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Hypergraph Colorings, Commutative Algebra, and Gröbner Bases

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HYPERGRAPH COLORINGS, COMMUTATIVE ALGEBRA, AND GRÖBNER BASES

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ABSTRACT

A uniform hypergraph is properly k-colorable if each vertex is colored by one of k colors and no edge is completely colored by one color. In 2008 Hillar and Windfeldt gave a complete characterization of the k-colorability of graphs through algebraic methods. We generalize their work and give a complete algebraic characterization of the k-colorability of $r-$uniform hypergraphs. In addition to general k colorability, we provide an alternate characterization for 2-colorability and apply this to some constructions of hypergraphs that are minimally non-2-colorable.

We also provide examples and verification of minimality for non-2-colorable 5- and 6-uniform hypergraphs. As a further application, we give a characterization for a uniform hypergraph to be conflict-free colorable.

Finally, we provide an improvement on the construction introduced by Abbott and Hanson in 1969, and improved upon by Seymour in 1974.

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CHAPTER 1

Introduction

1.1 Introduction

Many problems in combinatorics have elegant algebraic characterizations and many useful tools in combinatorics are based on algebraic methods. Such tools include the graph polynomial, the Combinatorial Nullstellensatz, and the Stanley-Reisner ideal. These tools allow alternative methods for analyzing combinatorial properties of graphs and hypergraphs by encoding them into polynomial ideals and algebraic varieties. The focus of this thesis is the use of tools from commutative algebra, namely Gröbner bases and radical ideals, to provide a complete algebraic characterization for general colorability of uniform hypergraphs. We also provide computationally supported bounds on specific types of hypergraph colorings. This thesis uses results from commutative algebra, algebraic geometry, enumerative combinatorics, and graph theory. We utilize ideas and tools including: polynomial ideals over algebraically closed fields, zero-dimensional algebraic varieties, Gröbner bases, integer partitions, and Hilbert's Nullstellensatz.

Definition 1.1. *Let $r \geq 2$ be a positive integer. An r-*uniform hypergraph*, $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, is a set of vertices $V(\mathcal{H})$, along with a collection of non-empty subsets of vertices each with cardinality r, called edges, $E(\mathcal{H})$.*

Traditional Graph Theory is the study of 2-uniform hypergraphs. Although some techniques and applications developed in graph theory have been generalized to hypergraphs many of the techniques and theorems do not directly generalize. Hypergraph Theory is an active area in combinatorics, both in direct research, and in application to other areas of mathematics. One of the most thoroughly studied area of graph theory is that of graph colorings. Thus it is natural to generalize the idea of a graph coloring to that of a hypergraph coloring.

Definition 1.2. *Let k be a positive integer and let $\mathcal{H}$ be a uniform hypergraph. A* proper $k-$coloring *of a hypergraph $\mathcal{H}$ is a map, c, from the vertex set of $\mathcal{H}$ to a set of k colors, $\mathcal{C}$:*

$$c : V(\mathcal{H}) \to \mathcal{C}, \quad |\mathcal{C}| = k,$$

where each vertex is assigned exactly one color, and no edge is colored completely by a single color.

There are many applications of hypergraph colorings in diverse areas such as Computer Science, Statistical Physics, and Mathematical Chemistry. It is known that deciding whether an r-uniform hypergraph is k-colorable is NP-hard unless $r = 2$ and $k = 2$ [1] [2].

We show that the k-colorability of an r-uniform hypergraph can be studied via an ideal of a polynomial ring which will be called the colorability ideal. The encoding of the colorability of a hypergraph is done through several different sets of polynomials which will be defined below. Using well known theorems and properties from commutative algebra and algebraic geometry, we show that the colorability ideal of a hypergraph can be decomposed into individual coloring ideals which allow one to test if any given hypergraph is colorable by any desired color scheme.

Colorability of graphs has a rich and extensive history and includes many different techniques. Studying colorability through algebraic methods has been addressed by several authors, including: Bayer, Alon, Tarsi, Lovász, de Loera, Hillar, and Windfeldt (cf. [3] [4] [5] [6] [7] [8] [9] [10]).

An interesting question concerning uniform hypergraph colorings is: "What is the smallest number of edges allowed by a non-k-colorable hypergraph?" In the case of $k = 2$ this is known as Property B, and was introduced by F. Bernstein in [11] and studied by E. W. Miller in [12]. P. Erdős and A. Hajnal later defined $m(r)$ to be the minimum number of edges allowed in an r-uniform hypergraph that is not 2-colorable [13]. Following Erdős and Hajnal, H. L. Abbott and D. Hanson modified this notation to include the number of vertices in the hypergraph, n, so $m_n(r)$ is the least number of edges allowed by a non-2-colorable r-uniform hypergraph on n vertices. As it turns out, determining these values is not an easy task and so far only $m(3) = m_7(3) = 7$ and $m(4)$ are known to be tight. In 1969 Abbott and Hanson gave bounds for $m_n(r)$ with recursive type inequalities in r and n. In 1974 P.D. Seymour improved upon one of these bounds and designed a hypergraph that shows $m(4) = m_{11}(4) \leq 23$ [14]. Recently it has been shown by P. Östergard that $m(4) = m_{11}(4) = 23$ proving that Seymour's construction is optimal [15].

We will show that a generalization of Abbott and Hanson's construction yields minimally non-2-colorable hypergraphs. In addition, we will provide some computational examples for upper bounds for $m_n(r)$, when $r = 5, 6$.

1.2 Algebraic Tools in Graph Theory

Using algebraic techniques to characterize graph theoretic properties dates back to the late 1800's when Hilbert began studying certain classes of homogeneous polynomials which turned out to be what is known as the graph polynomial.

Definition 1.3. *Let $G = (V, E)$ be a graph on $V = \{1, \ldots, n\}$ and define the* graph polynomial*, P_G to be:*

$$P_G = \prod (x_i - x_j) : i \text{ and } j \text{ are adjacent and } i < j.$$

The *Handbook of Combinatorics* contains some excellent surveys on the uses and consequences of the graph polynomial [16].

The techniques we use to address coloring problems are part of a relatively new approach to graph theory and combinatorics. With the development of more powerful computers and more efficient algorithms, it has become possible to address combinatorial problems through these algebraic means. In 1982 D. Bayer introduced a method of determining the 3-colorability of a graph by examining systems of polynomials and applying the division algorithm [3]. Ten years later, the work of N. Alon and M. Tarsi used polynomials to prove several conjectures about the chromatic number of a graph [5]. Also, they gave equivalent conditions for a graph to be not k-colorable; we will generalize this notion to uniform hypergraphs. In 1994 L. Lovász used polynomial ideals to characterize stable sets in graphs [6].

Later, J. de Loera and C. Hillar et al. produced results concerning the algebraic characterization of a graph's colorability [7] [10]. The main tools de Loera and Hillar use in their algebraic characterizations for the colorability of a graph are polynomial ideals and Gröbner bases. Gröbner bases were introduced by B. Buchberger in 1965 and have since become widely used in the study of polynomial ideals [17]. This thesis will generalize the above results to uniform hypergraphs, and will also utilize Gröbner bases.

The algebraic techniques developed by de Loera and Hillar et al. extended here give not only theoretical results, but also provide algorithms for solving specific problems. The process for determining the k-colorability of a hypergraph can be adjusted to detect specified color patterns required by an application. In particular, a coloring pattern known as a conflict-free coloring is addressed for k-colorings. Conflict-free colorings were introduced in connection to work on applications to cellular networks [18] [19].

It is worth noting another line of research utilizing polynomial ideals. Many authors have utilized the rich interplay between hypergraphs and certain monomial ideals to gain insight on the structure of these ideals.

Definition 1.4. *The* edge ideal*, I_E, of a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is defined to be:*

$$I_E = \left\langle \prod_{x_i \in e} x_i : e = (e_1, \ldots, e_r) \in E(\mathcal{H}) \right\rangle.$$

This ideal was first introduced by Villarreal [20]. By studying the associated hypergraph, many interesting results about the structure of the ideal can be achieved. This ideal has other names including the face ideal, or the facet ideal, and is also the Stanley-Reisner ideal of the appropriate simplicial complex. These ideals are not exclusive to hypergraphs; in [21] S. Jacques used tools from graph theory to study these monomial ideals.

1.3 Results

In this thesis we introduce algebraic characterizations of several different types of colorings for a uniform hypergraph $\mathcal{H}$. The main result of this thesis is a generalization of a result by Hillar and Windfeldt [10].

Let the ideals $J_{n,k}, I_{n,k}$, and $I_{G,k}$ be defined as in [10], where K_n denotes the complete graph on n vertices, that is:

$$J_{n,k} = \langle P_G : G = K_{k+1} \cup \{\text{a set of isolated vertices in } [n]\}\rangle.$$

$$I_{n,k} = \langle x_i^k - 1 : i \in [n]\rangle.$$

$$I_{G,k} = I_{n,k} + \left\langle x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1} : \{i,j\} \in E(G) \right\rangle.$$

Theorem 1.1 (Theorem 1.1, [10])**.** *The following statements are equivalent:*

(1) The graph G is not k-colorable.

(2) $\dim_{\mathbb{C}} R/I_{G,k} = 0$ as a vector space.

(3) The constant polynomial 1 belongs to the ideal $I_{G,k}$.

(4) The graph polynomial P_G belongs to the ideal $I_{n,k}$.

(5) The graph polynomial P_G belongs to the ideal $J_{n,k}$.

We give a theorem that generalizes parts 1 through 4 of the above theorem to r-uniform hypergraphs. For 2-colorings we prove the following.

Theorem 1.2. *Let $\mathcal{H}$ be a uniform hypergraph. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I_2(\mathcal{H})$ be the 2-colorability ideal of $\mathcal{H}$ and let $P_{\mathcal{H},2}$ be the 2-color hypergraph polynomial for $\mathcal{H}$. Then following are equivalent:*

(1) The hypergraph $\mathcal{H}$ is not 2-colorable.

(2) The constant 1 is an element of the ideal $I_2(\mathcal{H})$.

(3) $\dim_{\mathbb{C}} R/I_2(\mathcal{H}) = 0$ as a vector space.

(4) The hypergraph polynomial $P_{\mathcal{H},2}$ belongs to the ideal

$$\left\langle x_i^2 - 1 : i \in V(\mathcal{H}) \right\rangle .$$

These equivalent statements rely on an ideal which completely captures the 2-colorability of the hypergraph $\mathcal{H}$

Theorem 1.3. *The polynomials in the ideal $I_2(\mathcal{H})$ have a common solution if and only if $\mathcal{H}$ is properly 2-colorable. We call this ideal the 2-colorability ideal of $\mathcal{H}$.*

Both the 2-colorability ideal and the 2-color hypergraph polynomial will be examined further in Chapter 3. Moreover, in Chapter 4 we completely generalize Theorem 1.1 for uniform hypergraphs with the following theorem.

Theorem 1.4. *Let r, $k \geq 2$ be positive integers. Let $\mathcal{H}$ be an r-uniform hypergraph. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I(\mathcal{H}, k)$ be the k-colorability ideal for $\mathcal{H}$ and let $P_{\mathcal{H},k}$ be the k-color hypergraph polynomial for $\mathcal{H}$. Then following are equivalent:*

(1) The hypergraph $\mathcal{H}$ is not k-colorable.

(2) The constant 1 is an element of the ideal $I(\mathcal{H}, k)$.

(3) $\dim_{\mathbb{C}} R/I(\mathcal{H}, k) = 0$ *as a vector space.*

(4) The hypergraph polynomial $P_{\mathcal{H},k}$ belongs to the ideal C_k.

As with the 2-colorable case, Theorem 1.4 depends on the following ideal which we explore in Chapter 4 along with the k-color hypergraph polynomial.

Theorem 1.5. *The polynomials in the ideal $I(\mathcal{H}, k)$ have a common solution if and only if $\mathcal{H}$ is properly k-colorable. We call this ideal the k-colorability ideal of $\mathcal{H}$.*

We note that Theorems 1.4 and 1.5 generalize Theorems 1.2 and 1.3. The polynomials that define $I_2(\mathcal{H})$ and $I(\mathcal{H}, k)$ are quite different and deserve independent study.

In Chapter 5 we collect several computational results for uniform hypergraphs that follow from the theorems above. We show that the construction introduced by Abbott and Hanson and improved upon by Seymour for 4-uniform hypergraph can be generalized [22], [14]. We then use Theorem 1.3 to provide upper bounds for $m_n(5)$ and $m_n(6)$. Finally we apply our results to Stable Kneser hypergraphs and extend a conjecture by Alon, Drewnowski and Łuczak [23].

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CHAPTER 2

Background

In this chapter we provide some additional basic definitions and develop the algebraic machinery required for the proofs in Chapters 3 and 4. Many of the definitions given here are common in combinatorics, commutative algebra, and algebraic geometry. In particular, the results given in Section 2.2 are a collection of well known facts from commutative algebra and algebraic geometry that are vital to the results in later chapters.

2.1 Graph theory and combinatorics

First we introduce some tools and important definitions from combinatorics. The main tool we will need will be integer partitions. We will work with r-uniform hypergraphs on finite vertex sets. We start with an important example.

Example 2.1. The *Fano plane* is the 3-uniform hypergraph, $FP = (V, E)$, where,

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

and

$$E = \{\{1, 2, 5\}, \{1, 3, 7\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{2, 4, 7\}, \{5, 6, 7\}\}.$$

Here, each edge is comprised of exactly three distinct vertices from the vertex set. So, FP is a 3-uniform hypergraph on 7 vertices with 7 edges.

Recall that a coloring of a hypergraph is considered proper if no edge in the hypergraph is completely colored by a single color. This definition is the natural extension from proper colorings of graphs. The smallest number of colors required in a proper coloring of $\mathcal{H}$ is called the chromatic number, $\chi(\mathcal{H})$. We call a hypergraph, $\mathcal{H}$, *critical* if deleting any edge from $\mathcal{H}$ decreases its chromatic number.

Example 2.2. *Let $c_1 : V(\mathcal{H}) \to \{1, -1\}$ be a 2-coloring of the hypergraph:*

$$V(\mathcal{H}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E(\mathcal{H}) = \big\{\{1, 3, 5, 9\}, \{2, 4, 7, 8\}$$

$$\{1, 2, 6, 7\}, \{2, 3, 8, 9\}$$

$$\{1, 4, 5, 8\}\big\}$$

given by:

i	1	2	3	4	5	6	7	8	9
$c_1(i)$	1	1	1	1	-1	-1	-1	-1	-1

Since no edge is completely colored by a single color, c_1 is a proper 2-coloring of $\mathcal{H}$. However, if we impose further restrictions on a coloring we have a coloring which has some useful applications.

Definition 2.1. *A proper coloring of $\mathcal{H}$ is called a* conflict-free *coloring if each edge in $\mathcal{H}$ contains a vertex whose color is not repeated on any other vertex in the edge.*

Example 2.3. *Let $c_2 : V(\mathcal{H}) \to \{1, -1\}$ be a 2-coloring of the hypergraph given in Example 2, $\mathcal{H}$ given by:*

i	1	2	3	4	5	6	7	8	9
$c_2(i)$	1	1	-1	-1	-1	1	-1	-1	-1

c_2 is a conflict-free 2-coloring of $\mathcal{H}$.

The smallest number of colors required in a conflict-free coloring is called the conflict-free chromatic number of $\mathcal{H}$, $\chi_{\mathrm{CF}}(\mathcal{H})$. The conflict-free chromatic number was introduced by G. Even et al. [1] in 2003. In Chapter 3 we will give an algorithm for finding conflict free 2-colorings of r-uniform hypergraphs. We will examine conflict free colorings of the Fano plane explicitly in Chapter 5.

To introduce the generators of the colorability ideals $I_2(\mathcal{H})$ and $I(\mathcal{H}, k)$ we use partitions of integers.

Definition 2.2. *A* k-partition *of a positive integer, n, is a k-length sequence of not necessarily distinct positive integers, called* parts*, that sum to n;*

$$(x_1, x_2, \ldots, x_k), \ \textit{such that,} \ \sum_{j=1}^{k} x_j = n.$$

For our needs, the order of the parts of any particular partition is not important. Furthermore, we wish to add some additional criteria to our partitions. This definition is borrowed from integer compositions, which are similar to integer partitions.

Definition 2.3. *A* weak k-partition *of a positive integer, n, is a k-length sequence of not necessarily distinct non-negative integers, again called* parts*, that sum to n;*

$$(y_1, y_2, \ldots, y_k), \ \textit{such that,} \ \sum_{j=1}^{k} y_j = n.$$

For example, $(1, 1, 2, 3, 5)$ is a 5-partition of 12 and $(0, 0, 1, 2, 2, 4, 10)$ is a weak 7-partition of 19. Furthermore, we define a *proper k-partition* of n to be a weak k-partition of n which only involves integers less than n.

2.2 Polynomial Algebra

We will be associating each hypergraph to an ideal in a polynomial ring, since we only consider hypergraphs with finite vertex sets, we need only consider polynomial rings with a finite number of indeterminates.

For any field $\mathbb{K}$ and any associated polynomial ring $\mathbb{K}[x_1, ..., x_n]$, Hilbert's Basis Theorem states that any ideal of $\mathbb{K}$ or $\mathbb{K}[x_1, ..., x_n]$ will be finitely generated since $\mathbb{K}$ and $\mathbb{K}[x_1, ..., x_n]$ are finitely generated. Throughout this thesis we will be working over the field of complex numbers since it is algebraically closed and we let $R = \mathbb{C}[x_1, \ldots, x_n]$.

2.2.1 Gröbner bases

The generalizations that we prove in this thesis require the use of a powerful tool from commutative algebra provided by B. Buchberger in [2]. The theory of Gröbner bases has become a critical part of algebraic geometry, commutative algebra, and computational and algorithmic algebra, among others. In short, a Gröbner basis is unique set of generators for an ideal.

We begin by setting an ordering on the monomials in the ring R.

Definition 2.4. *A* monomial ordering *is a well ordering, $<$, on the set of monomials that satisfies $mm_1 < mm_2$ whenever $m_1 < m_2$ for monomials m, m_1, m_2. Equivalently, a monomial ordering may be specified by defining a well ordering on the n-tuples $\alpha = (a_1, ..., a_n) \in \mathbb{Z}^n$ of multidegrees of monomials $Ax_1^{a_1} \cdots x_n^{a_n}$ that satisfies $\alpha + \gamma < \beta + \gamma$ if $\alpha < \beta$.*

With an ordering established of the ring R the leading term for any polynomial $f \in R$ can be distinguished.

Definition 2.5. *The* leading term *of f, $LT(f)$, is the monomial that is largest with respect to the monomial ordering $<$. Similarly, the leading monomial of $f \in R$, $LM(f)$ is the leading term of f with monic coefficient.*

Definition 2.6. *The* multidegree *of f, denoted $\partial(f)$, to be the multidegree of the leading term of f.*

Leading terms and monomials are crucial to understanding many properties of an ideal I, its relationship with R and with any polynomial in R. The ideal generated by the leading terms of I play a major role in the structure of I.

Definition 2.7. *If I is an ideal in $\mathbb{C}[x_1, ..., x_n]$, the* ideal of leading terms*, denoted $LT(I)$, is the ideal generated by the leading terms of all the elements in the ideal,* *i.e.,*

$$LT(I) = \langle LT(f) \mid f \in I \rangle .$$

It is also important to examine those monomials that are not the leading term of any polynomial in an ideal I.

Definition 2.8. *Any monomial which is not a leading term of any polynomial in an ideal I is called a* standard monomial *and the set of all such monomials is denoted $\mathcal{B}_<(I)$.*

In order to utilize these important ideals, we must have a way to generate them. Thus we have the following important definition.

Definition 2.9. *A* Gröbner basis *for I is a finite set of generators,*

$$\{g_1, ..., g_m\}$$

for I whose leading terms generate the ideal of all leading terms in I, i.e.,

$$I = \langle g_1, ..., g_m \rangle \quad \textit{and} \quad LT(I) = \langle LT(g_1), ..., LT(g_m) \rangle.$$

Once a monomial ordering has been established on a polynomial ring, we can preform general polynomial division of a polynomial f by a set of polynomials $\{g_1, ..., g_m\}$. If there is a remainder r after division we write

$$f \equiv r \mod \{g_1, ..., g_m\}.$$

We introduce some notation from [3]. Let f_1, f_2 be polynomials in $\mathbb{C}[x_1, ..., x_n]$ and let M be the monic least common multiple of the monomial terms $LT(f_1)$ and $LT(f_2)$. Then define the *difference polynomial*, $S(f_1, f_2)$ to be:

$$S(f_1, f_2) = \frac{M}{LT(f_1)} f_1 - \frac{M}{LT(f_2)} f_2.$$

This notation allows us to introduce a condition for a set of generators of a polynomial ideal to be a Gröbner basis for that ideal.

Proposition 2.1 (*Buchburger's Criterion*, p 324, [3])**.** *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and fix a monomial ordering on R. If $I = (g_1, ..., g_m)$ is a non-zero ideal in R, then $G = \{g_1, ..., g_m\}$ is a Gröbner basis for I if and only if $S(g_i, g_j) \equiv 0$ mod G for $1 \leq i < j \leq m$.*

This criterion is the primary component in an algorithm used to create a Gröbner basis for any ideal. Buchberger's Algorithm (pp 324-325 [3]) uses the criterion in the following way. Given a set of generators $\{g_1, ..., g_m\}$ of an ideal I, if Buchburger's Criterion is satisfied, then $\{g_1, ..., g_m\}$ is a Gröbner basis. If not, then at least one $S(g_i, g_j)$ has a remainder $r \neq 0$. Set $g_{m+1} = r$ and add g_{m+1} to $\{g_1, ..., g_m\}$ creating $\{g_1, ..., g_m, g_{m+1}\}$. Repeat the process of checking Buchburger's Criterion on this new set of generators of I. After a finite number iterations, this process will terminate and the set of generators $G = \{g_1, ..., g_m, ..., g_{m+n}\}$ will satisfy Buchburger's Criterion and thus be a Gröbner basis for I.

Although Gröbner bases in general are not unique for an ideal, if a stronger condition is placed on the Gröbner basis, uniqueness is attained.

Definition 2.10. *Fix a monomial ordering on $R = \mathbb{C}[x_1, ...x_n]$. A Gröbner basis $\{g_1, ..., g_m\}$ for the non-zero ideal I in R is called a* reduced Gröbner basis *if*

(a) each g_i has a monic leading term, i.e., $LT(g_i)$ is monic for $i = 1, ..., m$, and

(b) no term in g_j is divisible by $LT(g_i)$ for $j \neq i$.

As stated above, reduced Gröbner bases are unique.

Theorem 2.2 (Theorem 27, p326, [3])**.** *Fix a monomial ordering on $R = \mathbb{C}[x_1, \ldots, x_n]$. Then there is a unique reduced Gröbner basis for every non-zero ideal I in R.*

Among many other uses, this gives us a tool for comparing ideals in a polynomial ring, and determining ideal membership.

Corollary 2.3 (Corollary 28, p326, [3]). *Let I and J be two ideals in $\mathbb{C}[x_1, ...x_n]$. Then $I = J$ if and only if I and J have the same reduced Gröbner basis with respect to any fixed monomial ordering on $\mathbb{C}[x_1, ...x_n]$.*

Gröbner bases and leading term ideals give a method for examining the vector space properties of quotient rings that are also $\mathbb{C}$-algebras. This allows us to give bounds on the number of generators of a $\mathbb{C}$-algebra, which in turn can give us information on the number of common solutions to a polynomial ideal. The following two theorems give the connection between the quotient ring of a polynomial ideal and its corresponding $\mathbb{C}$-algebra.

Theorem 2.4 (Proposition 1, p227, [4]). *Fix a monomial ordering, $<$, on $R = \mathbb{C}[x_1, \ldots, x_n]$ and let I be an ideal of R. Let $\langle LT(I)\rangle$ denote the ideal generated by the leading terms of I.*

(i) Every $f \in R$ is congruent modulo I to a unique polynomial r which is a $\mathbb{C}$-linear combination of the monomials in the complement of $\langle LT(I)\rangle$.

(ii) The elements of $\{x^\alpha : x^\alpha \notin \langle LT(I)\rangle\}$ are "linearly independent modulo I." That is, if

$$\sum_\alpha c_\alpha x^\alpha \equiv 0 \ mod\ I,$$

where the x^α appearing are all in the complement of $\langle LT(I)\rangle$, then $c_\alpha = 0$ for all α.

Theorem 2.5 (Proposition 4, p229, [4]). *Let $I \subset \mathbb{C}[x_1, \ldots x_n]$ be an ideal. Then $\mathbb{C}[x_1, \ldots x_n]/I$ is isomorphic as a $\mathbb{C}$-vector space to $S = span\{x^\alpha \notin \langle LT(I)\rangle\}$.*

2.2.2 Square-free, radical ideals

In what follows we collect several results from algebraic geometry and commutative algebra. To this point these results have not been assembled in a single work. The use of radical, square-free generated ideals greatly simplifies the computations involved with determining the variety of the ideal in question. Since our main goal is to determine which ideals give rise to desired varieties, i.e. the variety that contains all possible proper colorings of a hypergraph, we feel this section is crucial to the results of this thesis. The majority of the results in this section come from [4] and [5], we reproduce serveral proofs for completeness only.

Let I be an ideal of the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$.

Definition 2.11. *The* radical *of I, denoted $\sqrt{I}$, is the set:*

$$\sqrt{I} = \{f \in R : f^m \in I, \text{ for some } m \in \mathbb{Z}^+, \}$$

moreover, the ideal I is radical *if $I = \sqrt{I}$.*

Working with the radical of an ideal, or a radical ideal greatly simplifies computation and gives a more complete description of the structure of the ideal. It also simplifies the geometric structure associated with the ideal.

Definition 2.12. *The subset of $\mathbb{C}^n$ consisting of all of the solutions common to each polynomial in I is the* variety *of I, denoted $\mathcal{V}(I)$. Conversely, given a subset $V \subseteq \mathbb{C}^n$, the* vanishing ideal *is the set of all polynomials that vanish at every point in V, and is denoted: $\mathcal{I}(V)$. The two maps $\mathcal{V}$ and $\mathcal{I}$ are related by:*

$$\mathcal{V}(\mathcal{I}(V)) = V, \quad \text{and} \quad \mathcal{I}(\mathcal{V}(I)) = \sqrt{I},$$

and are also inclusion-reversing.

The second equality above is known as The Strong Nullstellensatz, a proof is given below. Algebraic varieties are a main focus in algebraic geometry and can be quite complicated, fortunately the varieties that are associated with the ideals used to encode the colorability of a hypergraph are rather simple.

Definition 2.13. *The ideal I is called* zero-dimensional *(as an ideal) if its variety $\mathcal{V}(I)$ contains only a finite number of points.*

Conditions given for an ideal to be zero-dimensional are given in The Finiteness Theorem below. Working with zero-dimensional ideals of polynomial rings with coefficients coming from algebraically closed fields allows us to compare two ideals through their varieties.

The following theorem is the primary tool used to decide if the polynomials in an ideal have a common solution. It has been used extensively by De Loera, Hillar, Windfeldt and others [6], [7], [8], [9].

Theorem 2.6 (Weak Nullstellensatz)**.** *If k is an algebraically closed field and I is an ideal in $k[x_1, \dots x_n]$ satisfying $\mathcal{V}(I) = \emptyset$. Then $I = k[x_1, \dots x_n]$.*

The contrapositive of this theorem states that if I is a proper ideal of $R = \mathbb{C}[x_1, \dots, x_n]$, then $\mathcal{V}(I) \neq \emptyset$. Moreover since a Gröbner basis is a set of generators for an ideal, to see if there are common solutions to the polynomials in an ideal we need only check to see if a Gröbner basis is equal to 1 or not.

Theorem 2.7 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field. If*

$$f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$$

are such that

$$f \in \mathcal{I}(\mathcal{V}(f_1, \dots, f_s)),$$

then there exists and integer $m \geq 1$ such that

$$f^m \in \langle f_1, \dots, f_s \rangle .$$

Theorem 2.8 (Strong Nullstellensatz)**.** *If k is an algebraically closed field and I is an ideal in $k[x_1, \dots x_n]$, then*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Theorem 2.9 (Finiteness Theorem). *Let I be an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$. The following are equivalent:*

1. *The algebra $A = \mathbb{C}[x_1, \ldots, x_n]/I$ is finite dimensional over $\mathbb{C}$.*

2. *The variety $\mathcal{V}(I) \subset \mathbb{C}^n$ is a finite set.*

3. *If $\mathcal{G}$ is a Gröbner basis for I, then for each i, $1 \leq i \leq n$, there is an $m_i \geq 0$ such that $x^{m_i} = LT(g)$ for some $g \in \mathcal{G}$.*

Recall that such an ideal is called a *zero-dimensional* ideal.

Corollary 2.10. *I is zero-dimensional if and only if there is a non-zero polynomial in $I \cap \mathbb{C}[x_i]$ for each $i = 1, \ldots, n$.*

Proof: Suppose I is zero-dimensional. Let $\mathcal{G}$ be a reduced Gröbner basis for any lexicographic ordering with x_i as the smallest variable. By part 3. of the Finiteness Theorem, there is some $g \in \mathcal{G}$ with $LT(g) = x_i^{m_i}$. This implies that $g \in \mathbb{C}[x_i]$ since the ordering on I is lexicographic. So g is the non-zero polynomial required, moreover g generates the ideal $I \cap \mathbb{C}[x_i]$. Note that $\mathcal{G}$ being a reduced Gröbner basis and the chosen ordering being lexicographic give us that $\langle g \rangle = I \cap \mathbb{C}[x_i]$. This is known as The Elimination Theorem, (theorem 3, pp 115 [4]).

Conversely, suppose $I \cap \mathbb{C}[x_i]$ is non-zero for each i, and let m_i be the degree of the unique monic generator of $I \cap \mathbb{C}[x_i]$. Then $x_i \in \langle LT(I) \rangle$ for any monomial order, so that all monomials not in $\langle LT(I) \rangle$ will contain x_i to a power strictly less than m_i. Hence the set of monomials $x^\alpha \notin \langle LT(I) \rangle$ is finite, and thus A is finite. $\square$

Definition 2.14. *The* square-free part *of a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$, is denoted p_{red} and has exactly the same roots as p, but all with multiplicity 1.*

Claim 2.11. *If p_{red} is the square-free part of $p \in \mathbb{C}[x]$, then $\sqrt{\langle p \rangle} = \langle p_{red} \rangle$.*

Proof: Note that by the Strong Nullstellensatz and the definition of the square-free part of p, p_{red}, we have:

$$\mathcal{V}(\langle p\rangle) = \mathcal{V}(\langle p_{red}\rangle) \Rightarrow \mathcal{I}(\mathcal{V}(\langle p\rangle)) = \mathcal{I}(\mathcal{V}(\langle p_{red}\rangle)) \Rightarrow \sqrt{p_{red}} = \sqrt{p}.$$

So, if $\sqrt{\langle p_{red}\rangle} = \langle p_{red}\rangle$, we are done.

The inclusion $\sqrt{\langle p_{red}\rangle} \supseteq \langle p_{red}\rangle$ follows from the definition of the radical of an ideal.

Let

$$p_{red} = \prod_{i=1}^{k}(x - a_i)$$

and suppose $f \in \sqrt{\langle p_{red}\rangle}$. Then f must be of the form

$$f = h\prod_{i=1}^{k}(x - a_i)^{\beta_i}$$

where $\beta_i \geq 1$ and $h \in \mathbb{C}[x]$. Then there exists some $h' \in \mathbb{C}[x]$ such that

$$f = h\prod_{i=1}^{k}(x - a_i)^{\beta_i} = h'\prod_{i=1}^{k}(x - a_i), \text{ namely, } h' = h\prod_{j=1}^{k'}(x - a_j)^{\beta_j - 1},$$

where the set $\{j = 1, \ldots, k'\} \subseteq \{i = 1, \ldots, k\}$ is such that $\beta_j > 1$.

Thus $f \in \langle p_{red}\rangle$. $\square$

For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ and any i, $I \cap \mathbb{C}[x_i]$ is a principle ideal domain, and thus contains a unique monic generator. Combining this with Claim 7 we have that if p_i is the unique monic generator of $I \cap \mathbb{C}[x_i]$, and $p_{i,red}$ is the square-free part of p_i then

$$\sqrt{I \cap \mathbb{C}[x_i]} = \langle p_{i,red}\rangle.$$

The following technical lemmas illustrate the relationship between square-free polynomial generators their ideals.

Lemma 2.12. *Let I be an ideal of $R = \mathbb{C}[x_1]$, and let $p = (x_1 - a_1)(x_1 - a_2)\cdots(x_1 - a_d)$, where the a_j are distinct. Then*

$$I + \langle p\rangle = \bigcap_j (I + \langle x_1 - a_j\rangle).$$

Proof: First we show

$$I + \langle p \rangle \subset \bigcap_j (I + \langle x_1 - a_j \rangle).$$

Let $f \in I + \langle p \rangle$, then

$$f = g + hp$$

for some $g \in I$ and $h \in \mathbb{C}[x_1]$. Since p vanishes at each a_j we have that

$$f = g + hp \in I + \langle x_1 - a_j \rangle \quad \forall\ j,$$

thus $f \in \bigcap_j (I + \langle x_1 - a_j \rangle)$.

For the opposite inclusion, consider the following polynomials:

$$p_j = \prod_{i \neq j} (x_1 - a_j).$$

Note that the collection of polynomials $\{p_1, \ldots, p_n\}$ do not have any common zeros, hence by the Weak Nullstellensatz, there exist polynomials h_j such that

$$\sum_{j=1}^{n} h_j p_j = 1.$$

Also, we show that

$$p_j (I + \langle x_1 - a_j \rangle) \subset I + \langle p \rangle\,.$$

Let $f \in p_j (I + \langle x_1 - a_j \rangle)$, then

$$f = f' p_j (g + h(x_1 - a_j)),\ \text{for}\ g \in I,\ \text{and}\ f', h \in \mathbb{C}[x_1].$$

Since $p_j (x_1 - a_j) = p$ we have:

$$f = f' p_j (g + h(x_1 - a_j)) = f' p_j g + hp \in I + \langle p \rangle\,.$$

To show the inclusion

$$\bigcap_j (I + \langle x_1 - a_j \rangle) \subset I + \langle p \rangle\,,$$

let $f \in \cap_j (I + \langle x_1 - a_j \rangle)$, then

$$f = g_j + f_j(x_1 - a_j),$$

where $g_j \in I$ and $f_j \in \mathbb{C}[x_1]$ for all j. Recall that there exist $h_j \in \mathbb{C}$ such that

$$\sum_{j=1}^{n} h_j p_j = 1 \Rightarrow \sum_{j=1}^{n} h_j p_j f = f.$$

Thus,

$$\begin{aligned} f &= \sum_{j=1}^{n} h_j p_j f \\ &= \sum_{j=1}^{n} h_j p_j (g_j + f_j(x_1 - a_j) \\ &= \sum_{j=1}^{n} h_j p_j g_j + h_j p_j f_j (x_1 - a_j) \\ &= \sum_{j=1}^{n} h_j p_j g_j + h_j f_j p. \end{aligned}$$

Since $g_j \in I$, and $h_j f_j p \in \langle p \rangle$ for all j, we have that $f \in I + \langle p \rangle$, hence

$$I + \langle p \rangle = \bigcap_j (I + \langle x_1 - a_j \rangle).$$

□

Lemma 2.13. *Let I be a zero-dimensional ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$. For each $i = 1, \ldots, n$, let p_i be the unique monic generator of $I \cap \mathbb{C}[x_i]$, and let $p_{i,red}$ be the square-free part of p_i. Then*

$$\sqrt{I} = I + \langle p_{1,red}, \ldots, p_{n,red} \rangle.$$

Proof: Let $J = I + \langle p_{1,red}, \ldots, p_{n,red} \rangle$. For each i, since $\mathbb{C}$ is closed, we can factor $p_{i,red}$ into distinct factors:

$$p_{i,red} = (x_i - a_{i1})(x_i - a_{i2}) \cdots (x_i - a_{id_i}).$$

Thus,

$$J = J + \langle p_{1,red} \rangle = \bigcap_j \left(J + \langle x_1 - a_{1j} \rangle \right),$$

where the first holds since $p_{1,red} \in J$ and the second holds by Lemma 2.12 since $p_{1,red}$ has distinct roots. Repeating this argument for $i = 2, \ldots, n$ we have

$$J = \bigcap_{j_1, \ldots, j_n} \left(J + \langle x_1 - a_{1j_1}, \ldots, x_n - a_{nj_n} \rangle \right).$$

Since $M = \langle x_1 - a_{1j_1}, \ldots, x_n - a_{nj_n} \rangle$ is a maximal ideal, $J + M$ is either M or the entire ring R. It follows that J is a finite intersection of maximal ideals. Since a maximal ideal is radical and an intersection of radical ideals is radical, J is radical.

It remains to see that $J = \sqrt{I}$. The inclusion $I \subset J$ is by definition of J, and the inclusion $J \subset \sqrt{I}$ follows from the Strong Nullstellensatz, since the square-free parts of the p_i vanish at all the points of $\mathcal{V}(I)$. Hence

$$I \subset J \subset \sqrt{I},$$

and taking radicals gives

$$\sqrt{I} \subset \sqrt{J} \subset \sqrt{\sqrt{I}} = \sqrt{I}.$$

so $\sqrt{I} = \sqrt{J}$ and since J is radical,

$$J = \sqrt{I}.$$

□

Lemma 2.14. *Let I be a zero-dimensional ideal in $R = \mathbb{C}[x_1, \ldots, x_n]$, and let $A = R/I$. Then $dim_{\mathbb{C}}(A)$ is greater than or equal to the number of points in $\mathcal{V}(I)$. Moreover, equality occurs if and only if I is a radical ideal.*

Proof: Let I be a zero-dimensional ideal. By the Finiteness Theorem, $\mathcal{V}(I)$ is a finite set in $\mathbb{C}^n$, say $\mathcal{V}(I) = \{p_1, \ldots, p_m\}$. Consider the mapping

$$\phi : A \to \mathbb{C}^m, \quad [f] \to (f(p_1), \ldots, f(p_m)),$$

which is well defined and linear. We show that ϕ is onto, and thus conclude that $dim_{\mathbb{C}}(A) \geq m = |\mathcal{V}(I)|$.

For $i = 1, \ldots, m$, let g_i be polynomials in R such that

$$g_i(p_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} .$$

Given an arbitrary $(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, let

$$f = \sum_{i=1}^{m} \lambda_i g_i$$

so that

$$\phi([f]) = (\lambda_1, \ldots, \lambda_m).$$

So ϕ is onto and hence $dim_{\mathbb{C}}(A) \geq m = |\mathcal{V}(I)|$.

Next, suppose that I is radical. If $[f] \in ker(\phi)$, then $f(p_i) = 0$ for all i, so that by the Strong Nullstellensatz, $f \in \mathcal{I}(\mathcal{V}(I)) = \sqrt{I} = I$. Thus $[f] = [0]$, which shows that ϕ is injective as well as onto. So ϕ is an isomorphism, which shows that $dim_{\mathbb{C}}(A) = m$ if I is radical.

Conversely, if $dim_{\mathbb{C}}(A) = m$, then ϕ is an isomorphism since it is an onto linear map between vector spaces of the same dimension. Hence ϕ is injective. Since $I \subset \sqrt{I}$ always, it suffices to show

$$f \in \sqrt{I} = \mathcal{I}(\mathcal{V}(I)) \Rightarrow f \in I.$$

If $f \in \sqrt{I}$, then

$$f(p_i) = 0 \ \forall \ i, \quad \Rightarrow \quad \phi([f]) = (0, \ldots, 0).$$

Since ϕ is injective, we have that $[f] = [0]$, so $f \in I$.

$\square$

The following is a statement of a lemma of Hillar and Windfeldt. We include it here and give a proof as it is crucial for many theorems in following sections.

Lemma 2.15 (Lemma 2.1, [9]). *Let I be a zero-dimensional ideal and fix a monomial ordering $<$. Then,*

$$\dim_{\mathbb{C}} R/I = |\mathcal{B}_<(I)| \geq |\mathcal{V}(I)| \quad \textit{(as a vector space).}$$

Moreover, the following are equivalent:

1. *I is a radical.*

2. *I contains a univariate square-free polynomial in each indeterminate.*

3. *$|\mathcal{B}_<(I)| = |\mathcal{V}(I)|$.*

Proof: The dimension condition follows from Theorem 2.14 and the fact that the standard monomials $\mathcal{B}_<(I)$ form a basis for the algebra R/I.

Furthermore, for each $i = 1, \ldots, n$, let p_i be the unique monic generator of $I \cap \mathbb{C}[x_i]$, and let $p_{i,red}$ be the square-free part of p_i.

1. $\Rightarrow$ 2. If I is radical then $I = \sqrt{I} = I + \langle p_{1,red}, \ldots, p_{n,red} \rangle$ by Proposition 2.13, so I contains a univariate square-free polynomial in the each indeterminate.

2. $\Rightarrow$ 1. By Proposition 2.13 and the fact that the univariate square-free polynomials in each indeterminate can be taken to be the square-free part of the unique monic generators of $I \cap \mathbb{C}[x_i]$. That is $p_{i,red} \in I$ for all i, so $\sqrt{I} = I + \langle p_{1,red}, \ldots, p_{n,red} \rangle = I$.

1. $\Leftrightarrow$ 3. by Theorem 2.14. □

Once we can switch between a zero-dimensional radical square-free ideal and its corresponding variety, we use the ideal quotient to compute the difference of two varieties. We conclude this section with some useful results from commutative algebra and algebraic geometry. Radical ideals and their varieties behave nicely under certain operations.

Theorem 2.16 (Section 8, Chapter 4 and Proposition 16 p 188, [4])**.** *Let I and J be ideals of R. If I and J are radical, then there is a one-to-one correspondence given by $\mathcal{I}$ and $\mathcal{V}$ such that:*

$$IJ \longleftrightarrow \mathcal{V}(I) \cup \mathcal{V}(J)$$

$$I \cap J \longleftrightarrow \mathcal{V}(I) \cup \mathcal{V}(J)$$

$$I + J \longleftrightarrow \mathcal{V}(I) \cap \mathcal{V}(J).$$

Moreover,

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Definition 2.15. *The* ideal quotient (or colon ideal) *of the ideals I and J of R is the ideal:*

$$I : J = \{f \in R : fg \in I, \ \forall \ g \in J\}.$$

Ideal quotients have a nice property when restricted to ideals of polynomial rings over algebraically closed fields.

Proposition 2.17. *Given two varieties $V, W \subset \mathbb{C}^n$, then*

$$\mathcal{I}(V) : \mathcal{I}(W) = \mathcal{I}(V \setminus W).$$

We include one final theorem concerning the structure of the variety of a radical ideal.

Theorem 2.18 (Corollary 2.6, p 143 [5])**.** *Let I be a zero dimensional ideal of the ring $R = \mathbb{C}[x_1, \ldots, x_n]$. Then I is radical if and only if each point in $\mathcal{V}(I)$ has multiplicity* 1*.*

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CHAPTER 3

2-Colorability

3.1 2-Colorability Results

The approach to the colorability problems we use involves translating properties of a combinatorial object, i.e. a hypergraph, into the language of commutative algebra, namely into ideals and varieties of polynomial rings. To address the encoding in the 2-colorability case, we will define a system of polynomials that will capture the colorability of the hypergraph $\mathcal{H}$. In addition we will provide polynomials that capture individual coloring schemes, and show how they relate to the overall colorability of the hypergraph. Here we collect all of the results concerning 2-coloring in this chapter, we provide proofs in Section 3.2.

For 2-colorings, we partially extend Theorem 1.1 to r-uniform hypergraphs. We define a 2-coloring as a map $c : V(\mathcal{H}) \to \{-1, 1\}$, and note that this formulation is equivalent to the definition above. We introduce some notation that will allow us to define our polynomials and ideals.

Let $\mathcal{H}$ be an r-uniform hypergraph on the vertex set $V(\mathcal{H}) = [n]$, with $m = |E(\mathcal{H})|$ edges. For each edge, e let:

$$e = (x_{e,1}, x_{e,2}, \ldots, x_{e,r}),$$

where (e, j) represents a vertex that belongs to e. Let $par(r, 2)$ be the set of all proper 2-integer partitions of r and let $p(r, 2) = |par(r, 2)|$. Let $d(r, 2)$ be the set of all differences of proper 2-integer partitions of r, that is:

$$\text{if } \{r_1, r_2\} \in par(r, 2),\ r_1 \geq r_2, \text{ then } r_1 - r_2 \in d(r, 2).$$

We can then define the f_e polynomials:

$$f_e = (c_1 x_{e,1} + x_{e,2}, \cdots + x_{e,r}) \cdots (x_{e,1} + x_{e,2}, \cdots + c_1 x_{e,r}) \cdot$$
$$(c_2 x_{e,1} + x_{e,2}, \cdots + x_{e,r}) \cdots (x_{e,1} + x_{e,2}, \cdots + c_2 x_{e,r}) \cdots$$
$$(c_{p(r,2)} x_{e,1} + x_{e,2}, \cdots + x_{e,r}) \cdots (x_{e,1} + x_{e,2}, \cdots + c_{p(r,2)} x_{e,r})$$

where $c_j - 1 \in d(r, 2)$, $1 \leq j \leq p(r, 2)$.

These polynomials are crucial in the definition of the 2-colorability ideal for $\mathcal{H}$, $I_2(\mathcal{H})$:

$$I_2(\mathcal{H}) = \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle f_e : e \in E(\mathcal{H}) \rangle.$$

As the name implies, the ideal $I_2(\mathcal{H})$ is closely tied with the proper colorings of $\mathcal{H}$. The following theorem introduces the 2-colorability ideal for $\mathcal{H}$

Theorem 3.1. *The polynomials in the ideal $I_2(\mathcal{H})$ have a common solution if and only if $\mathcal{H}$ is properly 2-colorable.*

As an analogue to the commonly used graph polynomial, define the *hypergraph polynomial for 2-colorability*, $P_2(\mathcal{H})$, by:

$$P_2(\mathcal{H}) = \prod_{j=1}^{m} \left[\left(\sum_{i \in e_j} x_i - r \right) \left(\sum_{i \in e_j} x_i + r \right) \right],$$

where $e_j \subset [n]$, $|e_j| = r$, is an edge in $\mathcal{H}$. A similar generalization of the graph polynomial was introduced in [1]. We state a theorem that captures the generalization for 2-colorability here and postpone the proof to the next section.

Theorem 3.2. *$\mathcal{H}$ is not properly 2-colorable if and only if $P_2(\mathcal{H})$ has a solution.*

We can now state our main result for 2-coloring r-uniform hypergraphs. It is a generalization of Hillar and Windfeldt's Theorem 2.1 in [2].

Theorem 3.3. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices and let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I_2(\mathcal{H})$ be the 2-coloring ideal of $\mathcal{H}$ and let $P_2(\mathcal{H})$ be the 2-color hypergraph polynomial for $\mathcal{H}$. Then following are equivalent:*

(1) The hypergraph $\mathcal{H}$ is not 2-colorable.

(2) The constant 1 is an element of the ideal $I_2(\mathcal{H})$.

(3) $\dim_{\mathbb{C}} R/I_2(\mathcal{H}) = 0$ *as a vector space.*

(4) The hypergraph polynomial $P_2(\mathcal{H})$ belongs to the ideal

$$\left\langle x_i^2 - 1 : i \in V(\mathcal{H}) \right\rangle .$$

Next, we give a decomposition of the possible proper 2-colorings into *coloring schemes*, represented by coloring ideals. First we define a useful tool for distinguishing 2-colorings of a hypergraph. The *edge signature* of an edge $e \in E(\mathcal{H})$, $a(e)$, is defined to be the sum of the values of the vertices in e:

$$a(e) = \sum_{i=1}^{r} x_{e,i}.$$

A *proper edge signature* is an edge signature that appears in a proper coloring of $\mathcal{H}$. Note that because we are only concerned with 2-colorings and thus restrict our colors to be represented by the second roots of unity, ± 1, we have that our (proper) edge signatures are uniquely determined. For an r-uniform hypergraph there are $r-1$ proper edge signatures and are found by examining the proper 2-integer partitions of r. These edge signatures allow us to characterize 2-colorings of $\mathcal{H}$.

Let $A = \{a_1, a_2, \ldots, a_{r-1}\}$ be the set of proper edge signatures for the r-uniform hypergraph $\mathcal{H}$. Let U be a non-empty subset of A.

Definition 3.1. *The* 2-coloring scheme *for the r-uniform hypergraph $\mathcal{H}$ given by U is the set of all colorings of $\mathcal{H}$ with the edge signatures in U.*

The 2-*coloring scheme ideal*, $J_2(U)$, is the ideal that encodes the colorability of $\mathcal{H}$ by the edge signatures in U:

$$J_2(U) = \langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle + \left\langle \prod_{a \in U} \left(\sum_{i=1}^{r} x_{e,i} - a \right) : e \in E(\mathcal{H}) \right\rangle.$$

These ideals play an important role in distinguishing one proper coloring pattern from another.

Theorem 3.4. *The polynomials in the ideal $J_2(U)$ have a common solution if and only if the hypergraph $\mathcal{H}$ can be colored by the edge signatures in U.*

These ideals also provide us with a complete characterization of the proper 2-colorings of $\mathcal{H}$, as stated here in our main result.

Theorem 3.5. *Let $A = \{a_1, a_2, \ldots a_{r-1}\}$ represent the possible proper edge signatures of $\mathcal{H}$. Then the colorability ideal $I_2(\mathcal{H})$ is the intersection of all of the coloring ideals $J_2(U)$:*

$$I_2(\mathcal{H}) = \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} J_2(U).$$

3.2 2-Colorability Proofs

In this section we let A be the set of all proper edge signatures for a 2-coloring of $\mathcal{H}$. We first establish the ideal characterization of a proper 2-coloring of $\mathcal{H}$ by proving that our f_e polynomials encode proper edge coloring.

Lemma 3.6. *The polynomial f_e vanishes if and only if the edge e is properly 2-colored.*

Proof: Let $e \in E(\mathcal{H})$. Let c be a 2-coloring of $\mathcal{H}$.

($\Leftarrow$) Assume c is a proper 2-coloring of $\mathcal{H}$. Then e has a proper edge signature, a, where by definition:

$$\sum_{i=1}^{r} x_{e,i} = a, \text{ and } a \in [-(r-2), r-2].$$

Then we have that:

$$\exists\ j,\ 1 \leq j \leq p(r,2) \text{ such that, } c_j - 1 = a.$$

Thus at least one of the factors,

$$(c_j x_{e,1} + x_{e,2} + \cdots + x_{e,r})(x_{e,1} + c_j x_{e,2} + \cdots + x_{e,r}) \cdots (x_{e,1} + x_{e,2} + \cdots + c_j x_{e,r})$$

of f_e will be zero. Hence $f_e(c) = 0$.

($\Rightarrow$) Assume $f_e(c) = 0$. Assume e is not properly colored. Then each factor in:

$$(c_j x_{e,1} + x_{e,2} + \cdots + x_{e,r})(x_{e,1} + c_j x_{e,2} + \cdots + x_{e,r}) \cdots (x_{e,1} + x_{e,2} + \cdots + c_j x_{e,r})$$

has sum either,

$$c_j + r - 1 \text{ or } -(c_j + r - 1).$$

Since $c_j - 1 \neq -r$, no factor in f_e can be zero, thus $f_e \neq 0$, a contradiction. So $\mathcal{H}$ is properly 2-colored by c. $\square$

With this fact we can characterize the 2-colorability of $\mathcal{H}$ in the ideal $I_2(\mathcal{H})$ with Theorem 1.3.

Proof: (Theorem 1.3)

($\Rightarrow$) Let $c \in V(I_2(\mathcal{H}))$, $c = (c_1, \ldots, c_n)$. Clearly the first n polynomials in $I_2(\mathcal{H})$ give us that $c_i = \pm 1$ for all $i \in [n]$. Hence, c is a 2-coloring of $\mathcal{H}$. Also, since $f_e(c) = 0$ for every $e \in E(\mathcal{H})$, we have that c must zero one or more factors of each f_e. Hence by Lemma 3.6, c must not color any edge with all 1's or all -1's.

($\Leftarrow$) Let $c = (c_1, \ldots, c_n)$ be a proper 2-coloring of $\mathcal{H}$. Then c does not color the vertices of any edge e with either all 1's or all -1's. Thus, by Lemma 3.6, c will zero every polynomial in $I_2(\mathcal{H})$. $\square$

Remark 1. A note on Theorem 1.3: in the r-even case, we need not cycle the c_j coefficient through r factors of f_e when $c_j = r_1 - r_2 + 1 = 1$ since all coefficients will be 1. Thus, in the even case, f_e will contain one factor of the form $(x_{e,1} + x_{e,2}, \cdots + x_{e,r})$ and the remaining $(r-1)$ factors where $c_j = 1$ are omitted. This does not change the variety $\mathcal{V}(I_2(\mathcal{H}))$, however it does simplify some computations within $I_2(\mathcal{H})$.

We now note how the hypergraph polynomial encodes a generalization of the graph polynomial.

Proof: (Theorem 3.2)

We note that $\mathcal{H}$ is not properly 2-colored if and only if there exists an edge with all vertices assigned the same color. This happens if and only if that edge has a vertex sum equal to r times the value of a single color. Since our colors have been restricted to ± 1, the following are equivalent:

- $\mathcal{H}$ is not properly 2-colored.
- $\exists$ an edge in $E(\mathcal{H})$ with vertices colored by all 1's or all -1's.
- $\exists$ an edge whose vertices sum to $\pm r$.

So, given a 2-coloring of $\mathcal{H}$ with the colors ± 1:

$$P_2(\mathcal{H}) = 0 \textit{ iff } \exists\, j \in [m] \text{ such that } \left(\sum_{i \in e_j} x_i - r\right)\left(\sum_{i \in e_j} x_i + r\right) = 0, \tag{1}$$

$$\textit{iff} \text{ either } \left(\sum_{i \in e_j} x_i - r\right) = 0 \text{ or } \left(\sum_{i \in e_j} x_i + r\right) = 0. \tag{2}$$

Since $x_i = \pm 1$, (2) happens *iff* $\exists\, e_j \in E(\mathcal{H})$ that has vertices colored by either all 1's or all -1's, that is,

$$P_2(\mathcal{H}) = 0 \textit{ iff } \mathcal{H} \text{ is not properly 2-colored.}$$

□

The following lemmas and their proofs are analogues of Lemmas 3.1 and 3.4 in [2].

Lemma 3.7. *The varieties $V(\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle)$, $V(I_2(\mathcal{H}))$, and $V(\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle + \langle P_2(\mathcal{H})\rangle)$ correspond to the sets of all 2 colorings of $\mathcal{H}$, the proper 2-colorings of $\mathcal{H}$, and the improper 2-colorings of $\mathcal{H}$, respectfully.*

Proof: Clearly, the set $V(\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle)$ is the set of all possible n-tuples of ± 1 which represent all possible 2-colorings of an n-vertex (hyper)graph.
By the construction of the f_e polynomials and the ideal $I_2(\mathcal{H})$ in Theorem 1.3, $V(I_2(\mathcal{H}))$ is the set of all proper 2-colorings of $\mathcal{H}$.

Finally, Theorem 3.2 states that the hypergraph polynomial $P_2(\mathcal{H})$ has a solution if and only if $\mathcal{H}$ is not properly 2-colored, thus $V(\langle x_i^2 - 1 : i \in [n]\rangle + \langle P_2(\mathcal{H})\rangle)$ is the set of all improper 2-colorings of $\mathcal{H}$. □

From this and the fact that $I_2(\mathcal{H})$ is radical we have the following lemma concerning *the hypergraph chromatic polynomial*, $\chi_{\mathcal{H}}(k)$, which is the univariate polynomial that counts the number of proper k-colorings for $\mathcal{H}$.

Lemma 3.8.

$$\chi_{\mathcal{H}}(2) = |V(I_2(\mathcal{H}))| = \dim_{\mathbb{C}} R/I_2(\mathcal{H}),$$

and

$$2^n - \chi_{\mathcal{H}}(2) = \dim_{\mathbb{C}} R/\left(\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle + \langle P_2(\mathcal{H})\rangle\right).$$

Proof: These statements follow from the fact that the ideals $I_2(\mathcal{H})$, and $\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle + \langle P_2(\mathcal{H})\rangle$ are radical and from Lemma 3.7. □

We may now prove our analogue of Hillar and Windfeldt's Theorem 1.1, [2].

Proof: (Theorem 1.2)

The equivalence of 1, 2, and 3 is given by Theorem 1.3.

($2 \Rightarrow 4$) Assume $1 \in I_2(\mathcal{H})$.

Then since $V(I_2(\mathcal{H})) = \emptyset$, and both $I_2(\mathcal{H})$ and $\langle x_i^2 - 1 : i \in V(\mathcal{H})\rangle$ are radical, the ideal quotient:

$$\left\langle x_i^2 - 1 : i \in V(\mathcal{H})\right\rangle : I_2(\mathcal{H}) = \left\langle x_i^2 - 1 : i \in V(\mathcal{H})\right\rangle.$$

Also, we have that

$$\begin{aligned}\left\langle x_i^2-1 : i \in V(\mathcal{H})\right\rangle : I_2(\mathcal{H}) &= \left\langle x_i^2-1 : i \in V(\mathcal{H})\right\rangle + \left\langle P_2(\mathcal{H})\right\rangle \\ &= \left\langle x_i^2-1 : i \in V(\mathcal{H})\right\rangle,\end{aligned}$$

so $P_2(\mathcal{H}) \in \langle x_i^2-1 : i \in V(\mathcal{H})\rangle$.

$(4 \Rightarrow 1)$, if $P_2(\mathcal{H}) \in \langle x_i^2-1 : i \in V(\mathcal{H})\rangle$, then Lemma 3.8 gives us that

$$\begin{aligned}2^n - \chi_{\mathcal{H}}(2) &= \dim_{\mathbb{C}} R/\left(\langle x_i^2-1 : i \in V(\mathcal{H})\rangle + \langle P_2(\mathcal{H})\rangle\right) \\ &= \dim_{\mathbb{C}} R/\langle x_i^2-1 : i \in V(\mathcal{H})\rangle \\ &= 2^n.\end{aligned}$$

So $\chi_{\mathcal{H}}(2) = 0$, and thus $\mathcal{H}$ is not 2-colorable. $\square$

Proof: (Theorem 3.4) Let $U \subseteq A$, U non-empty. Consider the ideal:

$$J_2(U) = \left\langle x_i^2-1 : i \in V(\mathcal{H})\right\rangle + \left\langle \prod_{a\in U}\left(\sum_{j=1}^{r} x_{e,j} - a\right) : e \in E(\mathcal{H})\right\rangle.$$

From the first set of polynomials we see that any common solution will be an n-tuple of 1's and -1's. Also, it is clear that for every edge, $e \in E(\mathcal{H})$:

$$\prod_{a\in U}\left(\sum_{j=1}^{r} x_{e,j} - a\right) = 0$$

if and only if one of the factors,

$$\sum_{j=1}^{r} x_{e,j} = a, \text{ for some } a \in U.$$

Since each factor is the sum of the values of the vertices in the edge e, this can happen if and only if the edge is colored by a signature in U. $\square$

Theorem 3.4 also gives us the following as a corollary. This is one of the key components to the proof of the decomposition Theorem 3.5.

Corollary 3.9.

$$\mathcal{V}(I_2(\mathcal{H})) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(J_2(U)).$$

Proof: This follows from Theorems 1.3 and 3.4. □

We can now prove our main decomposition theorem for 2-colorability.

Proof: (Theorem 3.5) Since the ideals $I_2(\mathcal{H})$ and $J_2(U)$ contain square-free univariate polynomials in each indeterminate, they are radical. Also, since

$$\mathcal{V}(I_2(\mathcal{H})) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(J_2(U)),$$

by Theorem 2.8, we have that:

$$\begin{aligned} I_2(\mathcal{H}) &= \mathcal{I}(\mathcal{V}(I_2(\mathcal{H}))) \\ &= \mathcal{I}\left(\bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(J_2(U)) \right) \\ &= \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{I}(\mathcal{V}(J_2(U))) \\ &= \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} J_2(U). \end{aligned}$$

□

As a first corollary to Theorem 3.5, we have that given some $U \subseteq A$, we can test to see if $I_2(\mathcal{H})$ can be colored by the edge colors/signatures in U.

Corollary 3.10. *Given $U \subseteq A$, $\mathcal{H}$ can be colored by the edge signatures in U if and only if $I_2(\mathcal{H}) \subseteq J_2(U)$.*

Proof: Let $U \subseteq A$. By Theorems 3.4 and 3.5, $\mathcal{H}$ can be colored by the edge signatures in U if and only if $\mathcal{V}(J_U) \subseteq \mathcal{V}(I_2(\mathcal{H}))$. Since both $I_2(\mathcal{H})$ and $J_2(U)$ are radical:

$$\mathcal{V}(J_2(U)) \subseteq \mathcal{V}(I_2(\mathcal{H})) \text{ if and only if } I_2(\mathcal{H}) \subseteq J_2(U).$$

□

A second result following Theorem 3.5 concerns the uniqueness of the vertex coloring of $\mathcal{H}$. A unique vertex 2-coloring is a proper 2-coloring of the vertex set that is unique up to permutation of the colors. For 2-colorings of the hypergraph $\mathcal{H}$, a unique vertex coloring corresponds to two distinct points in $\mathcal{V}(I_2(\mathcal{H}))$. These two points are the permutations of the colors on the vertices.

Corollary 3.11. *Let $\mathcal{H}$ be an r-uniform hypergraph and $I_2(\mathcal{H})$ be its associated coloring ideal. Then $\mathcal{H}$ is uniquely 2-colorable if and only if:*

$$I_2(\mathcal{H}) = J_{\{a_1\}} \cap J_{\{a_2\}},$$

where $a_1, a_2 \in A$ are proper edge signatures and $a_1 = -a_2$.

Proof: ($\Rightarrow$) Let $\mathcal{H}$ be uniquely 2-colorable.

Let a_1 and a_2 represent the two permutations of the colors on the vertex set. Then by Theorem 3.5,

$$I_2(\mathcal{H}) = J_{\{a_1\}} \cap J_{\{a_2\}}.$$

Moreover, since the two edge signatures a_1 and a_2 are permutations of each other, we have that $a_1 = -a_2$.

($\Leftarrow$) Assume that,

$$I_2(\mathcal{H}) = J_{\{a_1\}} \cap J_{\{a_2\}},$$

where $a_1, a_2 \in A$ are proper edge signatures and $a_1 = -a_2$.

Since $a_1 = -a_2$, we have that the signatures a_1 and a_2 are permutations of the colors assigned to the vertices. We also have that $|\mathcal{V}(I_2(\mathcal{H}))| = 2$ by Theorem 3.5. So $\mathcal{H}$ is 2-colorable. □

3.3 Conflict-free coloring

Our goal is to show how we can recognize hypergraphs with $\chi_{\mathrm{CF}}(\mathcal{H}) = 2$ and $\chi_{\mathrm{CF}}(\mathcal{H}) \neq 2$. First we note that clearly, $\chi(\mathcal{H}) \leq \chi_{\mathrm{CF}}(\mathcal{H})$, and we establish an equivalent condition for conflict free coloring.

For an r-uniform hypergraph, $\mathcal{H}$, with $\chi(\mathcal{H}) = 2$, the only edge signatures allowed in a conflict free coloring are the signatures

$$(1, 1, \ldots, 1, -1) \text{ and } (-1, -1, \ldots, -1, 1),$$

as these are the only signatures in which one of the colors is not repeated. Thus, if $\chi_{\mathrm{CF}}(\mathcal{H}) = 2$, then $\mathcal{V}(J_2(\{a_1, a_2\})) \neq \emptyset$. Also, if $\mathcal{V}(J_2(\{a_1, a_2\})) \neq \emptyset$, then $\mathcal{H}$ is properly 2-colored by the edge signatures a_1 and a_2, and since this is a conflict free coloring we have that $\chi_{\mathrm{CF}}(\mathcal{H}) \leq 2$. Since $\chi(\mathcal{H}) = 2$, in this second case we have that, $\chi_{\mathrm{CF}}(\mathcal{H}) = 2$. So,

$$\chi_{\mathrm{CF}}(\mathcal{H}) = 2 \text{ if and only if } \mathcal{V}(J_2(\{a_1, a_2\})) \neq \emptyset.$$

Now we may express this condition in terms of the ideals $I_2(\mathcal{H})$ and $J_2(\{a_1, a_2\})$.

Theorem 3.12. *Let a_1 and a_2 be the edge signature of the edge colorings $(1, 1, \ldots, 1, -1)$ and $(-1, -1, \ldots, -1, 1)$ respectively, that is $a_1 = r - 2$ and $a_2 = -(r - 2)$. Let the ideals $I_2(\mathcal{H})$ and $J_2(\{a_1, a_2\})$ be as in Theorem 3.5. Then $\chi_{\mathrm{CF}}(\mathcal{H}) = 2$ if and only if $I_2(\mathcal{H}) \subseteq J_2(\{a_1, a_2\})$.*

Proof: Assume $\chi(\mathcal{H}) = 2$ and let the ideals $I_2(\mathcal{H})$ and $J_2(\{a_1, a_2\})$ be as in Theorem 3.5. Then we have that $\mathcal{V}(I_2(\mathcal{H})) \neq \emptyset$.
Since $\chi_{\mathrm{CF}}(\mathcal{H}) = 2$ if and only if $\mathcal{V}(J_2(\{a_1, a_2\})) \neq \emptyset$, we have that:

$$\chi_{\mathrm{CF}}(\mathcal{H}) = 2 \text{ if and only if } \mathcal{V}(J_2(\{a_1, a_2\})) \subseteq \mathcal{V}(I_2(\mathcal{H})),$$

and since both $I_2(\mathcal{H})$ and $J_2(\{a_1, a_2\})$ are radical, we have

$$\mathcal{V}(J_2(\{a_1, a_2\})) \subseteq \mathcal{V}(I_2(\mathcal{H})) \text{ if and only if } J_2(\{a_1, a_2\}) \supseteq I_2(\mathcal{H}).$$

$\square$

See Chapter 5 for a detailed illustration of Theorem 3.12.

List of References

[1] S. Vishwanathan, "On 2-coloring certain r-uniform hypergraphs," *Journal of Combinatorial Theory Series A*, vol. 101, pp. 168 – 172, 2003.

[2] C. Hillar and T. Windfeldt, "Algebraic characterization of uniquely vertex colorable graphs," *Journal of Combinatorial Theory Series B*, vol. 98, pp. 400 – 414, 2008.

CHAPTER 4

k-Colorability

In this chapter we introduce our complete generalization of Hillar and Windfeldt's Theorem 1.1, along with the associated k-colorability results. We provide all statements of the k-colorability results in Section 4.1 and collect their proofs in Section 4.2. We also state and prove some theoretical results concerning list colorings of uniform hypergraphs and provide an algebraic characterization of the t-choosability of a uniform hypergraph. We end the chapter with an algebraic consequence of the structure of the ideals defined in this thesis.

4.1 k-Colorability Results

To address the k-colorability of an r-uniform hypergraph, we use similar techniques as in the 2-colorable case. Let $r \geq 2$ be an integer. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with m edges. For an integer $k \geq 1$, a *k-coloring* of the vertex set of $\mathcal{H}$ is defined to be a map, $c : V(\mathcal{H}) \to [k]$. Recall that a proper k-coloring is a k-coloring of $\mathcal{H}$ where no edge $e \in E(\mathcal{H})$ is monochromatic. For the 2-colorability case, we used the second roots of unity, ± 1 as our colors. However we cannot directly generalize this to the k-colorable case using primitive k^{th} roots of unity as was done in [1] and [2] for graph colorings. Instead, for the general k-colorability we will utilize prime numbers as our colors.

Let $k \geq 2$ be an integer, and let $\mathcal{P}_k$ be the set of the first k primes. Define a k-coloring of the uniform hypergraph $\mathcal{H}$ as a map,

$$c : V(\mathcal{H}) \to \mathcal{P}_k$$

Note that this is an equivalent definition of a k-coloring. For each edge, $e \in E(\mathcal{H})$

let:

$$e = (x_{e_1}, x_{e_2}, \dots, x_{e_r}),$$

where $e_j \in V(\mathcal{H}) = [n]$. Whereas the 2-colorability of a uniform hypergraph was characterized by proper edge sums, k-colorability relies on the product of the vertices in an edge. Define the *edge product* of an edge $e \in E(\mathcal{H})$ to be:

$$\prod_{i=1}^{r} x_{e_i}.$$

A proper coloring does not allow a monochromatic edge, thus we encode all of the non-monochromatic edges and use them to force a proper coloring.

Define the following as the set of *proper edge products*:

$$A = \left\{ a = \prod_{t=1}^{k} p_t^{\alpha_t} : p_t \in \mathcal{P}_k,\ \alpha_t \in [0, 1, \dots, r-1],\ \sum_{t=1}^{k} \alpha_t = r \right\}.$$

Note that for any color $p \in \mathcal{P}_k$,

$$a \neq p^r \quad \forall \quad a \in A.$$

The k-tuples of exponents, $(\alpha_1, \dots, \alpha_k)$, of the prime products in A are precisely the set of all proper k-integer partitions of r.

Consider the ideal:

$$C_k = \left\langle \prod_{p \in \mathcal{P}_k} (x_i - p) : i \in V(\mathcal{H}) \right\rangle.$$

Claim 4.1. *C_k is the ideal encoding all k-colorings of $\mathcal{H}$.*

Corollary 4.2. *$\mathcal{V}(C_k)$ is the set of all k-colorings of $\mathcal{H}$.*

The ideal C_k captures *every* k-coloring of $\mathcal{H}$, including the improper colorings. We therefore need some additional conditions to ensure the only the proper colorings are captured. We will utilize the set, A, of all proper edge products for $\mathcal{H}$. The polynomials below define the k-colorability ideal for $\mathcal{H}$, $I(\mathcal{H}, k)$:

$$f_{e,k} = \prod_{a \in A} \left[\left(\prod_{i=1}^{r} x_{e_i} \right) - a \right].$$

Define $I(\mathcal{H}, k)$ as:

$$I(\mathcal{H}, k) = C_k + \langle f_{e,k} : e \in E(\mathcal{H})\rangle.$$

We can now restate Theorem 5 from Chapter 1 which defines the k-colorability ideal for $\mathcal{H}$.

Theorem 4.3. *The polynomials in the ideal $I(\mathcal{H}, k)$ have a common solution if and only if $\mathcal{H}$ is properly $k-$colorable.*

Define the *hypergraph polynomial for k-colorability*, $P_{\mathcal{H},k}$, by:

$$P_{\mathcal{H},k} = \prod_{e \in E(\mathcal{H})} \prod_{t=1}^{k} \left[\left(\prod_{i=1}^{r} x_{e_i} \right) - p_t^r \right].$$

Theorem 4.4. *$\mathcal{H}$ is not properly $k-$colorable if and only if $\langle P_{\mathcal{H},k}\rangle + C_k$ has a solution.*

We can now state our complete generalization of Hillar and Windfeldt's Theorem 2.1 in [2]

Theorem 4.5. *Let r, $k \geq 2$ be positive integers and let $\mathcal{H}$ be an r-uniform hypergraph. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I(\mathcal{H}, k)$ be the k-colorability ideal for $\mathcal{H}$ and let $P_{\mathcal{H},k}$ be the k-color hypergraph polynomial for $\mathcal{H}$. Then following are equivalent:*

(1) The hypergraph $\mathcal{H}$ is not k-colorable.

(2) The constant 1 is an element of the ideal $I(\mathcal{H}, k)$.

(3) $\dim_{\mathbb{C}} R/I(\mathcal{H}, k) = 0$ as a vector space.

(4) The hypergraph polynomial $P_{\mathcal{H},k}$ belongs to the ideal C_k.

Moreover, by the structure of the ideal $I(\mathcal{H}, k)$, we can classify certain colorings of $\mathcal{H}$. Let U be a non-empty subset of A.

Definition 4.1. *The k-*coloring scheme *for the r-uniform hypergraph $\mathcal{H}$ given by U is the set of all colorings of $\mathcal{H}$ with the edge products in U.*

We define the *k-coloring scheme ideal*, $J(U,k)$, as the ideal that encodes the colorability of $\mathcal{H}$ by the edge products in U:

$$J(U,k) = C_k + \left\langle \prod_{a \in U} \left(\prod_{i=1}^{r} x_{e_i} \right) - a : e \in E(\mathcal{H}) \right\rangle .$$

Theorem 4.6. *The polynomials in the ideal $J(U,k)$ have a common solution if and only if the hypergraph $\mathcal{H}$ can be colored by the edge products in U.*

Define $I(\mathcal{H},k)$, $J(U,k)$, and A as above. Then the ideals and their associated varieties, $J(U,k)$, $I(\mathcal{H},k)$ and $\mathcal{V}(J(U,k))$, $\mathcal{V}(I(\mathcal{H},k))$, are related in the following ways.

Theorem 4.7. *Let r, $k \geq 2$ be integers and let $\mathcal{H}$ be an r-uniform hypergraph. Then,*

$$\mathcal{V}(I(\mathcal{H},k)) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(J(U,k)),$$

and

$$I(\mathcal{H},k) = \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} J(U,k).$$

Corollary 4.8. *Given $U \subseteq A$, $\mathcal{H}$ can be colored by the edge products in U if and only if $I(\mathcal{H},k) \subseteq J(U,k)$.*

As an illustration of Corollary 4.8, let U be the subset of all proper edge products that correspond to the particular k-integer partition of r, $\alpha = (\alpha_1, \ldots, \alpha_k)$. That is,

$$U = \{a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k} : p_j \in \mathcal{P}_k\},$$

where the $p_1, \ldots, p_k$ are permuted in all possible ways. Then the variety of the k-coloring scheme ideal $J(\mathcal{H}, U)$ contains the proper colorings of $\mathcal{H}$ in which α_j vertices share the same color in each edge, for $j = 1, 2, \ldots, k$. Hence each edge contains the same *color pattern* associated with the partition α, although the colors assigned to each part α_j may differ on distinct edges.

4.2 k-Colorability Proofs

In this section we let A be the set of all proper edge products for a k-coloring of $\mathcal{H}$. Also let $\mathcal{P}_k$ be the set of the first k prime numbers.

Proof: (Claim 4.1) For any $i \in V(\mathcal{H})$ the associated polynomial in C_k.

$$\prod_{p \in \mathcal{P}_k} (x_i - p),$$

vanishes if and only if $x_i = p$ for some $p \in \mathcal{P}_k$. That is, if and only if the vertex i is colored by a prime color in $\mathcal{P}_k$. □

Proof: (Corollary 4.2) Since the polynomials in C_k have a common solution c if and only if c is a k-coloring of $\mathcal{H}$, the variety $\mathcal{V}(C_k)$ is the set of all k-colorings of $\mathcal{H}$. □

Proof: (Theorem 1.5) ($\Rightarrow$) Let $c \in \mathcal{V}(I(\mathcal{H}, k))$. Since c zeros every polynomial in C_k, each vertex takes on a value in $\mathcal{P}_k$. Also, for each edge $e \in E(\mathcal{H})$, $f_{e,k}(c) = 0$, so the edge product of e is a value in A. Thus each edge is properly colored.

($\Leftarrow$) Let c be a proper $k-$coloring of $\mathcal{H}$. That is, c is an n-tuple of primes in $\mathcal{P}_k$ and c colors no edge monochromatically. Since each vertex is a prime in $\mathcal{P}_k$, c us a common solution of C_k. Also, since no edge is colored monochromatically, $f_{e,k}(c) = 0$ for every edge in $E(\mathcal{H})$. So c is a common solution of $I(\mathcal{H}, k)$. □

Proof: (Theorem 4.4) ($\Rightarrow$) Assume $\mathcal{H}$ is not properly $k-$colorable. Let c be any $k-$coloring of $\mathcal{H}$, c must monochromatically color an edge. Let e be the improperly colored edge, and without loss of generality let p_t be the color of each vertex in e.

The edge product of e is then p_t^r, thus c is a common solution of $\langle P_{\mathcal{H},k} \rangle + C_k$.

($\Leftarrow$) Assume $\langle P_{\mathcal{H},k} \rangle + C_k$ has a common solution, c. Since C_k vanishes at c, each vertex is assigned a single power of a prime in $\mathcal{P}_k$. Also, since $P_{\mathcal{H},k}(c) = 0$ there exists an edge $e \in E(\mathcal{H})$ and a prime $p_t \in \mathcal{P}_k$ such that:

$$\prod_{i=1}^{r} x_{e_i} - p_t^r = 0 \Rightarrow \prod_{i=1}^{r} x_{e_i} = p_t^r.$$

Thus the edge e has an edge product of p_t^r and is not properly $k-$colored. $\square$

Proof: (Theorem 4.6) Let U be a non-empty subset of A. ($\Rightarrow$) Assume c is a common solution to $J(U,k)$. By the definition of C_k, c is a $k-$coloring of $\mathcal{H}$. Moreover, since the product of the vertices in each edge is a value in U, $\mathcal{H}$ is properly colored by the edge products in U.

($\Leftarrow$) Assume $\mathcal{H}$ is colorable by the edge products in U. Let c be any such $k-$coloring. Then c assigns an edge product from U to each edge, thus c is a solution to $J(U,k)$. $\square$

Proof: (Theorem 4.7) Note that since C_k is contained in both $I(\mathcal{H},k)$ and $J(U,k)$, and C_k contains univariate squarefree polynomials in each indeterminate, C_k, and thus $I(\mathcal{H},k)$ and $J(U,k)$, are all radical by Theorem 2.15. Since,

$$A = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} U,$$

we have that,

$$\mathcal{V}(I(\mathcal{H},k)) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(J(U,k)).$$

Moreover, since $I(\mathcal{H},k)$ and $J(U,k)$ are radical, we have that:

$$I(\mathcal{H},k) = \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} J(U,k),$$

by Theorem 2.8. $\square$

Proof: (Corollary 4.8) Let U be a non-empty subset of A.

By Theorems 3.4 and 4.7, $\mathcal{H}$ can be colored by the edge products in U if and only if $\mathcal{V}(J(U,k)) \subseteq \mathcal{V}(I(\mathcal{H},k))$. Since both $I(\mathcal{H},k)$ and $J(U,k)$ are radical:

$$\mathcal{V}(J(U,k)) \subseteq \mathcal{V}(I(\mathcal{H},k)) \text{ if and only if } I(\mathcal{H},k) \subseteq J(U,k).$$

□

4.3 Conflict-free k-Colorings

In this section we address conflict-free colorings of uniform hypergraphs when using possibly more than two colors. As with conflict-free 2-colorings, a conflict-free k-coloring is a proper coloring in which each edge contains a vertex whose color is not repeated by any other vertex in the edge. We capture this coloring with the following edge products. Let U_{CF} be the subset of proper edge products, A, such that:

$$U_{CF} = \left\{ a = \prod_{t=1}^{k} p_t^{\alpha_t} \in A : \exists\ t \in [k] \text{ such that } \alpha_t = 1 \right\},$$

where the k-tuple $(\alpha_1, \ldots, \alpha_k)$ is a proper k-integer partition of r. Define U_{CF} to be the set of all *conflict-free edge products*. We can determine if a uniform hypergraph contains a conflict-free k-coloring via the following theorem.

Theorem 4.9. *Let $\mathcal{H}$ be an r-uniform hypergraph. Then $\mathcal{H}$ admits a conflict-free k-coloring if and only if*

$$I(\mathcal{H},k) \subseteq J(\mathcal{H},U_{CF}).$$

Proof: Let $U = U_{CF}$, the result follows from Corollary 4.8. □

4.4 List Colorings

In this section we introduce list colorings of uniform hypergraphs and provide an algebraic characterization of the t-choosability of a uniform hypergraph. We note that the characterization holds for graphs as well. As before we collect all results in Section 4.4 and postpone the proofs until Section 4.5.

Let $\mathcal{H}$ be an r-uniform hypergraph on the vertex set $[n]$. Let $\mathcal{P}_k$ be the set of the first k primes. For each vertex $v \in V(\mathcal{H})$ let

$$S_v = \{p_{v_1}, p_{v_2}, \ldots, p_{v_t}\}$$

be a given list of colors where each $p_{v_i} \in \mathcal{P}_k$.

Definition 4.2. *A* list coloring *of $\mathcal{H}$* by the lists *$S = \{S_v\}_{v\in V(\mathcal{H})}$ is a map $c : V(\mathcal{H}) \to \mathcal{P}_k$ such that $c(v) \in S_v$ for every $v \in V(\mathcal{H})$. If $\mathcal{H}$ admits a list coloring for any collection of lists $S = \{S_v\}_{v\in V(\mathcal{H})}$ where each list has length t, then $\mathcal{H}$ is* t-list choosable.

As in the k-Colorability section we define the following set as the *proper edge products* for an r-uniform hypergraph:

$$A = \left\{ a = \prod_{t=1}^{k} p_t^{\alpha_t} : p_t \in \mathcal{P}_k,\ \alpha_t \in [0, 1, \ldots, r-1],\ \sum_{t=1}^{k} \alpha_t = r \right\}.$$

We only restrict the possible colors to be the first k primes for computational considerations; the colors need only be distinct relatively prime elements from a unique factorization domain contained within an algebraically closed field of coefficients.

Let $S = \{S_v\}_{v\in V(\mathcal{H})}$ be a collection of lists of colors and consider the following ideals:

$$C_v = \left\langle \prod_{p_v \in S_v} (x_v - p_v) \right\rangle,$$

$$L(\mathcal{H}, S) = \sum_{v\in V(\mathcal{H})} C_v.$$

Proposition 4.10. *The polynomials in the ideal C_v have a common solution if and only if the vertex v is colored by a color in S_v.*

Theorem 4.11. *The polynomials in the ideal $L(\mathcal{H}, S)$ have a common solution if and only if the hypergraph $\mathcal{H}$ is colored by the lists in $S = \{S_v\}_{v \in V(\mathcal{H})}$.*

The corresponding variety is the collection of the appropriate list colorings.

Corollary 4.12. *$\mathcal{V}(L(\mathcal{H}, S))$ is the set of all possible list colorings of $\mathcal{H}$ by the colors $\mathcal{P}_k$ according to the lists in $S = \{S_v\}_{v \in V(\mathcal{H})}$.*

Theorem 4.13. *The ideal $I(\mathcal{H}, S) = I(\mathcal{H}, k) + L(\mathcal{H}, S)$ is an algebraic characterization of the proper list colorings of $\mathcal{H}$ by the collection of lists $S = \{S_v : v \in V(\mathcal{H})\}$. That is, the polynomials in $I(\mathcal{H}, S)$ have a common solution if and only if $\mathcal{H}$ is properly list colored by the lists in $S = \{S_{v \in V(\mathcal{H})}\}$.*

Theorem 4.14. *Let $k \geq 2$ be an integer and let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. The product of ideals*

$$\prod_S I(\mathcal{H}, S)$$

where $S = \{S_1, S_2, \ldots, S_n\}$ ranges over all collections of t-length lists S_i each a subset of $\mathcal{P}_k$, encodes the t-list colorability of the hypergraph $\mathcal{H}$. That is, the hypergraph $\mathcal{H}$ is t-list choosable if and only if

$$\mathcal{V}\left(\prod_S I(\mathcal{H}, S)\right) \neq \emptyset.$$

4.5 List Coloring Proofs

Proof: (Proposition 4.10) ($\Rightarrow$) Let v be any vertex in $\mathcal{H}$. Assume the v^{th} coordinate of $c \in \mathbb{C}^n$ is a solution to $\prod_{i=1}^{t}(x_v - p_{v_i})$. Then the v^{th} coordinate of c has a value $p_v \in S_v$. Thus the vertex v is colored by the list S_v.

($\Leftarrow$) Assume $c \in \mathbb{C}^n$ is a coloring of $\mathcal{H}$ in which the vertex v is colored by a member of the list $S_v = \{p_{v_1}, \ldots, p_{v_t}\}$. Then the v^{th} coordinate of c has a value

$p_{v_i} \in S_v$. Hence exactly one factor of the product

$$\prod_{i=1}^{t}(x_v - p_{v_i})$$

is zero, so c is a solution of the ideal C_v. $\square$

Proof: (Theorem 4.11) Since the argument in Proposition 4.10 holds for every $v \in V(\mathcal{H})$, i.e. for each indeterminate $x_i \in \mathbb{C}[x_1, \ldots, x_n]$, and

$$L(\mathcal{H}, S) = \sum_{v \in V(\mathcal{H})} C_v = \left\langle \prod_{p_1 \in S_1} (x_1 - p_1), \ldots, \prod_{p_n \in S_n} (x_n - p_n) \right\rangle,$$

we have that $c \in \mathbb{C}^n$ is a common solution of $L(\mathcal{H}, S)$ if and only if $\mathcal{H}$ is colored by the lists $S = \{S_v\}_{v \in V(\mathcal{H})}$. $\square$

Proof: (Corollary 4.12) By Theorem 4.11, an n-tuple $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ is in $\mathcal{V}(L(\mathcal{H}, S))$ if and only if c is a coloring of $\mathcal{H}$ with colors from $\mathcal{P}_k$ according to the lists $S = \{S_v\}_{v \in V(\mathcal{H})}$. Hence $\mathcal{V}(L(\mathcal{H}, S))$ is the collection of all such colorings. $\square$

Proof: (Theorem 4.13) As above, $c \in \mathbb{C}^n$ is a common solution of $I(\mathcal{H}, S) = I(\mathcal{H}, k) + L(\mathcal{H}, S)$ if and only if c is a common solution of $I(\mathcal{H}, k)$ and $L(\mathcal{H}, S)$. So by Theorem 1.5 and Proposition 4.10, c must be a proper coloring of $\mathcal{H}$ by the lists $S = \{S_1, \ldots, S_n\}$. $\square$

Proof: (Theorem 4.14) ($\Rightarrow$) Assume the hypergraph $\mathcal{H}$ is t-list choosable. Then there exists a set of lists, each of length t, $S = \{S_v\}_{v \in V(\mathcal{H})}$ of colors so that $\mathcal{H}$ admits a proper coloring by the lists in S. Then by Theorem 4.13, the polynomials in the ideal $I(\mathcal{H}, S)$ will have a common solution. Then, since the product of radical ideals has the same variety as the intersection of radical ideals by Theorem 2.16

$$\mathcal{V}\left(\prod_S I(\mathcal{H}, S)\right) = \mathcal{V}\left(\bigcap_S I(\mathcal{H}, S)\right) \neq \emptyset.$$

($\Leftarrow$) Assume,

$$\mathcal{V}\left(\prod_S I(\mathcal{H}, S)\right) \neq \emptyset,$$

then by Theorem 2.16:

$$\mathcal{V}\left(\bigcap_S I(\mathcal{H}, S)\right) \neq \emptyset.$$

and

$$\mathcal{V}\left(\bigcap_S I(\mathcal{H}, S)\right) = \bigcup_S \mathcal{V}(I(\mathcal{H}, S)),$$

by Theorem 2.8, we have that for some list S, $\mathcal{V}(I(\mathcal{H}, S)) \neq \emptyset$. Hence by Corollary 4.13, $\mathcal{H}$ is list-colorable by the list S, and since $|S| = t$, $\mathcal{H}$ is t-list choosable. □

4.6 Ideal Primary Decompositions

In this section we examine the relationship between the collection of proper colorings of a uniform hypergraph and the structure of the ideals that encode these colorings. The encodings for colorability given in Chapters 3 and 4 owe much of their ease to the structure of their ideals. Each colorability ideal is constructed so that its variety is the collection of all possible colorings that satisfy some condition. As a result, each variety is a finite collection of points in $\mathbb{C}^n$. Moreover, we can make a statement about the multiplicity of these solutions.

Theorem 4.15. *Each point in the varieties $\mathcal{V}(I_2(\mathcal{H}))$, $\mathcal{V}(I(\mathcal{H}, k))$ and $\mathcal{V}(I(\mathcal{H}, S))$ has multiplicity 1.*

Proof: By Theorem 2.15, $I_2(\mathcal{H})$, $I(\mathcal{H}, k)$ and $I(\mathcal{H}, S)$ are all radical. Thus by Theorem 2.18 each point $\mathcal{V}(I_2(\mathcal{H}))$, $\mathcal{V}(I(\mathcal{H}, k))$ or $\mathcal{V}(I(\mathcal{H}, k))$, has multiplicity 1. □

For each point c in any of the variety, set $V_c = \{c\}$, subsets of this form are also varieties and are known as *irreducible varieties*. Furthermore, for $c = (c_1, \ldots, c_n)$ define

$$I_c = \mathcal{I}(V_c) = \langle x_1 - c_1, \ldots, x_n - c_n \rangle .$$

Proposition 4.16 (Propositions 9 and 10, pp 198-199, [3]). *I_c is maximal and prime in $R = \mathbb{C}[x_1, \ldots, x_n]$.*

Prime ideals are important in determining the structure of an ideal. If an ideal I can be uniquely written as an intersection of distinct prime ideals Q_i then the intersection is called the *minimal primary decomposition* of I:

$$I = \bigcap_i Q_i.$$

Similarly, if a variety V can be written as union of disjoint irreducible varieties V_j the union is called the *minimal decomposition* of V:

$$V = \bigcup_j V_j.$$

For more information on primary decompositions of ideals and decompositions of varieties, see [3] or [4].

Theorem 4.17. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices and let $\mathbb{I}$ be any of $I_2(\mathcal{H})$, $I(\mathcal{H}, k)$ or $I(\mathcal{H}, S)$. Then,*

$$\mathcal{V}(\mathbb{I}) = \bigcup_{c \in \mathcal{V}(\mathbb{I})} V_c,$$

is the minimal decomposition of $\mathcal{V}(\mathbb{I})$, and

$$\mathbb{I} = \bigcap_{c \in \mathcal{V}(\mathbb{I})} I_c$$

is the minimal primary decomposition of $\mathbb{I}$.

Proof: Note that

$$\mathcal{V}(\mathbb{I}) = \bigcup_{c \in \mathcal{V}(\mathbb{I})} \{c\} = \bigcup_{c \in \mathcal{V}(\mathbb{I})} V_c.$$

Then since $\mathbb{I}$ is radical,

$$\begin{aligned}
\mathbb{I} &= \mathcal{I}(\mathcal{V}(\mathbb{I})) \\
&= \mathcal{I}\left(\bigcup_{c\in\mathcal{V}(\mathbb{I})} \{c\}\right) \\
&= \mathcal{I}\left(\bigcup_{c\in\mathcal{V}(\mathbb{I})} V_c\right) \\
&= \bigcap_{c\in\mathcal{V}(\mathbb{I})} \mathcal{I}(V_c) \\
&= \bigcap_{c\in\mathcal{V}(\mathbb{I})} I_c.
\end{aligned}$$

□

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[3] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*. New York, New York, United States of America: Springer-Verlag, 1997.

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CHAPTER 5

Computation

In this chapter we provide some detailed examples of applications of the theorems in Chapters 3 and 4. In the first section we address the coloring scheme ideals and conflict-free colorings of the Fano plane. In Section 5.2 we provide a technique for determining if a uniform hypergraph can be properly colored which we utilize in Section 5.3. In Section 5.3 we give a generalization of the construction given by Abbott and Hanson in [1], and improved on by Seymour in [2]. We end the chapter by providing some computations on the chromatic number of Stable Kneser hypergraphs.

5.1 The Fano plane

We begin with this simplest example of a non-2-colorable 3-uniform hypergraph. Recall the Fano plane, FP, is a 3-uniform hypergraph on 7 vertices with 7 edges:

$$V(FP) = \{1, 2, 3, 4, 5, 6, 7\}$$

and

$$E(FP) = \{\{1, 2, 5\}, \{1, 3, 7\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{2, 4, 7\}, \{5, 6, 7\}\}.$$

We examine the 2-colorability ideal, and check to see if we can modify the Fano plane so that it will admit a conflict-free coloring.

5.1.1 2-colorability

The corresponding 2-colorability ideal $I_2(FP)$ given by Theorem 1.3 is:

$$\langle x_i^2 - 1 : i \in [7] \rangle +$$

$$\langle (2x_1 + x_2 + x_5)(x_1 + 2x_2 + x_5)(x_1 + x_2 + 2x_5),$$

$$(2x_1 + x_3 + x_7)(x_1 + 2x_3 + x_7)(x_1 + x_3 + 2x_7),$$

$$(2x_1 + x_4 + x_6)(x_1 + 2x_4 + x_6)(x_1 + x_4 + 2x_6),$$

$$(2x_2 + x_3 + x_6)(x_2 + 2x_3 + x_6)(x_2 + x_3 + 2x_6),$$

$$(2x_3 + x_4 + x_5)(x_3 + 2x_4 + x_5)(x_3 + x_4 + 2x_5),$$

$$(2x_2 + x_4 + x_7)(x_2 + 2x_4 + x_7)(x_2 + x_4 + 2x_7),$$

$$(2x_5 + x_6 + x_7)(x_5 + 2x_6 + x_7)(x_5 + x_6 + 2x_7) \rangle$$

Since FP is 3-uniform, there are only 2 possible proper edge signatures for a 2-coloring of FP:

$$\text{the edge colored with two } 1's \text{ and one } -1 \Rightarrow a_1 = \sum_{i=1}^{3} x_{e,i} = 1,$$

and,

$$\text{the edge colored with one 1 and two } -1's \Rightarrow a_2 = \sum_{i=1}^{3} x_{e,i} = -1.$$

So $A = \{1, -1\}$. Thus the possible non-empty subsets, U, of A are:

$$\{1\},\ \{-1\} \text{ and } \{1, -1\}.$$

Which gives us the coloring scheme ideals:

$$
\begin{aligned}
J_2(\{1\}) = {} & \langle x_i^2 - 1 : i \in [7] \rangle + \\
& \langle x_1 + x_2 + x_5 - 1, x_1 + x_3 + x_7 - 1, x_1 + x_4 + x_6 - 1, \\
& x_2 + x_3 + x_6 - 1, x_3 + x_4 + x_5 - 1, x_2 + x_4 + x_7 - 1, \\
& x_5 + x_6 + x_7 - 1 \rangle, \\
J_2(\{-1\}) = {} & \langle x_i^2 - 1 : i \in [7] \rangle + \\
& \langle x_1 + x_2 + x_5 + 1, x_1 + x_3 + x_7 + 1, x_1 + x_4 + x_6 + 1, \\
& x_2 + x_3 + x_6 + 1, x_3 + x_4 + x_5 + 1, x_2 + x_4 + x_7 + 1, \\
& x_5 + x_6 + x_7 + 1 \rangle, \\
J_2(\{1, -1\}) = {} & \langle x_i^2 - 1 : i \in [7] \rangle + \\
& \langle (x_1 + x_2 + x_5 - 1)(x_1 + x_2 + x_5 + 1), \\
& (x_1 + x_3 + x_7 - 1)(x_1 + x_3 + x_7 + 1), \\
& (x_1 + x_4 + x_6 - 1)(x_1 + x_4 + x_6 + 1), \\
& (x_2 + x_3 + x_6 - 1)(x_2 + x_3 + x_6 + 1), \\
& (x_3 + x_4 + x_5 - 1)(x_3 + x_4 + x_5 + 1), \\
& (x_2 + x_4 + x_7 - 1)(x_2 + x_4 + x_7 + 1), \\
& (x_5 + x_6 + x_7 - 1)(x_5 + x_6 + x_7 + 1) \rangle.
\end{aligned}
$$

Using a Gröbner basis package in a computer algebra system like Mathematica or Singular we can show that all of the above ideals contain the constant 1 and thus by Theorems 1.3 and 3.5, the Fano plane is not 2-colorable.

5.1.2 Conflict Free Colorings of the Fano Plane

To further demonstrate Theorem 3.5 and give an example of a conflict-free coloring we consider the Fano plane with an edge removed. Note that,

$$\chi(FP \setminus \{any\ edge\}) = 2.$$

Let the Modified Fano plane be the hypergraph $FP' = FP \setminus \{1,2,5\}$ where:

$$V(FP') = \{1,2,3,4,5,6,7\}$$

and

$$E(FP') = \{\{1,3,7\},\{1,4,6\},\{2,3,6\},\{3,4,5\},\{2,4,7\},\{5,6,7\}\}.$$

The corresponding 2-colorability ideal is:

$$\begin{aligned} I_2(FP') =& \langle x_i^2 - 1 : i \in [7]\rangle + \\ & \langle (2x_1 + x_3 + x_7)(x_1 + 2x_3 + x_7)(x_1 + x_3 + 2x_7), \\ & (2x_1 + x_4 + x_6)(x_1 + 2x_4 + x_6)(x_1 + x_4 + 2x_6), \\ & (2x_2 + x_3 + x_6)(x_2 + 2x_3 + x_6)(x_2 + x_3 + 2x_6), \\ & (2x_3 + x_4 + x_5)(x_3 + 2x_4 + x_5)(x_3 + x_4 + 2x_5), \\ & (2x_2 + x_4 + x_7)(x_2 + 2x_4 + x_7)(x_2 + x_4 + 2x_7), \\ & (2x_5 + x_6 + x_7)(x_5 + 2x_6 + x_7)(x_5 + x_6 + 2x_7)\rangle. \end{aligned}$$

The reduced Gröbner basis for $I_2(FP')$ with respect to the monomial ordering $x_1 > x_2 > \cdots > x_7$ is:

$\{x_7^2 - 1,\ x_6^2 - 1,\ x_5x_6 + x_5x_7 + x_6x_7 + 1,\ x_5^2 - 1,\ x_4x_6 - x_4x_7 - 2x_5x_7 - x_6x_7 - 1,$
$x_4x_5 + x_4x_7 + x_5x_7 + 1,\ x_4^2 - 1,\ x_3x_6 - x_3x_7 - 2x_5x_7 - x_6x_7 - 1,$
$x_3x_5 + x_3x_7 + x_5x_7 + 1,\ x_3x_4 - x_3x_7 - x_4x_7 - 2x_5x_7 - 1,$
$x_3^2 - 1,\ x_2 - x_5,\ x_1 - x_5\}.$

This also tells us that $\chi_{\mathrm{CR}}(FP') = 2$.

Next we examine the coloring scheme ideals for FP':

$$\begin{aligned}
J_2(\{1\}) =& \langle x_i^2 - 1 : i \in [7]\rangle + \\
& \langle x_1 + x_3 + x_7 - 1, x_1 + x_4 + x_6 - 1, \\
& x_2 + x_3 + x_6 - 1, x_3 + x_4 + x_5 - 1, \\
& x_2 + x_4 + x_7 - 1, x_5 + x_6 + x_7 - 1\rangle, \\
J_2(\{-1\}) =& \langle x_i^2 - 1 : i \in [7]\rangle + \\
& \langle x_1 + x_3 + x_7 + 1, x_1 + x_4 + x_6 + 1, \\
& x_2 + x_3 + x_6 + 1, x_3 + x_4 + x_5 + 1, \\
& x_2 + x_4 + x_7 + 1, x_5 + x_6 + x_7 + 1\rangle, \\
J_2(\{1, -1\}) =& \langle x_i^2 - 1 : i \in [7]\rangle + \\
& \langle (x_1 + x_3 + x_7 - 1)(x_1 + x_3 + x_7 + 1), \\
& (x_1 + x_4 + x_6 - 1)(x_1 + x_4 + x_6 + 1), \\
& (x_2 + x_3 + x_6 - 1)(x_2 + x_3 + x_6 + 1), \\
& (x_3 + x_4 + x_5 - 1)(x_3 + x_4 + x_5 + 1), \\
& (x_2 + x_4 + x_7 - 1)(x_2 + x_4 + x_7 + 1), \\
& (x_5 + x_6 + x_7 - 1)(x_5 + x_6 + x_7 + 1)\rangle.
\end{aligned}$$

The reduced Gröbner bases for $J_2(\{1\}), J_2(\{-1\})$, and $J_2(\{1, -1\})$, with respect to the monomial ordering $x_1 > x_2 > \cdots > x_7$, are:

$$J_2(\{1\}) : \{x_7 - 1, x_6 - 1, x_5 + 1, x_4 - 1, x_3 - 1, x_2 + 1, x_1 + 1\}$$
$$J_2(\{-1\}) : \{x_7 + 1, x_6 + 1, x_5 - 1, x_4 + 1, x_3 + 1, x_2 - 1, x_1 - 1\}$$

$$
\begin{aligned}
J_2(\{1,-1\}) : \big\{ & x_7^2-1, x_6^2-1, x_5x_6+x_7x_6+x_5x_7+1, x_5^2-1, \\
& x_4x_6-x_7x_6-x_4x_7-2x_5x_7-1, x_4x_5+x_7x_5+x_4x_7+1, \\
& x_4^2-1, x_3x_6-x_7x_6-x_3x_7-2x_5x_7-1, x_3x_5+x_7x_5+x_3x_7+1, \\
& x_3x_4-x_7x_4-x_3x_7-2x_5x_7-1, x_3^2-1, x_2-x_5, x_1-x_5\big\}
\end{aligned}
$$

Moreover, we see that the Gröbner bases for $I_2(FP')$ and $J_2(\{1,-1\})$ with respect to the monomial ordering $x_1 > x_2 > \cdots > x_7$, are equal. In addition, the Gröbner basis for $J_2(\{1\}) \cap J_2(\{-1\}) \cap J_2(\{-1,1\})$ is the same also. Hence the 2-colorability ideal and the intersection of the color scheme ideals are identical and we conclude that the modified Fano plane FP' is 2-colorable, and also admits a conflict-free 2-coloring. Moreover, it can be similarly shown that the Fano plane with any single edge removed is properly 2-colorable and also has a conflict-free coloring.

5.2 Color Extensions

The coloring ideals given in Theorems 1.3 and 1.5 allow us to check if a given hypergraph, $\mathcal{H}$, is colorable by checking to see if the generating polynomials of $I_2(\mathcal{H})$ or $I(\mathcal{H}, k)$ have a common solution. If so, then each solution is a proper coloring of $\mathcal{H}$. The most complete method to answer this question is to compute the reduced Gröbner basis for the coloring ideal and apply the Weak Nullstellensatz. However, this is not always computationally feasible, as Gröbner basis computation is time comsuming.

To speed up this process we can partially color a hypergraph and use Gröbner bases to test if the partial coloring extends to a proper coloring. To do this, we choose a partial coloring of the given hypergraph $\mathcal{H}$, c_p. We then set the corresponding variables equal to the appropriate values in the coloring ideals $I(\mathcal{H}, k)$ or

$I_2(\mathcal{H})$. Let,

$$I(\mathcal{H},k)|_{c_p} \quad \text{and} \quad I_2(\mathcal{H})|_{c_p},$$

be the *partial coloring ideals* associated with partial coloring c_p for the hypergraph $\mathcal{H}$. We can then use the Weak Nullstellensatz as we have for Theorems 1.3 and 1.5.

Theorem 5.1. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. Let c_p be a partial coloring of $\mathcal{H}$. Let $I(\mathcal{H},k)|_{c_p}$ or $I_2(\mathcal{H})|_{c_p}$ be the partial coloring ideal associated with $\mathcal{H}$. Then c_p extends to a proper coloring of $\mathcal{H}$ if and only if $I(\mathcal{H},k)|_{c_p}$ or $I_2(\mathcal{H})|_{c_p}$ has a common solution.*

Proof: Without loss of generality, assume only x_1 is colored, we can iteratively apply the results if more vertices are colored. Also, assume that the color chosen is appropriate: ± 1 for $I_2(\mathcal{H})$ or some $p \in \mathcal{P}_k$ for $I(\mathcal{H},k)$.

Since appropriate colors are used, either the polynomial $x_1^2 - 1$ in $I_2(\mathcal{H})$ or $\prod_{p \in \mathcal{P}_k}(x_1 - p)$ in $I(\mathcal{H},k)$ will vanish. Moreover the f_e or $f_{e,k}$ polynomials in either ideal will have each incidence of x_1 replaced with the chosen color. This will require any common solution of either ideal to contain the chosen color in the first coordinate. Computing a Gröbner basis for the remaining polynomials determine if they have a common solution with the first coordinate fixed. Thus any common solution will be an extension of the partial coloring. $\square$

This technique allows us to quickly determine that a hypergraph is properly colorable withour knowing the complete Gröbner basis of the colorability ideal. If a partial coloring does not extend to a proper coloring, we cannot conclude that the hypergraph is not colorable, as a different partial coloring may extend to a proper coloring. Extending a partial coloring can be used to test for non-colorability, as long as all possible initial colorings are tested on the chosen vertices. Depending on the number of vertices in $\mathcal{H}$ and the uniformity of $\mathcal{H}$, coloring as few as 3 vertices can improve computing time.

5.3 Constructions

Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. We defined $m_n(r)$ be the least positive integer m such that: $|E(\mathcal{H})| = m$, and $\mathcal{H}$ does not have Property B. That is, $m_n(r)$ is the least number of edges in a non-2-colorable, r-uniform hypergraph on n vertices. Abbott and Hanson give (among others) the following inequalities for $m_n(r)$ in [1]:

$$m_{n+2r}(r) \leq \begin{cases} r \cdot m_n(r-2) + 2^{r-1}, & \text{if } r \text{ odd.} \\ r \cdot m_n(r-2) + 2^{r-1} + 2^{r-2}, & \text{if } r \text{ even.} \end{cases}$$

When $r = 4$ and $n = 3$, Seymour gives a construction which improves on the bound given by the inequality above, [2]. For the r odd case, a generalized version of Seymour's construction cannot improve on the bounds given by Abbott and Hanson in [1]. When r is even, however, we show that a generalization of Seymour's construction can improve these bounds. We specifically show the $r = 6$ construction and improve the bounds on $m_{23}(6)$ to 180.

5.3.1 Optimizing Seymour's Construction.

Let n and r be positive integers. The construction given in [2] generalizes to the following:

Take $S = [2r + n] = \{1, 2, \ldots, 2r + n\}$, and let A be a non-2-colorable hypergraph on

$$\{2r + 1, 2r + 2, \ldots, 2r + n\}$$

with $m_n(r - 2)$ edges.

Define,

$$B = \{\{1,2\},\{3,4\},\ldots,\{2r-1,2r\}\}$$
$$C = \{X \cup Y : X \in A, Y \in B\}$$
$$D = \{\{x_1,\ldots,x_r\} : x_i \in \{2i-1,2i\}, i = 1,\ldots,r\}.$$

Let E be a subset of D such that the following two conditions hold:

(i) If $X, Y \in D$ and $X \cup Y = [2r+n]$, then either X or Y is a member of E.

(ii) If $|Q| = r-1$, and Q is a subset of any member of D, then Q is a subset of a member of E.

Claim 5.2. *If $F = C \cup E$ is an r-uniform hypergraph on $2r+n$ edges, then F is not 2-colorable.*

Proof: Fix n and r in $\mathbb{Z}^+$. Let $F = C \cup E$.

Suppose that F is 2-colorable, or has property B; that is, suppose Z is a subset of S that intersects every member of F, but contains no member of F. Then $S \setminus Z$ is also a set intersecting each member of F that contains none, so we may assume that,

$$\left| Z \cap \{2r+1, 2r+2, \ldots, 2r+n\} \right| \leq \left\lfloor \frac{r}{2} \right\rfloor . \tag{3}$$

This follows from counting the maximum number of vertices in each edge that can be in either Z or $S \setminus Z$. Since A is not 2-colorable, (1) implies that Z does not intersect some member of A, else Z and $S \setminus Z$ would be a proper two coloring of A. Thus Z must intersect each member of B, since by hypothesis Z intersects each member of C. So for some $Y \in D$, $Y \subseteq Z$.

Note that $Z \cap [2r]$ intersects each member of F and so we may delete any other elements of Z and assume that $Z \subseteq [2r]$. Here we are focusing on the elements created from D and not the 'embedded' non-2-colorable hypergraph A. If $Y = Z$, then $Y \notin Z$, since F is assumed to have property B, and by (i),

$$\{[2r] \setminus Y\} \in E.$$

If $\{[2r] \setminus Y\} \in E$, and $Y = Z$, then

$$\{[2r] \setminus Y\} = \{[2r] \setminus Z\}, \text{ and } \{[2r] \setminus Z\} \cap Z = \emptyset,$$

which contradicts Z intersecting each member of F. Hence $Y \subset Z$, and $|Z| \geq r + 1$. This ensures that Z must contain at least on member of B; Without loss of generality, assume Z contains $\{1, 2\}$. Note that,

$$|Y \setminus \{1, 2\}| = r - 1, \text{ and, } Y \setminus \{1, 2\} \subset Y \in D;$$

by (ii),

$$\{Y \setminus \{1, 2\} \subset X, \text{ for some } X \in E.$$

Since $Y \subset Z$, and $\{1, 2\} \subset Z$, $X \subseteq Z$. This contradicts the hypothesis that Z contains no member of F, thus F does not have Property B. $\square$

Next we introduce the generalized cube graph, GQ_r as follows. First define the following set, let J be the set of all $r - 1$ element subsets of any element of D. Let GQ_r be a graph with vertex set $V(GQ_r) = D$, so each member of D is assigned a vertex in GQ_r. The edge set for GQ_r is defined as follows:

$$(v_1, v_2) \in E(GQ_r) \text{ if either } v_1 \cup v_2 = [2r], \text{ or, } v_1 \cap v_2 \in J.$$

Defined this way, the vertex, v_X, for any given element in $X \in D$ will be adjacent to the complement of X in $[2r]$, and all vertices associated with elements in D who share an $r - 1$ element subset with D.

The graph GQ_r is $r + 1$-regular hypercube on 2^r vertices with all antipodal diagonals as edges. A *vertex cover*, $V_C(G)$, of a graph is a subset of the vertex set of G such that every edge in $E(G)$ is incident to at least one vertex in $V_C(G)$. The following claim shows that there is a correspondence between the set E and $V_C(GQ_r)$.

Claim 5.3. *Let V_C be a vertex cover of G. The members of D that correspond to vertices in V_C satisfy conditions (i) and (ii).*

Proof: Let $v_X, v_{X^c} \in V_C$ be the vertices in $V(G)$ corresponding to

$$X, X^c = [2r] \setminus X \in D.$$

Since for all $X \in D$ the edge (v_X, v_{X^c}) is incident to at least one vertex in V_C, condition (i) is satisfied.

Since every edge in $E(G)$ is incident to a vertex in V_C, every $r-1$ element subset of any member of D is represented by at least one vertex in V_C, satisfying condition (ii). $\square$

Claim 5.3 asserts that any vertex cover will work in constructing a non-2-colorable hypergraph. By adding the members of D that are present in a vertex cover of G to C we will create a hypergraph that satisfies Claim 5.2. This leads us to utilize a *minimal* vertex cover. Adding a minimal vertex cover of GQ_r to C will create a non-2-colorable r-uniform hypergraph with the minimal number of edges allowed by Seymour's construction. We find minimal vertex covers of GQ_5 and GQ_6 using linear programming in Sections 5.2.2 and 5.2.3. We note that while this will improve on Abbott and Hanson's upper bound, it may not be the best possible construction. However, $m(6) \leq 180$ is the best known upper bound.

5.3.2 5-Uniform Construction

We can show minimality for the 5-uniform case, we begin with the 3-uniform, non-2-colorable hypergraph on 7 vertices, the Fano plane. Let $FP^* = (V(FP^*), E(FP^*))$ be the following:

$$V(FP^*) = \{11, 12, 13, 14, 15, 16, 17\}$$

and

$$E(FP^*) = \Big\{\{11, 12, 15\}, \{11, 13, 17\}, \{11, 14, 16\}, \{12, 13, 16\},$$
$$\{13, 14, 15\}, \{12, 14, 17\}, \{15, 16, 17\}\Big\}.$$

Note that this is the same hypergraph as above, with renamed vertices. We construct a 5-uniform hypergraph, following Abbott and Hanson in [1], as follows: we let

$$A = E(FP)$$

$$B = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}$$

$$C = \{X \cup Y : X \in A, Y \in B\}$$

$$D = \{\{x_1, \ldots, x_5\} : x_i \in \{2i - 1, 2i\}, i = 1, \ldots, 5\}.$$

Let $E \subset D$ satisfy conditions (i) and (ii) above and let $F_5 = C \cup E$. To choose E, we utilize the minimal vertex cover technique given in Claim 5.3. The set F_5 is given explicitly in Appendix 1.

Further, it can be shown computationally, by using the color extension technique described in Section 5.2, that removing any edge from F_5 will yield a 2-colorable hypergraph. That is, F_5 is critical.

Further, for any r odd, GQ_r is a bipartite graph and, hence, the minimal vertex cover for GQ_r has size 2^{r-1}. This agrees with the upper bound provided by Abbott and Hanson in [1].

5.3.3 6-Uniform Construction

For the 6-uniform case, we begin with a 4-uniform, non-2-colorable hypergraph on 11 vertices. Let $SE = (V(SE), E(SE))$ be the following:

$$V(SE) = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$$

and

$$\begin{aligned} E(SE) = \Big\{ & \{13, 14, 21, 22\}, \{13, 14, 21, 23\}, \{13, 14, 22, 23\}, \\ & \{13, 15, 17, 20\}, \{13, 15, 18, 19\}, \{13, 16, 17, 19\}, \\ & \{13, 16, 18, 19\}, \{13, 16, 18, 20\}, \{14, 15, 17, 19\}, \\ & \{14, 15, 18, 19\}, \{14, 15, 18, 20\}, \{14, 16, 17, 19\}, \\ & \{14, 16, 17, 20\}, \{14, 16, 18, 20\}, \{15, 16, 21, 22\}, \\ & \{15, 16, 21, 23\}, \{15, 16, 22, 23\}, \{17, 18, 21, 22\}, \\ & \{17, 18, 21, 23\}, \{17, 18, 22, 23\}, \{19, 20, 21, 22\}, \\ & \{19, 20, 21, 23\}, \{19, 20, 22, 23\} \Big\}. \end{aligned}$$

This is the hypergraph created by Seymour in [2]. We construct a 6-uniform hypergraph as follows: we let

$A = E(SE)$,

$B = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$,

$C = \{X \cup Y : X \in A, Y \in B\}$, and

$D = \{\{x_1, \dots, x_6\} : x_i \in \{2i - 1, 2i\}, i = 1, \dots, 6\}$.

Let $E \subset D$ satisfy conditions (i) and (ii) above and let $F_6 = C \cup E$. Again, we use Claim 5.3 to choose the edges in E. The set F_6 is given explicitly in Appendix 2.

It can be shown, again using the computation of color extensions, that removing any edge from F_6 will yield a 2-colorable hypergraph, hence F_6 is critical. Moreover, this construction improves on Abbott and Hanson's upper bound. Using the inequalities given at the beginning of this chapter, Abbott and Hanson guarantee a non-2-colorable 6-uniform hypergraph on 23 vertices with 196 edges. We have improved this to 180. In addition, Claim 5.3 implies that this is an optimal construction possible using conditions (i) and (ii).

5.4 The Chromatic Number of Stable Kneser Hypergraphs

As a further application of our method in this section we provide computation of chromatic numbers for some Stable Kneser hypergraphs. Let $r, n \geq 1$ be positive integers. A subset $S \subseteq [n]$ is *stable* if

$$r \leq |i-j| \leq n-r \quad \text{for all distinct } i, j \in S;$$

that is, any two elements of S are at least a 'distance of r apart' modulo n. For $r \geq 2$, $k \geq 2$, the r-stable k-uniform Kneser hypergraph,

$$KG^r\binom{[n]}{k}_{r-stab}$$

is the hypergraph with vertex set consisting of all r-stable k-element subsets of $[n]$. The edge set is formed by r-tuples $S_1, \ldots, S_r$ of pairwise disjoint vertices, i.e. of pairwise disjoint r-stable k-element subsets of $[n]$.

Stable Kneser hypergraphs generalize to Kneser hypergraphs introduced by M. Kneser in 1955, [3]. In 1978 Lovász proved Kneser's conjecture on the chromatic number of Kneser graphs, [4]. Later Alon, Frankl, and Lovász proved a conjecture of Erdős on the chromatic number of Kneser hypergraphs $KG^r\binom{[n]}{k}$, [5]. In [6], the authors conjecture that the chromatic numbers of Stable Kneser hypergraphs are the same as the chromatic numbers of Kneser hypergraphs.

Conjecture 5.4. *Let $n, k, r > 0$ be integers such that $n \geq rk$. Then*

$$\chi\left(KG^r\binom{[n]}{k}_{r-stab}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

The conjecture is known to hold only for r being a power of 2, [6]. Recently, F. Meunier supported the conjecture by computation. We extend Meunier's computation using the methods developed in this thesis. These computations are done using partial color extensions described in the previous section. We can conclude that the conjecture holds for:

$$\begin{aligned} r = 3, \quad & k = 4, \quad n \leq 15 \\ & k = 5, \quad n \leq 18 \\ & k = 6, \quad n \leq 21 \\ & k = 7, \quad n \leq 24 \\ r = 5, \quad & k = 2, \quad n \leq 14. \end{aligned}$$

We note that the Stable Kneser hypergraph $KG^r\binom{[n]}{k}_{r-stab}$ with $r = 3$, $k = 7$, and $n = 24$ has 288 vertices and 9568 edges.

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[4] L. Lovász, "Kneser's conjecture, chromatic number and homotopy," *Journal of Combinatorial Theory Series A*, vol. 25, pp. 319 – 324, 1978.

[5] N. Alon, P. Frankl, and L. Lovász, "The chromatic number of Kneser hypergraphs," *Transactions of the American Mathematical Society*, vol. 298, no. 1.

[6] N. Alon, L. Drewnowski, and T. Łuczak, "Stable Kneser hypergraphs and ideals in $\mathbb{N}$ with the Nikodým property," *Proceedings of the American Mathematical Society*, vol. 137, pp. 467 – 471, 2009.

CHAPTER 6

Conclusions

The techniques developed in this thesis extend the algebraic methods developed and implemented by authors such as Alon, De Loera, Hillar, and Lovász. Coloring of hypergraphs is the first application of these techniques. The results in this thesis provide new results on colorings of uniform hypergraph using polynomial ideals. The first possibility for generalization is to remove the uniformity condition.

Open Problem 1. *Extend the results of this thesis to the non-uniform case.*

In addition, we believe that these techniques can also extend computation. In Chapter 1 we mentioned the question posed by Miller, "What is the least number of edges allowed in a non-2-colorable uniform hypergraph?" Using Theorem 1.5 we can address a generalization of Miller's question. We set $m_n(r,k)$ to be the minimum number of edges allowed in a non-k-colorable r-uniform hypergraph on n vertices. Let $M_n(r,k)$ be the set of all positive integers m such that there exists an r-uniform hypergraph on n vertices, $\mathcal{H}$, that is critically non-k-colorable and where $|E(\mathcal{H})| = m$. Note that this set is related to the notation introduced by Erdős and Hajnal:

$$m(r) = \min_{n} \; M_n(r,2).$$

As an application of Theorem 1.5 we wish to examine these sets.

Open Problem 2. *Given positive integers r, k, and n, examine the structure of the set $M_n(r,k)$.*

In addition to coloring, using square-free generated radical ideals has potential for detecting certain subgraphs and subhypergraphs. It would be beneficial to find a polynomial ideal associated with a given graph or hypergraph which encodes the structure of the induced subgraphs present. Does there exist an ideal that satisfies the following question:

Open Problem 3. *Given hypergraphs $\mathcal{H}$, and $\mathcal{G}$. Find an ideal I which contains polynomials that have a common solution if and only if the hypergraph $\mathcal{H}$ contains G as an induced subhypergraph.*

The ideals constructed in theorems like Theorem 1.3 and 1.5 have an explicit structure that was designed to fit a specific need. Other ideals such as the Stanley-Reisner ideal for simplicial complices and edge ideals provide a more general interpretation of the structure of a graph or hypergraph. We propose further study to better understand the connection between properties of hypergraphs, the associated ideals, and their corresponding varieties.

Appendix 1

A 5-uniform hypergraph on 17 vertices with 51 edges that is not 2-colorable. Constructed using the generalized Seymour method, also given by Abbott and Hanson, [1].

$$F_5 = \Big\{\{1,2,11,12,13\},\{1,2,11,14,17\},\{1,2,11,15,16\},\{1,2,12,14,16\},$$
$$\{1,2,12,15,17\},\{1,2,13,14,15\},\{1,2,13,16,17\},\{1,3,5,7,10\},$$
$$\{1,3,5,8,9\},\{1,3,6,7,9\},\{1,3,6,8,10\},\{1,4,5,7,9\},$$
$$\{1,4,5,8,10\},\{1,4,6,7,10\},\{1,4,6,8,9\},\{2,3,5,7,9\},$$
$$\{2,3,5,8,10\},\{2,3,6,7,10\},\{2,3,6,8,9\},\{2,4,5,7,10\},$$
$$\{2,4,5,8,9\},\{2,4,6,7,9\},\{2,4,6,8,10\},\{3,4,11,12,13\},$$
$$\{3,4,11,14,17\},\{3,4,11,15,16\},\{3,4,12,14,16\},\{3,4,12,15,17\},$$
$$\{3,4,13,14,15\},\{3,4,13,16,17\},\{5,6,11,12,13\},\{5,6,11,14,17\},$$
$$\{5,6,11,15,16\},\{5,6,12,14,16\},\{5,6,12,15,17\},\{5,6,13,14,15\},$$
$$\{5,6,13,16,17\},\{7,8,11,12,13\},\{7,8,11,14,17\},\{7,8,11,15,16\},$$
$$\{7,8,12,14,16\},\{7,8,12,15,17\},\{7,8,13,14,15\},\{7,8,13,16,17\},$$
$$\{9,10,11,12,13\},\{9,10,11,14,17\},\{9,10,11,15,16\},\{9,10,12,14,16\},$$
$$\{9,10,12,15,17\},\{9,10,13,14,15\},\{9,10,13,16,17\}\Big\}$$

List of References

[1] H. L. Abbott and D. Hanson, "On a combinatorial problem of Erdős," *Canadian Mathematical Bulletin*, vol. 12, pp. 823–829, 1969.

Appendix 2

A 6-uniform hypergraph on 23 vertices with 180 edges that is not 2-colorable. Constructed using the generalized Seymour method.

$$
\begin{aligned}
F_6 = \big\{ &\{1,2,13,14,21,22\}, \{1,2,13,14,21,23\}, \{1,2,13,14,22,23\},\\
&\{1,2,13,15,17,20\}, \{1,2,13,15,18,19\}, \{1,2,13,16,17,19\},\\
&\{1,2,13,16,18,19\}, \{1,2,13,16,18,20\}, \{1,2,14,15,17,19\},\\
&\{1,2,14,15,18,19\}, \{1,2,14,15,18,20\}, \{1,2,14,16,17,19\},\\
&\{1,2,14,16,17,20\}, \{1,2,14,16,18,20\}, \{1,2,15,16,21,22\},\\
&\{1,2,15,16,21,23\}, \{1,2,15,16,22,23\}, \{1,2,17,18,21,22\},\\
&\{1,2,17,18,21,23\}, \{1,2,17,18,22,23\}, \{1,2,19,20,21,22\},\\
&\{1,2,19,20,21,23\}, \{1,2,19,20,22,23\}, \{1,3,5,7,9,12\},\\
&\{1,3,5,7,10,11\}, \{1,3,5,8,9,11\}, \{1,3,5,8,10,11\},\\
&\{1,3,5,8,10,12\}, \{1,3,6,7,9,11\}, \{1,3,6,7,10,11\},\\
&\{1,3,6,7,10,12\}, \{1,3,6,8,9,11\}, \{1,3,6,8,9,12\},\\
&\{1,3,6,8,10,12\}, \{1,4,5,7,9,11\}, \{1,4,5,7,10,12\},\\
&\{1,4,5,8,9,12\}, \{1,4,5,8,10,11\}, \{1,4,6,7,9,12\},\\
&\{1,4,6,7,10,11\}, \{1,4,6,8,9,11\}, \{1,4,6,8,10,11\},\\
&\{1,4,6,8,10,12\}, \{2,3,5,7,9,11\}, \{2,3,5,7,10,11\},\\
&\{2,3,5,7,10,12\}, \{2,3,5,8,9,11\}, \{2,3,5,8,9,12\},\\
&\{2,3,5,8,10,12\}, \{2,3,6,7,9,11\}, \{2,3,6,7,9,12\},\\
&\{2,3,6,7,10,12\}, \{2,3,6,8,9,12\}, \{2,3,6,8,10,11\},
\end{aligned}
$$

$\{2,4,5,7,9,12\},\{2,4,5,7,10,11\},\{2,4,5,8,9,11\},$

$\{2,4,5,8,10,11\},\{2,4,5,8,10,12\},\{2,4,6,7,9,11\},$

$\{2,4,6,7,10,11\},\{2,4,6,7,10,12\},\{2,4,6,8,9,11\},$

$\{2,4,6,8,9,12\},\{2,4,6,8,10,12\},\{3,4,13,14,21,22\},$

$\{3,4,13,14,21,23\},\{3,4,13,14,22,23\},\{3,4,13,15,17,20\},$

$\{3,4,13,15,18,19\},\{3,4,13,16,17,19\},\{3,4,13,16,18,19\},$

$\{3,4,13,16,18,20\},\{3,4,14,15,17,19\},\{3,4,14,15,18,19\},$

$\{3,4,14,15,18,20\},\{3,4,14,16,17,19\},\{3,4,14,16,17,20\},$

$\{3,4,14,16,18,20\},\{3,4,15,16,21,22\},\{3,4,15,16,21,23\},$

$\{3,4,15,16,22,23\},\{3,4,17,18,21,22\},\{3,4,17,18,21,23\},$

$\{3,4,17,18,22,23\},\{3,4,19,20,21,22\},\{3,4,19,20,21,23\},$

$\{3,4,19,20,22,23\},\{5,6,13,14,21,22\},\{5,6,13,14,21,23\},$

$\{5,6,13,14,22,23\},\{5,6,13,15,17,20\},\{5,6,13,15,18,19\},$

$\{5,6,13,16,17,19\},\{5,6,13,16,18,19\},\{5,6,13,16,18,20\},$

$\{5,6,14,15,17,19\},\{5,6,14,15,18,19\},\{5,6,14,15,18,20\},$

$\{5,6,14,16,17,19\},\{5,6,14,16,17,20\},\{5,6,14,16,18,20\},$

$\{5,6,15,16,21,22\},\{5,6,15,16,21,23\},\{5,6,15,16,22,23\},$

$\{5,6,17,18,21,22\},\{5,6,17,18,21,23\},\{5,6,17,18,22,23\},$

$\{5,6,19,20,21,22\},\{5,6,19,20,21,23\},\{5,6,19,20,22,23\},$

$\{7,8,13,14,21,22\},\{7,8,13,14,21,23\},\{7,8,13,14,22,23\},$

$\{7,8,13,15,17,20\},\{7,8,13,15,18,19\},\{7,8,13,16,17,19\},$

$\{7,8,13,16,18,19\},\{7,8,13,16,18,20\},\{7,8,14,15,17,19\},$

$\{7,8,14,15,18,19\},\{7,8,14,15,18,20\},\{7,8,14,16,17,19\},$

$\{7,8,14,16,17,20\},\{7,8,14,16,18,20\},\{7,8,15,16,21,22\},$

$\{7,8,15,16,21,23\},\{7,8,15,16,22,23\},\{7,8,17,18,21,22\},$

$\{7,8,17,18,21,23\},\{7,8,17,18,22,23\},\{7,8,19,20,21,22\},$

$\{7,8,19,20,21,23\},\{7,8,19,20,22,23\},\{9,10,13,14,21,22\},$

$\{9,10,13,14,22,23\},\{9,10,13,15,17,20\},\{9,10,13,15,18,19\},$

$\{9,10,13,16,18,19\},\{9,10,13,16,18,20\},\{9,10,14,15,17,19\},$

$\{9,10,14,15,18,20\},\{9,10,14,16,17,19\},\{9,10,14,16,17,20\},$

$\{9,10,15,16,21,22\},\{9,10,15,16,21,23\},\{9,10,15,16,22,23\},$

$\{9,10,17,18,21,23\},\{9,10,17,18,22,23\},\{9,10,19,20,21,22\},$

$\{9,10,19,20,21,23\},\{9,10,17,18,21,22\},\{9,10,14,16,18,20\},$

$\{9,10,14,15,18,19\},\{9,10,13,16,17,19\},\{9,10,13,14,21,23\},$

$\{9,10,19,20,22,23\},\{11,12,13,14,21,22\},\{11,12,13,14,21,23\},$

$\{11,12,13,15,17,20\},\{11,12,13,15,18,19\},\{11,12,13,16,17,19\},$

$\{11,12,13,16,18,20\},\{11,12,14,15,17,19\},\{11,12,14,15,18,19\},$

$\{11,12,14,16,17,19\},\{11,12,14,16,17,20\},\{11,12,14,16,18,20\},$

$\{11,12,15,16,21,23\},\{11,12,15,16,22,23\},\{11,12,17,18,21,22\},$

$\{11,12,17,18,22,23\},\{11,12,19,20,21,22\},\{11,12,19,20,21,23\},$

$\{11,12,13,14,22,23\},\{11,12,13,16,18,19\},\{11,12,14,15,18,20\},$

$\{11,12,19,20,22,23\},\{11,12,17,18,21,23\},\{11,12,15,16,21,22\}\}.$

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