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20. Relativistic Mechanics I

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Abstract
Part twenty of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc20]

20. Relativistic Mechanics I

Relativistic versus Newtonian Mechanics [mln49]

Common requirement:
The laws of mechanics are universal in inertial systems.

Newtonian mechanics

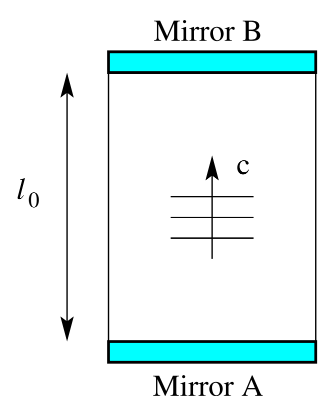
- Instantaneous signals are available.
- Rigid bodies (ruler, compass) exist to map out absolute (3D) Euclidean space.
- Universal clocks exist to measure absolute (1D) time.
- Clocks can be synchronized between any two points within the same inertial system and between two points anywhere in different inertial systems.

Relativistic mechanics

- The speed of light c is universal.
- The speed of any signal cannot exceed c.
- Absolute entities exist only in (4D) space-time. Projections onto (3D) space and (1D) time are relative.
- Instruments exist to measure proper times, lengths, and angles within any inertial system,
- Clocks can be synchronized between any two points within the same inertial system but only locally between points in relative motion.

Einstein's signature clock
uses a light signal
bouncing between flat mirrors.

Time period of cycle
(in rest frame of clock):
$\Delta\tau = 2\ell_0/c$.



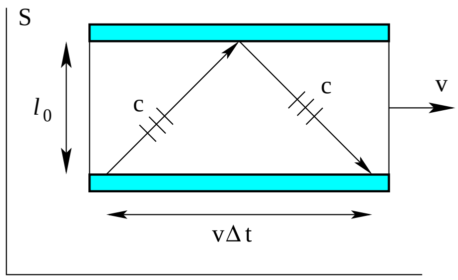
Relativity of Space and Time [mln50]

Frame S' moves with velocity v relative to frame S.
Clock is at rest in frame S'.
Distance traveled by signal in frame S': $2\ell_0$ (proper length ℓ_0).
Time period measured in frame S': $\Delta\tau = 2\ell_0/c$ (proper time).

Time dilation

Distance traveled by signal in S: $2\sqrt{\ell_0^2 + (v\Delta t/2)^2} = c\Delta t$.

Time period measured in S: $\Delta t = \dfrac{2\ell_0/c}{\sqrt{1-v^2/c^2}} = \dfrac{\Delta\tau}{\sqrt{1-v^2/c^2}}$.



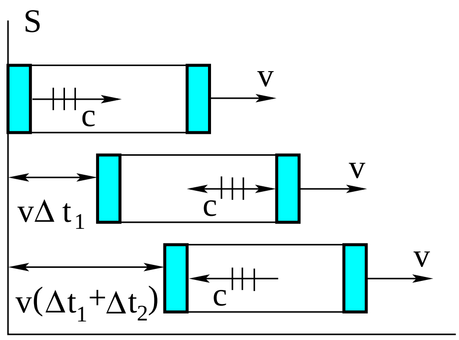
Length contraction

Distance traveled by signal in S: $c(\Delta t_1 + \Delta t_2) = (\ell + v\Delta t_1) + (\ell - v\Delta t_2)$.

Time period measured in S: $\Delta t = \Delta t_1 + \Delta t_2 = \dfrac{\ell}{c-v} + \dfrac{\ell}{c+v} = \dfrac{2\ell/c}{1-v^2/c^2}$.

Comparison with proper time and length: $\Delta t = \dfrac{\Delta\tau}{\sqrt{1-v^2/c^2}} = \dfrac{2\ell_0/c}{\sqrt{1-v^2/c^2}}$.

Length contraction: $\ell = \ell_0\sqrt{1-v^2/c^2}$.



Relativity of Simultaneity [mln51]

Light signal sent out to synchronized, equidistant clocks in frame S'.

Arrival times observed in S': $ct'_1 = ct'_2 = \ell_0/2 \quad \Rightarrow \; \Delta t' = 0$.

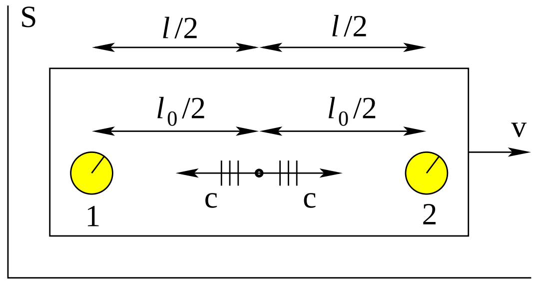
Arrival times observed in S: $ct_1 = \ell/2 - vt_1, \quad ct_2 = \ell/2 + vt_2$.

$$\Rightarrow \; \Delta t = t_2 - t_1 = \frac{\ell/2}{c-v} - \frac{\ell/2}{c+v} = \frac{\ell v/c^2}{1 - v^2/c^2} = \frac{\ell_0 v/c^2}{\sqrt{1-v^2/c^2}}.$$

Time difference translated from S to S': $\Delta t' = \Delta t\sqrt{1 - v^2/c^2} = \dfrac{\ell_0 v}{c^2}$.

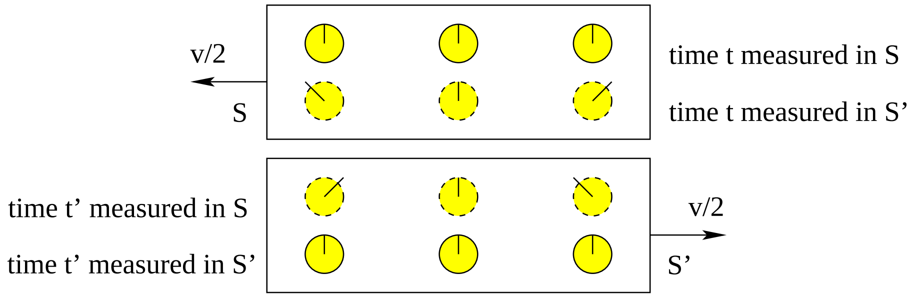
Result in conflict with actual observation $\Delta t' = 0$ made in S'.

Resolution of paradox:
When viewed from S, clock 1 was set ahead of clock 2 by $\ell_o v/c^2$.



Consider two arrays of synchronized clocks in relative motion as shown.

When an observer in frame S asks about the time t' in frame S', the answer depends on the position x'... and vice versa.



Time Dilation Paradox [mln52]

Consider two reference frames S and S' in relative motion. An observer in S determines that time in S' is slowed, whereas an observer in S' determines that time in S is slowed. How can both observers be right?

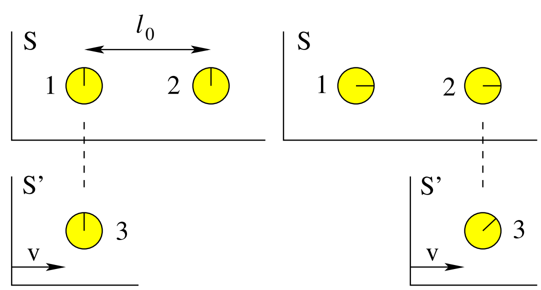
Clocks 1 and 2 are synchronized in S for all times.
Clocks 1 and 3 are synchronized at $t = 0$.

View from frame S:

Proper distance between clocks 1 and 2: ℓ_o.

Reading of clock 2 when clock 3 arrives there: $t = \ell_0/v$.

Time elapsed in S': $t' = (\ell_0/v)\sqrt{1 - v^2/c^2}$.



View from frame S':

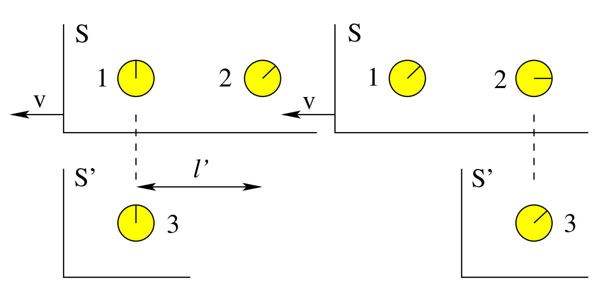
Distance between clocks 1 and 2: $\ell' = \ell_0\sqrt{1 - v^2/c^2}$.

Reading of clock 3 when it reaches clock 2: $t' = \ell'/v = (\ell_0/v)\sqrt{1 - v^2/c^2}$.

Time elapsed in S: $\Delta t = t'\sqrt{1 - v^2/c^2} = (\ell_0/v)(1 - v^2/c^2)$.

Initial reading of clock 2 as viewed from S': $t_i = \ell_0 v/c^2$.

Final reading of clock 2: $t_i + \Delta t = \ell_0/v$.



Length Contraction Paradox [mln53]

Consider two frames S and S' in relative motion with velocity v. Jack is at rest in S and Jill is at rest in S'.

Jack is holding a rod of proper length ℓ_0 in his hand. When Jill measures the length of Jack's rod on the move, how do we reconcile her result with that of Jack's own measurement?

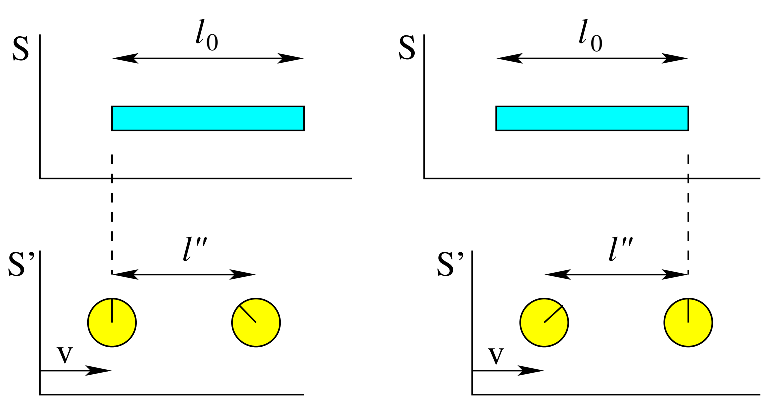
As the two pass by each other, Jill records her measurement by holding her hands a (contracted) distance $\ell' = \ell_0\sqrt{1-v^2/c^2}$ apart. Jack, in turn, measures the distance between Jill's hands, $\ell'' = \ell'\sqrt{1-v^2/c^2}$, which is further contracted and thus falls short of ℓ_0.

What is missing?
Relativity of simultaneity must be taken into account!

Let Jill mark her measurement by two synchronized clocks coinciding with the endpoints of Jack's rod at time t'_m. From Jill's perspective, the distance between the clocks is $\ell' = \ell_0\sqrt{1-v^2/c^2}$.

From Jack's perspective, the distance between the clocks is $\ell'' = \ell_0(1-v^2/c^2)$ and the clocks are out of sync by $\Delta t' = \ell' v/c^2$ in S'. This time lag is dilated to $\Delta t = \Delta t'/\sqrt{1-v^2/c^2} = \ell_0 v/c^2$ in S.

To determine the length of his own rod via Jill's measurement, Jack must mark the positions of Jill's clocks at instances when they record the same time as seen from S. The result, $\ell'' + v\Delta t = \ell_0$, is in agreement with the proper length of his rod.



[mex207] Hello Earth

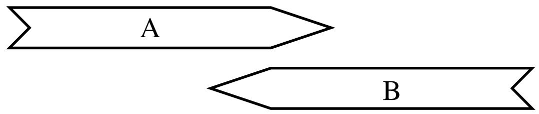
When a spaceship (frame S') passes Earth (frame S) at relative velocity $v = 0.6c$ (event 1), clocks are synchronized: $t_1 = t'_1 = 0$. At time $t_2 = 10\text{min}$ a light pulse is emitted from Earth toward the spaceship (event 2). At time t'_3 the light pulse is detected on the spaceship (event 3).
(a) Identify the proper time intervals among Δt_{12}, Δt_{13}, Δt_{23}, $\Delta t'_{12}$, $\Delta t'_{13}$, $\Delta t'_{23}$.
(b) Find the time of event 2 as recorded on the spaceship.
(c) Find the distance between Earth and spaceship at event 2 as seen in both frames.
(d) Find the time interval between events 2 and 3 as recorded on Earth and on the spaceship.
(e) Find the time of event 3 as recorded on Earth and on the spaceship.

Solution:

[mex208] Who passes more quickly?

Spaceships A and B, each having proper length $\ell_0 = 100\text{m}$, pass each other moving in opposite direction with relative velocity of $v_r = 7 \times 10^7\text{m/s}$. Each spaceship has synchronized clocks at both ends, front and rear.

According to clocks in spaceship B (A) the time it takes the front end of spaceship A to pass the entire length of spaceship B is t_1^B (t_1^A) and the time it takes the entire length of spaceship A to pass the front end of spaceship B is t_2^B (t_2^A). Determine these four times.

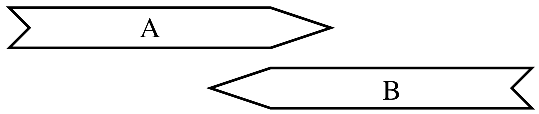


Solution:

[mex236] Time on the fly

Spaceships A and B, each having proper length $\ell_0 = 100\text{m}$, pass each other moving in opposite direction with relative velocity of $v_r = 7 \times 10^7\text{m/s}$. Each spaceship has synchronized clocks at both ends, front and rear.

The clocks at the front end of spaceship A and at the rear end of spaceship B happen to strike noon simultaneously, $t_{AF} = t_{BR} = 12:00:00.00000000$, when they are opposite one another. What are the readings t_{AR} and t_{BF} of the clocks at the rear end of spaceship and at the front end of the spaceship B, respectively, when they are opposite one another?



Solution:

[mex209] Pion decay in accelerator

Pions in their rest frame S' decay according to the empirical law,

$$\frac{N(t')}{N_0} = 2^{-t'/T},$$

where $T \simeq 1.8 \times 10^{-8}$s is the half-life for these particles. Researchers at Fermilab create high-energy pulses of pions and observe that two thirds of these particles reach a detector at a distance $\ell = 35$m (in the lab frame S) from the point where they were created (target).
(a) Find the velocity v of the pions in units of c.
(b) Find the distance ℓ' between target and detector in the rest frame of the particles.

Solution:

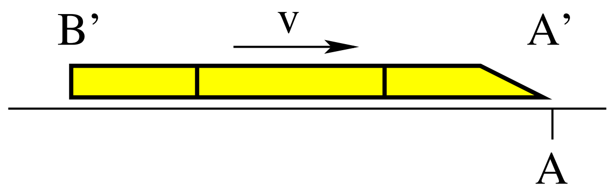
[mex210] Interstellar travel

The distance from Earth to a star is 12 light-years. Their relative velocity is negligible.
(a) How fast (in units of c) must a spaceship travel so that it reaches the star in 12 years according to the clocks on board?
(b) How long will the trip take according to the clocks on Earth?
(c) How many light-years is the distance between Earth and the star from the perspective of the cruising spaceship?

Solution:

[mex211] TGV

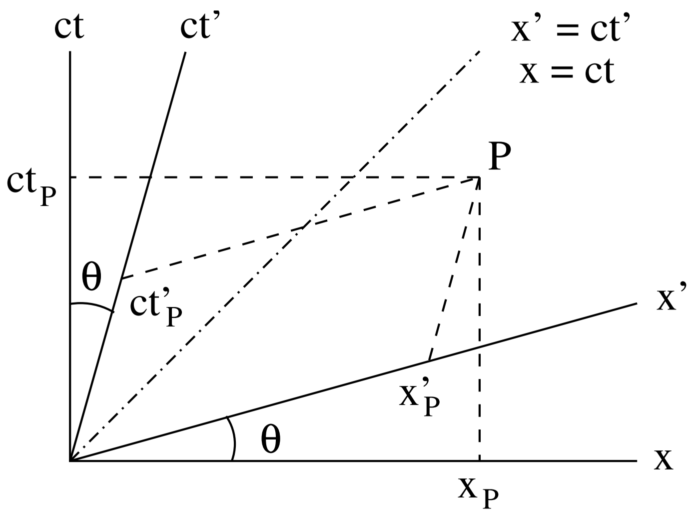
A French high-speed train of proper length ℓ_0 travels at constant velocity v relative to the ground. The front end A' of the train passes point A on the ground at $t = t' = 0$, when a light signal is sent from A' to the tail end B' of the train.
(a) Find the times t_1 and t'_1 when the signal reaches B' as measured in the two frames.
(b) Find the times t_2 and t'_2 when the point B' passes point A as measured in the two frames.



Solution:

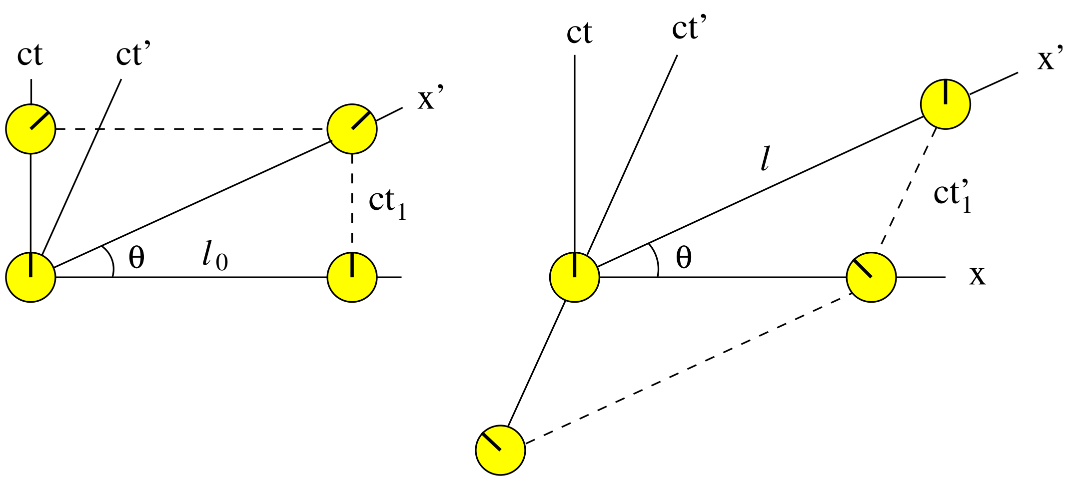
Minkowski Diagram I [mln54]

Consider two inertial systems moving with relative velocity v in x-direction. Clocks are synchronized at $t = t' = 0$ and $x = x' = 0$.



The axes of the moving frame are tilted by $\theta = \arctan(v/c)$. The (dot-dashed) world line of light is the same in both frames.

Relativity of simultaneity:



Left: synchronized clocks at $t = 0$ and $t = t_1 = \ell_0 v/c^2$.

Right: synchronized clocks at $t' = 0$ and $t' = t'_1 = -\ell v/c^2$.

Note: the moving clock that is spatially ahead lags in time.

Minkowski Diagram II [mln55]

Minkowski diagrams do not preserve angles and scales. Units on the primed and unprimed axes are related by the following scale factor:

$$\frac{S'}{S} = \sqrt{\left(1 + \frac{v^2}{c^2}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)}.$$

In the illustration below we use

$$\frac{v}{c} = 0.6 \quad \Rightarrow \sqrt{1 - v^2/c^2} = 0.8.$$

Length contraction:

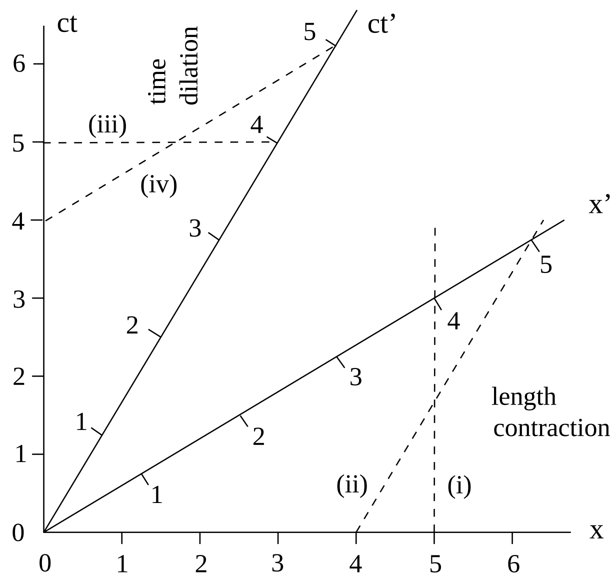
Moving rod viewed from S: $\ell = \ell' \sqrt{1 - v^2/c^2}$ [line (ii)].

Moving rod viewed from S': $\ell' = \ell \sqrt{1 - v^2/c^2}$ [line (i)].

Time dilation:

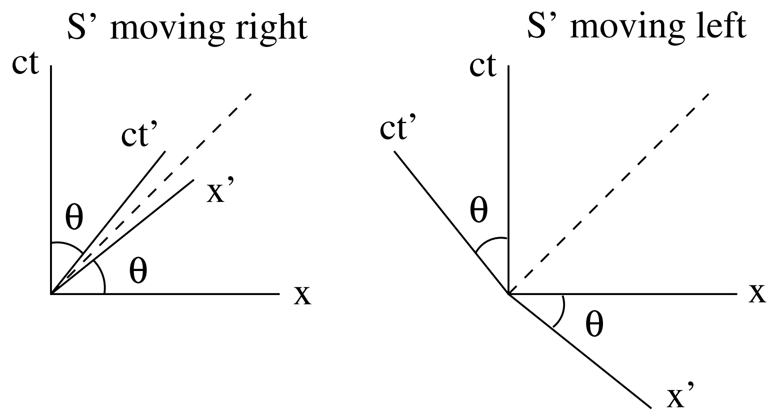
Moving clock viewed from S: $\Delta t = \dfrac{\Delta t'}{\sqrt{1 - v^2/c^2}}$ [line (iii)].

Moving clock viewed from S': $\Delta t' = \dfrac{\Delta t}{\sqrt{1 - v^2/c^2}}$ [line (iv)].



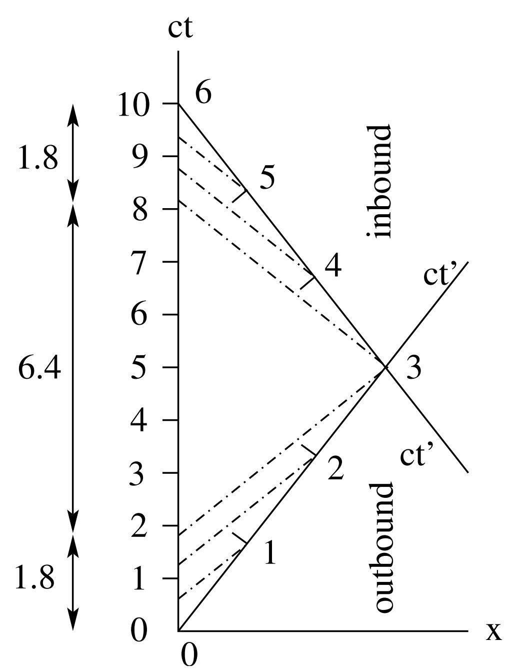
Twin Paradox [mln56]

Jack and Jill are twins. They synchronize their watches (and calendars) at $t = t' = 0$. Then Jill travels into space at $v = 0.8c$ for some time, turns around, and returns at the same speed.



Jack's perspective:

- While Jill is outbound
  she ages 3 years
  and he ages 5 years.
- Jill turns around quickly.
- While Jill is inbound
  she ages 3 years
  and he ages 5 years.



Jill's perspective:

- While outbound
  she ages 3 years
  and Jack ages 1.8 years.
- As she turns around quickly
  Jack ages 6.4 years.
- While inbound
  she ages 3 years
  and Jack ages 1.8 years.

Longitudinal Doppler Effect [mln57]

Sound:

Transmitter T and receiver R moving with relative velocity v toward each other. Sound wave propagates with velocity v_s through medium.
Transmitter emits $N_T = \nu_T \Delta t$ cycles at frequency ν_T in time Δt.

(a) Receiver at rest in medium:

Advance toward R of wave in time Δt: $N_T \lambda = (v_s - v)\Delta t$.
Frequency received: $\nu_R = \dfrac{v_s}{\lambda} = \dfrac{N_T}{\Delta t}\dfrac{v_s}{v_s - v} = \dfrac{\nu_T}{1 - v/v_s}$.

(b) Transmitter at rest in medium:

Distance travelled by wave relative to receiver in Δt: $(v_s + v)\Delta t$.
Distance travelled by wave relative to transmitter in Δt: $v_s \Delta t$.
Receiver detects $N_R = \nu_R \Delta t$ cycles at frequency ν_R in time Δt.

Relation between N_R and N_T: $N_R = N_T\,\dfrac{v_s + v}{v_s}$.
Frequency received: $\nu_R = \dfrac{N_R}{\Delta t} = \dfrac{N_T}{\Delta t}\dfrac{v_s + v}{v_s} = \nu_T(1 + v/v_s)$.

Light:

Transmitter T and receiver R moving with relative velocity v toward each other. Light wave propagates with velocity c relative to T and relative to R.

Transmitter emits $N_T = \nu_T \Delta t_T$ cycles at frequency ν_T in time Δt_T.
Advance toward R of wave in time Δt_R: $N_T \lambda = (c - v)\Delta t_R$.
Time dilation: $\Delta t_R = \Delta t_T / \sqrt{1 - v^2/c^2}$.
Frequency received: $\nu_R = \dfrac{c}{\lambda} = \nu_T \sqrt{\dfrac{1 + v/c}{1 - v/c}}$.

[mex212] Optical birthday cards

Jack and Jill are twins. They synchronize their watches (and calendars) to $t = t' = 0$ at noon on their 20th birthday. Then Jill travels into space at $v = 0.8c$, turns around, and returns at the same speed. When they are reuinited Jack has aged 10 years and Jill 6 years as analyzed in [mln56].
While they are separated, each twin sends a light signal at noon on their birthday according to the local calendar. When Jack is about to send his 10th card and Jill her 6th, they find themselves reunited.
(a) Use the Doppler effect to determine the dates (measured in units of local years) when each birthday card arrives.
(b) Use the Minkowski diagram of [mln56] to draw the world lines traced by each birthday card in confirmation of the result obtained via the Doppler effect.

Solution:

[mex213] Two views of an event

Inertial frame S' is moving with velocity $v = +0.5c$ relative to inertial frame S. Consider the events P at $x = 1.8\text{m}$, $ct = 1.4\text{m}$ and event Q at $x' = 1.0\text{m}$, $ct' = 2.0\text{m}$. Draw a Minkowski diagram to scale on graph paper and determine the coordinates of both event in the other frame by graphical construction.

Solution:

[mex214] Hello Earth again

When a spaceship (frame S') passes Earth (frame S) at relative velocity $v = 0.6c$ (event 1), clocks are synchronized: $t_1 = t'_1 = 0$. At time $t_2 = 10\text{min}$ a light pulse is emitted from Earth toward the spaceship (event 2). At time t'_3 the light pulse is detected on the spaceship (event 3).
(a) Draw a Minkowski diagram with axes (x, t) and (x', t') to scale on graph paper (with time measured in minutes and distance in light-minutes). Then locate the events 1, 2, 3 in the diagram.
(b) Determine the coordinates of all three event in both frames by graphical construction. compare the results with those found in [mex207].

Solution: