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## 21. Relativistic Mechanics II

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### Abstract

Part twenty-one of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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## Contents of this Document [mtc21]

### 21. Relativistic Mechanics II

- Coordinate transformations [mln58]
- Relative and absolute [mln59]
- Lorentz transformation I [mex215]
- Lorentz transformation II [mex216]
- Observing transverse motion of meter stick [mln60]
- Skater's paradox [mln61]
- Skate mail fallacy [mex217]
- Interstellar speed control [mex218]
- Mass and energy [mln62]
- Relativistic momentum [mln63]
- Momentum conservation [mex221]
- Relativistic mass [mex222]
- Relativistic energy I [mln64]
- Relativistic energy II [mln65]
- Photon rocket [mex223]
- Photon absorption and photon emission [mex224]
- K meson decay [mex225]

# Coordinate Transformations [mln58]

**Galilei transformation:**

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

**Lorentz transformation:**

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$

- Check time dilation:  $t'_2 - t'_1 = \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}}$  for  $x_1 = x_2$ .  
Proper time interval:  $\Delta\tau \doteq t_2 - t_1$ .
- Check length contraction:  $x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$  for  $t_1 = t_2$ .  
Proper length:  $\ell_0 \doteq x'_2 - x'_1$ .
- Check relativity of simultaneity:  $t' = t(x) = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$ .  
Clocks synchronized at  $x = x' = 0$ .

**Longitudinal velocity addition:**

Substitute  $x = v_p t$  and  $x' = v'_p t'$  into transformation equations.

- Nonrelativistic:  $v_p = v'_p + v \Rightarrow v'_p = v_p - v$ .
- Relativistic:  $v_p = \frac{v'_p + v}{1 + v'_p v/c^2} \Rightarrow v'_p = \frac{v_p - v}{1 - v_p v/c^2}$ .

**Transverse velocity addition:**

Set  $y' = u'_p t'$  and  $y = u_p t$  and  $x = v_p t$ ,  $x' = v'_p t'$ .  
Then use  $y = y'$  and time dilation.

- Nonrelativistic:  $u'_p = u_p$ .
- Relativistic:  $u'_p = \frac{u_p \sqrt{1 - v^2/c^2}}{1 - v_p v/c^2} \Rightarrow u_p = \frac{u'_p \sqrt{1 - v^2/c^2}}{1 + v'_p v/c^2}$ .

# Relative and Absolute [mln59]

## Lorentz invariant:

Consider frames  $S$  and  $S'$  in relative motion with velocity  $v$ .  
A clock at rest in  $S$  signals a proper time interval  $\Delta\tau$ .

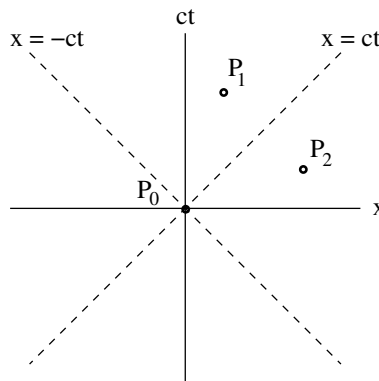
Time interval measured in  $S'$ :  $\Delta t' = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}}$ .

Displacement of clock measured in  $S'$ :  $\Delta x' = -v\Delta t' = -\frac{v\Delta\tau}{\sqrt{1 - v^2/c^2}}$ .

$\Rightarrow (c\Delta t')^2 - (\Delta x')^2 = (c\Delta\tau)^2$  independent of  $v$ .

Invariant quantity:  $(\Delta s)^2 \doteq (c\Delta t)^2 - (\Delta x)^2$ .

## Light cone:



Events  $P_0$  and  $P_1$ :

- time-like relation,
- causally related,
- at the same position in some frame.

Events  $P_0$  and  $P_2$ :

- space-like relation,
- not causally related,
- simultaneous in some frame.

**[mex215] Lorentz transformation I**

Two simultaneous events in frame  $S$  are separated by a distance  $\Delta x = 2\text{m}$ . In frame  $S'$  they are separated by  $\Delta x' = 4\text{m}$ .

- (a) Find the time difference  $c\Delta t'$  between the events in frame  $S'$ .
- (b) Find the relative velocity  $v/c$  between the two frames.
- (c) Sketch a Minkowski diagram for the two events in frames  $S$  and  $S'$ . Then establish the relationship between  $\Delta x$ ,  $\Delta x'$ , and  $c\Delta t'$  by geometric means from the diagram.

**Solution:**

**[mex216] Lorentz transformation II**

Consider the two events with coordinates  $x_1 = 1\text{ly}$ ,  $t_1 = 1\text{y}$  and  $x_2 = 2\text{ly}$ ,  $t_2 = 0.5\text{y}$  in frame  $S$ . (Here “ly” stands for light-years and “y” for years.) The two events are simultaneous in frame  $S'$ .

- (a) Find the relative velocity  $v$  between frames  $S$  and  $S'$ .
- (b) Find the time  $t'_1 = t'_2$ .

**Solution:**

## Observing Transverse Motion of Meter Stick [mln60]

Consider a meter stick aligned in  $x$ -direction of frame  $S$  and moving with velocity  $v_y$ . Frame  $S'$  moves in  $x$ -direction with velocity  $v$  relative to  $S$ . The center of the stick passes the point  $x = x' = 0$ ,  $y = y' = 0$  at  $t = t' = 0$ .

Viewed from  $S$  the two ends of the stick reach  $y = y' = 0$  simultaneously.

Viewed from  $S'$  the right end of the stick goes through  $y = y' = 0$  before the left end does. This is a consequence of the relativity of simultaneity. Hence the stick appears tilted in  $S'$  as shown.

Event 1: right end of stick as it crosses  $x'$ -axis.

$$x'_1 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}} = \frac{(0.5\text{m})}{\sqrt{1 - v^2/c^2}}, \quad y'_1 = 0.$$

$$t'_1 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} = -\frac{(0.5\text{m})v/c^2}{\sqrt{1 - v^2/c^2}}.$$

Event 2: center of stick as it crosses  $x'$ -axis.

$$x'_2 = 0, \quad y'_2 = 0, \quad t'_2 = 0.$$

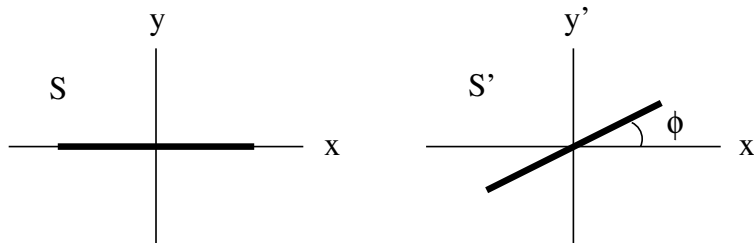
Event 3: right end of stick as center crosses  $x'$ -axis.

$$x'_3 = x'_1 + v'_x(t'_3 - t'_1), \quad y'_3 = y'_1 + v'_y(t'_3 - t'_1), \quad t'_3 = 0.$$

Velocity of stick in  $S'$ :  $v'_x = -v$ ,  $v'_y = v_y\sqrt{1 - v^2/c^2}$ .

$$\Rightarrow x'_3 = (0.5\text{m})\sqrt{1 - v^2/c^2}, \quad y'_3 = (0.5\text{m})\frac{vv_y}{c^2}.$$

Tilt angle:  $\tan \phi = \frac{y'_3}{x'_3} = \frac{vv_y/c^2}{\sqrt{1 - v^2/c^2}}$ .



## Skater's Paradox [mln61]

A skater with blades of proper length  $\ell_0 = 15\text{in}$  on his skates moves with velocity  $v = 0.8c$  relative to a flat ice surface, approaching a gap of proper width  $d_0 = 10\text{in}$ .

### Skater's perspective (frame $S'$ ):

The gap in the ice is Lorentz contracted to a width  $d = d_0\sqrt{1 - (0.8)^2} = 6\text{in}$ , which is shorter than the length  $\ell_0 = 15\text{in}$  of his blades. Therefore, the front end of the blade will gain support on the far side of the gap before the back end loses support on the near side. The skater concludes that he will make it across the gap without accident.

### Spectator's perspective (frame $S$ ):

The blades are Lorentz contracted to length  $\ell = \ell_0\sqrt{1 - (0.8)^2} = 9\text{in}$ , which is shorter than the length  $d_0 = 10\text{in}$  of the gap in the ice. Therefore, the back end of the blade loses support on the near side of the gap before the front end is able to gain support on the far side. The spectator concludes that the skater will not make it across the gap without accident.

### Analysis:

Event 1: Back end of blade enters gap.

Event 2: Front end of blade exits gap.

Check Lorentz invariant quantity  $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$  between events.

- Frame  $S'$ :  $x_1 = 0$ ,  $x_2 = 15\text{in}$ ,  $t_2 = 0$ ,  $t_1 = (9\text{in})/v$ .  
 $\Rightarrow (\Delta s)^2 = (11.25\text{in})^2 - (15\text{in})^2 = -(98.4375\text{in})^2$ .
- Frame  $S$ :  $x_1 = 0$ ,  $x_2 = 10\text{in}$ ,  $t_1 = 0$ ,  $t_2 = (1\text{in})/v$ .  
 $\Rightarrow (\Delta s)^2 = (1.25\text{in})^2 - (10\text{in})^2 = -(98.4375\text{in})^2$ .

### Conclusion:

Events 1 and 2 have a space-like relationship. They have no definite time-ordering as demonstrated. Such events cannot be causally related.

The skater implies a causal relation, which is fallacious. He treats the blade as a rigid body, transporting information across the gap faster than the speed of light. He sinks into the gap and falls.  $\Rightarrow$  [mex217].



**[mex217] Skate mail fallacy**

A skater with blades of proper length  $\ell_0 = 15\text{in}$  on his skates moves with velocity  $v = 0.8c$  relative to a flat ice surface, approaching a gap of (Lorentz contracted) width  $d = 6\text{in}$ . He argues that the front end of the blade (point  $A'$ ) will gain support on the far side of the gap (point  $C$ ) before the back end of the blade (point  $B'$ ) loses support on the near side of the gap (point  $D$ ). Therefore, he concludes, he will make it across the gap without accident.

This conclusion is based on the assumption that the blade can be regarded as a rigid body. Demonstrate the fallacy of this assumption as follows: If that assumption were true, point  $B'$  would know that point  $A'$  has reached safety at point  $C$  before a light signal from  $A'$  to  $B'$  could confirm that message, which is impossible.

**Solution:**

### [mex218] Interstellar speed control

In some science fiction novel it takes space cops to enforce the universal speed limit  $c$ . Every spaceship has a built-in device that sends out signature light signals in all directions when it passes certain markers arrayed in space. The markers are stationary relative to police headquarters.

One inbound spaceship traveling at constant speed  $v$  sends out a signal at marker  $A$  three light-years away from police headquarters and then another signal at marker  $B$  two light-years away. The two signals arrive at headquarters just four months apart, which draws the attention of a young policeman (still in training). Does he have evidence for issuing a speeding ticket? What is the speed of the spaceship relative to the markers?

**Solution:**

# Mass and Energy [mln62]

Consider an autonomous system (internal forces only).

- The postulates of Newtonian mechanics imply separate conservation laws for the total momentum and the total mass.
- The postulates of relativistic mechanics are incompatible with separate conservation laws for mass and momentum.

## Einstein's thought experiment:

Consider a macroscopic rectangular box of mass  $M$  and length  $L$  at rest in the inertial frame of the observer. A light pulse of momentum  $p$  and energy  $E = pc$  is emitted at one end and absorbed at the other end.

According to Newtonian mechanics, energy is transferred from one end to the other end of the box, but no mass is transferred.

The box recoils a distance  $\Delta x$  during the time of flight  $\Delta t$  of the light pulse in accordance with momentum conservation.

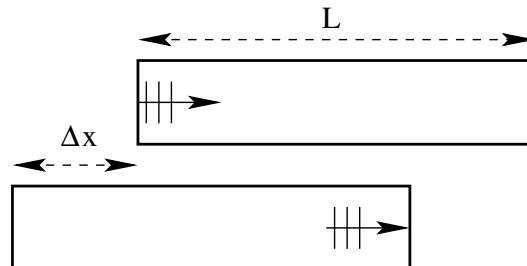
$$Mv = -\frac{E}{c}, \quad \Delta x = v\Delta t, \quad \Delta t = \frac{L}{c} \quad \Rightarrow \quad \Delta x = -\frac{EL}{Mc^2}.$$

Problem: the nonzero displacement  $\Delta x$  of the box is incompatible with zero mass transfer. To reconcile momentum conservation with zero shift of the center of mass of the box, we must attribute a mass  $m$  to the transferred energy:

$$\Delta x_{\text{cm}} = 0 \quad \Rightarrow \quad mL + M\Delta x = 0 \quad \Rightarrow \quad m = \frac{E}{c^2} \quad \Rightarrow \quad E = mc^2.$$

Correct for shortened distance traveled by light and for mass loss of box:

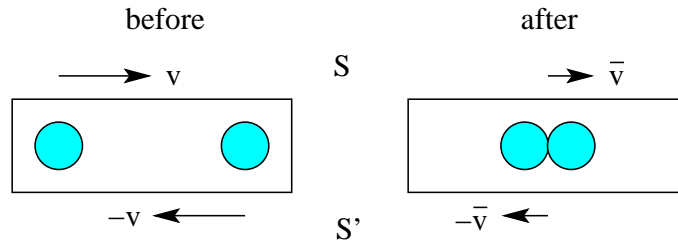
$$m(L + \Delta x) + (M - m)\Delta x = 0 \quad \Rightarrow \quad E = mc^2 \quad (\text{no change!})$$



# Relativistic Momentum [mln63]

Ansatz for relativistic momentum:  $\mathbf{p} = m(v)\mathbf{v}$ .

Two particles with equal masses  $m$  as measured when at rest are undergoing an inelastic collision as shown in the lab frame  $S$  and in the frame  $S'$  moving with velocity  $v$  to the right.



1. Relation between  $v$  and  $\bar{v}$  from [mln58] and symmetry:

$$\bar{v} = \frac{-\bar{v} + v}{1 - \bar{v}v/c^2} \Rightarrow v = \frac{2\bar{v}}{1 + \bar{v}^2/c^2}.$$

2. Conservation of total momentum:

$$m(v)v + m(0)0 = M(\bar{v})\bar{v}.$$

3. Lorentz invariance of momentum conservation implies [mex221]:

$$M(\bar{v}) = m(v) + m(0).$$

Relativistic mass from 1.-3. [mex222]:

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where  $m_0 = m(0)$  is called the *rest mass*.

Relativistic momentum:

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}.$$

### [mex221] Momentum conservation

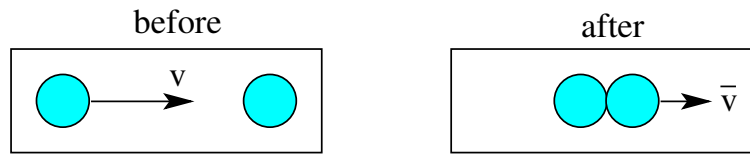
Two particles with equal masses  $m$  as measured when at rest are undergoing an inelastic collision as shown in the lab frame  $S$ . Conservation of total momentum implies

$$m(v)\mathbf{v} + m(0)0 = M(\bar{v})\bar{\mathbf{v}},$$

where  $v = 2\bar{v}/(1 + \bar{v}^2/c^2)$ . Use the requirement that the total momentum is also conserved in frame  $S'$  that moves with relative velocity  $\mathbf{u}$  perpendicular to  $\mathbf{v}$  to infer the relation

$$m(v')\mathbf{u} + m(u)\mathbf{u} = M(\bar{v}')\mathbf{u},$$

where  $v' = \sqrt{v^2 + u^2(1 - v^2/c^2)}$  and  $\bar{v}' = \sqrt{\bar{v}^2 + u^2(1 - \bar{v}^2/c^2)}$ . In the limit  $u \rightarrow 0$ , this becomes a relativistic relation between individual masses and composite mass.



**Solution:**

### [mex222] Relativistic mass

Two particles with equal masses  $m$  as measured when at rest are undergoing an inelastic collision as shown in the lab frame  $S$ . From the conservation of total momentum in frame  $S$ ,

$$m(v)\mathbf{v} + m(0)0 = M(\bar{v})\bar{\mathbf{v}}, \quad v = 2\bar{v}/(1 + \bar{v}^2/c^2),$$

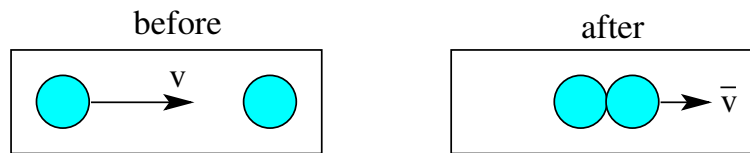
and the conservation of transverse total momentum in frame  $S'$  that moves with relative velocity  $\mathbf{u}$  perpendicular to  $\mathbf{v}$  (as shown in [mex221]),

$$m(v')\mathbf{u} + m(u)\mathbf{u} = M(\bar{v}')\mathbf{u}, \quad v' = \sqrt{v^2 + u^2(1 - v^2/c^2)}, \quad \bar{v}' = \sqrt{\bar{v}^2 + u^2(1 - \bar{v}^2/c^2)},$$

derive the expression

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

for the relativistic mass, where  $m_0 = m(0)$  is called the rest mass.



**Solution:**

# Relativistic Energy I [mln64]

The two colliding particles of equal mass viewed from the frame in which the total momentum is zero.



Relativistic mass before and after the collision  
(inferred from momentum conservation):

$$M = m(\vec{v}) + m(-\vec{v}) = 2m(\vec{v}) = \frac{2m_0}{\sqrt{1 - \vec{v}^2/c^2}}.$$

Increase in rest mass (after collision):

$$\Delta M = M - 2m_0 = 2m_0 \left( \frac{1}{\sqrt{1 - \vec{v}^2/c^2}} - 1 \right) \simeq \frac{m_0 \vec{v}^2}{c^2}.$$

Relativistic energy (in general):

$$E \doteq m(v)c^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}.$$

Conservation of relativistic energy (in collision):

$$\Delta E = Mc^2 - 2m(\vec{v})c^2 = 0.$$

Relativistic kinetic energy (in general):

$$T \doteq E - m_0 c^2 = m_0 c^2 \left( \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \simeq \frac{1}{2} m_0 v^2.$$

Kinetic energy converted into thermal energy (during collision):

$$\Delta Q = -\Delta T = \Delta M c^2 \simeq 2 \left( \frac{1}{2} m_0 \vec{v}^2 \right).$$

# Relativistic Energy II [mln65]

Relativistic adaptation of Newton's equation of motion:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad \mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}}.$$

Conservative force:  $\mathbf{F} = -\nabla U$ .

Work and potential energy:  $W_{12} = \int_1^2 d\mathbf{r} \cdot \mathbf{F} = -(U_2 - U_1)$ .

Work and relativistic energy:

$$W_{12} = \int_1^2 dt \mathbf{v} \cdot \frac{d}{dt} \left( \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) = \int_1^2 dt \frac{d}{dt} \left( \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \right) = T_2 - T_1.$$

Energy conservation:  $T_1 + U_1 = T_2 + U_2$ .

Space-time four-vector:  $x_\mu \doteq (ct, x_1, x_2, x_3)$ .

Energy-momentum four-vector:  $p_\mu \doteq (E/c, p_1, p_2, p_3)$ .

Lorentz transformation:

$$x'_1 = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad t' = \frac{t - (v/c^2)x_1}{\sqrt{1 - v^2/c^2}}.$$
$$p'_1 = \frac{p_1 - (v/c^2)E}{\sqrt{1 - v^2/c^2}}, \quad p'_2 = p_2, \quad p'_3 = p_3, \quad E' = \frac{E - vp_1}{\sqrt{1 - v^2/c^2}}.$$

Transformation of radiant energy ( $p_1 = E/c$ ):  $E' = E \sqrt{\frac{1 - v/c}{1 + v/c}}$ .

Invariant quantity:  $E^2/c^2 - \mathbf{p}^2 = m_0^2 c^2$ .

Relativistic energy-momentum relation:  $E = \sqrt{\mathbf{p}^2 c^2 + m_0^2 c^4}$ .

- Nonrelativistic limit:  $E \simeq m_0 c^2 + \frac{\mathbf{p}^2}{2m_0}$ .
- Ultrarelativistic limit:  $E \simeq pc$ .



### [mex223] Photon rocket

When a conventional rocket is launched from rest in a force-free environment, it acquires the speed  $v_c = u \ln(m_i/m_f)$ , where  $m_i$  is the initial mass,  $m_f$  the final mass, and  $u$  the speed of the exhaust gases relative to the rocket (see [mex17]).

Now calculate the speed  $v_p$  which a rocket acquires if the difference between its initial rest mass  $m_i$  and its final rest mass  $m_f$  is converted into radiant energy.

**Solution:**

**[mex224] Photon absorption and photon emission**

- (a) A photon of energy  $E$  is absorbed by a stationary atom of rest mass  $m_0$  in a force-free environment. By what amount  $\Delta m_0$  has the rest mass of the atom changed when it is in the excited state?
- (b) A photon of energy  $E'$  is emitted by a stationary atom of rest mass  $m'_0$  in a force-free environment. By what amount  $\Delta m'_0$  has the mass of the atom changed when it is back in the ground state?
- (c) Express the energy of each photon in units of  $m_0 c^2$  as a function of  $\Delta m_0/m_0$  expanded two second order.

**Solution:**

**[mex225]  $K$  meson decay**

A  $K$  meson (rest energy 494 MeV) decays into two  $\pi$  mesons (rest energy 137 MeV). In the frame where one of the  $\pi$  mesons is at rest find (a) the energy of the other  $\pi$  meson and (b) the energy of the original  $K$  meson.

**Solution:**