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27. Motional EMF. Faraday's law with applications.

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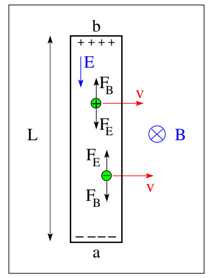
PHY204 Lecture 27 [rln27]

Motional EMF

Conducting rod moving across region of uniform magnetic field

- moving charge carriers
- magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$
- charge separation
- electric field $\vec{E}$
- electric force $\vec{F}_E = q\vec{E}$



Equilibrium between electric and magnetic force:

$$F_E = F_B \quad \Rightarrow \; qE = qvB \quad \Rightarrow \; E = vB$$

Potential difference induced between endpoints of rod:

$$V_{ab} \equiv V_b - V_a = EL \quad \Rightarrow \; V_{ab} = vBL \quad \text{(motional EMF)}$$

tsl246

In this lecture our goal is to understand Faraday's law, the last of the four Maxwell's equations to be introduced in this course. Getting a good grip on Faraday's law in its full glory is bit challenging when you are new to it. Yet, a thorough understanding of Faraday's law is important for what comes later. Therefore, let us progress gently from the concrete to the more abstract.

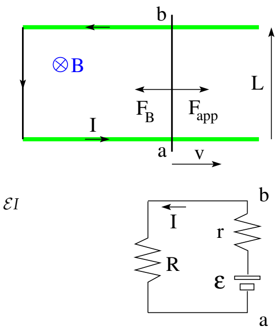
Consider a conducting rod moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$ as shown. Mobile charge carriers inside the conductor, as the move along, are being pushed by the magnetic force up (down) if their charge is positive (negative). The result is a surplus of positive (negative) charge at point b (a). This charge separation builds up an electric field inside the conductor directed ↓.

The induced electric field, as it builds up, causes an electric force on the mobile charge carriers which, eventually, balances the magnetic force. The balancing condition allows us to determine the strength of the electric field that the motion of the rod effectively generates (see slide). Associated with the electric field $\vec{E} = \vec{v} \times \vec{B}$ inside the rod is a potential difference $V_{ab} = vBL$ between the ends of the rod.

In summary, a conducting rod moving in a magnetic field acts like a battery with a voltage V_{ab} between its terminals. The voltage is named *motional EMF*. There are ways to connect this source of EMF to a circuit such that it drives a current. On the next page we show one simple way.

Current Produced by Motional EMF

- Motional EMF: $\mathcal{E} = vBL$
- Terminal voltage: $V_{ab} = \mathcal{E} - Ir$
- Electric current: $\mathcal{E} - Ir - IR = 0 \;\Rightarrow\; I = \dfrac{\mathcal{E}}{r+R}$
- Applied mechanical force: $\vec{F}_{app}$
- Magnetic force: $\vec{F}_B = I\vec{L} \times \vec{B}$
- Motion at constant velocity: $\vec{F}_{app} = -\vec{F}_B$
- Electrical power generated: $P_{gen} = \mathcal{E}I$
- Mechanical power input: $P_{in} = Fv = (ILB)v = (vBL)I = \mathcal{E}I$
- Electrical power output: $P_{out} = V_{ab}I = \mathcal{E}I - I^2r$



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When we force the conducting rod to slide across a conducting frame as shown, we effectively have a one-loop circuit with internal resistance r denoting the resistance of the rod and external resistance R denoting the resistance of the frame. The symbolic circuit is shown underneath the physical circuit.

We know from the previous page that point b is the positive terminal and point a the negative terminal of the motional EMF. From the loop rule we infer the current, which is positive in the ccw direction.

Where does the energy of the EMF source come from? In a car battery, chemical energy is converted into electrical energy. In the motional EMF, the source is mechanical energy instead.

Forcing the rod to move at constant speed v requires an applied force that balances the magnetic force caused by the current. The mechanical power input, Fv, generated by the applied force is shown (on the slide) to be equal to electrical power, $\mathcal{E}I$, generated in the motional EMF.

Part of that power, I^2r, is dissipated in the source and the remainder $V_{ab}I$ counts as power output, which, in this case is power I^2R dissipated in the conducting frame.

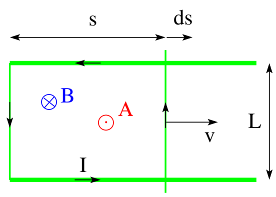
What we have described on this page is one concrete realization of Faraday's law in action. We have done it without using the conceptual framework needed to formalte the law.

On the next page we describe the same situation in more abstract terms. It is those more abstract terms that can be transcribed to a large variety of realizations of Faraday's law in action.

Faraday's Law of Induction (1)

Prototype: motional EMF reformulated.

- Choose area vector $\vec{A}$ for current loop: $A = Ls\ \odot$.
- Magnetic flux: $\Phi_B = \int \vec{B} \cdot d\vec{A}$. Here $\Phi_B = -BLs$.
- Motional EMF: $\mathcal{E} = vBL$.
- Change in area of loop: $dA = Lds$.
- Change in magnetic flux: $d\Phi_B = -BdA = -BLds$.
- SI unit of magnetic flux: 1Wb=1Tm^2 (Weber).
- Rate of change of flux: $\frac{d\Phi_B}{dt} = -BL\frac{ds}{dt} = -vBL$.
- Faraday's law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$.



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We are familiar with magnetic flux Φ_B through closed surfaces from Gauss's law (lecture 25). We found that the magnetic flux through closed surfaces always vanishes.

In the context of Faraday's law, we are dealing with magnetic flux Φ_B through open surfaces, here through the rectangular loop bounded by the frame and the rod. Note the SI unit [Wb] for magnetic flux. Note also that we must choose area vectors pointing to one or the other side of an open surface before we can attribute a definite value to flux.

At the instant shown, the area is $A = Ls$ and we pick $\odot$ for the direction of $\vec{A}$. Associated with that choice via the right-hand rule is that ccw is the positive direction around the loop.

The magnetic flux Φ_B comes out to be negative and getting more negative as the rod moves to the right. The time rate at which the magnetic flux changes, $d\Phi_B/dt$, is worked out on the slide. It is a negative quantity and recognizably equal in magnitude and opposite in sign to the motional EMF, $\mathcal{E} = vBL$, calculated previously.

It turns out that what we have established here in a particular context, is true more generally. Faraday's law, as stated in the last line of the slide, holds irrespective of how we choose a loop and irrespective of the means by which the magnetic flux through the loop changes.

The biggest challenge is to get the directions right. We must adopt a convention and then stick with that convention for the entire application. What counts as positive or negative depends on the convention adopted, but all physical results, e.g. whether an induced current flows cw or ccw, are identical under any convention.

Let us demonstrate that the physical result established on the previous page is independent of the choice of area vector. In the table below, two different conventions (line 1) are being adopted. Associated with the convention by right-hand rule is the positive loop direction (line 2).

The direction of the magnetic field (line 3) is given in this application. The quantities on lines 4-6 have signs (±) that depend on the convention adopted. The physical result on line 7 (current direction) is independent of convention. The positive (negative) EMF in the first (second) convention both produce a ccw current around the loop.

1	area vector	$\vec{A} \odot$	$\vec{A} \otimes$
2	positive loop direction	ccw	cw
3	magnetic field	$\vec{B} \otimes$	$\vec{B} \otimes$
4	magnetic flux	$\Phi_B < 0$	$\Phi_B > 0$
5	rate of flux change	$\frac{d\Phi_B}{dt} < 0$	$\frac{d\Phi_B}{dt} > 0$
6	induced EMF	$\mathcal{E} > 0$	$\mathcal{E} < 0$
7	induced current	I ccw	I ccw

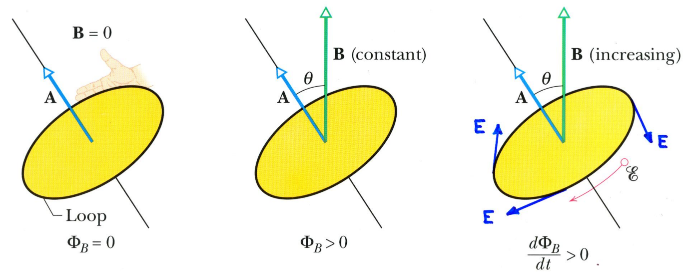
If, on the other hand, we consider situations that are physically different, e.g. by choosing opposite directions of the magnetic field, then the physical result (current direction) is different as shown in the table below.

1	area vector	$\vec{A} \otimes$	$\vec{A} \otimes$
2	positive loop direction	cw	cw
3	magnetic field	$\vec{B} \odot$	$\vec{B} \otimes$
4	magnetic flux	$\Phi_B < 0$	$\Phi_B > 0$
5	rate of flux change	$\frac{d\Phi_B}{dt} < 0$	$\frac{d\Phi_B}{dt} > 0$
6	induced EMF	$\mathcal{E} > 0$	$\mathcal{E} < 0$
7	induced current	I cw	I ccw

If we switch the area vector (line 1) and the associated loop direction (line 2) for either of these two physical situations in the second table, the physical result (line 7) will remain unchanged.

Area – Field – Flux – EMF (1)

$$\Phi_B = \int \vec{B} \cdot d\vec{A}, \qquad \mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$



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On this page, the level of abstraction is moved up yet another notch. The loop under consideration in applications of Faraday's law does not need to be a conductor in which a current is induced. Any line, real or imagined, will do as long as it is closed in itself.

On the slide we place or imagine three circular loops as shown. For each loop we adopt the same convention by choosing the area vector as shown. The vector $\vec{A}$ must be perpendicular to the plane of the loop. The convention adopted is one of two possible choices. The chosen direction of $\vec{A}$, makes ccw the positive loop direction (by right-hand rule).

The loop on the left is positioned in a region of space with no magnetic field. Hence there is zero magnetic flux through it at all times. Hence we have $d\Phi_B/dt = 0$ as well. Then Faraday's law implies $\mathcal{E} = 0$ (no induced EMF).

The loop in the middle is positioned in a constant magnetic field directed vertically up, implying a positive and constant magnetic flux. A flux that does not change implies $d\Phi_B/dt = 0$ and, according to Faraday's law, $\mathcal{E} = 0$ (still no induced EMF).

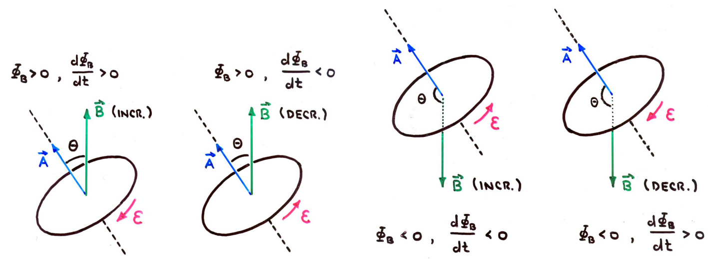
The loop on the right is positioned in a region of a magnetic field that is also directed vertically up but gradually increasing in strength. The magnetic flux is positive and increasing with time, now implying $d\Phi_B/dt > 0$ and $\mathcal{E} < 0$.

In general, the induced EMF is associated with an induced electric field in the region of time-varying magnetic field. The induced EMF is related to the induced electric field by the loop integral stated on slide. The integration is carried out in the positive loop direction (here ccw).

Knowing that $\mathcal{E} < 0$, we conclude that the dot product $\vec{E} \cdot d\vec{l}$ is mostly negative, which implies that the directions of $\vec{E}$ and $d\vec{l}$ tend to be opposite (as indicated on the slide).

Area – Field – Flux – EMF (2)

$$\Phi_B = \int \vec{B} \cdot d\vec{A}, \qquad \mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$



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This page presents variations of the theme discussed on the previous page.

The same convention is adopted for all four loops. The choice of area vector implies that ccw is the positive loop direction.

Given that convention, the magnetic flux is positive through the first two loops and negative through the last two. Why? Because the angle between vectors $\vec{A}$ and $\vec{B}$ is acute on the left and obtuse on the right, implying a positive and a negative cosine, respectively.

However, whether the flux itself is positive or negative does not determine the sign of the induced EMF. What matters is whether the flux increases or decreases.

The flux increases when it is positive and gets more positive or when it is negative and gets less negative. That is the case in the first and fourth loop, respectively.

Conversely, the flux decreases when it is positive and gets less positive or when it is negative and gets more negative, which is the case in the second and third loop, respectively.

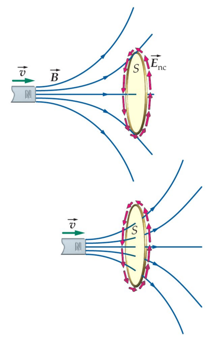
It is this attribute that determines whether the induced EMF is negative (clockwise) as realized in loops 1 and 4 or positive (counterclockwise) as realized in loops 2 and 3.

Recall the convention adopted at the beginning: positive is ccw and negative cw. In loops 1 and 4 the induced EMF is negative (meaning cw) because $d\Phi_B/dt$ is positive. In loops 2 and 3 the opposite is the case.

Faraday's Law of Induction (2)

Here the change in magnetic flux Φ_B is caused by a moving bar magnet.

- Assume area vector $\vec{A}$ of loop pointing right.
  Hence positive direction around loop is clockwise.
- Motion of bar magnet causes $\frac{d\Phi_B}{dt} > 0$.
- Faraday's law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$.
- Induced EMF is in negative direction, $\mathcal{E} < 0$,
  which is counterclockwise.
- Induced EMF reflects induced electric field: $\mathcal{E} = \oint_C \vec{E} \cdot d\vec{\ell}$.
- Field lines of induced electric field are closed.
- Faraday's law is a dynamics relation between electric and magnetic fields: $\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}$.



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A change in strength of magnetic field as used on the previous page can be implemented by a moving bar magnet as shown on this slide. It is not a uniform magnetic field in this case but, as mentioned earlier, the only thing that matters for the determination of the induced EMF is how the magnetic flux through a loop of our choice changes in time.

Given the convention adopted on the slide, the magnetic flux through the ring increases when the bar magnet is moved toward the right. Hence the induced EMF is ccw, which is the physical meaning of negative for that convention.

If the ring were made of copper, an induced current would flow in ccw direction. If the ring were to have a resistance R and the induced EMF inferred from Faraday's law were $\mathcal{E}$, then the induced current would be $I = \mathcal{E}/R$.

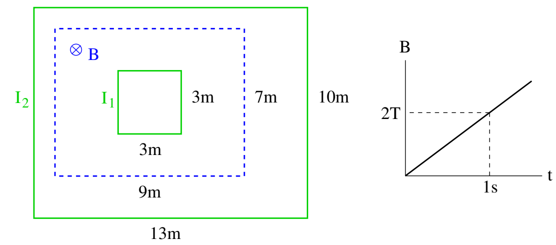
Let us emphasize again that the primary quantity that is being induced is an electric field $\vec{E}$. This happens wherever there is a time-varying magnetic field $\vec{B}$. The induced EMF is determined from the induced electric field via the loop integral shown on the slide.

We will learn later that the converse holds true as well. In any region where there is a time-varying electric field, a magnetic field is being induced. This effect will be covered in the completion of Ampère's law announced earlier.

Magnetic Induction: Application (3)

A uniform magnetic field $\vec{B}$ pointing into the plane and increasing in magnitude as shown in the graph exists inside the dashed rectangle.

- Find the magnitude (in amps) and the direction (cw/ccw) of the currents I_1, I_2 induced in the small conducting square and in the big conducting rectangle, respectively. Each conducting loop has a resistance $R = 9\Omega$



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This applications of Faraday's law involves a time-varying magnetic field of uniform strength across the dashed rectangle and zero field outside, a situation that can be well approximated by a long solenoid with rectangular cross section.

A conducting square is embedded in the region of field and a bigger conducting rectangle surrounds the region of field.

The main point to made in this application is that we must take Faraday's law seriously in the following sense. The induced EMF in a loop only depends on the rate of magnetic flux change through the loop, irrespective of whether or not the loop is in direct contact with the magnetic field.

For both conductors the rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = A\frac{dB}{dt}, \quad \frac{dB}{dT} = 2\text{T/s},$$

where A is the area inside the loop where a magnetic field is present. For the square conductor we must use $A_S = 9\text{m}^2$ and for the rectangular conductor $A_R = 63\text{m}^2$.

If we choose an area vector pointing $\otimes$, i.e. in the direction of the magnetic field $\vec{B}$, then the flux is positive and the cw is the positive loop direction. The induced EMF and current in each conductor thus become,

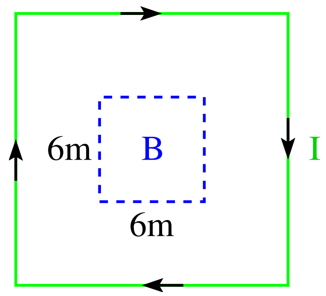
$$\mathcal{E}_S = -\frac{d\Phi_B^{(S)}}{dt} = -(9\text{m}^2)(2\text{T/s}) = -18\text{V}, \quad I_S = \frac{\mathcal{E}_S}{R} = -2\text{A} \quad (\text{ccw}).$$

$$\mathcal{E}_R = -\frac{d\Phi_B^{(R)}}{dt} = -(63\text{m}^2)(2\text{T/s}) = -126\text{V}, \quad , \quad I_R = \frac{\mathcal{E}_R}{R} = -14\text{A} \quad (\text{ccw}).$$

Magnetic Induction: Application (4)

A magnetic field $\vec{B}$ of increasing strength and directed perpendicular to the plane exists inside the dashed square. It induces a constant clockwise current $I = 8\text{A}$ in the large conducting square with resistance $R = 9\Omega$.

- If $\vec{B} = 0$ at time $t = 0$, find the direction ($\odot, \otimes$) and magnitude of $\vec{B}$ at time $t = 5\text{s}$.



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The situation here is very similar, but here we must reason backward.

We are told that a uniform and time-varying magnetic field inside the dashed border induces a steady and clockwise current in conducting square surrounding the region.

We begin by adopting the convention that the given cw direction of the current is positive. This implies, by right-hand rule, that the area vector is directed $\otimes$. The induced EMF that drives the current must then be positive and, according to Faraday's law, the rate of flux change through the loop negative:

$$\mathcal{E} = RI = (9\Omega)(8\text{A}) = 72\text{V} \quad \Rightarrow \frac{d\Phi_B}{dt} = A\frac{dB}{dt} = -72\text{Wb/s}.$$

Note that V and Wb/s are equivalent SI units.

Given that the relevant area is $A = (6\text{m})^2 = 36\text{m}^2$, we can infer the (constant) rate at which the magnetic field changes and, by integration, the magnetic field as a function of time:

$$\frac{dB}{dt} = -2\text{T/s} \quad \Rightarrow \; B(t) = B(0) - 2t\,\text{T/s}.$$

If the initial field is zero, its value at 5 seconds is $B(5\text{s}) = -10\text{T}$. Now we recall that positive means $\otimes$ in the convention adopted. Hence the magnetic field direction is $\odot$.

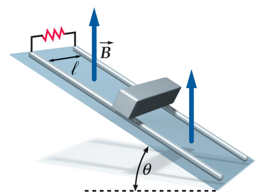
Magnetic Induction: Application (13)

A rod of length ℓ, mass m, and negligible resistance slides without friction down a pair of parallel conducting rails, which are connected at the top of the incline by a resistor with resistance R. A uniform vertical magnetic field $\vec{B}$ exists throughout the region.

(a) Identify the forces acting on the rod when it slides down with velocity v.
(b) Determine the velocity for which all forces acting on the rod are in balance.

Determine the direction of the induced current from

(c) the magnetic force acting on the charge carriers in the rod,
(d) the change in magnetic flux through the conducting loop,
(e) Lenz's law.



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(a) There are three forces acting on the rod: (i) the weight $\vec{w} = m\vec{g}$ pointing vertically down, a normal force $\vec{N}$ perpendicular to the plane of the rails toward the upper right, and a magnetic force $\vec{F} = I\vec{l} \times \vec{B}$ in the plane of the rails toward the upper left.

For the last force, we have assumed that the motion of the rod induces a cw current, implying that the vector $\vec{l}$ is along the rod toward the front. Let us justify this assertion and calculate the induced current I before we continue with part (b).

(d) Consider the magnetic flux Φ_B through the rectangle. The area increases at the rate $dA/dt = vl$. If we choose the area vector $\vec{A}$ up and right (like $\vec{N}$ is), then ccw is the positive loop direction and the flux increases at the rate $d\Phi_B/dt = vBl\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

Faraday's law then dictates that the induced current is cw and has the value $I = vBl\cos\theta/R$. This determines the direction of $\vec{l}$ toward the front as asserted earlier.

(b) Balancing the three forces parallel and perpendicular to the plane of the rails requires that $N = mg\cos\theta$ and $IBl\cos\theta = mg\sin\theta$ with $I = vBl\cos\theta/R$. We conclude that the rod will slide down at constant velocity,

$$v = \frac{mgR\sin\theta}{B^2l^2\cos^2\theta}.$$

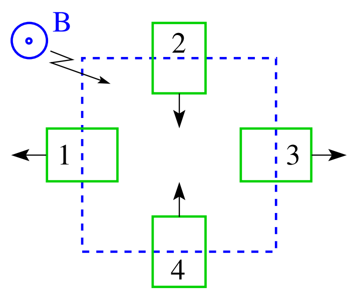
(c) The current direction can be determined directly from motional EMF: positive mobile charge carriers inside the sliding rod will be pushed toward the front by the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$. That causes the cw current.

(e) We postpone this part to page 2 of the next lecture, where we introduce Lenz's rule.

Magnetic Induction: Application (5)

A uniform magnetic field $\vec{B}$ pointing out of the plane exists inside the dashed square. Four conducting rectangles 1,2,3,4 move in the directions indicated.

- Find the direction (cw,ccw) of the current induced in each rectangle.



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This is the quiz for lecture 27.

Hint: Start with adopting a convention. There are two options:

1. $\vec{A}$ is directed $\odot$, implying ccw is positive.
2. $\vec{A}$ is directed $\otimes$, implying cw is positive.

In convention 1 (2) the flux through each rectangle is positive (negative). In either case, what matters is not the flux itself but whether the flux increases or decreases.

It increases when it is positive and gets more positive or when it is negative and gets less negative. Conversely, it decreases when it is positive and gets less positive or when it is negative and gets more negative.

Why does the flux change? Not because the field changes as in the previous applications but because the rectangle moves into or out of the region of magnetic field.

The result for the direction (cw, ccw) of the induced current in each rectangle is independent of the convention you adopt.