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25. Gauss's law for the magnetic field. Ampere's law with applications

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PHY204 Lecture 25 [rln25]

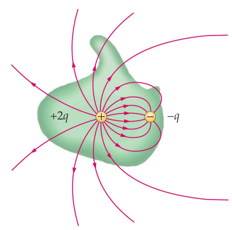
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \qquad \text{i.e.} \qquad \Phi_E = \frac{Q_{in}}{\epsilon_0} \qquad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12}\mathrm{C^2N^{-1}m^{-2}}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



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We begin this lecture by reviewing one Maxwell equation (Gauss's law for the electric field) and then continue on the next two pages by introducing a second Maxwell equation (Gauss's law for the magnetic field) plus a restricted version of a third Maxwell equation (Ampère's law).

The fourth Maxwell equation (Faraday's law) will be introduced later, in lecture 27. The full set of Maxwell's equations including the general version of Ampère's law will be presented in lecture 35.

Recall the nature of electric flux. It is not a vector. electric flux is constructed from dot products, $\vec{E} \cdot d\vec{A}$ of the electric-field vector and an element-of-area vector.

Gauss's law for the electric field states a relationship between the electric flux through a closed surface of our choice and the net electric charge inside. The area vectors $d\vec{A}$ always point toward the outside of a closed surface.

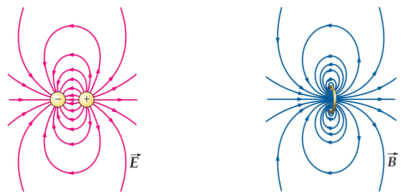
Gauss's Law for Magnetic Field

The net magnetic flux Φ_B through any closed surface is equal to zero:

$$\oint \vec{B} \cdot d\vec{A} = 0.$$

There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

The figures below illustrate Gauss's laws for the electric and magnetic fields in the context of an electric dipole (left) and a magnetic dipole (right).



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Gauss's law for the magnetic field states a corresponding relationship but magnetic charges (monopoles) have never been found. Presumably they do not exist (like unicorns). Therefore, the magnetic flux through a closed surface always vanishes.

We denote magnetic flux by the symbol Φ_B to distinguish it from the electric flux Φ_E. Magnetic flux through open surfaces plays an important part in what comes later (magnetic induction).

All magnetic field lines such as those shown on the lower right of the slide are always closed in themselves. In this instance, the source of the magnetic field is a magnetic dipole in the form of a current ring.

The electric field lines associated with an electric dipole, as shown on the lower left, behave differently. They begin at the positive charge and end at the negative charge.

Not all electric field lines begin or end in electric charges. There are electric field lines that close in themselves as do all magnetic field lines. We will first encounter such electric field lines in the context of Faraday's law (lecture 27).

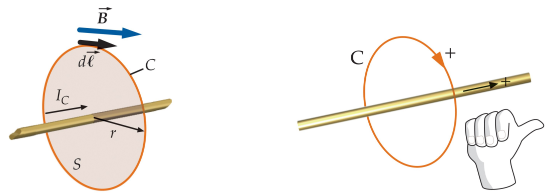
Ampère's Law (Restricted Version)

The circulation integral of the magnetic field $\vec{B}$ around any closed curve (loop) C is equal to the net electric current I_C flowing through the loop:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C, \qquad \text{with} \quad \mu_0 = 4\pi \times 10^{-7}\mathrm{Tm/A}$$

The symbol $\oint$ denotes an integral over a closed curve in this context.
Note: Only the component of $\vec{B}$ tangential to the loop contributes to the integral.

The positive current direction through the loop is determined by the right-hand rule.



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The integral $\oint \vec{B} \cdot d\vec{A}$ in Gauss's law and the integral $\oint \vec{B} \cdot d\vec{l}$ in Ampère's law may look similar in symbolic notation, but they mean very different things.

The former is an integral over a closed surface and the latter over a closed line (a loop). In the former case, the vector $d\vec{A}$ is perpendicular to the surface, in the latter case, the vector $d\vec{l}$ is tangential to the loop.

Ampère's law states a relationship between the loop integral and the net electric current that flows through the loop.

Just as electric charges inside Gaussian surfaces can be positive or negative, there are positive and negative currents flowing through the loop. Which is which is determined by yet another rendition of the right-hand rule as illustrated on the slide.

Curl the fingers of your right hand in the direction of the loop integration, then any current that flows in the direction of your thumb must be counted positively and all currents that flow in the opposite direction must be counted negatively.

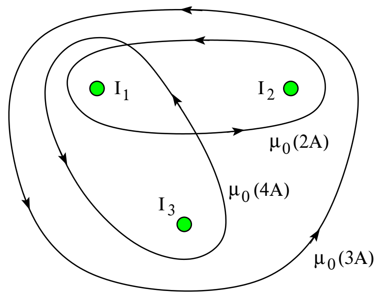
In what sense is this version of Ampère's law restricted? The full version has a second term on the right-hand side, associated with time-varying phenomena, as we shall see later. The restricted version is valid for steady currents and static fields.

Note that Ampère's law works both ways, just as Gauss's law does. We can use it for the calculation of the magnetic field generated a specific current configuration or for determining the current flowing through a conductor surrounded by a known magnetic field.

Ampère's Law: Application (1)

The line integrals $\oint \vec{B} \cdot d\vec{s}$ along the three Amperian loops are as indicated.

- Find the direction ($\odot, \otimes$) and the magnitude of the currents I_1, I_2, I_3.



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Here we consider the magnetic field generated by three unknown currents I_1, I_2, I_3 flowing perpendicular to the page. This magnetic field has been mapped out experimentally for points on the plane of the page.

Using these magnetic-field data, we calculate loop integrals, $\oint \vec{B} \cdot d\vec{s}$, for the three loops shown. The result of each integral is a number in units Tesla meter [Tm], which we conveniently write as a product of a current [A] and the permeability constant [Tm/A].

We now use Ampère's law to determine the three currents. Note that all three loop integrals have been carried out in counterclockwise direction, implying that currents directed out of the page ($\odot$) are counted positively and currents directed into the page ($\otimes$) are counted negatively.

Relating loop integral to currents flowing through the loop thus yields three linear equations,

$$I_1 + I_2 + I_3 = 3\text{A}, \quad I_1 + I_2 = 2\text{A}, \quad I_1 + I_3 = 4\text{A}.$$

The first equation pertains to the big loop that envelops all three currents. We have three equations for three unknowns. They are readily worked out. The solution reads,

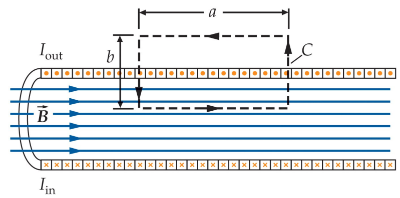
$$I_1 = 3\text{A}, \quad I_2 = -1\text{A}, \quad I_3 = 1\text{A}.$$

The negative I_2 means that -1A is flowing out of the page or, equivalently, that $+1$A is flowing into the page.

Ampère's Law: Magnetic Field Inside a Long Solenoid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the rectangular Amperian loop shown.

- Magnetic field inside: strong, uniform, directed along axis.
- Magnetic field outside: negligibly weak.
- Number of turns per unit length: n.
- Total current through Amperian loop: $I_C = nIa$ (I is the current in the wire).
- Ampère's law applied to rectangular loop: $Ba = \mu_0 nIa$.
- Magnetic field inside: $B = \mu_0 nI$.



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The magnetic field inside a very long solenoid is strong and uniform. It is very weak on the outside except near the ends of the coil. If we wish to calculate the strength of $\vec{B}$ inside the solenoid, Ampère's law provides the quickest answer.

The slide shows a cut through the solenoid along its axis. The current I in the tightly wound wire is directed out of the page on top and into the page at the bottom. This produces, according the the right-hand rule applied to one nearly circular turn of wire, a magnetic field directed from left to right inside the solenoid.

In order to evaluate the left-hand side of Ampère's law we must pick a loop around which we integrate $\vec{B} \cdot d\vec{l}$. We pick the dashed rectangle shown and integrate counterclockwise around it. The horizontal side on top does not contribute because there is no magnetic field. The two vertical sides do not contribute because there is either no field or $\vec{B}$ is perpendicular to $d\vec{l}$. The contribution from the horizontal side inside the solenoid is Ba.

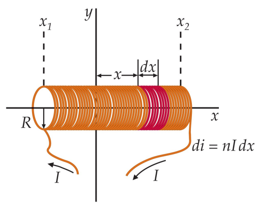
For the right-hand side of Ampère's law we need to know the net current that flows through the loop. One specification of a solenoid is the number n of turns of wire per unit length. Each turn inside the rectangle contributes a positive current I to I_C. The number of turns inside the rectangle is na.

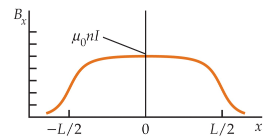
Ampère's law for the rectangular loop is stated as the fifth item on the slide. It can be solved for the unknown magnetic-field strength B as is done in the last item. Note that the result remains the same if the rectangle is moved up or down a bit or left and right, thus confirming that the field inside is uniform.

Magnetic Field on the Axis of a Solenoid

- Number of turns per unit length: $n = N/L$
- Current circulating in ring of width dx': $nIdx'$
- Magnetic field on axis of ring: $dB_x = \frac{\mu_0(nIdx')}{2} \frac{R^2}{[(x-x')^2+R^2]^{3/2}}$
- Magnetic field on axis of solenoid:

$$B_x = \frac{\mu_0 nI}{2} R^2 \int_{x_1}^{x_2} \frac{dx'}{[(x-x')^2+R^2]^{3/2}} = \frac{\mu_0 nI}{2}\left(\frac{x-x_1}{\sqrt{(x-x_1)^2+R^2}} - \frac{x-x_2}{\sqrt{(x-x_2)^2+R^2}}\right)$$





tsl215

If we wish to calculate the field inside a solenoid of finite length, we can do that by applying the law of Biot and Savart from lecture 23. We limit our goal to calculating the field at points on the axis of the coil (the x-axis).

We use the result from page 5 in lecture 23 for a circular current and apply it to an infinitesimally short segment of solenoid. Then we add up the contributions, which amounts to an integration from one end to the other end of the solenoid.

The result looks complicated. It is a function of x, which is the field point on the axis. The coordinates x_1, x_2 mark the ends of the solenoid.

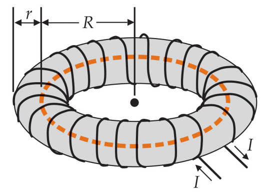
The graph on the lower right shows how the field strength various along the axis of the solenoid. Its maximum value is at the center between the two ends. At either end, the field becomes weaker quickly.

If we make the solenoid longer and longer, then B inside varies less and less. The curve becomes flatter and flatter. The nearly uniform field approaches the value $B = \mu_0 nI$. This is the value found on the previous page by a different method for an infinitely long solenoid.

Ampère's Law: Magnetic Field Inside a Toroid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular Amperian loop shown.

- Magnetic field inside: directed tangentially with magnitude depending on R only.
- Magnetic field outside: negligibly weak.
- Number of turns: N.
- Total current through Amperian loop: $I_C = NI$ (I is the current in the wire).
- Ampère's law applied to circular loop: $B(2\pi R) = \mu_0 NI$.
- Magnetic field inside: $B = \frac{\mu_0 NI}{2\pi R}$.



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When we bend a solenoid into a circle such that the two ends meet, we have what is called a toroid. This configuration avoids the stray magnetic fields that are present at the ends of the solenoid. The magnetic field is now completely contained but it is no longer uniform.

The magnetic field lines are now circles inside the turns of the wire. The dashed circle is one field line. The direction of the field depends on the direction of the current sent through the turns wound around the toroid.

Calculating the strength of the magnetic field at radius R from the center of the toroid is another simple application of Ampère's law. In this case we choose the circular loop along the dashed field line in magnetic-field direction.

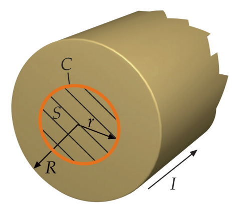
The last item on the slide states the the dependence of the magnetic-field strength on the radius R. This result holds, course only for values R representing field points inside the torus.

Ampère's Law: Magnetic Field Inside a Wire

Consider a long, straight wire of radius R.
The current I is distributed uniformly over the cross section.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r < R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Line integral: $\oint \vec{B} \cdot d\vec{\ell} = B(2\pi r)$.
- Fraction of current inside loop: $\dfrac{I_C}{I} = \dfrac{\pi r^2}{\pi R^2}$.
- Magnetic field at radius $r < R$: $B = \dfrac{\mu_0 I_C}{2\pi r} = \dfrac{\mu_0 I r}{2\pi R^2}$.
- B increases linearly with r from zero at the center.
- Magnetic field at the perimeter: $B = \dfrac{\mu_0 I}{2\pi R}$.



tsl243

On this page and the next, we use Ampère's law to calculate the magnetic field generated by the current in a long, straight wire. We already know the answer if the field point is outside a thin wire. We found that answer by an application of the law of Biot and Savart (see page 7 on lecture 23).

The slide on this page walks us through the derivation for the case where the (circular) loop C is inside the wire. The result is presented as the fourth item.

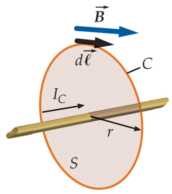
The result tells us that the field is zero at the center of the wire and increases linearly with radial distance r from the center, reaching the value quoted as the last item when r has reached the perimeter at radius R.

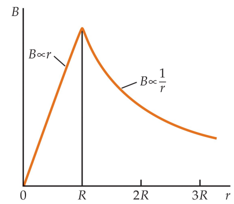
Ampère's Law: Magnetic Field Outside a Wire

Consider a long, straight wire of radius R with current I.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r > R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Current inside loop: $I_C = I$.
- Ampère's law applied: $B(2\pi r) = \mu_0 I$.
- Magnetic field at radius $r > R$: $B = \dfrac{\mu_0 I}{2\pi r}$.





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This slide continues the derivation for field points outside the wire. Now the loop radius r is larger than the radius R of the wire.

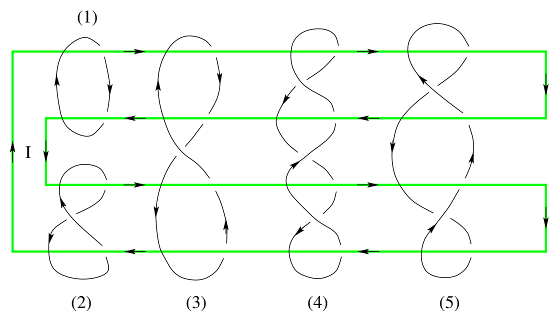
The main difference from the previous case is that as we increase the radius r of the loop, the amount of current that flows through the loop does no longer change. The result, presented as the fourth item, is the familar result from lecture 23 (page 7).

The graph on the lower right of the slide shows the field strength as a function of radial distance r from the center of the wire. The strongest field is realized at the surface of the wire.

Ampère's Law: Application (2)

An electric current I flows through the wire in the direction indicated.

- Determine for each of the five Amperian loops whether the line integral $\oint \vec{B} \cdot d\vec{s}$ is positive, negative, or zero.



tsl240

This is the quiz for lecture 25.

This mental exercise is also a physical exercise for your right hand. It requires some acrobatics, especially for the twisted loops.

You must follow the fingers of your right hand along the outside of the loop in the direction of the arrows. While you are doing this, watch the direction of your thumb. It tells you in every region of the loop which currents to count positively and which to count negatively.

Of course, it's always the same current I flowing though the green wire. However, in some loop vicinities it contributes positively and other loop vicinities negatively.