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21. Torque acting on current loops. Magnetic dipole moment

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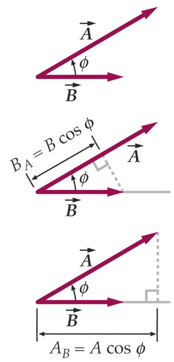
PHY204 Lecture 21 [rln21]

Dot Product Between Vectors

Consider two vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.

- $\vec{A}\cdot\vec{B} = AB\cos\phi = AB_A = BA_B$.
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.
- $\vec{A}\cdot\vec{B} = AB$ if $\vec{A} \parallel \vec{B}$.
- $\vec{A}\cdot\vec{B} = 0$ if $\vec{A} \perp \vec{B}$.
- $$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k}) \\ &\quad + A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k}) \\ &\quad + A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k}).\end{aligned}$$
- Use $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$,
  $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.
- $\Rightarrow\ \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.



tsl229

We begin this lecture with two slides elaborating on two distinct products of vectors: the dot product and the cross product.

The *dot product*, also named *scalar product*, yields a scalar i.e. a number.

The slide shows how we evaluate the dot product when the product vectors are given in components. It also shows how we calculate the dot product if we know the magnitude of each factor vector and the angle between them.

It is useful to recall that the dot product yields a positive number if the angle between the two factor vectors is acute ($< 90°$), a negative number if that angle is obtuse ($> 90°$), and zero if the two vectors are perpendicular to each other (at right angle, $90°$).

The dot product is commutative, meaning that the sequencing of the factors makes no difference.

We have have used the dot product for calculating electric flux $\Phi_E = \int \vec{E}\cdot d\vec{A}$.

The dot product has also made its appearance in the calculation of electric potential from an electric field via a path integral: $V = -\int \vec{E}\cdot d\vec{s}$.

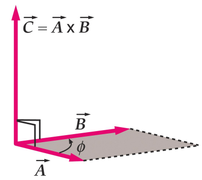
Moreover, we have introduced the electric current through a wire cross section as the flux of electric current density: $I = \int \vec{J}\cdot d\vec{A}$.

Further applications of dot products will be coming soon.

Cross Product Between Vectors

Consider two vectors $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$.

- $\vec{A} \times \vec{B} = AB\sin\phi\,\hat{n}$.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.
- $\vec{A} \times \vec{A} = 0$.
- $\vec{A} \times \vec{B} = AB\,\hat{n}$ if $\vec{A} \perp \vec{B}$.
- $\vec{A} \times \vec{B} = 0$ if $\vec{A} \parallel \vec{B}$.
- $\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$
  $= A_xB_x(\hat{\imath}\times\hat{\imath}) + A_xB_y(\hat{\imath}\times\hat{\jmath}) + A_xB_z(\hat{\imath}\times\hat{k})$
  $+A_yB_x(\hat{\jmath}\times\hat{\imath}) + A_yB_y(\hat{\jmath}\times\hat{\jmath}) + A_yB_z(\hat{\jmath}\times\hat{k})$
  $+A_zB_x(\hat{k}\times\hat{\imath}) + A_zB_y(\hat{k}\times\hat{\jmath}) + A_zB_z(\hat{k}\times\hat{k})$.
- Use $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$,
  $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$.
- $\Rightarrow$ $\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$.



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The *cross product*, also named *vector product*, yields a vector. The slide shows how we evaluate the cross product when the product vectors are given in components.

If we know the magnitude of each factor vector and their orientation in space we can construct the product vector as shown and already explained in lecture 19.

When the two factor vectors are parallel or anti-parallel, then the cross product vanishes. Otherwise, the orientations of the factor vectors span a plane of unique orientation. The product vector is perpendicular to that plane. There are two such orientations. The right-hand rule (see lecture 19) determines which one to pick.

The cross product is non-commutative, meaning that the sequencing of the factors does matter. Interchanging the factor vectors inverts the direction of the product vector.

The cross product features prominently in the expressions of magnetic force, $\vec{F} = q\vec{v} \times \vec{B}$ or $\vec{F} = I\vec{L} \times \vec{B}$, as we have already seen. Further applications are just around the corner (on the next page).

Magnetic Dipole Moment of Current Loop

N: number of turns

I: current through wire

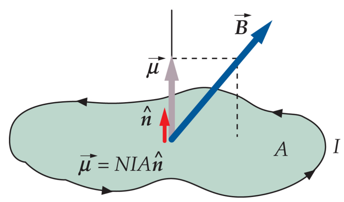
A: area of loop

$\hat{n}$: unit vector perpendicular to plane of loop

$\vec{\mu} = NIA\hat{n}$: magnetic dipole moment

$\vec{B}$: magnetic field

$\vec{\tau} = \vec{\mu} \times \vec{B}$: torque acting on current loop



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A current loop such as the one shown on the slide is a realization of a magnetic dipole. A magnetic dipole is characterized by a vector quantity, named magnetic dipole moment $\vec{\mu}$.

The list on the slide explains how to determine the vector $\vec{\mu}$ from the specifications of a flat loop with arbitrary shape.

The vector $\vec{\mu}$ is perpendicular to the plane of the loop. That leaves two possible directions. A new rendition of the right-hand rule tells us how to pick the right one. Curl the fingers of right hand in the direction of the current around the loop. Then the thumb points in the direction of $\vec{\mu}$.

For loops that are not flat, the construction $\vec{\mu}$ is a bit more involved. An example will be analyzed later.

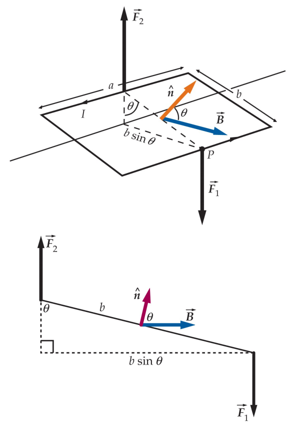
There are magnetic dipoles that are not current loops. For example the electron and the proton are electric dipoles. Attempts to explain the dipole moments of the electron and the proton as manifestations of rotating charge have utterly failed. Even the neutron, which does not carry a net charge, does have a magnetic dipole moment.

When a magnetic dipole moment $\vec{\mu}$ is positioned into a magnetic field $\vec{B}$ it experiences a torque $\tau = \vec{\mu} \times \vec{B}$. The torque is an agent of reorientation just as the force is an agent of relocation.

The torque $\vec{\tau}$ aims to align the vector $\vec{\mu}$ with the vector $\vec{B}$. It is strongest when $\vec{\mu}$ is perpendicular to $\vec{B}$ and vanishes when $\vec{\mu}$ is aligned with $\vec{B}$.

Torque on Current Loop

- magnetic field: $\vec{B}$ (horizontal)
- area of loop: $A = ab$
- unit vector $\perp$ to plane of loop: $\hat{n}$
- right-hand rule: $\hat{n}$ points up.
- forces on sides a: $F = IaB$ (vertical)
- forces on sides b: $F = IbB$ (horizontal, not shown)
- torque: $\tau = Fb \sin\theta = IAB \sin\theta$
- magnetic moment: $\vec{\mu} = IA\hat{n}$
- torque (vector): $\vec{\tau} = \vec{\mu} \times \vec{B}$



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This slide elaborates on the construction of the torque for a rectangular loop from first principles i.e. from forces and lever arms, familiar concepts from mechanics.

A rectangular loop is positioned in a uniform magnetic field as shown from two different perspectives.

The forces on the two sides of length b are opposite in direction and acting along the same line. Therefore, they do not contribute to any torque.

The forces on the two sides of length a are also opposite in direction but acting along different lines as shown. Therefore, they do produce a torque.

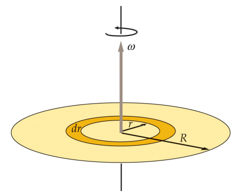
The end result as worked out on the slide agrees with the more general result from the previous slide.

Magnetic Moment of a Rotating Disk

Consider a nonconducting disk of radius R with a uniform surface charge density σ. The disk rotates with angular velocity $\vec{\omega}$.

Calculation of the magnetic moment $\vec{\mu}$:

- Total charge on disk: $Q = \sigma(\pi R^2)$.
- Divide the disk into concentric rings of width dr.
- Period of rotation: $T = \dfrac{2\pi}{\omega}$.
- Current within ring: $dI = \dfrac{dQ}{T} = \sigma(2\pi r dr)\dfrac{\omega}{2\pi} = \sigma\omega r dr$.
- Magnetic moment of ring: $d\mu = dI(\pi r^2) = \pi\sigma\omega r^3 dr$.
- Magnetic moment of disk: $\mu = \displaystyle\int_0^R \pi\sigma\omega r^3 dr = \frac{\pi}{4}\sigma R^4\omega$.
- Vector relation: $\vec{\mu} = \dfrac{\pi}{4}\sigma R^4\vec{\omega} = \dfrac{1}{4}QR^2\vec{\omega}$.



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Here we consider a magnetic dipole that is not a simple current loop, where all current flow is on the perimeter.

We are looking at a spinning disk. The disk is uniformly charged with charge density σ. The rotating charge within concentric rings of radius r and width dr is reinterpreted as the current of a loop.

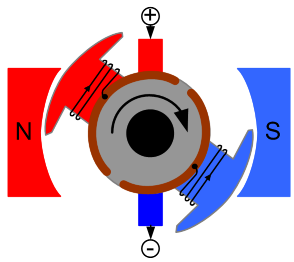
Each loop then produces a magnetic dipole moment $d\mu$ according to the recipe from two slides ago. Summing up the contributions from all rings amounts to an integral across the range $0 < r < R$ of radius.

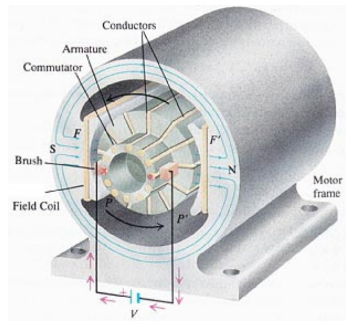
Note that the directions of the angular velocity vector $\vec{\omega}$ and $\vec{\mu}$ are both determined by the right-hand rule:

- Curl the fingers of your right hand in the direction of the spinning motion. Then the thumb points in the direction of $\vec{\omega}$.
- Curl the fingers of you right hand in the direction of the (effective) current. Then the thumb points in the direction of $\vec{\mu}$.

If the charge Q on the disk is positive, then $\vec{\mu}$ has the same direction as $\vec{\omega}$. Making Q negative switches the direction of $\vec{\mu}$ only.

Direct-Current Motor





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How can we convert the torque on a current loop into an agent of continuous rotating motion? On this slide you see two primitive designs for direct-current motors.

In the torque expression, $\tau = \mu B \sin\theta$, the factor $\sin\theta$ changes sign after every half turn. The trick is to undo this change of sign by changing the current direction in the loop after every half turn, which will also change the direction of the magnetic moment vector $\vec{\mu}$ after every half turn.

In both designs this switch of current direction is accomplished by brushes and commutator contacts.

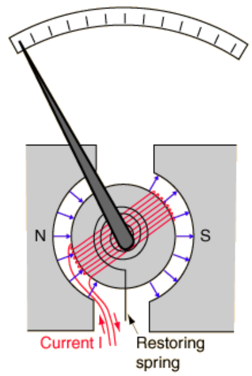
The consequence of this trick is that the torque maintains the same direction. The loop will keep accelerating or keep driving a load in steady rotating motion.

Later in this course, we will see that alternating currents (ac) offer many advantages over direct currents (dc). One advantage of ac motors is that they do not require sliding electrical contacts.

Galvanometer

Measuring direct currents.

- magnetic moment $\vec{\mu}$ (along needle)
- magnetic field $\vec{B}$ (toward right)
- torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ (into plane)



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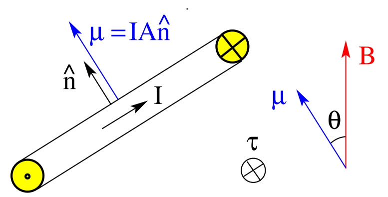
The galvanometer is the prototypical analog device for measuring direct currents. A current loop is mounted inside a magnet such that the plane of the loop can rotate relative to the direction of the magnetic field. The loop is held in a reference orientation by a spring as shown.

When a current I flows through the loop, it produces a magnetic dipole moment $\vec{\mu}$ in the direction of the needle. The magnetic field $\vec{B}$ points to the right ($\rightarrow$). Therefore the torque, $\vec{\tau} = \vec{\mu} \times \vec{B}$ points into the plane ($\otimes$).

Use your right hand and point the thumb in the direction of $\vec{\tau}$. Then your fingers curl clockwise, meaning that the torque aims to reorient the needle in that direction. Its tip on the scale will move to the right.

The restoring elastic force provided by the spring counteracts that angular displacement with a torque in the opposite direction. The needle will settle at a certain angle away from the reference orientation. The angular displacement is proportional to the current I. Once the instrument is calibrated, the current I can be read off the scale.

- Magnetic dipole moment: $\vec{\mu} = IA\hat{n}$
- Torque exerted by magnetic field: $\vec{\tau} = \vec{\mu} \times \vec{B}$
- Potential energy: $U = -\vec{\mu} \cdot \vec{B}$
  $U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = \mu B \int_{\pi/2}^{\theta} \sin\theta d\theta = -\mu B \cos\theta$
  Note: $\tau(\theta)$ and $d\theta$ have opposite sign.



tsl198

On this page and the next, we review some basic attributes of electric and magnetic dipole moments in comparison. The importance of both kinds of dipole moments in all branches of natural science and engineering can hardly be overemphasized.

Here the focus is on the magnetic dipole moment. We use the familiar realization of a flat current loop, here positioned perpendicular to the plane of the page. The current flows out of the page on the left and into the page on the right. The arrow shows the direction at the front of the loop.

The torque aims to align the dipole moment $\vec{\mu}$ with the magnetic field $\vec{B}$. In the configuration shown, the torque $\vec{\tau}$ is directed into the page ($\otimes$).

Rotating the orientation of $\vec{\mu}$ out of alignment with $\vec{B}$ requires work. The differential of rotational work is $\tau d\theta$ in correspondence to Fdx for translational work.

The orientational potential energy U of a magnetic dipole can then be expressed as a dot product between the vectors $\vec{\mu}$ and $\vec{B}$ as shown. It assumes the lowest value when the two vectors are aligned.

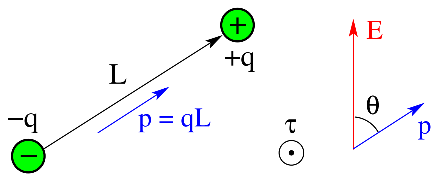
Many materials contain microscopic magnetic dipoles (mostly arising from electrons). An external magnetic field tends to align these dipoles. The material thus becomes magnetized. In ferromagnets the magnetization persists when the external field is turned off.

Electric Dipole in Uniform Electric Field

- Electric dipole moment: $\vec{p} = q\vec{L}$
- Torque exerted by electric field: $\vec{\tau} = \vec{p} \times \vec{E}$
- Potential energy: $U = -\vec{p} \cdot \vec{E}$

  $U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = pE \int_{\pi/2}^{\theta} \sin\theta d\theta = -pE \cos\theta$

  Note: $\tau(\theta)$ and $d\theta$ have opposite sign.



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Here we switch focus to the electric dipole moment. It is an electrically neutral object. Equal amounts of positive and negative charge are slightly displaced from each other.

The simplest realization of an electric dipole, as shown on the slide, are two opposite point charges at a distance L from each other. The direction of the electric dipole moment $\vec{p}$ points toward the positive charge.

The torque expression reads "moment cross field" and the potential-energy expression reads "minus moment dot field" just as in the magnetic case.

Torque aims to align the dipole moment with the field. The aligned state has the lowest potential energy.

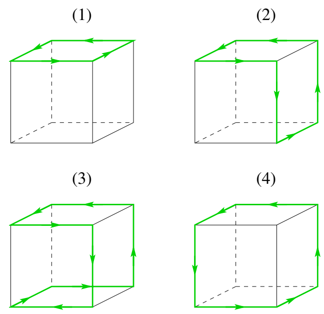
Note that here the cross product $\vec{p} \times \vec{E}$ produces a torque $\vec{\tau}$ pointing out of the page ($\odot$).

Many molecules are permanent electric dipoles, the most prominent case being H_2O as illustrated and explained in an earlier lecture.

External electric fields induce an electric polarization in dielectric materials as discussed earlier in the context of capacitors. On the microscopic level, this means that the (positively charged) nuclei and the surrounding shells of (negatively charged) electrons of atoms are being pulled in opposite directions. In consequence they are slightly displaced by the external electric field. Each atom becomes an induced electric dipole.

Magnetic Force Application (11)

If the magnetic moment of the current loop (1) is $\mu_1 = 1\mathrm{Am}^2$, what are the magnetic moments μ_2, μ_3, μ_4 of the current loops (2), (3), (4), respectively?



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Let us begin by establishing a coordinate system with $\hat{\mathbf{i}}$ pointing toward the front, $\hat{\mathbf{j}}$ pointing toward the right, and $\hat{\mathbf{k}}$ pointing up.

Then the flat square loop (1) has magnetic dipole moment,

$$\vec{\mu}_1 = 1\mathrm{Am}^2\,\hat{\mathbf{k}}.$$

Loop (2), which is not flat, can be thought of as made of two flat square loops. The two currents along the shared side cancel. The magnetic dipole moment is

$$\vec{\mu}_2 = 1\mathrm{Am}^2\,\hat{\mathbf{j}} + 1\mathrm{Am}^2\,\hat{\mathbf{k}},$$

with magnitude $\mu_2 = \sqrt{2}\mathrm{Am}^2$.

In like manner, loop (3) can be interpreted as consisting of three flat square loops with two shared sides of zero current. The magnetic dipole moment is

$$\vec{\mu}_3 = 1\mathrm{Am}^2\,\hat{\mathbf{k}} + 1\mathrm{Am}^2\,\hat{\mathbf{j}} - 1\mathrm{Am}^2\,\hat{\mathbf{k}} = 1\mathrm{Am}^2\,\hat{\mathbf{j}},$$

with magnitude $\mu_3 = 1\mathrm{Am}^2$.

Loop (4), finally, can be interpreted as consisting of three flat square loops with three shared sides of zero current. The magnetic dipole moment is

$$\vec{\mu}_4 = 1\mathrm{Am}^2\,\hat{\mathbf{i}} + 1\mathrm{Am}^2\,\hat{\mathbf{j}} + 1\mathrm{Am}^2\,\hat{\mathbf{k}},$$

with magnitude $\mu_3 = \sqrt{3}\mathrm{Am}^2$.

A loop positioned on an arbitrarily curved surface can thus be divided into a grid of loops small enough to be essentially flat and with shared sides of zero current. The vector sum of the magnetic moments generated by the small loops on the grid then represents the magnetic moment of the loop on the curved surface.

Unit Exam III: Problem #1 (Spring '12)

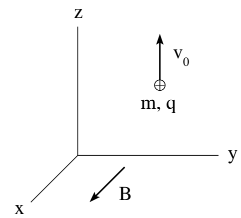
In a region of uniform magnetic field $\mathbf{B} = 5\mathrm{mT}\hat{\mathbf{i}}$, a proton $(m = 1.67 \times 10^{-27}\mathrm{kg},\ q = 1.60 \times 10^{-19}\mathrm{C})$ is launched with velocity $\mathbf{v}_0 = 4000\mathrm{m/s}\hat{\mathbf{k}}$.

(a) Calculate the magnitude F of the magnetic force that keeps the proton on a circular path.

(b) Calculate the radius r of the circular path.

(c) Calculate the time T it takes the proton to go around that circle once.

(d) Sketch the circular path of the proton in the graph.



Solution:

(a) $F = qv_0B = 3.2 \times 10^{-18}\mathrm{N}.$

(b) $\dfrac{mv_0^2}{r} = qv_0B \quad \Rightarrow\ r = \dfrac{mv_0}{qB} = 8.35\mathrm{mm}.$

(c) $T = \dfrac{2\pi r}{v_0} = \dfrac{2\pi m}{qB} = 13.1\mu\mathrm{s}.$

(d) Center of circle to the right of proton's initial position (cw motion).

tsl435

We conclude this lecture with an application of a particle moving in circular motion through a region of uniform magnetic field.

The solution as given on the slide is easy to reproduce. It is a bit more challenging to come up with the solution without peeking at the solution.

The best preparation for an exam (and for a professional life in science and engineering) is to use your mental power, creativity, and imagination as the only resources.

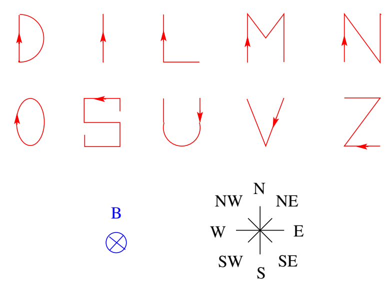
The mental tools that you must be able to recall in this application are the following:

- The structure of magnetic force acting on a charged particle in motion.
- The concept of centripetal force, here provided magnetically.
- The circumference of a circle.
- The right-hand rule.

Magnetic Force Application (6)

An electric current flows through each of the letter-shaped wires in a region of uniform magnetic field pointing into the plane.

- Find the direction of the resultant magnetic force on each letter.



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This is the quiz for lecture 21.

Here we have another application of the vector equation $\vec{F} = I\vec{L} \times \vec{B}$.

Possible answers are either no force or one of the eight compass points indicated.

Hint: Remember what we said about the net force on a current loop in a uniform magnetic field.