i}$ m$^2$,  
   $\Phi_{E}^{(1)} = \vec{E} \cdot \vec{A}_1 = (2\text{N/C})(6\text{m}^2) = 12\text{Nm}^2/\text{C}$.

(b) $\vec{A}_2 = 12\hat{k}$ m$^2$,  
   $\Phi_{E}^{(2)} = \vec{E} \cdot \vec{A}_2 = (-3\text{N/C})(12\text{m}^2) = -36\text{Nm}^2/\text{C}$.