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24. Magnetic field of current configurations. A glimpse of relativity

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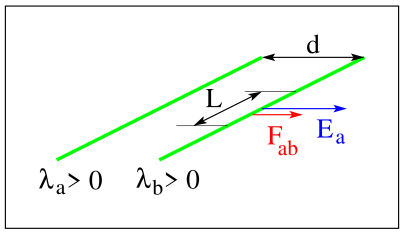
PHY204 Lecture 24 [rln24]

Force Between Parallel Lines of Electric Charge

- Electric charge densities: λ_a, λ_b
- Electric field generated by line a: $E_a = \frac{1}{2\pi\epsilon_0}\frac{\lambda_a}{d}$
- Electric force on segment of line b: $F_{ab} = \lambda_b L E_a$
- Electric force per unit length (repulsive): $\frac{F_{ab}}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_a\lambda_b}{d}$



tsl231

We begin this lecture by examining some consequences of the laws of electricity and magnetism introduced earlier. The conclusions will force us to rethink the concept of motion in space.

We begin by considering two very long, thin, parallel rods, a distance d apart and both uniformly charged. The charge densities λ_a and λ_b are positive, which causes the rods to experience a repulsive interaction force.

We calculate the force per unit length exerted by rod a on rod b in two steps. The second line on the slide reproduces the result from early in the course for the electric field E_a generated by rod a at radius d from it. The electric field E_a has the direction shown: away from the positively charged rod a.

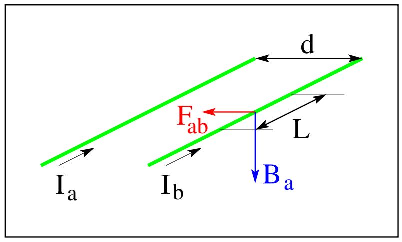
The third line is the electric force experienced by a segment of length L on rod b due to the electric field E_a generated by rod a.

When we combine the results from lines 2 and 3, we obtain, on the fourth line, an expression for the repulsive electric force per unit length between the two rods.

This simple result is far less innocent than might be apparent. It is a result of electrostatics, meaning that the charged rods are at rest. At rest relative to each other, certainly. At rest relative to what else? The observer who measures the force?

Force Between Parallel Lines of Electric Current

- Electric currents: I_a, I_b
- Magnetic field generated by line a: $B_a = \frac{\mu_0}{2\pi}\frac{I_a}{d}$
- Magnetic force on segment of line b: $F_{ab} = I_b L B_a$
- Magnetic force per unit length (attractive): $\frac{F_{ab}}{L} = \frac{\mu_0}{2\pi}\frac{I_a I_b}{d}$



tsl232

Next we consider two very long, thin, parallel, conducting rods, a distance d apart, both carrying currents. The currents I_a and I_b flow in the same direction, which causes the rods to experience an attractive interaction force.

We again calculate the force per unit length exerted by rod a on rod b in two steps. The second line reproduces the result from lecture 23 for the magnetic field B_a in the direction shown generated by rod at radius d.

The third line is the magnetic force on a segment of length L on rod b due to the electric field generated by rod a.

When we combine the results from lines 2 and 3, we obtain, on the fourth line, an expression for the attractive magnetic force per unit length between the two rods.

If we replace the conducting rods, which are electrically neutral, by the charged rods from the previous page and have them move at constant velocity v in the direction of the arrows, the moving charges represent currents $I_a = \lambda_a v$ and $I_b = \lambda_b v$, respectively, in the two rods. Now the repulsive electrical force is weakened by an attractive magnetic force:

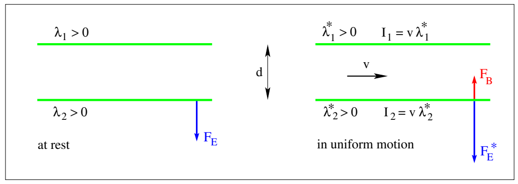
$$\frac{F_{ab}}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_a\lambda_b}{d} - \frac{\mu_0}{2\pi}\frac{(\lambda_a v)(\lambda_b v)}{d}.$$

Consider two charged rails at rest and a jogger running at velocity $-v$ relative to them. In the coordinate system where the rods are at rest, only the electrical force is present, but in the coordinate system where the jogger is at rest, the rods move with velocity v and both forces are present.

The net force between the rods, a measurable quantity, is different depending on whether the rails move or the jogger moves. Can that be? It would imply that we can measure absolute motion. If we deny this possibility, as Einstein did more than a century ago, something has to give.

Is There Absolute Motion?

Forces between two long, parallel, charged rods



- $\frac{F_E}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1\lambda_2}{d}$ (left), $\frac{F_E^*}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1^*\lambda_2^*}{d}$, $\frac{F_B}{L} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$, (right)
- $\frac{F_E^* - F_B}{L} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1^*\lambda_2^*}{d}\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{2\pi\epsilon_0}\frac{\lambda_1\lambda_2}{d}$
- $c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.998 \times 10^8\mathrm{ms}^{-1}$ (speed of light)
- $\lambda_1^* = \frac{\lambda_1}{\sqrt{1 - v^2/c^2}}$, $\lambda_2^* = \frac{\lambda_2}{\sqrt{1 - v^2/c^2}}$ (due to length contraction)

tsl211

What has to give (among other things) is the concept of absolute space, which takes for granted the existence of rigid measuring sticks.

If we insist that the force between the two rods is the same as measured by the first observer, who sees them at rest, and by the second observer, who sees them in motion, we must allow the rails to contract in length when seen in motion. The contraction increases the charge per unit length from λ_1 and λ_2 to λ_1^* and λ_2^*, respectively.

The first item on the slide gives (on the left) the electric force F_E/L measured by an observer who sees the rails at rest. It also gives (on the right) the electric force F_E/L and the magnetic force F_B/L measured by an observer who sees the two rails in motion.

In the magnetic force, we use $I_1 = \lambda_1 v$, $I_2 = \lambda_2 v$ and the substitution $\mu_0 = 1/(\epsilon_0 c^2)$, where c is the speed of light (see third item). This brings the combined force into the form shown on the second item.

If we equate the combined force (measured by the second observer) with electric force (measured by the first observer), we find the change in the charge densities as given in the last item.

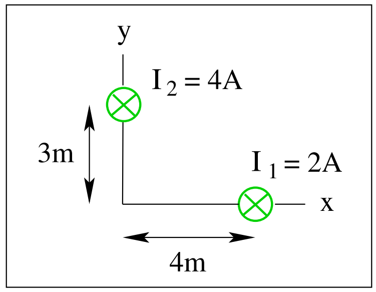
Augmented charge densities arise when the same charge is spread over shorter distances. The length contraction as seen by the second observer,

$$\Delta l^* = \Delta l\sqrt{1 - \frac{v^2}{c^2}},$$

does not amount to much unless the velocity v is a significant fraction of the speed of light c. However, length contraction is a universal phenomenon (one of several) to which objects seen in motion are subjected to. Another universal phenomenon is the speed limit, $v < c$, for objects with mass, which we will discuss in the last lecture of this course.

Consider two infinitely long straight currents I_1 and I_2 as shown.

- Find the components B_x and B_y of the magnetic field at the origin of the coordinate system.



tsl234

Returning to quantitative applications of magnetic-field calculations for simple configurations, we consider, on this page and the next, two very long straight wires oriented perpendicular to the xy-plane.

We see that both currents I_1 and I_2 are directed into the plane. At a generic point in the xy-plane, each current generates a magnetic field with x- and y-components, but no z-component. This much we can tell up front.

At the origin of the coordinate system, the magnetic field generated by current I_1 is vertical and pointing $\uparrow$. At the same location, the magnetic field generated by current I_2 is horizontal and pointing $\leftarrow$. We draw these conclusions by employing the right-hand rule with our thum pointing in current direction, which is into the page.

Each of the two Cartesian magnetic-field components B_x and B_y are thus generated by only one current, the former by I_2 and the latter by I_1.

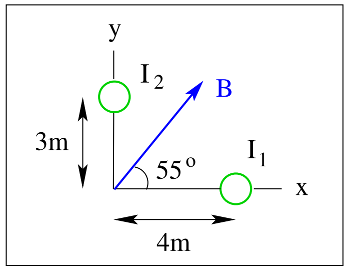
With the expression for the magnetic field generated by a long straight current at our fingertips, we can write,

$$B_x = -\frac{\mu_0}{2\pi}\frac{4\text{A}}{3\text{m}} = -2.67 \times 10^{-7}\text{T}, \quad B_y = \frac{\mu_0}{2\pi}\frac{2\text{A}}{4\text{m}} = 1.00 \times 10^{-7}\text{T}.$$

The magnetic field $\vec{B} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}$ is directed left and up.

Two straight electric currents I_1 and I_2 of infinite length directed perpendicular to the xy-plane generate a magnetic field of magnitude $B = 6.4 \times 10^{-7}$T in the direction shown.

- Find the magnitude and direction $(\odot, \otimes)$ of each current.



tsl235

Here is the reverse of the application from the previous page. The orientation of the two wires and their positions in the xy-plane are the same. However, we do not know the two currents I_1 and I_2, not even if they are directed in $(\odot)$ or out $(\otimes)$.

We know instead the magnitude and direction of the magnetic field **B** the two currents generate in combination at the origin of the coordinate system. The direction must be in the xy-plane. We can write $\vec{B} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}$.

Just as on the previous page, current I_1 only produces a vertical magnetic field and current I_2 only a horizontal field at the field point.

Hence we can write,

$$B_x = B\cos 55^\circ = 3.67 \times 10^{-7}\mathrm{T} = \frac{\mu_0}{2\pi}\frac{I_2}{3\mathrm{m}},$$

$$B_y = 5.24 \times 10^{-7}\mathrm{T} = B\sin 55^\circ = \frac{\mu_0}{2\pi}\frac{I_1}{4\mathrm{m}}.$$

The two equations are readily solved for the unknown currents I_2 and I_1.

The two current direction are determined by the right-hand rule. Curl the fingers of your right hand around circle marked I_2 such that they point to the right at the field point. Your thumb will point out of the page.

Then curl the fingers of your right hand around circle marked I_1 such that they point up at the field point. Your thumb will point into the page.

Thus confirm the final results,

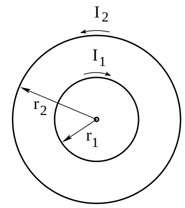
$$I_2 = 5.51\mathrm{A} \quad \odot, \qquad I_1 = 10.5\mathrm{A} \quad \otimes\,.$$

Unit Exam III: Problem #1 (Spring '08)

Consider two circular currents $I_1 = 3\text{A}$ at radius $r_1 = 2\text{m}$ and $I_2 = 5\text{A}$ at radius $r_2 = 4\text{m}$ in the directions shown.

(a) Find magnitude B and direction ($\odot, \otimes$) of the resultant magnetic field at the center.

(b) Find magnitude μ and direction ($\odot, \otimes$) of the magnetic dipole moment generated by the two currents.



Solution:

(a) $B = \dfrac{\mu_0(3\text{A})}{2(2\text{m})} - \dfrac{\mu_0(5\text{A})}{2(4\text{m})} = (9.42 - 7.85) \times 10^{-7}\text{T}$

$\Rightarrow \ B = 1.57 \times 10^{-7}\text{T} \quad \otimes$

(b) $\mu = \pi(4\text{m})^2(5\text{A}) - \pi(2\text{m})^2(3\text{A}) = (251 - 38)\text{Am}^2$

$\Rightarrow \ \mu = 213\text{Am}^2 \quad \odot$

tsl381

You have seen this slide before two lectures ago. We were ready to solve part (b) then. Now we are ready to also solve part (a).

In the previous lecture we have calculated the magnetic field generated by a current I along a circular wire of radius R, using the law of Biot and Savart. The magnitude of the field at the center of the circle is,

$$B = \frac{\mu_0 I}{2R}.$$

The direction is perpendicular to the plane, out of the page ($\odot$) if the current is counterclockwise (ccw) and into the page ($\otimes$) if it is clockwise (cw).

In this application we have two circular currents in opposite direction, I_1 cw and I_2 ccw. In consequence, the two fields have opposite direction as well, $\vec{B}_1 \otimes$ and $\vec{B}_2 \odot$.

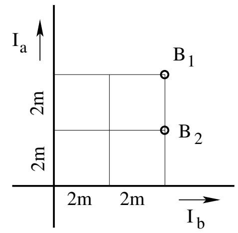
It turns out that $\vec{B}_1$ is stronger than $\vec{B}_2$, implying that the resultant field is directed $\otimes$.

On the slide, $\otimes$ is (tacitly) declared to be the positive direction for part (a). The result is independent of that convention.

Intermediate Exam III: Problem #1 (Spring '06)

Consider two infinitely long, straight wires with currents of equal magnitude $I_a = I_b = 5\text{A}$ in the directions shown.

Find the direction (in/out) and the magnitude of the magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ at the points marked in the graph.



Solution:

- $B_1 = \frac{\mu_0}{2\pi}\left(\frac{5\text{A}}{4\text{m}} - \frac{5\text{A}}{4\text{m}}\right) = 0$ (no direction).
- $B_2 = \frac{\mu_0}{2\pi}\left(\frac{5\text{A}}{2\text{m}} - \frac{5\text{A}}{4\text{m}}\right) = 0.25\mu\text{T}$ (out of plane).

tsl355

In this application, we have two very long wires, perpendicular to each other and positioned in the plane of the paper as shown. Both wires carry a current of 5A in the directions shown.

We know from earlier that the direction of magnetic field generated by any current that flows in the plane of the page has to be perpendicular to that plane, either $\odot$ (out) or $\otimes$ (in). We use the right-hand rule to determine which is the case. The current I_a generates a field in ($\otimes$) to its right. The current I_b generates a field out ($\odot$) above it.

For a long, thin, and straight wire the strength of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi r},$$

where r is the distance of the field point from the wire.

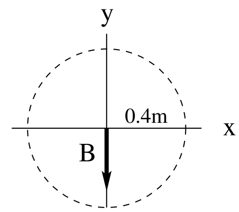
With this information, it is straightforward to determine the fields $\vec{B}_1$ and $\vec{B}_2$ generated by the combination of two currents. The results are worked out on the slide.

Intermediate Exam III: Problem #1 (Spring '05)

An infinitely long straight current of magnitude $I = 6$A is directed into the plane ($\otimes$) and located a distance $d = 0.4$m from the coordinate origin (somewhere on the dashed circle). The magnetic field $\vec{B}$ generated by this current is in the negative y-direction as shown.

(a) Find the magnitude B of the magnetic field.

(b) Mark the location of the position of the current $\otimes$ on the dashed circle.



Solution:

(a) $B = \frac{\mu_0}{2\pi}\frac{I}{d} = 3\mu$T.

(b) Position of current $\otimes$ is at $y = 0,\ x = -0.4$m.

tsl342

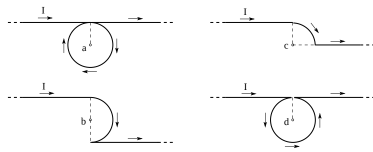
This is an even simpler application of a long, thin, and straight wire carrying a current.

In part (a) we reason forward using the relevant distance d between current and field point.

In part (b) we reason backward using the right-hand rule.

Unit Exam III: Problem #2 (Spring '09)

Two semi-infinite straight wires are connected to a curved wire in the form of a full circle, quarter circle, or half circle of radius $R = 1\text{m}$ in four different configurations. A current $I = 1\text{A}$ flows in the directions shown. Find magnitude B_a, B_b, B_c, B_d and direction ($\odot/\otimes$) of the magnetic field thus generated at the points a, b, c, d.



Solution:

$$B_a = \left|\frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R}\right| = |100\text{nT} + 628\text{nT} + 100\text{nT}| = 828\text{nT} \quad \otimes$$

$$B_b = \left|\frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4R} - \frac{\mu_0 I}{4\pi R}\right| = |100\text{nT} + 314\text{nT} - 100\text{nT}| = 314\text{nT} \quad \otimes$$

$$B_c = \left|\frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{8R} + 0\right| = |100\text{nT} + 157\text{nT}| = 257\text{nT} \quad \otimes$$

$$B_d = \left|\frac{\mu_0 I}{4\pi R} - \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R}\right| = |100\text{nT} - 628\text{nT} + 100\text{nT}| = 428\text{nT} \quad \odot$$

tsl396

From earlier we know that a thin wire bent into a circle of radius R generates a magnetic field $B_\circ$ at its center and a very long, thin straight wire generates a magnetic field $B_|$ a distance r from it if the wires carry a current I:

$$B_\circ = \frac{\mu_0 I}{2R}, \qquad B_| = \frac{\mu_0 I}{2\pi r}.$$

In the application on this slide, we see infinite and semi-infinite wires, as well as wires bent into full circles, half circles and quarter circles.

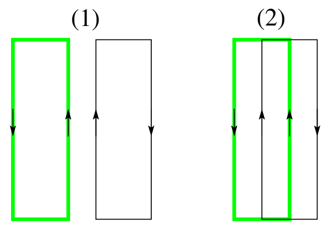
When we go back and examine our derivations of the above expressions via the law of Biot and Savart we realize at once that the field generated by a semi-infinite portion is just half the field generated by the infinitely long wire. Likewise, a semicircle generates half the field of a full circle at the center and a quarter circle generates a half the field of the semicircle.

With this knowledge we can construct the field at points a through d from the parts that contribute. We must heed the fact though that some parts contribute fields out and other parts fields in. The right-hand rule applies to each part.

The convention used in the solution shown is that fields in are counted positively and fields out negatively.

Consider two pairs of rectangular electric currents flowing in the directions indicated.

(a) What is the direction (→, ←) of the magnetic force experienced by the black rectangle in each case?
(b) Which black rectangle experiences the stronger magnetic force?



tsl238

We know from earlier that wires carrying currents in the same direction exert an attractive force on each other and wires carrying currents in opposite direction a repulsive force.

We also know that the force is inversely proportional to the distance between two current-carrying wires.

In this application, we only consider the forces exerted by the long sides of the green rectangle on the long sides of the black rectangle. There are four such forces in each configuration.

Let us name the magnitude of force between closest sides in each configuration F_0. All other distances are either the same or multiples of that distance.

Furthermore, let us count forces to the right (→) positively and forces to the left (←) negatively.

With these conventions in place, the net force on the black rectangle in two configurations becomes,

$$F_1 = F_0\left(-1 + \frac{1}{3} + \frac{1}{3} - \frac{1}{5}\right) \simeq -0.53F_0 \quad \leftarrow$$

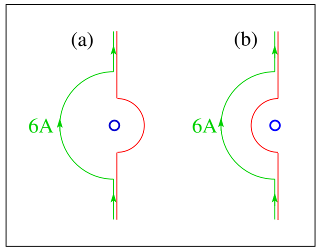
$$F_2 = F_0\left(1 + 1 + 1 - \frac{1}{3}\right) \simeq 2.7F_0 \quad \rightarrow$$

Note that not all attractive (or repulsive) forces are in the same direction.

Magnetic Field Application (9)

Two wires of infinite length contain concentric semicircular segments of radii 1m and 2m, respectively.

- If one of the wires carries a 6A current in the direction indicated, what must be the direction (↑, ↓) and magnitude of the current in the other wire such that the magnetic field at the center of the semicircles vanishes?



tsl226

The magnetic field at the center of a circular current I of radius R is

$$B_c = \frac{\mu_0 I}{2R},$$

directed perpendicular to the plane of the circle in a direction dictated by the right-hand rule.

The 6A semicircular current shown in parts (a) and (b) thus generates a magnetic field $\frac{1}{2}B_c$ directed into the page ($\otimes$).

In part (a) this magnetic field must be compensated by a semi circular current looping around the same field point on the opposite side with half the radius. A semicircle with half the radius produces the same magnetic field if the half the current flows. If a 3A current flows up (↑) through the small semicircle, then it produces a magnetic field $\frac{1}{2}B_c$ directed out of the page ($\odot$).

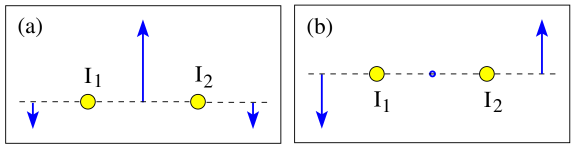
In part (b) the same magnetic field generated by the large semicircle must be compensated by a semicircular current looping around the same field point on the same side with half the radius. A magnetic field of equal magnitude at half radius again requires half the current. If a 3A current flows down (↓) through the small semicircle, then it produces a magnetic field $\frac{1}{2}B_c$ directed out of the page ($\odot$).

The problem statement obscures a simpler way of looking at the situation. The current in the larger semicircle is clockwise. Hence the current in the smaller semicircle must be counterclockwise to have a compensating effect. That holds for both parts (a) and (b).

Magnetic Field Application (2)

The currents I_1, I_2 in two long straight wires have equal magnitude and generate a magnetic field $\vec{B}$ as shown at three points in space.

- Find the directions ($\odot, \otimes$) for I_1, I_2 in configurations (a) and (b).



tsl218

This is the quiz for lecture 24.

Recall the right-hand rule for the magnetic field direction around long, straight currents.

Recall that the strength of magnetic field weakens with distance from the current.